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On the well-posedness and asymptotic behaviour of the generalized Korteweg–de Vries–Burgers equation

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In this paper we are concerned with the well-posedness and the exponential stabilization of the generalized Korteweg–de Vries–Burgers equation, posed on the whole real line, under the effect of a damping term. Both problems are investigated when the exponent p in the nonlinear term ranges over the interval $[1, 5)$. We first prove the global well-posedness in $H^s(\mathbb{R})$ for $0 \leqslant s \leqslant 3$ and $1 \leqslant p < 2$, and in $H^3(\mathbb{R})$ when $p \geqslant 2$. For $2 \leqslant p < 5$, we prove the existence of global solutions in the L^2-setting. Then, by using multiplier techniques and interpolation theory, the exponential stabilization is obtained with an indefinite damping term and $1 \leqslant p < 2$. Under the effect of a localized damping term the result is obtained when $2 \leqslant p < 5$. Combining multiplier techniques and compactness arguments, we show that the problem of exponential decay is reduced to proving the unique continuation property of weak solutions. Here, the unique continuation is obtained via the usual Carleman estimate.

Keywords: KdV–Burgers equation; stabilization; Carleman estimate; unique continuation property

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1. Introduction

It is common knowledge that many physical problems, e.g. nonlinear shallow-water waves and wave motion in plasmas, can be described by the family of Korteweg–de Vries (KdV) equations. The KdV-type equations have also been used to describe a wide range of important physical phenomena related to acoustic waves in a harmonic crystals, quantum field theory, plasma physics and solid-state physics. In the study of wave propagation in a tube filled with viscous fluid or of the flow of a fluid containing gas bubbles, for example, the control equation can be reduced to the so-called KdV–Burgers equation [23]. This is commonly obtained from the KdV equation by adding a viscous term, and combines nonlinearity, linear dissipation and dispersion terms:

$$u_t + \delta u_{xxx} - \nu u_{xx} + u u_x = 0, \quad t > 0,\ x \in \mathbb{R}.$$

Since δ and ν are positive numbers, the model can be viewed as a generalization of the KdV and Burgers equations. In particular, the Burgers equation is a simple model equation for a variety of diffusion/dissipative processes in convection dominated systems, which include the formation of weak shocks, traffic flow and turbulence. If, besides the convective nonlinearity and dissipation/diffusion mechanism, dispersion also plays a role over the spatial and temporal scales of interest, then the simplest nonlinear partial differential equation (PDE) governing the wave dynamics is a combination of both the KdV and Burgers equations, known as the KdV–Burgers equation.

In this work we are concerned with the generalized KdV–Burgers (GKdVB) equation under the effect of a damping term represented by a function $b = b(x)$; more precisely,

$$\left.\begin{aligned} u_t + u_{xxx} - u_{xx} + a(u)u_x + b(x)u = 0 \quad & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x) \quad & \text{in } \mathbb{R}. \end{aligned}\right\} \tag{1.1}$$

Our main aim is to address two mathematical issues connected with the initial-value problem (1.1): global well-posedness and large-time behaviour of solutions. More precisely, we establish the well-posedness and the exponential decay of solutions in the classical Sobolev spaces H^s. Therefore, as usual, let us first consider the energy associated with the model, given by

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} u^2(x,t)\, \mathrm{d}x.$$

Thus, at least formally, the solutions of (1.1) should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = - \int_{\mathbb{R}} u_x^2\, \mathrm{d}x - \int_{\mathbb{R}} b(x) u^2\, \mathrm{d}x \tag{1.2}$$

for any positive t. Then, if we assume that $b(x) \geqslant b_0$ for some $b_0 > 0$, it is straightforward to infer that $E(t)$ converges to zero exponentially. By contrast, when the damping function b is allowed to change sign or is effective on a subset of the domain, the problem is much more subtle. Moreover, whether (1.2) generates a flow that can be continued indefinitely in the temporal variable, defining a solution valid for all $t \geqslant 0$, is a non-trivial question.

To obtain the tools with which to handle both problems, we assume that $a = a(x)$ is a positive real-valued function that satisfies the growth conditions

$$\left.\begin{aligned} |a^{(j)}(\mu)| \leqslant C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R},\ \text{for some } C > 0, \\ j = \begin{cases} 0, 1 & \text{if } 1 \leqslant p < 2, \\ 0, 1, 2 & \text{if } p \geqslant 2, \end{cases} \end{aligned}\right\} \tag{1.3}$$

except when u_0 belongs to $L^2(\mathbb{R})$ and $2 \leqslant p < 5$ (see theorem 2.14 and remark 2.15).

Moreover, in order to obtain the exponential stability in the $1 \leqslant p < 2$ case, we take an indefinite damping satisfying the following, where 'a.e.' denotes 'almost everywhere':

$$b \in H^1(\mathbb{R}) \text{ and } b(x) \geqslant \lambda_0 + \lambda_1(x) \text{ a.e. for some } \lambda_0 > 0 \text{ and } \lambda_1 \in L^p(\mathbb{R}),$$
$$\text{such that } \|\lambda_1\|_{L^p(\mathbb{R})} < \left(\frac{\lambda_0}{c_p}\right)^{1-1/2p}, \quad c_p = \left(1 - \frac{1}{2p}\right)\left(\frac{2}{p}\right)^{1/(2p-1)}. \quad (1.4)$$

Concerning the $p \geqslant 2$ case, we consider a localized damping that acts everywhere but on a bounded subset of the line; more precisely,

$$b \in H^1(\mathbb{R}) \text{ is non-negative and } b(x) \geqslant \lambda_0 > 0 \text{ a.e. in } (-\infty, \alpha) \cup (\beta, \infty),$$
$$\text{for some } \alpha, \beta \in \mathbb{R}, \text{ with } \alpha < \beta. \quad (1.5)$$

Our analysis was inspired by the results obtained by Cavalcanti *et al.* for the KdV–Burgers equation [10] and by Rosier and Zhang for the generalized KdV equation posed on a bounded domain [19] (see also [14]). In this context, we refer the reader to the survey [20] for a review on the state of the art.

When $1 \leqslant p < 2$ and $0 \leqslant s \leqslant 3$, we obtain the global well-posedness in the class $B_{s,T} = C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}(\mathbb{R}))$ and prove that the solution decays exponentially to zero in $H^s(\mathbb{R})$, where H^s denotes the classical Sobolev spaces. As in the theory of dispersive wave equations, the results depend on the local theory, on the *a priori* estimates satisfied by the solutions and also on linear theory. Indeed, we combine the Duhamel formula and a contraction-mapping principle to prove the local well-posedness directly. In order to get the global result we derive energy-type inequalities and make use of interpolation arguments. Those *a priori* estimates are sufficient to yield the global stabilization result and a strong smoothing property for solutions $u \in C([\varepsilon, T]; H^s(\mathbb{R})) \cap L^2(\varepsilon, T; H^{s+1}(\mathbb{R}))$ for any $\varepsilon > 0$. Our analysis extends the results obtained in [10], from which we borrow some ideas involved in our proofs.

When $p \geqslant 2$ we can use the same approach to prove that the global well-posedness also holds in $B_{3,T}$. In order to obtain the result in a stronger/weaker norm, we need *a priori* global estimates. However, the only available *a priori* estimate for (1.1) is that provided by (1.2), which does not guarantee the existence of global-in-time solutions. In fact, we do not know if the problem is locally well-posed in the energy space. Therefore, we restrict ourselves to the $2 \leqslant p < 5$ case to prove that the estimate provided by the energy dissipation law holds, and establish the existence of global solutions in the space $C_\omega([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$.

The uniqueness remains an open problem. The main difficulty in this context comes from the structure of nonlinearities and the lack of regularity of the solutions we are dealing with. Concerning the asymptotic behaviour, we prove the exponential decay in the L^2-setting by following the approach in [19]. This combines multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions. To overcome this problem we develop a Carleman inequality by modifying (slightly) a Carleman estimate obtained by Rosier [18] to study the controllability properties of the KdV equation. It allows us to prove the unique continuation property directly.

We remark that, by using numerical simulations, in [7] Bona *et al*. studied the blow-up and decay for periodic solutions of the GKdVB equation. They conjectured that, for $p \geqslant 4$ and sufficiently large initial data, the solutions become unstable and blow-up. Later, in [8], they considered the initial-value and periodic boundary-value problems for the generalized Korteweg–de Vries equation

$$u_t + u_{xxx} + u^p u_x = 0,$$

and studied the effect of a dissipative term on the global well-posedness of the solutions. Actually, they considered two different dissipative terms: a Burgers-type one $-\delta u_{xx}$ and a zeroth-order term σu. In both cases, they showed that for $p \geqslant 4$ there exist critical values δ_c and σ_c such that if $\delta > \delta_c$ or $\sigma > \sigma_c$, the solution is globally well defined. However, the solution blows-up when the damping is too weak, as with the KdV equation. In contrast, it was proved by Rosier and Zhang [19] that the generalized KdV is exponentially stable for $1 \leqslant p < 4$.

With this information in hand and following the ideas in [19], we get a solution of the initial-value problem associated with the GKdVB equation that decays exponentially for $p \geqslant 4$, without any restriction on the initial data. More precisely, we get a solution of the initial-value problem for $1 \leqslant p < 2$ and $2 \leqslant p < 5$, under an indefinite damping and a localized damping, respectively.

Our work was carried out for the particular choice of damping effect appearing in (1.1) and aims to establish as a fact that such a model predicts the interesting qualitative properties initially observed for the KdV–Burgers-type equations. Consideration of this issue for nonlinear dispersive equations, particularly the problems on the time decay rate, has received considerable attention. In this respect, it is important to point out that the approach used here was successfully applied in the context of the KdV equation posed on $\mathbb{R}^+$ and $\mathbb{R}$ under the effect of a localized damping term [9, 15, 17]. We also remark that the stabilization problem in the absence of the damping term b was addressed by Bona and Luo [3, 4], complementing the earlier studies developed in [1, 2, 11] and deriving sharp polynomial decay rates for the solutions. Later on, Bona and Luo [5] and Said-Houari [21] improved upon such a theory. The asymptotic behaviour has also been discussed by Dlotko and Sun in the language of global attractors [12, 13]. More precisely, Dlotko and Sun studied the large-time behaviour of the corresponding semigroup in constructing a global attractor.

The analysis described above is organized into two sections: in § 2 we establish the global well-posedness results, while § 3 is devoted to the stabilization problem. In both sections we split the results into several steps for clarity.

2. Well-posedness

First, we consider the corresponding linear inhomogeneous initial-value problem:

$$\left.\begin{aligned} u_t - u_{xx} + u_{xxx} + b(x)u = f, &\qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), &\qquad x \in \mathbb{R}. \end{aligned}\right\} \tag{2.1}$$

Setting

$$A_b := \partial_x^2 - \partial_x^3 - bI \quad \text{and} \quad D(A_b) = H^3(\mathbb{R}), \quad b \in L^\infty(\mathbb{R}),$$

(2.1) can be written in the form

$$u_t = A_b u + f,$$
$$u(0) = u_0.$$

According to [10], A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ of contractions in $L^2(\mathbb{R})$. Hence, if we consider the Banach space

$$\left.\begin{aligned} B_{s,T} &:= C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}(\mathbb{R})),\\ \|u\|_{s,T} &= \sup_{t\in[0,T]} \|u(t)\|_{H^s(\mathbb{R})} + \|\partial_x^{s+1} u\|_{L^2(0,T;L^2(\mathbb{R}))}, \end{aligned}\right\} \tag{2.2}$$

the following result holds.

PROPOSITION 2.1. *Let $T > 0$. If $u_0 \in L^2(\mathbb{R})$ and $f \in L^1(0,T; L^2(\mathbb{R}))$, (2.1) has a unique mild solution $u \in B_{0,T}$, and*

$$\|u\|_{0,T} \leqslant C_T\{\|u_0\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))}\}, \quad \text{with } C_T = 2\mathrm{e}^{T\|b\|_\infty}.$$

Furthermore, the following energy identity holds for all $t \in [0,T]$:

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 \,\mathrm{d}s + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s = \|u_0\|_2^2 + 2\int_0^t \int_{\mathbb{R}} f(x,s)u(x,s)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.3}$$

Proof. See [10, proposition 4.1]. □

2.1. The $1 \leqslant p < 2$ case

In order to establish the well-posedness of (1.1) we need the following technical lemmas, which will play an important role in the proofs.

LEMMA 2.2 (generalized Hölder inequality). *Suppose $f_i \in L^{p_i}$ and $\sum_{i=1}^n 1/p_i = 1$ for $i = 1, 2, \ldots, n$. Then,*

$$\|f_1 \cdot f_2 \cdots f_n\|_{L^1} \leqslant \prod_{i=1}^n \|f_i\|_{L^{p_i}}. \tag{2.4}$$

LEMMA 2.3. *Let $a \in C^0(\mathbb{R})$ be a function satisfying*

$$|a(\mu)| \leqslant C(1 + |\mu|^p), \quad \forall \mu \in \mathbb{R}, \tag{2.5}$$

with $0 \leqslant p < 2$. Then, there exists a positive constant C such that for any $T > 0$ and $u, v \in B_{0,T}$ we have

$$\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant 2^{p/2} C T^{(2-p)/4} \|u\|_{0,T}^p \|v\|_{0,T} + CT^{1/2}\|v\|_{0,T}.$$

Proof. Recall that $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and

$$\|u\|_\infty^2 \leqslant 2\|u\|_2 \|u_x\|_2 \tag{2.6}$$

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for all $u \in H^1(\mathbb{R})$. On the other hand, by (2.5),

$$\begin{aligned}\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leqslant C\int_0^T \|(1+|u(t)|^p)v_x(t)\|_2\,\mathrm{d}t\\ &\leqslant C\int_0^T \|v_x(t)\|_2\,\mathrm{d}t + C\int_0^T \|u(t)\|_\infty^p\|v_x(t)\|_2\,\mathrm{d}t.\end{aligned}$$

Using the Hölder inequality (2.4) and (2.6), we have

$$\begin{aligned}\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leqslant CT^{1/2}\|v_x\|_{L^2(0,T;L^2)}\\ &\quad + 2^{p/2}C\int_0^T \|u(t)\|_2^{p/2}\|u_x(t)\|_2^{p/2}\|v_x(t)\|_2\,\mathrm{d}t\\ &\leqslant CT^{1/2}\|v_x\|_{L^2(0,T;L^2)}\\ &\quad + 2^{p/2}C\|u\|_{C([0,T];L^2)}^{p/2}\int_0^T \|u_x(t)\|_2^{p/2}\|v_x(t)\|_2\,\mathrm{d}t.\end{aligned}$$

Applying lemma 2.2 with $\frac{1}{4}p$, $\frac{1}{4}(2-p)$ and $\frac{1}{2}$, it follows that

$$\begin{aligned}\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leqslant CT^{1/2}\|v\|_{0,T}\\ &\quad + 2^{p/2}CT^{(2-p)/4}\|u\|_{0,T}^{p/2}\|u_x\|_{L^2(0,T;L^2(\mathbb{R}))}^{p/2}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\leqslant 2^{p/2}CT^{(2-p)/4}\|u\|_{0,T}^p\|v\|_{0,T} + CT^{1/2}\|v\|_{0,T}.\end{aligned}$$

□

Lemma 2.4. *For any $T>0$, $b\in L^\infty(\mathbb{R})$ and $u,v,w\in B_{0,T}$, we have*

(i) $\|bu\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant T^{1/2}\|b\|_\infty\|u\|_{0,T}$,

(ii) $\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant 2^{1/2}T^{1/4}\|u\|_{0,T}\|w\|_{0,T}$.

If $1\leqslant p<2$, we have that

(iii) $\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant 2^{p/2}T^{(2-p)/4}\|u\|_{0,T}\|w\|_{0,T}\|v\|_{0,T}^{p-1}$.

(iv) *for the map $M\colon B_{0,T}\to L^1(0,T;L^2(\mathbb{R}))$ defined by $Mu := a(u)u_x$, M is locally Lipschitz continuous and*

$$\begin{aligned}&\|Mu-Mv\||_{L^1(0,T;L^2(\mathbb{R}))}\\ &\quad\leqslant C\{2^{1/2}T^{1/4}\|u\|_{0,T}\\ &\qquad + 2^{p/2}T^{(2-p)/4}(\|u\|_{0,T}^p + \|u\|_{0,T}\|v\|_{0,T}^{p-1} + \|v\|_{0,T}^p) + T^{1/2}\}\|u-v\|_{0,T},\end{aligned}$$

where C is a positive constant.

