Feedback

On the Teaching Note: The integral of $1/x$: Douglas W. Mitchell writes: Glyn George has given intuition as to why $\ln x + C$ 'fills the gap' at $n = -1$ in the formula for $\int x^n dx$, which for all other n equals $x^{n+1}/(n+1) + C$, by noting via a Maclaurin expansion that $\int_1^b t^n dt = \begin{cases} (b^{n+1} - 1)/(n+1), & n \neq -1 \\ \ln b, & n = -1 \end{cases}$

and by showing the similarity of the plots of

$$y(x) = \begin{cases} (x^{n+1} - 1)/(n+1), & n \neq -1 \\ \ln x, & n = -1 \end{cases}$$

for $n = -1.2$, -1 and -0.8. I would like to make two observations about this gap.

First, while the three plots look very similar in the limited range for x shown, in fact there is a discontinuity in the dynamics of the plot as n goes through $n = -1$. For $n < -1$, the graph is bounded above (at $-1/(n+1) > 0$) as $x \to \infty$ but unbounded below as $x \to 0^+$. At $n = -1$ the graph is unbounded both above and below in $(0, \infty)$. Then for $n > -1$ a *lower* bound (at $-1/(n+1) < 0$) appears at $x = 0$ but there is no *upper* bound as $x \to \infty$. So the doubly unbounded $\ln x$ marks the transition between having just an upper bound and having just a lower bound.

Second, this gap-filling can be viewed from a different perspective. The above function $y(x)$ with $n < 0$ is widely used by financial and other economists, in a context of choice under uncertainty about x, as a utility (preference) function whose expected value is to be maximized. The degree of 'relative risk aversion' exhibited by any utility function is given by $RRA(x) = \dfrac{-xy''(x)}{y'(x)}$, which is one measure of curvature. For $y(x)$ as above, when $n \neq -1$, we have $RRA(x) = -n > 0$. This ranges from ∞ to 1^+ as n ranges through $(-\infty, -1)$ and from 1^- to 0^+ as n ranges through $(-1,0)$, thus giving a 'gap' at $n = -1$, corresponding to $RRA(x) = 1$. For $n = -1$, since $\dfrac{d \ln x}{dx} = x^{-1}$ (equivalently, $\int x^{-1} dx = \ln x + C$), $\ln x$ gives the gap-filling result $RRA(x) = 1$.

On 95.08: Nick Lord writes: I enjoyed Michael Hirschhorn's use of algebraic identities to prove some particular cases of the AM-GM inequality, (1) to (4) on page 83. Clearing fractions and writing $x = \frac{b}{a}$, the general case amounts to the identity

$$(n - 1 + x)^n - n^n x = (x - 1)^2 \left(a_0 + a_1 x + \ldots + a_{n-2} x^{n-2}\right)$$

where $a_0 = (n-1)^n > a_1 > \ldots > a_{n-2} = 1$ are all positive integers. This can be shown as follows. If $f(x) = (n - 1 + x)^n - n^n x$, then $f(1) = 0$ and $f'(1) = 0$ so that $(x - 1)^2$ is a factor of $f(x)$ and $a_0 = (n-1)^n$, $a_{n-2} = 1$

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from the coefficients of x^0 and x^n in $f(x)$. Equating coefficients in the expansion of $(1-x)^{-2}\lfloor(n-1+x)^n - n^n x\rfloor = a_0 + a_1 x + \ldots + a_{n-2}x^{n-2}$ gives $a_r = \sum_{k=0}^{r}\binom{n}{k}(n-1)^{n-k}(r+1-k) - n^n r$ so that

$$a_r > \sum_{k=0}^{n}\binom{n}{k}(n-1)^{n-k}(r+1-k) - n^n r = (r+1)n^n - n^n - n^n r = 0,$$

since

$$\sum_{k=0}^{n}\binom{n}{k}(n-1)^{n-k} = n^n = \sum_{k=0}^{n}\binom{n}{k}(n-1)^{n-k}k.$$

Also,

$$a_r - a_{r+1} = n^n - \sum_{k=0}^{r+1}\binom{n}{k}(n-1)^{n-k} = \sum_{k=r+2}^{n}\binom{n}{k}(n-1)^{n-k} > 0$$

as claimed.