Proof.
(i) Using the Hölder inequality, we have

$$\|bu\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant T^{1/2}\|b\|_\infty\|u\|_{L^2(0,T;L^2)} \leqslant T^{1/2}\|b\|_\infty\|u\|_{0,T}.$$

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(ii) Combining (2.6) and lemma 2.2 with 1/2, 1/4 and 1/4, it follows that

$$\begin{aligned}
\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \\
&\leqslant \int_0^T \|u(t)\|_\infty \|w_x(t)\|_2 \,\mathrm{d}t \\
&\leqslant 2^{1/2} \int_0^T \|u(t)\|_2^{1/2} \|u_x(t)\|_2^{1/2} \|w_x(t)\|_2 \,\mathrm{d}t \\
&\leqslant 2^{1/2} \|u(t)\|_{C([0,T];L^2)}^{1/2} \left(\int_0^T \|u_x(t)\|_2^2 \,\mathrm{d}t \right)^{1/4} \left(\int_0^T \|w_x(t)\|_2^2 \,\mathrm{d}t \right)^{1/2} T^{1/4} \\
&\leqslant 2^{1/2} T^{1/4} \|u\|_{0,T} \|w_x\|_{0,T}.
\end{aligned}$$

(iii) We proceed as in (i), combining (2.6) and lemma 2.2 with $\frac{1}{4}$, $\frac{1}{4}(p-1)$, $\frac{1}{4}(2-p)$ and $\frac{1}{2}$ to obtain

$$\begin{aligned}
\|u|v|^{p-1} w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \\
&\leqslant \int_0^T \|u(t)\|_\infty \|v(t)\|_\infty^{p-1} \|w_x(t)\|_2 \,\mathrm{d}t \\
&\leqslant 2^{p/2} \int_0^T \|u(t)\|_2^{1/2} \|u_x(t)\|_2^{1/2} \|v(t)\|_2^{(p-1)/2} \|v_x(t)\|_2^{(p-1)/2} \|w_x(t)\|_2 \,\mathrm{d}t \\
&\leqslant 2^{p/2} \|u\|_{0,T}^{1/2} \|v\|_{0,T}^{(p-1)/2} \int_0^T \|u_x(t)\|_2^{1/2} \|v_x(t)\|_2^{(p-1)/2} \|w_x(t)\|_2 \,\mathrm{d}t \\
&\leqslant 2^{p/2} \|u\|_{0,T}^{1/2} \|v\|_{0,T}^{(p-1)/2} \\
&\qquad \times \left(\int_0^T \|u_x\|_2^2 \,\mathrm{d}t \right)^{1/4} \left(\int_0^T \|v_x\|_2^2 \,\mathrm{d}t \right)^{(p-1)/4} \left(\int_0^T \|w_x\|_2^2 \,\mathrm{d}t \right)^{1/2} T^{(2-p)/4} \\
&\leqslant 2^{p/2} T^{(2-p)/4} \|u\|_{0,T}^{1/2} \|v\|_{0,T}^{(p-1)/2} \|u\|_{0,T}^{1/2} \|v\|_{0,T}^{(p-1)/2} \|w\|_{0,T},
\end{aligned}$$

which allows us to conclude the result.

(iv) Note that

$$\begin{aligned}
\|Mu - Mv\|_{L^1(0,T;L^2(\mathbb{R}))} \leqslant{}& \|(a(u) - a(v))u_x\|_{L^1(0,T;L^2(\mathbb{R}))} \\
&+ \|a(v)(u-v)_x\|_{L^1(0,T;L^2(\mathbb{R}))}.
\end{aligned}$$

Using the mean-value theorem, (ii), (iii) and lemma 2.3, we have

$$\begin{aligned}
\|Mu - Mv\|_{L^1(0,T;L^2(\mathbb{R}))} &\leqslant C\|(1 + |u|^{p-1} + |v|^{p-1})|u-v|u_x\|_{L^1(0,T;L^2)} \\
&\quad + \|a(v)(u-v)_x\|_{L^1(0,T;L^2)} \\
&\leqslant C\{2^{1/2} T^{1/4} \|u-v\|_{0,T} \|u\|_{0,T} \\
&\qquad + 2^{p/2} T^{(2-p)/4} \|u-v\|_{0,T} \|u\|_{0,T}^p \\
&\qquad + 2^{p/2} T^{(2-p)/4} \|u-v\|_{0,T} \|u\|_{0,T} \|v\|_{0,T}^{p-1} \\
&\qquad + 2^{p/2} T^{(2-p)/4} \|u-v\|_{0,T} \|v\|_{0,T}^p + T^{1/2} \|u-v\|_{0,T}\}.
\end{aligned}$$

□

The above estimates lead to the following local existence result and *a priori* estimate.

PROPOSITION 2.5. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leqslant C(1+|\mu|^p) \quad \text{and} \quad |a'(\mu)| \leqslant C(1+|\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leqslant p < 2$. Let $b \in L^\infty(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$. Then, there exist $T > 0$ and a unique mild solution $u \in B_{0,T}$ of (1.1). Moreover,

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 \, \mathrm{d}s + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s = \|u_0\|_2^2, \quad \forall t \in [0,T]. \tag{2.7}$$

Proof. Let $T > 0$ be as determined later. For each $u \in B_{0,T}$ consider the problem

$$\left.\begin{aligned} v_t &= A_b v - Mu, \\ v(0) &= u_0, \end{aligned}\right\} \tag{2.8}$$

where $A_b v = \partial_x^2 v - \partial_x^3 v - bv$ and $Mu = a(u)u_x$. Since A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ of contractions in $L^2(\mathbb{R})$, lemma 2.3 and proposition 2.1 allow us to conclude that (2.8) has a unique mild solution $v \in B_{0,T}$, such that

$$\|v\|_{0,T} \leqslant C_T\{\|u_0\|_2 + \|Mu\|_{L^1(0,T;L^2(\mathbb{R}))}\}, \tag{2.9}$$

where $C_T = 2\mathrm{e}^{T\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma \colon B_{0,T} \to B_{0,T} \quad \text{given by } \Gamma(u) = v.$$

By using lemma 2.3 and (2.9), we have

$$\|\Gamma u\|_{0,T} \leqslant C_T\{\|u_0\|_2 + 2^{p/2} C T^{(2-p)/4} \|u\|_{0,T}^{p+1} + C T^{1/2} \|u\|_{0,T}\}.$$

Thus, for $u \in B_R(0) := \{u \in B_{0,T} \colon \|u\|_{B_{0,T}} \leqslant R\}$, it follows that

$$\|\Gamma u\|_{0,T} \leqslant C_T\{\|u_0\|_2 + 2^{p/2} C T^{(2-p)/4} R^{p+1} + C T^{1/2} R\}.$$

Choosing $R = 2C_T\|u_0\|_2$, we obtain the following estimate:

$$\|\Gamma u\|_{0,T} \leqslant (K_1 + \tfrac{1}{2})R,$$

where $K_1 = K_1(T) = 2^{p/2} C_T C T^{(2-p)/4} R^p + C_T C T^{1/2}$. On the other hand, note that $\Gamma u - \Gamma w$ is a solution of

$$\begin{aligned} v_t &= A_b v - (Mu - Mw), \\ v(0) &= 0. \end{aligned}$$

Again, by applying proposition 2.1, we have

$$\|\Gamma u - \Gamma w\|_{0,T} \leqslant C_T \|Mu - Mw\|_{L^1(0,T;L^2)},$$

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and estimate (iv) in lemma 2.4 allows us to conclude that

$$\begin{aligned}\|\Gamma u-\Gamma w\|_{0,T}\\ &\leqslant C_T C\{2^{1/2}T^{1/4}\|u\|_{0,T}\\ &\qquad +2^{p/2}T^{(2-p)/4}(\|u\|^p_{0,T}+\|u\|_{0,T}\|w\|^{p-1}_{0,T}+\|w\|^p_{0,T})+T^{1/2}\}\|u-w\|_{0,T}.\end{aligned}$$

Suppose that $u,w\in B_R(0)$ as defined above. Then,

$$\|\Gamma u-\Gamma w\|_{B_{0,T}}\leqslant K_2\|u-w\|_{B_{0,T}},$$

where $K_2=K_2(T)=C_T C\{2^{1/2}T^{1/4}R+3(2^{p/2})T^{(2-p)/4}R^p+T^{1/2}\}$. Since $K_1\leqslant K_2$, we can choose $T>0$ to obtain $K_2<\frac{1}{2}$ and

$$\left.\begin{aligned}\|\Gamma u\|_{B_{0,T}}&\leqslant R,\\ \|\Gamma u-\Gamma w\|_{B_{0,T}}&<\tfrac{1}{2}\|u-w\|_{B_{0,T}},\end{aligned}\right\}\quad \forall u,w\in B_R(0)\subset B_{0,T}.$$

Hence, $\Gamma\colon B_R(0)\to B_R(0)$ is a contraction, and by the Banach fixed-point theorem we obtain a unique $u\in B_R(0)$ such that $\Gamma(u)=u$. Consequently, u is a unique local mild solution of (1.1) and

$$\|u\|_{B_{0,T}}\leqslant 2C_T\|u_0\|_2. \tag{2.10}$$

In order to prove (2.7), consider $v_n=\Gamma v_{n-1}$, $n\geqslant 1$. Since Γ is a contraction, we have

$$\lim_{n\to\infty} v_n=u\quad \text{in } B_{0,T}.$$

On the other hand, by (2.3), v_n verifies the following identity:

$$\begin{aligned}\|v_n(t)\|_2^2+2\int_0^t\|v_{nx}(s)\|_2^2\,\mathrm{d}s+2\int_0^t\int_{\mathbb{R}}b(x)|v(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s\\ =\|u_0\|_2^2+2\int_0^t\int_{\mathbb{R}}Mv_{n-1}(x,s)v_n(x,s)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Then, taking the limit as $n\to\infty$, we get

$$\|u(t)\|_2^2+2\int_0^t\|u_x(s)\|_2^2\,\mathrm{d}s+2\int_0^t\int_{\mathbb{R}}b(x)|u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s=\|u_0\|_2^2$$

since the limit of the last term is

$$\int_0^t\int_{\mathbb{R}}Mu(x,s)u(x,s)\,\mathrm{d}x\,\mathrm{d}s=0.$$

In fact,

$$\int_{\mathbb{R}}a(u(x))u_x(x)\,\mathrm{d}x=\int_{\mathbb{R}}[A(u(x))]_x\,\mathrm{d}x,\quad A(v)=\int_0^v a(s)\,\mathrm{d}s.$$

□

From proposition 2.5 we obtain our first global-in-time existence result.

Theorem 2.6. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leqslant C(1+|\mu|^p) \quad \text{and} \quad |a'(\mu)| \leqslant C(1+|\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leqslant p < 2$. Let $b \in L^\infty(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$. Then, there exists a unique global mild solution u of (1.1) such that for each $T > 0$ there exists a non-decreasing continuous function $\beta_0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies

$$\|u\|_{0,T} \leqslant \beta_0(\|u_0\|_2)\|u_0\|_2. \tag{2.11}$$

Moreover, the following energy identity holds for all $t \geqslant 0$:

$$\|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 \,\mathrm{d}s + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s = \|u_0\|_2^2. \tag{2.12}$$

Proof. By proposition 2.5, there exists a unique mild solution $u \in B_{0,T}$ for all $T < T_{\max} \leqslant \infty$. Moreover,

$$\|u\|_{0,T} \leqslant 4\mathrm{e}^{\|b\|_\infty t}\|u_0\|_2, \quad \forall t \in [0, T_{\max}),$$

which implies that u is a global mild solution of (1.1). On the other hand, (2.10) implies (2.11) with $\beta_0(s) = 2C_T$. The identity (2.12) is a direct consequence of (2.7) in proposition 2.5. □

It follows from theorem 2.6 that for each fixed $T > 0$ the solution map

$$\mathcal{A} \colon L^2(\mathbb{R}) \to B_{0,T}, \quad \mathcal{A}u_0 = u, \tag{2.13}$$

is well defined. Moreover, we have the following result.

Proposition 2.7. *The solution map (2.13) is locally Lipschitz continuous, i.e. there exists a continuous function $C_0 \colon \mathbb{R}^+ \times (0,\infty) \to \mathbb{R}^+$, non-decreasing in its first variable, such that for all $u_0, v_0 \in L^2(\mathbb{R})$ we have*

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,T} \leqslant C_0(\|u_0\|_2 + \|v_0\|_2, T)\|u_0 - v_0\|_2.$$

Proof. Let $0 < \theta \leqslant T$ and $n = [T/\theta]$. By theorem 2.6, we have

$$\|\mathcal{A}u_0\|_{0,\theta} \leqslant 2C_\theta\|u_0\|_2 \tag{2.14}$$

and

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \leqslant C_\theta\{\|u_0 - v_0\|_2 + \|M(\mathcal{A}u_0) - M(\mathcal{A}v_0)\|_{L^1(0,\theta;L^2(\mathbb{R}))}\},$$

where $C_\theta = 2\mathrm{e}^{\theta\|b\|_\infty}$. By lemma 2.4, we have

$$\begin{aligned}
&\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\
&\quad \leqslant C_\theta\|u_0 - v_0\|_2 \\
&\qquad + C_\theta C\{2^{1/2}\theta^{1/4}\|\mathcal{A}u_0\|_{0,\theta} \\
&\qquad\qquad + 2^{p/2}\theta^{(2-p)/4}(\|\mathcal{A}u_0\|_{0,\theta}^p + \|\mathcal{A}u_0\|_{0,\theta}\|\mathcal{A}v_0\|_{0,\theta}^{p-1} + \|\mathcal{A}v_0\|_{0,\theta}^p) + \theta^{1/2}\} \\
&\qquad \times \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta},
\end{aligned}$$

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and, applying (2.14), it follows that

$$\begin{aligned}
&\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\
&\quad \leqslant C_\theta \|u_0 - v_0\|_2 \\
&\qquad + C_\theta C\{2^{3/2}\theta^{1/4} C_\theta \|u_0\|_2 \\
&\qquad\qquad + 2^{3p/2}\theta^{(2-p)/4} C_\theta^p (\|u_0\|_2^p + \|u_0\|_2 \|v_0\|_2^{p-1} + \|v_0\|_2^p) + \theta^{1/2}\} \\
&\qquad \times \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\
&\quad \leqslant C_T \|u_0 - v_0\|_2 \\
&\qquad + C_T C \theta^{(2-p)/4}\{2^{3/2}\theta^{(p-1)/4} C_T (\|u_0\|_2 + \|v_0\|_2) \\
&\qquad\qquad + 2^{3p/2} C_T^p (\|u_0\|_2 + \|v_0\|_2)^p + \theta^{p/4}\} \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \\
&\quad \leqslant C_T \|u_0 - v_0\|_2 \\
&\qquad + C_T C \theta^{(2-p)/4}\{2^{5/2} T^{(p-1)/4} C_T^2 (\|u_0\|_2 + \|v_0\|_2) \\
&\qquad\qquad + 2^{5p/2} C_T^{2p} (\|u_0\|_2 + \|v_0\|_2)^p + T^{p/4}\} \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta}.
\end{aligned}$$

Choosing θ sufficiently small such that

$$\begin{aligned}
\theta < [2C_T C\{2^{5/2} T^{(p-1)/4} C_T \|u_0\|_2 \\
+ \|v_0\|_2 + 2^{5p/2} C_T^{2p} (\|u_0\|_2 + \|v_0\|_2)^p + T^{p/4}\}]^{-4/(2-p)}
\end{aligned} \tag{2.15}$$

yields

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \leqslant 2C_T \|u_0 - v_0\|_2. \tag{2.16}$$

Analogously, we can deduce that

$$\|\mathcal{A}u_0\|_{0,[k\theta,(k+1)\theta]} \leqslant 2C_\theta \|u(k\theta)\|_2, \quad k = 0, 1, \dots, n-1,$$

where $\|\cdot\|_{0,[k\theta,(k+1)\theta]}$ denotes the norm of

$$B_{0,[k\theta,(k+1)\theta]} := C([k\theta,(k+1)\theta]; L^2(\mathbb{R})) \cap L^2(k\theta,(k+1)\theta; H^1(\mathbb{R})).$$

Moreover, by using the same arguments, we have

$$\begin{aligned}
&\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \\
&\quad \leqslant C_T \|u(k\theta) - v(k\theta)\|_2 \\
&\qquad + C_T C \theta^{(2-p)/4}\{2^{3/2} T^{(p-1)/4} C_T (\|u(k\theta)\|_2 + \|v(k\theta)\|_2) \\
&\qquad\qquad + 2^{3p/2} C_T^p (\|u(k\theta)\|_2 + \|v(k\theta)\|_2)^p + T^{p/4}\} \\
&\qquad \times \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}.
\end{aligned}$$

Combining (2.14) and the above estimate, it follows that

$$\begin{aligned}
&\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \\
&\quad \leqslant C_T \|u(k\theta) - v(k\theta)\|_2 \\
&\qquad + C_T C \theta^{(2-p)/4}\{2^{5/2} T^{(p-1)/4} C_T^2 (\|u_0\|_2 + \|v_0\|_2) \\
&\qquad\qquad + 2^{5p/2} C_T^{2p} (\|u_0\|_2 + \|v_0\|_2)^p + T^{p/4}\} \\
&\qquad \times \|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]}.
\end{aligned}$$

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Finally, from (2.15), we get

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \leqslant 2C_T\|u(k\theta) - v(k\theta)\|_2, \quad k = 0, 1, \ldots, n-1. \tag{2.17}$$

On the other hand, note that (2.16) and (2.17) imply that

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \leqslant 2^k C_T^k\|u_0 - v_0\|_2, \quad k = 0, 1, \ldots, n-1,$$

and therefore

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \leqslant 2^n C_T^n\|u_0 - v_0\|_2.$$

Finally,

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,T} &\leqslant \sum_{k=0}^{n-1}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,[k\theta,(k+1)\theta]} \leqslant \sum_{k=0}^{n-1} 2^n C_T^n\|u_0 - v_0\|_2 \\ &\leqslant 2^n C_T^n n\|u_0 - v_0\|_2 \leqslant C_0(\|u_0\|_2 + \|v_0\|_2)\|u_0 - v_0\|_2,\end{aligned}$$

where

$$C_0(s) = \frac{T}{\theta(s)}[2C_T]^{T/\theta(s)}.$$

□

Next, we shall show well-posedness in $B_{3,T}$ with $1 \leqslant p < 2$. Therefore, let us first consider the following linearized problem:

$$\left.\begin{aligned} v_t + v_{xxx} - v_{xx} + [a(u)v]_x + bv = 0 \quad &\text{in } \mathbb{R}\times(0,\infty),\\ v(0) = v_0 \quad &\text{in } \mathbb{R}\times(0,\infty).\end{aligned}\right\} \tag{2.18}$$

Then, we can establish the following proposition.