(An amusing consequence of $a_1 = 2(n-1)^n + n(n-1)^{n-1} - n^n > 0$ is that $2\left(1-\frac{1}{n}\right)^n + \left(1-\frac{1}{n}\right)^{n-1} - 1 > 0$: letting $n \to \infty$ shows that $\frac{3}{e} \geqslant 1$ or $e \leqslant 3$!)

A similar proof shows that

$$(n-r+rx)^n - n^n x^r = (x-1)^2\left(c_0 + c_1 x + \ldots + c_{n-2}x^{n-2}\right)$$

with the c_i positive integers: this corresponds to the identity

$$a^{n-r}b^r = \left[\frac{(n-r)a + rb}{n}\right]^n - \frac{1}{n^n}\left(c_0 a^{n-2} + c_1 a^{n-3}b + \ldots + c_{n-2}b^{n-2}\right).$$

This provides an identity-based proof of the AM-GM inequality for the specific case where there are at most two different values of the variables. It is worth noting that Hurwitz has given a beautiful and somewhat analogous identity-based proof of the general AM-GM inequality, [1, pp. 8-9].

Reference

1. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer (1983).

On 95.09: Nick Lord writes: I greatly enjoyed the author's unusual proof of the AM-GM inequality for 3 variables from that for 2 variables, but it is worth noting that the step from 2 to 3 variables may be replicated to provide the inductive step from $n-1$ to n variables and thus prove the general AM-GM inequality. Since the steps match so closely those in the article, I shall omit some of the detail below. For $a_1, \ldots, a_n > 0$

$$\frac{a_1 + a_2 + \ldots + a_n}{n}$$

$$= \frac{\frac{1}{n-1}(a_1 + \ldots + a_{n-1}) + \frac{1}{n-1}(a_2 + \ldots + a_n) + \ldots + \frac{1}{n-1}(a_n + a_1 + \ldots + a_{n-2})}{n}$$

$$\geqslant \frac{(a_1 \ldots a_{n-1})^{1/(n-1)} + (a_2 \ldots a_n)^{1/(n-1)} + \ldots + (a_n a_1 \ldots a_{n-2})^{1/(n-1)}}{n}$$

(by the inductive hypothesis)

$$\geqslant (a_1 \ldots a_n)^{(n-2)/(n-1)^2} \left(\frac{a_1^{1/(n-1)^2} + \ldots + a_n^{1/(n-1)^2}}{n} \right),$$

on repeating the same step as above.

Iterating the argument gives, on letting the number of iterations tend to infinity,

$$\frac{a_1 + \ldots + a_n}{n} \geqslant (a_1 \ldots a_n)^{\frac{(n-2)}{(n-1)^2}\left[1 + \frac{1}{(n-1)^2} + \frac{1}{(n-1)^4} + \ldots\right]} = (a_1 \ldots a_n)^{\frac{1}{n}},$$

as required. (The case of equality is readily dealt with as in the article.)

On Note 95.14: Michael de Villiers writes: The dual results in the above note for semi-regular angle- and side-gons, which respectively are generalisations of a rectangle and a rhombus, can be further generalised as follows.

Theorem 1: A cyclic $2n$-gon has n distinct pairs of adjacent angles equal if, and only if, one set of alternate sides are equal.

Theorem 2: A circumscribed $2n$-gon has n distinct pairs of adjacent sides equal if, and only if, one set of alternate angles are equal.

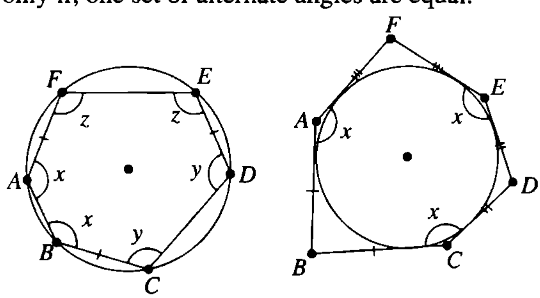

FIGURE 1

Both results are illustrated in Figure 1, the first theorem by the cyclic hexagon and the second theorem by the circumscribed hexagon. Both results can be proved in exactly the same way as those of the original note. Note that the first theorem is a generalisation of an isosceles trapezium while the second one is a generalisation of its dual, which is a kite.

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