Proposition 2.8. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leqslant C(1 + |\mu|^p) \quad \text{and} \quad |a'(\mu)| \leqslant C(1 + |\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leqslant p < 2$. Let $T > 0$, $b \in L^\infty(\mathbb{R})$, $u \in B_{0,T}$ and $v_0 \in L^2(\mathbb{R})$. Then, problem (2.18) admits a unique solution $v \in B_{0,T}$ such that

$$\|v\|_{0,T} \leqslant \sigma(\|u\|_{0,T})\|v_0\|_2,$$

where $\sigma\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function.

Proof. Let $0 < \theta \leqslant T$ and $u \in B_{0,T}$. The proof of the existence follows the steps of proposition 2.5 and theorem 2.6. Therefore, we shall omit the details. First, note that lemmas 2.3 and 2.4 imply that $Nw := [a(u)w]_x \in L^1(0,\theta;L^2(\mathbb{R}))$ for all $w \in B_{0,\theta}$. Hence,

$$\begin{aligned}\|Nw\|_{L^1(0,\theta;L^2(\mathbb{R}))} \leqslant C\{&2^{1/2}\theta^{1/4}\|u\|_{0,\theta}\|w\|_{0,\theta} \\ &+ 2^{(p+2)/2}\theta^{(2-p)/4}\|u\|_{0,\theta}^p\|w\|_{0,\theta} + \theta^{1/2}\|w\|_{0,\theta}\}.\end{aligned} \tag{2.19}$$

With the notation above, problem (2.18) takes the form

$$\left.\begin{aligned} v_t &= A_b v - Nw,\\ v(0) &= u_0,\end{aligned}\right\} \tag{2.20}$$

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where $A_b v = \partial_x^2 v - \partial_x^3 v - bv$. Since A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ of contractions in $L^2(\mathbb{R})$, by proposition 2.1, (2.20) has a unique mild solution $v \in B_{0,\theta}$, such that

$$\|v\|_{0,\theta} \leqslant C_\theta\{\|v_0\|_2 + \|Nw\|_{L^1(0,\theta;L^2(\mathbb{R}))}\},$$

where $C_\theta = 2\mathrm{e}^{\theta\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma\colon B_{0,T} \to B_{0,T} \quad \text{given by } \Gamma(w) = v.$$

Let $R > 0$ be a constant to be determined later and let $w \in B_R(0) := \{w \in B_{0,\theta}\colon \|w\|_{B_{0,\theta}} \leqslant R\}$. Thus,

$$\|\Gamma w\|_{0,\theta} \leqslant C_T\{\|v_0\|_2 + (2^{1/2}C\theta^{1/4}\|u\|_{0,T} + 2^{(p+2)/2}C\theta^{(2-p)/4}\|u\|_{0,T}^p + \theta^{1/2}C)R\}.$$

By choosing $R = 2C_T\|v_0\|_2$, we have

$$\|\Gamma u\|_{0,\theta} \leqslant (K_1 + \tfrac{1}{2})R,$$

where $K_1 = C_T C(2^{1/2}C\theta^{1/4}\|u\|_{0,T} + 2^{(p+2)/2}C\theta^{(2-p)/4}\|u\|_{0,T}^p + \theta^{1/2})$. On the other hand, note that $\Gamma s - \Gamma w$ solves the following problem:

$$v_t = A_b v - (Ns - Nw),$$
$$v(0) = 0.$$

Thus,

$$\|\Gamma s - \Gamma w\|_{0,\theta} \leqslant K_1\|s - w\|_{0,\theta}.$$

Choosing $\theta > 0$ such that $K_1 = K_1(\theta) < \frac{1}{2}$, we have

$$\left.\begin{aligned} \|\Gamma w\|_{B_{0,\theta}} &\leqslant R, \\ |\Gamma s - \Gamma w\|_{B_{0,\theta}} &< \tfrac{1}{2}\|s - w\|_{0,\theta}, \end{aligned}\right\} \quad \forall s, w \in B_R(0) \subset B_{0,\theta}.$$

Hence, $\Gamma\colon B_R(0) \to B_R(0)$ is a contraction and, by the Banach fixed-point theorem, we obtain a unique $v \in B_R(0)$ such that $\Gamma(v) = v$. Consequently, v is a unique local mild solution of problem (2.18) and

$$\|v\|_{B_{0,\theta}} \leqslant 2C_T\|v_0\|_2.$$

Then, using standard arguments we may extend θ to T. Finally, the proof is completed by defining $\sigma(s) = 2C_T$. □

Our second global-in-time existence result is proved below. We make use of proposition 2.8 and classical energy-type estimates.

THEOREM 2.9. *Let a be a function $C^1(\mathbb{R})$ satisfying*

$$|a(\mu)| \leqslant C(1 + |\mu|^p) \quad \text{and} \quad |a'(\mu)| \leqslant C(1 + |\mu|^{p-1}), \quad \forall \mu \in \mathbb{R}, \tag{2.21}$$

with $1 \leqslant p < 2$. Let $T > 0$, $b \in H^1(\mathbb{R})$ and $u_0 \in H^3(\mathbb{R})$. Then, there exists a unique mild solution $u \in B_{3,T}$ of (1.1) such that

$$\|u\|_{3,T} \leqslant \beta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\beta_3\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing continuous function.

Proof. For clarity of exposition, the proof will be carried out in several steps.

STEP 1 ($u \in L^2(0,T;H^3(\mathbb{R}))$). Since $u_0 \in H^3(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, by theorem 2.6 there exists a unique solution $u \in B_{0,T}$ such that

$$\|u\|_{0,T} \leqslant \beta_0(\|u_0\|_2)\|u_0\|_2. \tag{2.22}$$

We shall show that $u \in B_{3,T}$. Let $v = u_t$. Then, v solves the problem

$$\begin{aligned} v_t + v_{xxx} - v_{xx} + [a(u)v]_x + bv = 0, \\ v(0,x) = v_0, \end{aligned}$$

where $v_0 = -\partial_x^3 u_0 + \partial_x^2 u_0 - a(u_0)\partial_x u_0 - bu_0$. Note that $v \in L^2(\mathbb{R})$ and there exists $C = C(\|u_0\|_2)$ satisfying

$$\|v_0\|_2 \leqslant C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}.$$

In fact, from (2.6) we can bound v_0 as follows:

$$\begin{aligned} \|v_0\|_2 &\leqslant \|\partial_x^3 u_0\|_2 + \|\partial_x^2 u_0\|_2 + \|a(u_0)\partial_x u_0\|_2 + \|bu_0\|_2 \\ &\leqslant C_1\{(1 + \|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{p/2}\|\partial_x u_0\|_2^{(p+2)/2}\}. \end{aligned}$$

Recall the Gagliardo–Nirenberg inequality:

$$\|\partial_x^j u_0\|_2 \leqslant C\|\partial_x^m u_0\|_2^{j/m}\|u_0\|_2^{1-j/m}, \quad j,m = 0,1,2,3,\ j \leqslant m. \tag{2.23}$$

Applying (2.23) with $j = 1$ and $m = 2$, we have

$$\|v_0\|_2 \leqslant C_2\{(1 + \|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{(3p+2)/4}\|\partial_x^2 u_0\|_2^{(p+2)/4}\}.$$

Then, Young's inequality guarantees that

$$\|v_0\|_2 \leqslant C_3\{(1 + \|b\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^3(\mathbb{R})} + \|u_0\|_2^{4p/(2-p)}\|u_0\|_2 + \|\partial_x^2 u_0\|_2\}.$$

Consequently, this gives

$$\|v_0\|_2 \leqslant C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{2.24}$$

where $C(s) = C_3\{2 + \|b\|_{L^\infty(\mathbb{R})} + s^{4p/(2-p)}\}$. Using proposition 2.8, we see that $v \in B_{0,T}$ and

$$\|v\|_{0,T} \leqslant \sigma(\|u\|_{0,T})\|v_0\|_2,$$

where $\sigma(s) = 2C_T$. Combining (2.22) and (2.24), we get

$$\|v\|_{0,T} \leqslant \sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}. \tag{2.25}$$

Then,

$$u, u_t \in L^2(0,T;H^1(\mathbb{R})), \tag{2.26}$$

and therefore

$$u \in C([0,T];H^1(\mathbb{R})) \hookrightarrow C([0,T];C(\mathbb{R})). \tag{2.27}$$

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On the other hand, note that $a(u)u_x, bu \in L^2(0,T;L^2(\mathbb{R}))$. In fact, from (2.27) it follows that

$$\|a(u)u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} \leqslant C\bigg\{\int_0^T \|u_x\|_2^2\,\mathrm{d}x + \int_0^T \||u|^p u_x\|_2^2\,\mathrm{d}x\bigg\}$$
$$\leqslant C\{1 + \|u\|^{2p}_{C(0,T;C(\mathbb{R}))}\}\|u\|^2_{0,T}$$

and

$$\|bu\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant \|b\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))}.$$

Moreover, $u_{xxx} - u_{xx} = -u_t - a(u)u_x - bu$ in $D'(0,T,\mathbb{R})$. Hence,

$$u_{xxx} - u_{xx} = f \in L^2(0,T;L^2(\mathbb{R})), \quad \text{where } f := -u_t - a(u)u_x - bu.$$

Taking the Fourier transform, we have

$$\hat{u} = \frac{\hat{f} + \hat{u}}{[1 + \xi^2 - \mathrm{i}\xi^3]} \tag{2.28}$$

and

$$\|u(t)\|^2_{H^3(\mathbb{R})} \leqslant C_3\{\|f(t)\|_2^2 + \|u(t)\|_2^2\}, \tag{2.29}$$

where $C_3 = 2\sup_{\xi\in\mathbb{R}}(1+\xi^2)^3/((1+\xi^2)^2+\xi^6)$. Integrating (2.29) over $[0,T]$, we deduce that

$$u \in L^2(0,T;H^3(\mathbb{R})). \tag{2.30}$$

Step 2 ($u \in B_{3,T}$). First, observe that, according to (2.26) and (2.30), we can apply [16, theorem 2.3] to obtain

$$u \in C([0,T];H^2(\mathbb{R})).$$

This further implies

$$u_{xx}, bu \in C([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R})). \tag{2.31}$$

On the other hand, note that

$$\begin{aligned}
&\|a(u(t))u_x(t) - a(u(t_0))u_x(t_0)\|_2 \\
&\quad\leqslant \|[a(u(t)) - a(u(t_0))]u_x(t)\|_2 + \|a(u(t_0))[u_x(t) - u_x(t_0)]\|_2 \\
&\quad\leqslant C\{\|(1 + |u(t)|^{p-1} + |u(t_0)|^{p-1})|u(t) - u(t_0)|u_x(t)\|_2 \\
&\qquad\qquad + \|(1 + |u(t_0)|^p)|u_x(t) - u_x(t_0)|\|_2\} \\
&\quad\leqslant C\{(1 + \|u(t)\|^{p-1}_\infty + \|u(t_0)\|^{p-1}_\infty)\|u(t) - u(t_0)\|_\infty\|u_x(t)\|_2 \\
&\qquad\qquad + (1 + \|u(t_0)\|^p_\infty)\|u_x(t) - u_x(t_0)\|_2\}.
\end{aligned}$$

Then, by (2.27) we have

$$\lim_{t\to t_0}\|a(u(t))u_x(t) - a(u(t_0))u_x(t_0)\|_2 = 0,$$

and therefore $a(u)u_x \in C([0,T];L^2(\mathbb{R}))$. The results above also guarantee that

$$a(u)u_x \in C([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R})). \tag{2.32}$$

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Indeed, it is sufficient to combine (2.27), (2.30) and the estimates

$$\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant C\{(1+\|u\|^{p-1}_{C([0,T];C(\mathbb{R}))})\|u_x\|_{C([0,T];C(\mathbb{R}))} \times \|u_x\|_{L^2([0,T];L^2(\mathbb{R}))}\}$$

and

$$\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant C\{(1+\|u\|^{p}_{C([0,T];C(\mathbb{R}))})\|u_{xx}\|_{L^2([0,T];L^2(\mathbb{R}))}\}.$$

Since

$$u_{xxx} = -u_t + u_{xx} - a(u)u_x - bu,$$

using the fact that $u_t \in C([0,T], L^2(\mathbb{R}))$ and (2.26), (2.31) and (2.32) we obtain

$$u_{xxx} \in C([0,T], L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{2.33}$$

Moreover, since $u \in B_{0,T}$, it follows from (2.33) that $u \in B_{3,T}$.

STEP 3 ($\|u\|_{C([0,T];H^3(\mathbb{R}))} \leqslant \sigma_1(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}$). First, note that, due to (2.29), the following estimate holds:

$$\|u(t)\|_{H^3(\mathbb{R})} \leqslant C_4\{\|u_t(t)\|_2 + \|a(u(t))u_x(t)\|_2 + \|bu(t)\|_2 + \|u(t)\|_2\}. \tag{2.34}$$

Next, we combine (2.21), (2.6) and (2.23), with $j=1$ and $m=2$, to obtain

$$\begin{aligned}\|a(u(t))u_x(t)\|_2 &\leqslant C\{\|u_x(t)\|_2 + \|u(t)\|_2^{p/2}\|u_x(t)\|_2^{(p+2)/2}\} \\ &\leqslant C\{\|u_{xx}(t)\|_2^{1/2}\|u(t)\|_2^{1/2} + \|u(t)\|_2^{(3p+2)/4}\|u_{xx}(t)\|_2^{(p+2)/4}\}.\end{aligned}$$

Moreover, Young's inequality gives

$$\|a(u(t))u_x(t)\|_2 \leqslant C_5(\|u(t)\|_2 + \|u(t)\|_2^{(3p+2)/(2-p)}) + \frac{1}{2C_4}\|u(t)\|_{H^3(\mathbb{R})}.$$

Replacing the estimate above in (2.34) and taking the supremum in $[0,T]$, we get

$$\|u\|_{C([0,T];H^3(\mathbb{R}))} \leqslant 2C_4\{\|u_t\|_{0,T} + (C_6 + \|b\|_\infty)\|u\|_{0,T} + C_5\|u\|_{0,T}^{(3p+2)/(2-p)}\}.$$

Then, using (2.22) and (2.25), it follows that

$$\begin{aligned}\|u\|_{C([0,T];H^3(\mathbb{R}))} &\leqslant 2C_4\{\sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)C(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})} \\ &\quad + (C_6 + \|b\|_\infty)\beta_0(\|u_0\|_2)\|u_0\|_2 \\ &\quad + C_5\beta_0^{(3p+2)/(2-p)}(\|u_0\|_2)\|u_0\|_2^{4p/(2-p)}\|u_0\|_2\} \\ &= \sigma_1(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},\end{aligned} \tag{2.35}$$

where

$$\sigma_1(s) = 2C_4\{\sigma(\beta_0(s)s)C(s) + (C_6 + \|b\|_\infty)\beta_0(s) + C_5\beta_0^{(3p+2)/(2-p)}(s)s^{4p/(2-p)}\}.$$

STEP 4 ($\|u_{xxxx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant \sigma_5(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}$). We know that $u \in L^2(0,T; H^4(\mathbb{R}))$ by (2.33). To prove the desired result, we differentiate the equation with

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respect to x to obtain

$$\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leqslant \|v\|_{0,T} + T^{1/2}\|u\|_{C(0;T,H^3(\mathbb{R}))} + \|a'(u)u_x^2\|_{L^2(0;T,L^2(\mathbb{R}))} + \|a(u)u_{xx}\|_{L^2(0;T,L^2(\mathbb{R}))} + \|[bu]_x\|_{L^2(0;T,L^2(\mathbb{R}))}. \quad (2.36)$$

Next, we estimate the terms on the right side of (2.36). First, observe that

$$\begin{aligned}\|[bu]_x\|_{L^2(0;T,L^2(\mathbb{R}))} &\leqslant \|b\|_{H^1(\mathbb{R})}\|u\|_{L^2(0;T,H^1(\mathbb{R}))} + \|b\|_{H^1(\mathbb{R})}\|u_x\|_{L^2(0;T,L^2(\mathbb{R}))} \\ &\leqslant 2\|b\|_{H^1(\mathbb{R})}\|u\|_{0,T}.\end{aligned}$$

Then, from (2.22), (2.25) and (2.35), we obtain

$$\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leqslant \sigma_2(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})} + \|a'(u)u_x^2\|_{L^2(0;T,L^2(\mathbb{R}))} + \|a(u)u_{xx}\|_{L^2(0;T,L^2(\mathbb{R}))}, \quad (2.37)$$

where $\sigma_2(s) = \sigma(\beta_0(s)s)C(s) + T^{1/2}\sigma_1(s) + 2\|b\|_{H^1(\mathbb{R})}\beta_0(s)$. Moreover, using (2.6) it follows that

$$\begin{aligned}&\|a'(u(t))u_x^2(t)\|_2 \\ &\quad\leqslant C\{\|u_x^2(t)\|_2 + \||u(t)|^{p-1}u_x^2(t)\|_2\} \\ &\quad\leqslant C_7\{\|u_x(t)\|_2^{3/2}\|u_{xx}(t)\|_2^{1/2} + \|u(t)\|_2^{(p-1)/2}\|u_x(t)\|_2^{(p+2)/2}\|u_{xx}(t)\|_2^{1/2}\} \\ &\quad\leqslant C_7\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{(p-1)/2}\|u_x(t)\|_2^{(p+2)/2}\|u_{xx}(t)\|_2^{1/2}\}.\end{aligned}$$

Then, the Gagliardo–Nirenberg inequality (2.23) with $j = 1$ and $m = 3$ leads to

$$\|a'(u(t))u_x^2(t)\|_2 \leqslant C_8\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{(5p+2)/6}\|u_{xxx}(t)\|_2^{(p+4)/6}\}.$$

Moreover, Young's inequality gives

$$\|a'(u(t))u_x^2(t)\|_2^2 \leqslant C_9\{\|u_x(t)\|_2\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{(5p+2)/(2-p)} + \|u_{xxx}(t)\|_2\},$$

which allows us to conclude that

$$\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant C_{10}\{\|u\|_{C([0,T];H^3(\mathbb{R}))}\|u\|_{0,T} + T^{1/2}\|u\|_{0,T}^{(5p+2)/(2-p)} + T^{1/2}\|u\|_{C([0,T];H^3(\mathbb{R}))}\}.$$

Hence,

$$\|a'(u)u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant \sigma_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})} \quad (2.38)$$

with

$$\sigma_3(s) = C_{10}\{\sigma_1(s)\beta_0(s)s + T^{1/2}\beta_0^{(5p+2)/(2-p)}(s)s^{((5p+2)/(2-p))-1} + T^{1/2}\sigma_1(s)\}.$$

On the other hand, (2.6) yields

$$\begin{aligned}\|a(u(t))u_{xx}(t)\|_2 &\leqslant C_{11}\{\|u(t)\|_{H^3(\mathbb{R})} + \|u(t)\|_2^{p/2}\|u_x(t)\|_2^{p/2}\|u_{xx}(t)\|_2\} \\ &\leqslant C_{11}\{\|u\|_{C([0,T];H^3(\mathbb{R}))} + \|u\|_{0,T}^{p/2}\|u\|_{C([0,T];H^3(\mathbb{R}))}\|u_x(t)\|_2^{p/2}\}.\end{aligned}$$

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It transpires that

$$\begin{aligned}\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} &\leqslant C_{12}\left\{T^{1/2}\|u\|_{C([0,T];H^3(\mathbb{R}))} + \|u\|_{0,T}^{p/2}\|u\|_{C([0,T];H^3(\mathbb{R}))}\left(\int_0^T \|u_x(t)\|_2^p\right)^{1/2}\right\}\\ &\leqslant C_{12}\{T^{1/2}\|u\|_{C([0,T];H^3(\mathbb{R}))} + T^{(2-p)/4}\|u\|_{0,T}^p\|u\|_{C([0,T];H^3(\mathbb{R}))}\},\end{aligned}$$

from which one obtains the inequality

$$\|a(u)u_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant \sigma_4(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{2.39}$$

with $\sigma_4(s) = C_{12}\{T^{1/2}\sigma_1(s) + T^{(2-p)/4}\beta_0^p(s)\sigma_1(s)s^p\}$. Consequently, (2.37)–(2.39) lead to

$$\|u_{xxxx}\|_{L^2(0;T,L^2(\mathbb{R}))} \leqslant \sigma_5(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}, \tag{2.40}$$

where $\sigma_5(s) = \sigma_2(s) + \sigma_3(s) + \sigma_4(s)$. Finally, using (2.35) and (2.40), we conclude that $u \in L^(0,T;H^4(\mathbb{R}))$ and

$$\|u\|_{3,T} \leqslant \beta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\beta_3(s) = \sigma_1(s) + \sigma_5(s)$. □

Next, we shall show the well-posedness of the initial-value problem (IVP) (1.1) in the space $H^s(\mathbb{R})$ for $0 \leqslant s \leqslant 3$ and $1 \leqslant p < 2$. To do this, we shall use a method introduced by Tartar [24] and adapted by Bona and Scott [6, theorem 4.3] to prove the global well-posedness of the pure initial-value problem for the KdV equation on the whole line in fractional order Sobolev spaces $H^s(\mathbb{R})$.

Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$, with the inclusion map being continuous. For $f \in B_0$ and $t \geqslant 0$, let

$$K(f,t) = \inf_{g\in B_1}\{\|f-g\|_{B_0} + t\|g\|_{B_1}\}.$$

For $0 < \theta < 1$ and $1 \leqslant p \leqslant +\infty$, define

$$\mathbb{B}_{\theta,p} := [B_0,B_1]_{\theta,p} = \left\{f \in B_0\colon \|f\|_{\theta,p} := \left(\int_0^\infty K(f,t)t^{-\theta p-1}\,\mathrm{d}t\right)^{1/p} < \infty\right\}$$

with the usual modification for the $p = \infty$ case. Then, $B_{\theta,p}$ is a Banach space with norm $\|\cdot\|_{\theta,p}$. Given two pairs, (θ_1,p_1) and (θ_2,p_2), as above, we write $(\theta_1,p_1) \prec (\theta_2,p_2)$ when

$$\theta_1 < \theta_2 \quad \text{or} \quad \theta_1 = \theta_2 \text{ and } p_1 > p_2.$$

If $(\theta_1,p_1) \prec (\theta_2,p_2)$, then $\mathbb{B}_{\theta_2,p_2} \subset \mathbb{B}_{\theta_1,p_1}$ with the inclusion map being continuous.

Then, the following result holds.

Theorem 2.10. *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$, for $j = 1,2$, with continuous inclusion mappings. Let α and q lie in the ranges $0 < \alpha < 1$ and $1 \leqslant q \leqslant \infty$. Suppose that $\mathcal{A}$ is a mapping satisfying*

(i) $\mathcal{A}\colon \mathbb{B}_{\alpha,q}^1 \to B_0^2$ *and, for* $f,g \in \mathbb{B}_{\alpha,q}^1$,

$$\|\mathcal{A}f - \mathcal{A}g\|_{B_0^2} \leqslant C_0(\|f\|_{\mathbb{B}_{\alpha,q}^1} + \|g\|_{\mathbb{B}_{\alpha,q}^1})\|f-g\|_{B_0^1},$$

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(ii) $\mathcal{A}\colon B_1^1 \to B_1^2$ *and, for* $h \in \mathbb{B}_1^1$,

$$\|\mathcal{A}h\|_{B_1^2} \leqslant C_1(\|h\|_{\mathbb{B}^1_{\alpha,q}})\|h\|_{B_1^1},$$

where $C_j\colon \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous non-decreasing functions for* $j = 0, 1$. *Then, if* $(\theta, p) \geqslant (\alpha, q)$, $\mathcal{A}$ *maps* $\mathbb{B}^1_{\theta,p}$ *into* $\mathbb{B}^2_{\theta,p}$ *and for* $f \in \mathbb{B}^1_{\theta,p}$ *we have*

$$\|\mathcal{A}f\|_{\mathbb{B}^2_{\theta,p}} \leqslant C(\|f\|_{\mathbb{B}^1_{\alpha,q}})\|f\|_{\mathbb{B}^1_{\theta,p}},$$

where $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^{\theta}$, $r > 0$.

Proof. See [6, theorem 4.3]. □

This theorem leads to the main result of this section.

THEOREM 2.11. *Let* a *be a* $C^1(\mathbb{R})$*-function satisfying*

$$|a(\mu)| \leqslant C(1 + |\mu|^p), \quad |a'(\mu)| \leqslant C(1 + |\mu|^{p-1}), \quad \forall \mu \in \mathbb{R},$$

with $1 \leqslant p < 2$, *and let* $T > 0$ *and* $0 \leqslant s \leqslant 3$ *be given. In addition, assume that* $b \in L^\infty(\mathbb{R})$ *when* $s = 0$ *and* $b \in H^1(\mathbb{R})$ *when* $s > 0$. *Then, for any* $u_0 \in H^s(\mathbb{R})$, *the IVP (1.1) admits a unique solution* $u \in B_{s,T}$. *Moreover, there exists a non-decreasing continuous function* $\beta_s\colon \mathbb{R}^+ \to \mathbb{R}^+$, *such that*

$$\|u\|_{B_{s,T}} \leqslant \beta_s(\|u_0\|_2)\|u_0\|_{H^s(\mathbb{R})}.$$

Proof. We define

$$B_0^1 = L^2(\mathbb{R}), \quad B_0^2 = B_{0,T}, \quad B_1^1 = H^3(\mathbb{R}) \quad \text{and} \quad B_1^2 = B_{3,T}.$$

Thus,

$$\mathbb{B}^1_{s/3,2} = [L^2(\mathbb{R}), H^3(\mathbb{R})]_{s/3,2} = H^s(\mathbb{R}) \quad \text{and} \quad \mathbb{B}^2_{s/3,2} = [B_{0,T}, B_{3,T}]_{s/3,2} = B_{s,T}.$$

Combining proposition 2.7 and theorem 2.9, we obtain (i) and (ii) in theorem 2.10. Then, theorem 2.10 yields the result. □

Theorem 2.11 gives a strong smoothing property for the solutions of the problem.

COROLLARY 2.12. *Under the assumptions of theorem 2.11, for any* $u_0 \in L^2(\mathbb{R})$ *the corresponding solution* u *of (1.1) belongs to*

$$B_{3,[\varepsilon,T]} = C([\varepsilon, T]; H^3(\mathbb{R})) \cap L^2(\varepsilon, T; H^4(\mathbb{R}))$$

for every $T > 0$ *and* $0 < \varepsilon < T$.

Proof. The same result was obtained for the generalized KdV and the KdV–Burgers equations in [19] and [10], respectively. Since the proof is analogous and follows from classical arguments, we omit it here. □

2.2. The $p \geqslant 2$ case

We first restrict ourselves to the $2 \leqslant p < 5$ case in order to obtain the existence of solutions in the L^2-setting, i.e. finite energy solutions. Next, we prove the global well-posedness in the space $B_{3,T}$.

First, we recall the following result, which follows from the Egoroff theorem.

LEMMA 2.13. *Let Ω be an open set in $\mathbb{R}^N$, $N \geqslant 1$, and let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ with $1 < p < \infty$, such that $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $f_n(x) \to g(x)$ a.e. Then $f(x) = g(x)$ a.e.*

Unlike the $1 \leqslant p < 2$ case, the next result is not obtained by combining semigroup theory and fixed-point arguments. Here, due to some technical problems, the solution is obtained as a limit of the regular problems. We follow the ideas in [19].

THEOREM 2.14. *Let a be a $C^1(\mathbb{R})$-function satisfying*

$$|a(\mu)| \leqslant C|\mu|^p \qquad |a'(\mu)| \leqslant C|\mu|^{p-1}, \quad \forall \mu \in \mathbb{R}, \tag{2.41}$$

with $2 \leqslant p < 5$. Then, for any $u_0 \in L^2(\mathbb{R})$, problem (1.1) admits at least one solution u, such that

$$u \in C_w([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})), \quad \forall T > 0.$$

Proof. Consider a sequence $\{a_n\} \in C_0^\infty(\mathbb{R})$ such that

$$|a_n^{(j)}(\mu)| \leqslant C(1 + |\mu|^{p-j}), \quad \forall \mu \in \mathbb{R},\ j = 0, 1, \tag{2.42}$$

$$a_n \to a \quad \text{uniformly in each compact set in } \mathbb{R}. \tag{2.43}$$

Note that $|a_n(\mu)| \leqslant C_n(1 + |\mu|)$ and $|a_n'(\mu)| \leqslant C_n$. Then, for each n, theorem 2.6 guarantees the existence of a function $u_n \in B_{0,T}$ as a solution of

$$\left.\begin{aligned} \partial_t u_n + \partial_x^3 u_n - \partial_x^2 u_n + a_n(u_n)\partial_x u_n + b(x)u_n = 0,\\ u_n(0,x) = u_0(x), \end{aligned}\right\} \tag{2.44}$$

with $\|u_n\|_{0,T} \leqslant 2C_T\|u_0\|_{L^2(\mathbb{R})}$. Hence,

$$\{u_n\} \text{ is bounded in } C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})). \tag{2.45}$$

From (2.45) we obtain a function u and a subsequence, still denoted by the same index n, such that

$$u_n \rightharpoonup u \quad \text{in } L^\infty(0,T; L^2(\mathbb{R})) \text{ weak}^*, \tag{2.46}$$

$$u_n \rightharpoonup u \quad \text{in } L^2(0,T; H^1(\mathbb{R})) \text{ weak}. \tag{2.47}$$

In order to analyse the nonlinear term $a_n(u_n)\partial_x u_n$, we consider the functions

$$A(u) := \int_0^u a(v)\,\mathrm{d}v \quad \text{and} \quad A_n(u) := \int_0^u a_n(v)\,\mathrm{d}v. \tag{2.48}$$

Note that $a_n(u_n)\partial_x u_n = \partial_x[A_n(u_n)]$. Then, taking $\alpha \in (1, 6/(p+1))$ and proceeding as in the proof of [19, theorem 2.14], we deduce that, for each interval $I \subset \mathbb{R}$, the

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sequence $\{A_n(u_n)\}$ is bounded in $L^\alpha([0,T]\times I)$. Indeed,

$$|A_n(u)|^\alpha \leqslant C\left(2|u| + \frac{|u|^{p+1}}{p+1}\right)^\alpha \leqslant C'(|u|^\alpha + |u|^{\alpha(p+1)}), \tag{2.49}$$

where C and C' denote some positive constants depending only on p and α. Therefore,

$$\begin{aligned}
&\|A_n(u_n)\|^\alpha_{L^\alpha((0,T)\times I)}\\
&\quad\leqslant C''\left(\|u_n\|^\alpha_{L^2(0,T;L^2(I))} + \int_0^T \|u_n(t)\|_\infty^{\alpha(p+1)-2}\|u_n(t)\|_2^2\,\mathrm{d}t\right)\\
&\quad\leqslant C''\left(\|u_n\|^\alpha_{0,T} + 2^{(\alpha(p+1)-2)/2}\int_0^T \|u_n(t)\|_2^{(\alpha(p+1)+2)/2}\|u_{nx}(t)\|_2^{(\alpha(p+1)-2)/2}\,\mathrm{d}t\right)\\
&\quad\leqslant C''(\|u_n\|^\alpha_{0,T} + 2^{(\alpha(p+1)-2)/2}T^{(6-\alpha(p+1))/4}\|u_n\|_{0,T}^{(\alpha(p+1)+2)/2}\|u_n\|_{0,T}^{(\alpha(p+1)-2)/2})\\
&\quad\leqslant C''(\|u_n\|^\alpha_{0,T} + \|u_n\|_{0,T}^{\alpha(p+1)})\\
&\quad\leqslant \tilde{C}(\|u_0\|_2^\alpha + \|u_0\|_2^{\alpha(p+1)}),
\end{aligned} \tag{2.50}$$

where $\tilde{C}$ is a positive constant. Consequently,

$$\{A_n(u_n)\}\quad \text{is bounded in } L^\alpha(0,T;H^{-1}(I))\quad (\text{since } L^\alpha(I)\hookrightarrow H^{-1}(I))$$

and

$$\{a_n(u_n)\partial_x u_n\} = \{\partial_x[A_n(u_n)]\}\quad \text{is bounded in } L^\alpha(0,T;H^{-2}(I)). \tag{2.51}$$

Moreover, (2.45) and the fact that $1<\alpha\leqslant 2$ allow us to conclude that

$$\{\partial_x^3 u_n\}, \{\partial_x^2 u_n\} \text{ and } \{bu_n\} \text{ are bounded in } L^2(0,T;H^{-2}(\mathbb{R})) \subset L^\alpha(0,T;H^{-2}(\mathbb{R})),$$

and therefore

$$\partial_t u_n = -\partial_x^3 u_n + \partial_x^2 u_n - a_n(u_n)\partial_x u_n - bu_n \text{ is bounded in } L^\alpha(0,T;H^{-2}(I)). \tag{2.52}$$

Since $\{u_n\}$ is bounded in $L^\alpha(0,T;H^1(\mathbb{R}))$ and the first embedding in $H^1(I)\hookrightarrow L^2(I)\hookrightarrow H^{-2}(I)$ is compact, we can apply [22, corollary 4, p. 85] to conclude that $\{u_n\}$ is relatively compact in $L^2(0,T;L^2(I))$. Using a diagonal process, we obtain a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \to u\quad \text{in } L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R})) \text{ strongly and a.e.} \tag{2.53}$$

Moreover, by (2.47),

$$u_n \rightharpoonup u\quad \text{weak in } L^2(0,T;L^2(\mathbb{R})) \equiv L^2(\mathbb{R}\times(0,T))$$

and by applying lemma 2.13 we obtain

$$u_n \to u\quad \text{a.e in } \mathbb{R}\times(0,T). \tag{2.54}$$

Then, using (2.43), (2.48) and (2.54), it is easy to see that

$$A_n(u_n(x,t)) \to A(u(x,t))\quad \text{a.e in } \mathbb{R}\times(0,T).$$

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Next, proceeding as in the previous steps and by applying lemma 2.13, the following convergence holds:

$$A_n(u_n) \rightharpoonup A(u) \quad \text{weak in } L^\alpha(0,T;L^\alpha_{\text{loc}}(\mathbb{R})).$$

Therefore, $A_n(u_n) \to A(u)$ in $D'(\mathbb{R}\times(0,T))$ and, by taking the partial derivative, we obtain

$$a_n(u_n)\partial_x u_n \to a(u)\partial_x u \quad \text{in } D'(\mathbb{R}\times(0,T)). \tag{2.55}$$

From (2.53) and (2.55), we can take the limit in (2.44) to conclude that u solves (1.1) in the sense of distribution, i.e.

$$u_t + u_{xxx} - u_{xx} + a(u)u_x + bu = 0 \quad \text{in } D'(\mathbb{R}\times(0,T)). \tag{2.56}$$

On the other hand, by (2.45) and (2.52), we infer from [22, corollary 4, p. 85] that $\{u_n\}$ is relatively compact in $C([0,T];H^{-1}_{\text{loc}}(\mathbb{R}))$. Therefore, there exists a subsequence (denoted by $\{u_n\}$), such that

$$u_n \to u \quad \text{in } C([0,T];H^{-1}_{\text{loc}}(\mathbb{R})). \tag{2.57}$$

In particular, $u(x,0) = \lim_{n\to\infty} u_n(x,0) = u_0(x)$. Now, note that (2.47) yields

$$\begin{aligned}
u_{xxx} &\in L^2(0,T;H^{-2}(\mathbb{R})) \hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})),\\
u_{xx} &\in L^2(0,T;H^{-1}(\mathbb{R})) \hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})),\\
bu &\in L^2(0,T;H^{1}(\mathbb{R})) \hookrightarrow L^\alpha(0,T;H^{-2}(\mathbb{R})).
\end{aligned}$$

Finally, we claim that

$$a(u)u_x = [A(u)]_x \in L^\alpha(0,T;H^{-2}(\mathbb{R})) \tag{2.58}$$

for any $\alpha \in (1, 6/(p+1))$. In fact, first note that $\beta = \frac{1}{2}\alpha(p+1) - 2 < 2$, then (2.46) and (2.47) imply that $u \in L^\infty(0,T,L^2(\mathbb{R})) \cap L^2(0,T,H^1(\mathbb{R})) \subset L^\infty(0,T,L^2(\mathbb{R})) \cap L^\beta(0,T,H^1(\mathbb{R}))$. Moreover, by using (2.41) and (2.48), there exists $C = C(\alpha,p) > 0$ such that $|A(u)|^\alpha \leqslant C|u|^{\alpha(p+1)}$. Thus, we obtain

$$\begin{aligned}
\|A(u)\|^\alpha_{L^\alpha((0,T)\times\mathbb{R})} &= \int_0^T\int_{\mathbb{R}} |A(u)(x,t)|^\alpha \,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_0^T\int_{\mathbb{R}} |u(x,t)|^{\alpha(p+1)}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\int_0^T \|u(t)\|_\infty^{\alpha(p+1)-2}\|u(t)\|_2^2\,\mathrm{d}t\\
&\leqslant 2^{(\alpha(p+1)-2)/2}C\int_0^T \|u(t)\|_2^{(\alpha(p+1)+2)/2}\|u_x(t)\|_2^{(\alpha(p+1)-2)/2}\,\mathrm{d}t\\
&\leqslant 2^{(\alpha(p+1)-2)/2}C\|u\|^{(\alpha(p+1)+2)/2}_{L^\infty(0,T,L^2(\mathbb{R}))}\|u\|^\beta_{L^\beta(0,T,H^1(\mathbb{R}))}.
\end{aligned}$$

This yields that

$$A(u) \in L^\alpha(0,T,L^\alpha(\mathbb{R})). \tag{2.59}$$

Furthermore, since $\alpha \in (1,2)$, it is easy to see that $L^\alpha(\mathbb{R}) \subset H^{-1}(\mathbb{R})$. Indeed, taking $v \in L^\alpha(\mathbb{R})$ with $1 < \alpha < 2$ for any $q > 1$, it thus follows that

$$\|v\|^2_{H^{-1}(\mathbb{R})} = \int_{\mathbb{R}} (1+|\xi|^2)^{-1}|\hat{v}(\xi)|^2\,\mathrm{d}\xi \leqslant K\|\hat{v}\|^2_{L^{2q}(\mathbb{R})},$$

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where

$$K = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^{-q/(q-1)} \, \mathrm{d}\xi \right)^{(q-1)/q}.$$

In order to obtain finite K, we take $q = \alpha/2(\alpha-1) > 1$ and, applying the Hausdorff–Young inequality

$$\|\hat{v}\|_{L^{\alpha/(\alpha-1)}(\mathbb{R})} \leqslant C_\alpha \|v\|_{L^\alpha(\mathbb{R})} \quad \text{for some } C_\alpha > 0,$$

we obtain

$$\|v\|^2_{H^{-1}(\mathbb{R})} \leqslant M \|v\|^2_{L^\alpha(\mathbb{R})},$$

where M is a positive constant, which proves that $L^\alpha(\mathbb{R}) \subset H^{-1}(\mathbb{R})$. Thus, (2.59) implies that $A(u) \in L^\alpha(0, T, H^{-1}(\mathbb{R}))$, proving the claim (2.58). Now, by (2.56), we deduce that $u_t \in L^\alpha(0, T; H^{-2}(\mathbb{R}))$, and then $u \in W^{1,\alpha}(0, T; H^{-2}(\mathbb{R}))$. Since $\alpha > 1$, we conclude that $u \in C([0, T]; H^{-2}(\mathbb{R}))$. In particular, we obtain

$$u \in L^\infty(0, T; L^2(\mathbb{R})) \cap C_w([0, T]; H^{-2}(\mathbb{R})),$$

and from [25, ch. III, lemma 1.4] it follows that $u \in C_w([0, T]; L^2(\mathbb{R}))$. □

Remark 2.15. When $2 \leqslant p < 4$ we can prove theorem 2.14 with more general assumptions on the function $a(\cdot)$. More precisely,

$$|a(\mu)| \leqslant C(1 + |\mu|^p), \quad |a'(\mu)| \leqslant C(1 + |\mu|^{p-1}), \quad \forall \mu \in \mathbb{R}.$$

The proof follows the same steps as those above, except for (2.59). Indeed, we first claim that there exists $\alpha \in (1, 6/(p + 1))$ such that

$$A(u) \in L^\alpha(0, T, L^2(\mathbb{R})). \tag{2.60}$$

To prove it, note that

$$\begin{aligned}
\|A(u)\|^\alpha_{L^\alpha(0,T,L^2(\mathbb{R}))}
&= \int_0^T \left(\int_{\mathbb{R}} |A(u)(x, t)|^2 \, \mathrm{d}x \right)^{\alpha/2} \mathrm{d}t \\
&\leqslant C \int_0^T \left(\int_{\mathbb{R}} (|u(x, t)|^2 + |u(x, t)|^{2(p+1)} \right) \mathrm{d}x)^{\alpha/2} \, \mathrm{d}t \\
&\leqslant C \left(\int_0^T \|u(t)\|^\alpha_{L^2(\mathbb{R})} \, \mathrm{d}t + \int_0^T \left(\int_{\mathbb{R}} |u(x, t)|^{2(p+1)} \, \mathrm{d}x \right)^{\alpha/2} \mathrm{d}t \right) \\
&\leqslant C \left(\|u\|^\alpha_{L^\alpha(0,T;L^2(\mathbb{R}))} + \int_0^T \|u(t)\|^{\alpha p}_{L^\infty(\mathbb{R})} \|u(t)\|^\alpha_{L^2(\mathbb{R})} \, \mathrm{d}t \right) \\
&\leqslant C \left(\|u\|^\alpha_{L^\alpha(0,T;L^2(\mathbb{R}))} + 2^{\alpha p/2} \int_0^T \|u(t)\|_2^{\alpha(p+2)/2} \|u_x(t)\|_2^{\alpha p/2} \, \mathrm{d}t \right).
\end{aligned} \tag{2.61}$$

Since $1 < 4/p \leqslant 6/(p+1)$, by picking any $\alpha \in (1, 4/p)$ and by using (2.47) we obtain $u \in L^\infty(0, T, L^2(\mathbb{R})) \cap L^2(0, T, H^1(\mathbb{R})) \subset L^\alpha(0, T, L^2(\mathbb{R})) \cap L^{\alpha p/2}(0, T, H^1(\mathbb{R}))$.

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Hence, (2.61) implies that

$$\|A(u)\|^{\alpha}_{L^{\alpha}(0,T,L^2(\mathbb{R}))} \leqslant C\big(\|u\|^{\alpha}_{L^{\alpha}(0,T;L^2(\mathbb{R}))} + 2^{\alpha p/2}\|u\|^{\alpha(p+2)/2}_{L^{\infty}(0,T,L^2(\mathbb{R}))}\|u\|^{\alpha p/2}_{L^{\alpha p/2}(0,T,H^1(\mathbb{R}))}\big),$$

which proves (2.60). Consequently, we obtain (2.58) provided that $2 \leqslant p < 4$.

DEFINITION 2.16. Let $T > 0$. A function $u \in C_w([0,T]; L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R}))$ is said to be a weak solution of (1.1) if there exist a sequence $\{a_n\}$ of functions in $C_0^{\infty}(\mathbb{R})$ satisfying (2.42) and (2.43) and a sequence of strong solutions u_n to (2.44) such that (2.46), (2.47), (2.54) and (2.57) hold.

The proof of theorem 2.19 requires the following adaptation of lemma 2.4.

LEMMA 2.17. *For any $T > 0$, $p \geqslant 1$ and $u, v, w \in B_{3,T}$ such that $u_t, v_t, w_t \in B_{0,T}$, the following hold.*

(i) *We have*

$$\|(a(u)v_x)_x\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant CT^{1/2}\{\|u\|_{3,T}\|v\|_{3,T} + 2\|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\}.$$

(ii) *We have*

$$\begin{aligned}\|a(u)v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} &\leqslant CT^{1/2}\{(\|v\|_{3,T} + \|v_t\|_{0,T}) + \|u\|^p_{3,T}(\|v\|_{3,T} + \|v_t\|_{0,T}) \\ &\quad + \|u_t\|_{0,T}\|v\|_{3,T} + \|u_t\|_{0,T}\|v\|_{3,T}\|u\|^{p-1}_{3,T}\},\end{aligned}$$

(iii) *We have*

$$\|uw_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} \leqslant 2^{1/2}T^{1/4}\{\|u\|_{3,T}\|w\|_{3,T} + \|u_t\|_{0,T}\|w\|_{3,T} + \|u\|_{3,T}\|w_t\|_{0,T}\}.$$

(iv) *If $p \geqslant 2$, then*

$$\begin{aligned}\|u|w|^{p-1}v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} &\leqslant T^{1/2}\{\|u\|_{3,T}\|w\|^{p-1}_{3,T}(\|v\|_{3,T} + \|v_t\|_{0,T}) \\ &\quad + \|v\|_{3,T}\|w\|^{p-1}_{3,T}\|u_t\|_{0,T} + (p-1)\|u\|_{3,T}\|w\|^{p-2}_{3,T}\|v\|_{3,T}\|w_t\|_{0,T}\}.\end{aligned}$$

Proof. First, note that if $u \in B_{3,T}$, then we have

$$\left.\begin{aligned}\partial_x^j u \in C([0,T];H^{3-j}(\mathbb{R})) \hookrightarrow C([0,T];C(\mathbb{R})),\\ \|\partial_x^j u\|_{C([0,T];C(\mathbb{R}))} \leqslant C\|u\|_{3,T},\end{aligned}\right\} \quad j = 0,1,2, \tag{2.62 a}$$

$$\left.\begin{aligned}\partial_x^j u \in L^2([0,T];H^{4-j}(\mathbb{R})) \hookrightarrow L^2([0,T];L^2(\mathbb{R})),\\ \|\partial_x^j u\|_{L^2([0,T];L^2(\mathbb{R}))} \leqslant C\|u\|_{3,T},\end{aligned}\right\} \quad j = 0,1,2,3. \tag{2.62 b}$$

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(i) Formulae (1.3) and (2.62) imply that

$$
\begin{aligned}
&\|(a(u)v_x)_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\
&\quad\leqslant C\{T^{1/2}\|u_x\|_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C(0,T;L^2(\mathbb{R}))}\\
&\qquad\quad + T^{1/2}\|u\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|u_x\|_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C(0,T;L^2(\mathbb{R}))}\\
&\qquad\quad + T^{1/2}\|v_{xx}\|_{C([0,T];L^2(\mathbb{R}))} + T^{1/2}\|u\|^{p}_{C([0,T];C(\mathbb{R}))}\|v_{xx}\|_{C(0,T;L^2(\mathbb{R}))}\}\\
&\quad\leqslant CT^{1/2}\{\|u\|_{3,T}\|v\|_{3,T} + 2\|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\}.
\end{aligned}
$$

(ii) By (1.3), the Hölder inequality and (2.62) we get

$$
\begin{aligned}
&\|a(u)v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))}\\
&\quad\leqslant C\{\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2} + \|u\|^p_{C([0,T];C(\mathbb{R}))}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\qquad\quad + \|v_x\|_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\qquad\quad + \|u\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\qquad\quad + T^{1/2}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))} + \|u\|^p_{C([0,T];C(\mathbb{R}))}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\}\\
&\quad\leqslant CT^{1/2}\{\|v\|_{3,T} + \|u\|^p_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\|u_t\|_{0,T}\\
&\qquad\qquad + \|u\|^{p-1}_{3,T}\|v\|_{3,T}\|u_t\|_{0,T} + \|v_t\|_{0,T} + \|u\|^p_{3,T}\|v_t\|_{0,T}\}.
\end{aligned}
$$

(iii) This is a consequence of lemma 2.4(ii).

(iv) The Hölder inequality and (2.62) lead to the desired result:

$$
\begin{aligned}
&\|u|w|^{p-1}v_x\|_{W^{1,1}(0,T;L^2(\mathbb{R}))}\\
&\leqslant \|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\quad + \|v_x\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|u_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\quad + (p-1)\|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-2}_{C([0,T];C(\mathbb{R}))}\|v_x\|_{C([0,T];C(\mathbb{R}))}\|w_t\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\quad + \|u\|_{C([0,T];C(\mathbb{R}))}\|w\|^{p-1}_{C([0,T];C(\mathbb{R}))}\|v_{tx}\|_{L^2(0,T;L^2(\mathbb{R}))}T^{1/2}\\
&\leqslant T^{1/2}\{\|u\|_{3,T}\|w\|^{p-1}_{3,T}\|v\|_{3,T} + \|v\|_{3,T}\|w\|^{p-1}_{3,T}\|u_t\|_{0,T}\\
&\quad + (p-1)\|u\|_{3,T}\|w\|^{p-2}_{3,T}\|v\|_{3,T}\|w_t\|_{0,T} + \|u\|_{3,T}\|w\|^{p-1}_{3,T}\|v_t\|_{0,T}\}.
\end{aligned}
$$

□

Proposition 2.1 asserts that the inhomogeneous linear problem (2.1) is well posed, and we obtain the existence of a mild solution. However, we can have a regular solution as shown by the following.

PROPOSITION 2.18. *Let $T > 0$, $b \in H^1(\mathbb{R})$ and $u_0 \in H^3(\mathbb{R})$. If $f \in W^{1,1}(0,T; L^2(\mathbb{R}))$ and $f_x \in L^2(0,T;L^2(\mathbb{R}))$, the inhomogeneous linear problem (2.1) has a unique regular solution $u \in B_{3,T}$ such that*

$$\|u\|_{3,T} \leqslant C_{3,T}\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}, \tag{2.63}$$

$$\|u_t\|_{0,T} \leqslant C_{0,T}\{\|u_0\|_{H^3(\mathbb{R})} + \|f(0)\|_{L^2(\mathbb{R})} + \|f_t\|_{L^1(0,T;L^2(\mathbb{R}))}\} \tag{2.64}$$

and $u_t \in B_{0,T}$, where $C_{3,T} = 2C\mathrm{e}^{\|b\|_\infty T}$ and $C_{0,T} = 2\mathrm{e}^{\|b\|_\infty T}$.

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Proof. By using semigroup theory and the previous results, we obtain a unique regular solution $u \in C([0,T]; H^3(\mathbb{R}))$. Therefore, we shall prove that $u \in L^2(0,T, H^4(\mathbb{R}))$. Indeed, first note that $u_0 \in H^3(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$. Hence, by applying proposition 2.1 it follows that $u \in B_{0,T}$ and

$$\|u\|_{0,T} \leqslant C_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}, \tag{2.65}$$

where $C_T = 2\mathrm{e}^{\|b\|_\infty T}$. On the other hand, note that u_t solves the following problem:

$$\begin{aligned} v_t - v_{xx} + v_{xxx} + bv &= f_t \quad \text{in } \mathbb{R} \times (0,\infty),\\ v(0) &= v_0 \quad \text{in } \mathbb{R} \times (0,\infty), \end{aligned}$$

where $v_0 = \partial_x^2 u_0 - \partial_x^3 u_0 - bu_0 + f(\cdot,0) \in L^2(\mathbb{R})$. Then, by applying proposition 2.1, we have $u_t \in B_{0,T}$ and

$$\|u_t\|_{0,T} \leqslant C_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f(0)\|_2 + \|f_t\|_{L^1(0,T;L^2(\mathbb{R}))}\}, \tag{2.66}$$

which yield (2.64). Moreover,

$$\begin{aligned} \|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))} &\leqslant \|b_x\|_2 \|u\|_{L^2(0,T;L^\infty(\mathbb{R}))} + \|b\|_\infty \|u_x\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\leqslant C\|b\|_{H^1(\mathbb{R})}\|u\|_{0,T}, \end{aligned} \tag{2.67}$$

where C is the embedding constant of $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Since

$$\partial_x^4 u = \partial_x^3 u - \partial_x u_t - \partial_x(bu) + \partial_x f \quad \text{in } D'(\mathbb{R}), \quad \forall t > 0,$$

we have that $\partial_x^4 u \in L^2(0,T;L^2(\mathbb{R}))$, i.e. $u \in L^2(0,T;H^4(\mathbb{R}))$ and $u \in B_{3,T}$. In order to prove (2.63), we need some estimates. Note that from (2.65) we get

$$\sup_{t\in[0,T]} \|u(t)\|_2 \leqslant C_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \tag{2.68}$$

Multiplying the equation in (2.1) by u_{xx} and integrating in $\mathbb{R}$, one obtains the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(t)\|_2^2 + \|u_{xx}(t)\|_2^2 \leqslant \{\|f(t)\|_2 + \|bu(t)\|_2\}\|u_{xx}(t)\|_2.$$

Then, Young's inequality leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(t)\|_2^2 + \tfrac{1}{2}\|u_{xx}(t)\|_2^2 \leqslant C\{\|f(t)\|_2^2 + \|b\|_\infty^2\|u(t)\|_2^2\}.$$

By integrating on $[0,T]$, using (2.65) and the embedding $W^{1,1}(0,T)(0,T;L^2(\mathbb{R})) \hookrightarrow L^\infty(0,T;L^2(\mathbb{R}))$, the solution can be estimated as follows:

$$\sup_{t\in[0,T]} \|u_x(t)\|_2 \leqslant CC_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \tag{2.69}$$

A similar estimate is obtained by multiplying the equation by $\partial_x^4 u$, integrating in $\mathbb{R}$ and using Young's inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{xx}(t)\|_2^2 + \tfrac{1}{2}\|u_{xxx}(t)\|_2^2 \leqslant C\{\|f_x(t)\|_2^2 + \|(bu)_x(t)\|_2^2\}.$$

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Integrating on $[0,T]$ and using (2.67) and (2.65) yields

$$\sup_{t\in[0,T]} \|u_{xx}(t)\|_2 \leqslant CC_T\{\|u_0\|_{H^3(\mathbb{R})}+\|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))}+\|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \quad (2.70)$$

Since

$$\|u_{xxx}(t)\|_2 \leqslant \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + \|bu(t)\|_2 + \|f(t)\|_2,$$

using (2.65), (2.66), (2.70) and the embedding above, we conclude that

$$\sup_{t\in[0,T]} \|u_{xxx}(t)\|_2 \leqslant CC_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \quad (2.71)$$

Putting together (2.68), (2.69), (2.70) and (2.71), we obtain

$$\|u\|_{C([0,T];H^3(\mathbb{R}))} \leqslant CC_T\{\|u_0\|_{H^3(\mathbb{R})}+\|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))}+\|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \quad (2.72)$$

On the other hand,

$$\begin{aligned}\|\partial_x^4 u\|_{L^2(0,T;L^2(\mathbb{R}))} &\leqslant \|u_{xxx}\|_{L^2(0,T;L^2(\mathbb{R}))} + \|\partial_x u_t\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\quad + \|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\leqslant T^{1/2}\|u\|_{C([0,T];H^3(\mathbb{R}))} + \|u_t\|_{0,T} + \|(bu)_x\|_{L^2(0,T;L^2(\mathbb{R}))} \\ &\quad + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}.\end{aligned}$$

The above inequality, (2.65)–(2.67) and (2.71) allow us to conclude that

$$\|\partial_x^4 u\|_{L^2(0,T;L^2(\mathbb{R}))} \leqslant CC_T\{\|u_0\|_{H^3(\mathbb{R})} + \|f\|_{W^{1,1}(0,T;L^2(\mathbb{R}))} + \|f_x\|_{L^2(0,T;L^2(\mathbb{R}))}\}. \quad (2.73)$$

Estimates (2.72) and (2.73) imply (2.63). □

THEOREM 2.19. *Let $b \in H^1(\mathbb{R})$ and $a \in C^2(\mathbb{R})$ satisfy*

$$|a(\mu)| \leqslant C(1+|\mu|^p), \quad |a'(\mu)| \leqslant C(1+|\mu|^{p-1}) \quad \text{and} \quad |a''(\mu)| \leqslant C(1+|\mu|^{p-2}), \quad \forall \mu \in \mathbb{R}, \quad (2.74)$$

with $p \geqslant 2$. Let $T > 0$ and $u_0 \in H^3(\mathbb{R})$. Then, there exists a unique solution $u \in B_{3,T}$ of (1.1) *such that*

$$\|u\|_{3,T} \leqslant \eta_3(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})},$$

where $\eta_3 \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing continuous function.

Proof. Let $0 < \theta \leqslant T$ and $R > 0$ to be a constant to be determined later. Consider

$$S_{\theta,R} := \{(u,v) \in B_{3,\theta} \times B_{0,\theta} \colon v = u_t, \|(u,v)\|_{B_{3,\theta}\times B_{0,\theta}} := \|u\|_{3,\theta} + \|v\|_{0,\theta} \leqslant R\}.$$

Then, for each $(u,u_t) \in S_{\theta,R} \subset B_{3,\theta} \times B_{0,\theta}$, consider the problems

$$\left.\begin{aligned} v_t &= A_b v - a(u)u_x, \\ v(0) &= u_0 \end{aligned}\right\} \quad (2.75)$$

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and

$$\left.\begin{aligned} z_t &= A_b z - [a(u)u_x]_t, \\ z(0) &= z_0, \end{aligned}\right\} \tag{2.76}$$

with $z_0 = -u_{0xxx} + u_{0xx} - bu_0 - a(u_0)u_{0x} \in L^2(\mathbb{R})$ and $A_b v = \partial_x^2 v - \partial_x^3 v - bv$. Recall that A_b generates a strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ of contractions in $L^2(\mathbb{R})$. Moreover, by lemma 2.17(i), (ii), $a(u)u_x \in W^{1,1}(0,\theta;L^2(\mathbb{R}))$ and $[a(u)u_x]_x \in L^2(0,\theta;L^2(\mathbb{R}))$. Then, by proposition 2.18, problems (2.75) and (2.76) have a unique mild solution v such that $(v, v_t) \in B_{3,\theta} \times B_{0,\theta}$ and

$$\|(v,v_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leqslant C_\theta\{\|u_0\|_{H^3(\mathbb{R})} + \|a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} + \|[a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}, \tag{2.77}$$

where $C_\theta = 2\mathrm{e}^{\theta\|b\|_\infty}$. Thus, we can define the operator

$$\Gamma\colon S_{\theta,R} \subset B_{3,\theta} \times B_{0,\theta} \to B_{3,\theta} \times B_{0,\theta} \quad \text{by } \Gamma(u,u_t) = (v,v_t).$$

Since $C_\theta \leqslant C_T$, from (2.77) and lemma 2.17, we have

$$\begin{aligned} \|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}} &\leqslant C_T\|u_0\|_{H^3(\mathbb{R})} \\ &\quad + C_T C\theta^{1/2}\{(\|u\|_{3,\theta} + \|u_t\|_{0,\theta}) + \|u\|_{3,\theta}^p(\|u\|_{3,\theta} + \|u_t\|_{0,\theta}) \\ &\qquad + \|u_t\|_{0,\theta}\|u\|_{3,\theta} + \|u_t\|_{0,\theta}\|u\|_{3,\theta}^p\} \\ &\quad + C_T C\theta^{1/2}\{\|u\|_{3,\theta}^2 + 2\|u\|_{3,\theta}^{p+1} + \|u\|_{3,T}\} \\ &\leqslant C_T\|u_0\|_{H^3(\mathbb{R})} + C_T C\theta^{1/2}\{4R^{p+1} + 2R^2 + 2R\}. \end{aligned}$$

Choosing $R = 2C_T\|u_0\|_{H^3(\mathbb{R})}$, it follows that

$$\|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}} \leqslant (K_1 + \tfrac{1}{2})R,$$

where $K_1(\theta) = C_T C\theta^{1/2}\{4R^p + 2R + 2\}$. On the other hand, let $(u,u_t),(w,w_t) \in S_{\theta,R}$ and note that $\Gamma(u,u_t) - \Gamma(w,w_t)$ is solutions of

$$\begin{aligned} v_t &= A_b v + [a(w)w_x - a(u)u_x], \\ v(0) &= 0 \end{aligned}$$

and

$$\begin{aligned} z_t &= A_b z + [a(w)w_x - a(u)u_x]_t, \\ z(0) &= 0. \end{aligned}$$

Hence, by lemma 2.17, the following estimate holds:

$$\begin{aligned} \|\Gamma(u,u_t) - \Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}} &\leqslant C_T\{\|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} \\ &\quad + \|[a(w)w_x - a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}. \end{aligned} \tag{2.78}$$

The next steps are devoted to estimating the terms on the right-hand side of (2.78), i.e.

$$\begin{aligned} \|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} &\leqslant \|(a(w) - a(u))w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}. \end{aligned}$$

By using the mean-valued theorem and lemma 2.17, we have

$$
\begin{aligned}
&\|a(w)w_x - a(u)u_x\|_{W^{1,1}(0,\theta;L^2)}\\
&\quad\leqslant C\{\|(1+|u|^{p-1}+|w|^{p-1})|w-u|w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}\\
&\qquad + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))} + \||w|^{p-1}|w-u|w_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}\\
&\qquad + \|a(u)(w-u)_x\|_{W^{1,1}(0,\theta;L^2(\mathbb{R}))}\}\\
&\quad\leqslant C\{2^{1/2}\theta^{1/4}\{\|w-u\|_{3,\theta}\|w\|_{3,\theta} + \|(w-u)_t\|_{0,\theta}\|w\|_{3,\theta} + \|w-u\|_{3,\theta}\|w_t\|_{0,\theta}\}\\
&\qquad + \theta^{1/2}\{\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-1}(\|w\|_{3,\theta} + \|w_t\|_{0,\theta})\\
&\qquad\qquad + \|(w-u)_t\|_{0,\theta}\|w\|_{3,\theta}\|u\|_{3,\theta}^{p-1}\\
&\qquad\qquad + (p-1)\|u_t\|_{0,\theta}\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}\}\\
&\qquad + \theta^{1/2}\{\|w-u\|_{3,\theta}\|w\|_{3,\theta}^{p-1}(\|w\|_{3,\theta} + \|w_t\|_{0,\theta})\\
&\qquad\qquad + \|w\|_{3,\theta}^{p}\|[w-u]_t\|_{0,\theta} + (p-1)\|w-u\|_{3,\theta}\|w\|_{3,\theta}^{p-1}\|w_t\|_{0,\theta}\}\\
&\qquad + C\theta^{1/2}\{(\|w-u\|_{3,\theta} + \|(w-u)_t\|_{0,\theta})\\
&\qquad\qquad + \|u\|_{3,\theta}^{p}(\|w-u\|_{3,\theta} + \|(w-u)_t\|_{0,\theta})\\
&\qquad\qquad + \|u_t\|_{0,\theta}\|w-u\|_{3,\theta} + \|u_t\|_{0,\theta}\|w-u\|_{3,\theta}\|u\|_{3,\theta}^{p-1}\}\\
&\quad\leqslant K_2\|(w-u,(w-u)_t)\|_{B_{3,\theta}\times B_{0,\theta}},
\end{aligned}
\tag{2.79}
$$

where $K_2(\theta) = C\{(2^{3/2}\theta^{1/4} + \theta^{1/2})R + 2(p+1)\theta^{1/2}R^p + \theta^{1/2}\}$. To estimate the second term, note that

$$
\begin{aligned}
[a(w)w_x - a(u)u_x]_x = [a'(w) - a'(u)]w_x^2 + a'(u)[w-u]_x[w+u]_x&\\
+ [a(w)-a(u)]w_{xx} + a(u)[w_{xx} - u_{xx}].&
\end{aligned}
\tag{2.80}
$$

Then, using the mean-value theorem, (2.62) and (2.74), we have the following estimates:

$$
\begin{aligned}
&\|[a'(w) - a'(u)]w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\quad\leqslant C\|[1+|w|^{p-2}+|u|^{p-2}]|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\quad\leqslant C\{\||w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \||w|^{p-2}|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\qquad\qquad + \||u|^{p-2}|w-u|w_x^2\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}\\
&\quad\leqslant C\theta^{1/2}\{\|w\|_{3,\theta}^2\|w-u\|_{3,\theta} + \|w\|_{3,\theta}^p\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}^2\|w-u\|_{3,\theta}\},
\end{aligned}
$$

$$
\begin{aligned}
&\|a'(u)[w-u]_x[w+u]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\quad\leqslant \|a'(u)[w-u]_xw_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \|a'(u)[w-u]_xu_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\quad\leqslant C\{\|[w-u]_xw_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \|[w-u]_xu_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\qquad + \|u|^{p-1}|[w-u]_xw_x\|_{L^2(0,\theta;L^2(\mathbb{R}))} + \||u|^{p-1}[w-u]_xu_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\}\\
&\quad\leqslant C\theta^{1/2}\{\|w\|_{3,\theta}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}\|w-u\|_{3,\theta}\\
&\qquad\qquad + \|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta}\|w-u\|_{3,\theta} + \|u\|_{3,\theta}^p\|w-u\|_{3,\theta}\}
\end{aligned}
$$

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and

$$\begin{aligned}
&\|[a(w)-a(u)]w_{xx}\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\qquad\leqslant C\theta^{1/2}\{\|w\|_{3,\theta}\|w-u\|_{3,\theta}\\
&\qquad\qquad+\|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta}\|w-u\|_{3,\theta}+\|w\|_{3,\theta}^{p}\|w-u\|_{3,\theta}\},\\
&\|a(u)[w_{xx}-u_{xx}]\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\qquad\leqslant C\theta^{1/2}\{\|w-u\|_{3,\theta}+\|u\|_{3,\theta}^{p}\|w-u\|_{3,\theta}\}.
\end{aligned}$$

The above estimates and (2.80), show that

$$\begin{aligned}
&\|[a(w)w_x-a(u)u_x]_x\|_{L^2(0,\theta;L^2(\mathbb{R}))}\\
&\quad\leqslant C\theta^{1/2}\{2\|w\|_{3,\theta}^{p}+\|w\|_{3,\theta}^{2}+2\|w\|_{3,\theta}+2\|u\|_{3,\theta}^{p}+\|u\|_{3,\theta}\\
&\qquad\qquad+2\|u\|_{3,\theta}^{p-1}\|w\|_{3,\theta}+\|u\|_{3,\theta}^{p-2}\|w\|_{3,\theta}^{2}+1\}\|w-u\|_{3,\theta}\\
&\quad\leqslant K_3\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}},
\end{aligned}\tag{2.81}$$

where $K_3(\theta)=C\theta^{1/2}\{7R^p+R^2+3R+1\}$. From (2.78), (2.79) and (2.81), we get

$$\|\Gamma(u,u_t)-\Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}}\leqslant K_4\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}},$$

where

$$\begin{aligned}
K_4&=C_T(K_2+K_3)\\
&=C_TC\theta^{1/2}\{2(p+9)R^p+R^{p-1}+R^2+4R+2\}+2^{3/2}C_TC\theta^{1/4}R.
\end{aligned}$$

Note that $K_1\leqslant K_4$. Therefore, choosing $\theta>0$ such that $K_4<\frac{1}{2}$, it follows that

$$\left.\begin{aligned}
\|\Gamma(u,u_t)\|_{B_{3,\theta}\times B_{0,\theta}}&\leqslant R,\\
\|\Gamma(u,u_t)-\Gamma(w,w_t)\|_{B_{3,\theta}\times B_{0,\theta}}&\leqslant \tfrac{1}{2}\|(w-u,[w-u]_t)\|_{B_{3,\theta}\times B_{0,\theta}},
\end{aligned}\right\}$$
$$\forall (u,u_t),(w,w_t)\in S_{\theta,R}\subset B_{3,\theta}\times B_{0,\theta}.$$

Hence, $\Gamma\colon S_{\theta,R}\to S_{\theta,R}$ is a contraction, and by the Banach fixed-point theorem we obtain a unique pair $(u,u_t)\in S_{\theta,R}$ such that $\Gamma(u,u_t)=(u,u_t)$. Thus, u is a unique local mild solution to problem (1.1) and satisfies

$$\|u\|_{3,\theta}\leqslant 2C_T\|u_0\|_{H^3(\mathbb{R})}.\tag{2.82}$$

Moreover, (2.82) implies the solution does not blow-up in finite time, and by using standard arguments we can extend θ to $[0,T]$. Finally, by defining $\eta_3(s)=2C_T$, the proof is complete. □

3. Exponential stability

This section is devoted to proving the exponential decay of the solutions under the assumptions (1.4) and (1.5). We consider two cases: $1\leqslant p<2$ and $2\leqslant p<5$.

3.1. The $1 \leqslant p < 2$ case

In order to make our work self-contained, we prove the following proposition, which is similar to [10, theorem 5.1].

PROPOSITION 3.1. *Let b satisfy (1.4). Then, for any $u_0 \in L^2(\mathbb{R})$ and $1 \leqslant p < 2$, the corresponding solution u of (1.1) is exponentially stable and satisfies the decay estimate*

$$\|u(t)\|_2 \leqslant \mathrm{e}^{-2\lambda_0 t}\|u_0\|_2, \quad \forall t \geqslant 0. \tag{3.1}$$

Proof. We first consider $u_0 \in H^3(\mathbb{R})$ and the corresponding smooth solution, u. Multiplying the equation in (1.1) by u and integrating in $\mathbb{R}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + 2\|u_x(t)\|_2^2 = -2\int_{\mathbb{R}} b(x)|u(x,t)|^2\,\mathrm{d}x.$$

Hence, proceeding as in [10, theorem 5.1], we obtain

$$\|u(t)\|_2 \leqslant \mathrm{e}^{-2\lambda_0 t}\|u_0\|_2.$$

Now, let $u_0 \in L^2(\mathbb{R})$ and let u be the corresponding mild solution given by theorem 2.6. Consider $\{u_{n,0}\} \in H^3(\mathbb{R})$, such that

$$u_{n,0} \to u_0 \quad \text{in } L^2(\mathbb{R}).$$

Then, the corresponding strong solutions u_n satisfy the estimate

$$\|u_n(t)\|_2 \leqslant \mathrm{e}^{-2\lambda_0 t}\|u_{n,0}\|_2. \tag{3.2}$$

On the other hand, note that the identity (2.12) in theorem 2.6 implies that, for all $t \geqslant 0$,

$$u_n \to u \quad \text{in } L^2(\mathbb{R}).$$

On taking the limit in (3.2), we obtain (3.1). □

COROLLARY 3.2. *Let $T > 0$, $u_0 \in L^2(\mathbb{R})$ and b satisfy (1.4). Then there exists a non-decreasing continuous function $\alpha_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that the corresponding solution, u, of (1.1) with $1 \leqslant p < 2$ satisfies*

$$\|u\|_{0,[t,t+T]} \leqslant \alpha_0(\|u_0\|_2)\mathrm{e}^{-2\lambda_0 t}, \quad \forall t \geqslant 0.$$

Proof. Note that, after a change of variable, the restriction of u to $[t, t+T]$ is a solution of problem (1.1) with respect to the initial data $u(t)$. Then, by theorem 2.11 and proposition 3.1 we have

$$\begin{aligned}\|u\|_{0,[t,t+T]} &\leqslant \beta_0(\|u(t)\|_2)\|u(t)\|_2 \\ &\leqslant \beta_0(\mathrm{e}^{-2\lambda_0 t}\|u_0\|_2)\|u_0\|_2\mathrm{e}^{-2\lambda_0 t} \\ &\leqslant \alpha_0(\|u_0\|_2)\mathrm{e}^{-2\lambda_0 t},\end{aligned}$$

where $\alpha_0(s) = \beta_0(s)s$. □

The next result was inspired by the ideas introduced in the proof of [10, theorem 6.1] and in [19, proposition 3.9].

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PROPOSITION 3.3. *Let $T > 0$, $1 \leqslant p < 2$, $a(0) = 0$ and b satisfy (1.4). Then, there exist $\gamma > 0$, $T_0 > 0$ and a non-negative continuous function $\alpha_3 \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $u_0 \in H^3(\mathbb{R})$ the corresponding solution, u, satisfies*

$$\|u(t)\|_{H^3(\mathbb{R})} \leqslant \alpha_3(\|u_0\|_2, T_0)\|u_0\|_{H^3(\mathbb{R})}\mathrm{e}^{-\gamma t}, \quad \forall t \geqslant T_0. \tag{3.3}$$

Proof. Let $v = u_t$. Then, by proposition 2.8, v solves the linearized equation (2.18) with $v_0 = -\partial_x^3 u_0 + \partial_x^2 u_0 - a(u_0)\partial_x u_0 - bu_0$ and satisfies

$$\|v\|_{0,T} \leqslant \sigma(\|u\|_{0,T})\|v_0\|_2. \tag{3.4}$$

After a change of variable, the restriction of v to $[t, t+T]$ is a solution of problem (2.18) with respect to the initial data $v(t)$ and

$$\|v\|_{0,[t,t+T]} \leqslant \sigma(\|u\|_{0,[t,t+T]})\|v(t)\|_2.$$

Applying corollary 3.2, it follows that

$$\|v\|_{0,[t,t+T]} \leqslant \sigma(\alpha_0(\|u_0\|_2)\mathrm{e}^{-2\lambda_0 t})\|v(t)\|_2 \leqslant \sigma(\alpha_0(\|u_0\|_2))\|v(t)\|_2. \tag{3.5}$$

On the other hand, the solution v may be written as

$$v(t) = S(t)v_0 - \int_0^t S(t-s)[a(u(s))v(s)]_x \,\mathrm{d}s,$$

where $S(t)$ is a C_0-semigroup of contraction in $L^2(\mathbb{R})$ generated by the operator A_b. Note that $v_1(t) = S(t)v_0$ is solution of problem (2.18) with $a(u) = 0$. Then, proceeding as in the proof of proposition 3.1, we have

$$\|v_1(t)\|_2 \leqslant \|v_0\|\mathrm{e}^{-2\lambda_0 t}, \quad \forall t \geqslant 0. \tag{3.6}$$

Let us now define

$$v_2(t) = \int_0^t S(t-s)[a(u(s))v(s)]_x \,\mathrm{d}s.$$

Note that

$$\|v_2(T)\|_2 \leqslant \|a'(u)u_x v\|_{L^1(0,T;L^2(\mathbb{R}))} + \|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))}.$$

Moreover, $a(0) = 0$ implies that $|a(u)| \leqslant C(1 + |u|^{p-1})|u|$ for some $C > 0$. Thus, by using lemma 2.4, the following holds:

$$\begin{aligned}\|v_2(T)\|_2 &\leqslant C\{\|(1+|u|^{p-1})u_x v\|_{L^1(0,T;L^2(\mathbb{R}))} + \|(1+|u|^{p-1})|u|v_x\|_{L^1(0,T;L^2(\mathbb{R}))}\} \\ &\leqslant 2C\{2^{1/2}T^{1/4}\|u\|_{0,T}\|v\|_{0,T} + 2^{p/2}T^{(2-p)/4}\|u\|_{0,T}^p\|v\|_{0,T}\}.\end{aligned} \tag{3.7}$$

Using (3.4), (3.6) and (3.7), we obtain a positive constant K_T such that

$$\|v(T)\|_2 \leqslant (\mathrm{e}^{-2\lambda_0 T} + K_T(1 + \|u\|_{0,T}^{p-1})\|u\|_{0,T}\sigma(\|u\|_{0,T}))\|v_0\|_2.$$

With the notation introduced above, we consider the sequence $y_n(\cdot) = v(\cdot, nT)$ and introduce $w_n(\cdot, t) = v(\cdot, t + nT)$. For $t \in [0, T]$, w_n solves the problem

$$\begin{aligned}\partial_t w_n + \partial_x^3 w_n - \partial_x^2 w_n + [a(u(\cdot + nT))w_n]_x + bw_n &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \\ w_n(0) &= y_n \quad \text{in } \mathbb{R}.\end{aligned}$$

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First, observe that for y_n we can obtain an estimate similar to that for $v(T)$:

$$\begin{aligned}
&\|y_{n+1}\|_2 \\
&\quad = \|w_n(T)\|_2 \\
&\quad \leqslant \mathrm{e}^{-2\lambda_0 T}\|w_0\|_2 + K_T(1 + \|u(\cdot + nT)\|_{0,T}^{p-1})\|u(\cdot + nT)\|_{0,T}\|w_n\|_{0,T} \\
&\quad \leqslant \{\mathrm{e}^{-2\lambda_0 T} + K_T(1 + \|u\|_{0,[nT,(n+1)T]}^{p-1})\|u\|_{0,[nT,(n+1)T]}\sigma(\|u\|_{0,[nT,(n+1)T]})\}\|y_n\|_2.
\end{aligned} \tag{3.8}$$

On the other hand, we can take $\beta > 0$ sufficiently small such that

$$\mathrm{e}^{-2\lambda_0 T} + K_T(1 + \beta^{p-1})\beta\sigma(\beta) < 1.$$

With this choice of β, corollary 3.2 allows us to choose $N > 0$ sufficiently large, which satisfies

$$\|u\|_{0,[nT,(n+1)T]} \leqslant \alpha_0(\|u_0\|_2)\mathrm{e}^{-2\lambda_0 nT} \leqslant \alpha_0(\|u_0\|_2)\mathrm{e}^{-2\lambda_0 NT} \leqslant \beta, \quad \forall n > N.$$

Thus, from (3.8) we obtain the following estimate:

$$\|y_{n+1}\|_2 \leqslant r\|y_n\|_2, \quad \forall n \geqslant N,\ 0 < r < 1,$$

which implies

$$\|v((n+k)T)\|_2 \leqslant r^k\|v(nT)\|_2, \quad \forall n \geqslant N. \tag{3.9}$$

Let $T_0 = NT$ and $t \geqslant T_0$. Then, there exist $k \in \mathbb{N}$ and $\theta \in [0, T]$ satisfying

$$t = (N + k)T + \theta.$$

Then, from (3.5) and (3.9), we find that

$$\begin{aligned}
\|v(t)\|_2 \leqslant \|v\|_{0,[(N+k)T,(N+k+1)T]} &\leqslant \sigma(\alpha_0(\|u_0\|_2))\|v((N+k)T)\|_2 \\
&\leqslant \sigma(\alpha_0(\|u_0\|_2))r^{(t-NT-\theta)/T}\|v(T_0)\|_2 \\
&\leqslant \sigma(\alpha_0(\|u_0\|_2))r^{(t-NT-\theta)/T}\sigma(\alpha_0(T_0, \|u_0\|_2))\|v(0)\|_2 \\
&\leqslant \eta_1(\|u_0\|)\mathrm{e}^{-\delta_1 t}\|v_0\|_2,
\end{aligned}$$

where

$$\delta_1 = \frac{1}{T}\ln\left(\frac{1}{r}\right) \quad \text{and} \quad \eta_1(s) = \sigma(\alpha_0(s))\sigma(\alpha_0(T_0, s))r^{-(N+1)}.$$

Invoking the estimate (2.24) in theorem 2.9, and bearing in mind that $v = u_t$, we get

$$\|u_t(t)\|_2 \leqslant \eta_2(\|u_0\|_2)\|u_0\|_{H^3(\mathbb{R})}\mathrm{e}^{-\delta_1 t}, \quad \forall t \geqslant T_0, \tag{3.10}$$

where $\eta_2(s) = \eta_1(s)C(s)$. On the other hand, note that

$$\|u_{xxx}(t)\|_2 \leqslant \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + \|a(u(t))u(t)\|_2 + \|b\|_\infty\|u(t)\|_2. \tag{3.11}$$

Estimating the nonlinear term as in the proof of lemma 2.3, i.e.

$$\begin{aligned}
\|a(u(t))u(t)\|_2 &= \|u(t)^{p+1}u_x(t)\|_2 \\
&\leqslant \|u(t)^{p+1}\|_\infty\|u_x(t)\|_2 \\
&\leqslant 2^{(p+1)/2}\|u(t)\|_2^{(p+1)/2}\|u_x(t)\|_2^{(p+3)/2},
\end{aligned}$$

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we obtain from (3.11) that

$$\|u_{xxx}(t)\|_2 \leqslant \|u_t(t)\|_2 + \|u_{xx}(t)\|_2 + 2^{(p+1)/2}\|u(t)\|_2^{(p+1)/2}\|u_x(t)\|_2^{(p+3)/2} + \|b\|_\infty\|u(t)\|_2.$$

Using the Gagliardo–Nirenberg and Young inequalities, it follows that

$$\begin{aligned}\|u_{xxx}(t)\|_2 &\leqslant \|u_t(t)\|_2 + C\|u_{xxx}(t)\|_2^{2/3}\|u(t)\|_2^{1/3} \\ &\quad + 2^{(p+1)/2}C\|u(t)\|_2^{(5p+9)/6}\|u_{xxx}(t)\|_2^{(p+3)/6} + \|b\|_\infty\|u(t)\|_2 \\ &\leqslant \|u_t(t)\|_2 + \left(\frac{C}{3\varepsilon} + \|b\|_\infty\right)\|u(t)\|_2 \\ &\quad + \frac{2^{(p+1)/2}(3-p)C}{6\varepsilon}\|u(t)\|_2^{(5p+9)/(3-p)} + \left(\frac{p+7}{6}\right)C\varepsilon\|u_{xxx}(t)\|_2.\end{aligned}$$

Choosing $\varepsilon = 3/C(p+7)$, we have

$$\|u_{xxx}(t)\|_2 \leqslant 2\|u_t(t)\|_2 + 2\left(\frac{C^2(p+7)}{9} + \|b\|_\infty\right)\|u(t)\|_2 + \frac{2^{(p+1)/2}(3-p)(p+7)C^2}{18}\|u(t)\|_2^{(5p+9)/(3-p)}.$$

By applying proposition 3.1 and estimate (3.10), the following decay estimate holds:

$$\|u_{xxx}(t)\|_2 \leqslant \eta_3(\|u_0\|)\|u_0\|_{H^3(\mathbb{R})}\mathrm{e}^{-\gamma t}, \quad \forall t \geqslant T_0, \tag{3.12}$$

where

$$\eta_3(s) = 2\eta_2(s) + \tfrac{2}{9}C^2(p+7) + 2\|b\|_\infty + \tfrac{1}{3}(2^{(p+1)/2}C^2(3-p)(p+7)s^{(6p-1)/(3-p)})$$

and $\gamma = \min\{\delta_1, 2\lambda_0\}$. Now, using Gagliardo–Nirenberg and Young inequalities it is easy to obtain

$$\|u(t)\|_{H^3(\mathbb{R})} \leqslant C_1(\|u(t)\|_2 + \|u_{xxx}(t)\|_2).$$

Finally, by proposition 3.1 and (3.12) we obtain (3.3) with $\alpha_3(s) = C_1(1+\eta_3(s))$. □

Propositions 3.1 and 3.3, together with corollary 2.12 and interpolation arguments, give the main result of this section.

Theorem 3.4. *Let $T > 0$, $1 \leqslant p < 2$, $a(0) = 0$ and b satisfying (1.4). Then, there exist positive constants γ, ε_0 and a continuous non-negative function $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$, such that, for every $u_0 \in H^s(\mathbb{R})$, with $0 \leqslant s \leqslant 3$, the corresponding solution u satisfies*

$$\|u(t)\|_{H^s(\mathbb{R})} \leqslant \alpha(T_0, \|u_0\|_2)\|u_0\|_{H^s(\mathbb{R})}\mathrm{e}^{-\lambda t}, \quad \forall t \geqslant T_0. \tag{3.13}$$

Proof. By corollary 2.12, the corresponding solution u belongs to $B_{0,[\varepsilon,T]}$ for all $\varepsilon \in (0,T]$. In particular, we choose $\varepsilon \leqslant T_0$, where T_0 is given by proposition 3.3. Then, by using the interpolation inequality (2.43) in [16, p. 19], we have

$$\|u(t)\|_{H^s(\mathbb{R})} = \|u(t)\|_{[L^2(\mathbb{R}),H^3(\mathbb{R})]_{2,s/3}} \leqslant C\|u(t)\|_2^{1-s/3}\|u(t)\|_{H^3(\mathbb{R})}^{s/3}, \quad \forall t \geqslant \varepsilon.$$

Finally, propositions 3.1 and 3.3 give us that

$$\|u(t)\|_{H^s(\mathbb{R})} \leqslant C\mathrm{e}^{-2(1-s/3)\lambda_0 t}\|u_0\|_2^{(1-s/3)}\alpha_3^{s/3}(\|u_0\|_2, T_0)\mathrm{e}^{-s\gamma t/3}, \quad \forall t \geqslant T_0.$$

Observe that, by construction, $\gamma \leqslant 2\lambda_0$. Therefore we obtain (3.13) with $\alpha(s, T_0) = C\alpha_3^{s/3}(s, T_0)$. □

3.2. The $2 \leqslant p < 5$ case

Throughout this section we assume that the damping function $b = b(x)$ does not change sign and satisfies (1.5). Under this condition, we prove the exponential decay of the solutions in the L^2-norm by using the so-called compactness–uniqueness argument. The key is to establish the unique continuation property for the solution of the GKdVB equation. The proof of this unique continuation property is mainly based on a Carleman estimate.

The following Carleman estimate is based on the global Carleman inequality obtained for the KdV equation in [18].

Lemma 3.5 (Carleman's estimate). *Let T and L be positive numbers. Then, there exist a smooth positive function ψ on $[-L, L]$ (which depends on L) and positive constants $s_0 = s_0(L, T)$ and $C = C(L, T)$ such that, for all $s \geqslant s_0$ and any*

$$q \in L^2(0, T; H^3(-L, L)) \cap H^1(0, T; L^2(-L, L)) \tag{3.14}$$

satisfying

$$q(t, \pm L) = q_x(t, \pm L) = q_{xx}(t, \pm L) = 0 \quad \text{for } 0 \leqslant t \leqslant T, \tag{3.15}$$

we have

$$\int_0^T \int_{-L}^L \left\{ \frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|q_x|^2 + \frac{s}{t(T-t)}|q_{xx}|^2 \right\} \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right) \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C \int_0^T \int_{-L}^L |q_t - q_{xx} + q_{xxx}|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right) \mathrm{d}x\,\mathrm{d}t.$$

The Carleman estimate in lemma 3.5 does not require a proof. Indeed, it is well known that the second-order term $-q_{xx}$ and the first-order term q_x can be absorbed by choosing s sufficiently large and increasing the constant C in the Carleman estimate in [18].

Lemma 3.6 (unique continuation property). *Let T be a positive number. If $u \in L^\infty(0, T; H^1(\mathbb{R}))$ solves*

$$u_t - u_{xx} + u_{xxx} + a(u)u_x = 0 \quad \text{in } \mathbb{R} \times (0, T),$$
$$u \equiv 0 \quad \text{in } (-\infty, -L) \cup (L, \infty) \times (0, T),$$

for some $L > 0$, with $a \in C(\mathbb{R})$ satisfying (1.3), then $u \equiv 0$ in $\mathbb{R} \times (0, T)$.

Proof. For $h > 0$, consider

$$u^h(x, t) = \frac{1}{h}\int_t^{t+h} u(x, s)\,\mathrm{d}s.$$

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Then, $u^h \in W^{1,\infty}(0,T',H^1(\mathbb{R}))$ and

$$u^h \to u \quad \text{in } L^\infty(0,T';H^1(\mathbb{R})) \tag{3.16}$$

for any $T' < T$. Moreover, u^h solves

$$\left.\begin{aligned} u_t^h - u_{xx}^h + u_{xxx}^h + (a(u)u_x)^h = 0 \quad &\text{in } \mathbb{R}\times(0,T'),\\ u^h \equiv 0 \quad &\text{in } (-\infty,-L)\cup(L,\infty)\times(0,T).\end{aligned}\right\} \tag{3.17}$$

On the other hand, note that $u \in L^\infty(0,T,H^1(\mathbb{R}))$ implies $a(u)u_x \in L^\infty(0,T,L^2(\mathbb{R}))$. Indeed, since

$$\|a(u)u_x\|_{L^\infty(0,T,L^2(\mathbb{R}))} \leqslant C\{\|u\|_{L^\infty(0,T,H^1(\mathbb{R}))} + \|u\|^p_{L^\infty(0,T,L^\infty(\mathbb{R}))}\|u\|_{L^\infty(0,T,H^1(\mathbb{R}))}\},$$

$(a(u)u_x)^h \in L^\infty(0,T,L^2(\mathbb{R}))$. Then, proceeding as in the proof of theorem 2.9, we have

$$u^h \in L^\infty(0,T',H_0^3(-L,L)) \cap H^1(0,T',L^2(-L,L)).$$

Invoking lemma 3.5, we obtain $C, s_0 > 0$ and a positive function ψ such that

$$\int_Q \left\{\frac{s^5|u^h|^2}{t^5(T-t)^5} + \frac{s^3|u_x^h|^2}{t^3(T-t)^3} + \frac{s|u_{xx}^h|^2}{t(T-t)}\right\}\exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t \leqslant C\int_Q |u_t^h - u_{xx}^h + u_{xxx}^h|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t$$

for all $s > s_0$ and $Q = (0,T')\times(-L,L)$. By (3.17),

$$\begin{aligned}\int_Q |u_t^h - u_{xx}^h + u_{xxx}^h|^2 &\exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q |(a(u)u_x)^h|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_Q |a(u)u_x^h|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t + \int_Q |(a(u)u_x)^h - a(u)u_x^h|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \|a(u)\|^2_{L^\infty(Q)}\int_Q |u_x^h|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t + \|(a(u)u_x)^h - a(u)u_x^h\|^2_{L^2(Q)}.\end{aligned}$$

Hence,

$$\begin{aligned}0 < \int_Q &\left\{\frac{s^5}{t^5(T-t)^5}|u^h|^2 + \left(\frac{s^3}{t^3(T-t)^3} - C\|a(u)\|^2_{L^\infty(Q)}\right)|u_x^h|^2 + \frac{s}{t(T-t)}|u_{xx}^h|^2\right\}\\ &\times \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\|(a(u)u_x)^h - a(u)u_x^h\|^2_{L^2(Q)},\end{aligned} \tag{3.18}$$

since, for s large enough, we obtain

$$\frac{s^3}{t^3(T-t)^3} - C\|a(u)\|^2_{L^\infty(Q)} > 0.$$

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Note that (3.16) guarantees that $a(u)u_x^h \to a(u)u_x$ in $L^2(0,T;L^2(-L.L))$, since $a(u) \in L^\infty(0,T',L^\infty(-L,L))$. Moreover, as $a(u)u_x \in L^2(0,T',L^2(-L,L))$ we have that

$$(a(u)u_x)^h \in W^{1,\infty}(0,T',L^2(-L,L)),$$
$$(a(u)u_x)^h \to a(u)u_x \in L^2(0,T',L^2(-L,L)).$$

Thus, passing to the limit in (3.18), we obtain that $u \equiv 0$ in $(-L,L)\times(0,T')$. Using (3.17), and since T' may be taken arbitrarily close to T, we have $u \equiv 0$ in $\mathbb{R}\times(0,T)$. □

Now we show that any weak solution of (1.1) decays exponentially to zero in the space $L^2(\mathbb{R})$.

THEOREM 3.7. *Let a be a $C^2(\mathbb{R})$-function satisfying (2.41) with $1 \leqslant p < 5$ and b satisfying (1.5). Then, system (1.1) is semi-globally uniformly exponentially stable in $L^2(\mathbb{R})$, i.e. for any $r > 0$ there exist two constants $C > 0$ and $\eta = \eta(r) > 0$ such that, for any $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_{L^2(\mathbb{R})} < r$, and any weak solution u of (1.1),*

$$\|u(t)\|_{L^2(\mathbb{R})} \leqslant C\|u_0\|_{L^2(\mathbb{R})}\mathrm{e}^{-\eta t}, \quad t \geqslant 0.$$

Proof. First, note that the corresponding solution u of (1.1) satisfies the following estimate:

$$\|u(t)\|^2_{L^2(\mathbb{R})} + 2\|u_x\|^2_{L^2(0,t;L^2(\mathbb{R}))} + 2\int_0^t \int_{\mathbb{R}} b(x)|u(x,\tau)|^2 \,\mathrm{d}x d\tau = \|u_0\|_2^2. \tag{3.19}$$

On the other hand, by multiplying the equation in (1.1) by $(T-t)u$ and integrating on $\mathbb{R}\times[0,T]$, we obtain

$$\tfrac{1}{2}T\|u_0\|_2^2 = \tfrac{1}{2}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} (T-t)|u_x(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} (T-t)b(x)|u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{3.20}$$

which implies that

$$\|u_0\|_2^2 \leqslant \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))} + 2\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + 2\int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3.21}$$

CLAIM 3.8. *For any $T > 0$ and $r > 0$ there exist $C = C(r,T)$ such that the following estimate holds for any weak solution u of (1.1) with $\|u_0\|_2 \leqslant r$:*

$$\int_0^T \int_\alpha^\beta |u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C\left(\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.22}$$

Proof. We argue by contradiction and suppose that (3.22) does not hold. Hence, there exists a sequence $\{u_n\}$ of weak solutions in $C_w([0,T];L^2(\mathbb{R}))\cap L^2(0,T;H^1(\mathbb{R}))$ satisfying

$$\|u_n(0)\|_2 \leqslant r$$

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and such that

$$\lim_{n\to\infty} \|u_n\|^2_{L^2(0,T;L^2(\alpha,\beta))}\left(\|\partial_x u_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T\int_{\mathbb{R}} b(x)|u_n|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{-1} = +\infty. \tag{3.23}$$

Define

$$\lambda_n := \|u_n\|_{L^2(0,T;L^2(\alpha,\beta))} \quad \text{and} \quad v_n(x,t) := \frac{u_n(x,t)}{\lambda_n}.$$

Then, v_n satisfies

$$\|v_n\|_{L^2(0,T;L^2(\alpha,\beta))} = 1, \quad \forall n\in\mathbb{N}, \tag{3.24}$$

and is a weak solution of

$$\partial_t v_n + \partial_x^3 v_n - \partial_x^2 v_n + a(\lambda_n v_n)\partial_x v_n + b v_n = 0,$$
$$v_n(x,0) = \frac{u_n(x,0)}{\lambda_n}.$$

Moreover, from (3.20), we get

$$\lambda_n := \|u_n\|_{L^2(0,T;L^2(\alpha,\beta))} \leqslant T^{1/2}\|u_n(0)\|_2 \leqslant T^{1/2} r, \tag{3.25}$$

and (3.23) implies that

$$\lim_{n\to\infty}\|\partial_x v_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}} b(x)|v_n|^2\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.26}$$

Furthermore, by (3.25) we obtain a subsequence, denoted by the same index n, and $\lambda \geqslant 0$, such that

$$\lambda_n \to \lambda.$$

On the other hand, note that

$$|a(\lambda_n \mu)| \leqslant C'(1+|\mu|^p)$$

and $v_n(x,0)$ is bounded in $L^2(\mathbb{R})$. In fact, by (3.21) and (3.26) we obtain that

$$\|v_n(0)\|_2^2 \leqslant \frac{2}{T} + \frac{2}{\lambda_n^2}\left\{\|\partial_x u_n\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T\int_{\mathbb{R}} b(x)|u_n(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right\} \tag{3.27}$$

for n sufficiently large. Combining (3.26), (3.27) and (3.19), we conclude that $\{v_n\}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))\cap L^2(0,T;H^1(\mathbb{R}))$. Hence, extracting a subsequence if needed, we have

$$v_n \rightharpoonup v \quad \text{in } L^\infty([0,T];L^2(\mathbb{R})) \text{ weakly}^*,$$
$$v_n \rightharpoonup v \quad \text{in } L^2([0,T];H^1(\mathbb{R})) \text{ weakly},$$

as $n\to\infty$. In order to analyse the nonlinear term, we consider the function

$$A(v) := \int_0^v a(\lambda u)\,\mathrm{d}u, \qquad A_n(v) := \int_0^v a(\lambda_n u)\,\mathrm{d}u.$$

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Proceeding as in the proof of theorem 2.14, it is easy to see that $a(\lambda_n v_n)\partial_x v_n = \partial_x[A_n(v_n)]$ is bounded in $L^\alpha([0,T];H^{-2}_{\mathrm{loc}}(\mathbb{R}))$, for $\alpha \in (1, 6/(p+1))$, and $\partial_t v_n = -\partial_x^3 v_n + \partial_x^2 v_n - a(\lambda_n v_n)\partial_x v_n - bv_n$ is bounded in $L^\alpha([0,T];H^{-2}_{\mathrm{loc}}(\mathbb{R})) \hookrightarrow L^1(0,T; H^{-2}_{\mathrm{loc}}(\mathbb{R}))$. Since $\{v_n\}$ is bounded in $L^2([0,T];H^1(\mathbb{R}))$, using the Aubin–Lions theorem (see [16]), we obtain a subsequence such that

$$v_n \to v \quad \text{strongly in } L^2((\alpha,\beta)\times(0,T)). \tag{3.28}$$

On the other hand, by (3.26) it follows that $v_n \to 0$ strongly in $L^2((\mathbb{R}\setminus(\alpha,\beta))\times(0,T))$. Therefore,

$$v_n \to v \quad \text{strongly in } L^2(\mathbb{R}\times(0,T)), \tag{3.29}$$

with

$$v \equiv 0 \quad \text{on } \omega\times[0,T], \quad \omega = \mathbb{R}\setminus(\alpha,\beta) \tag{3.30}$$

and

$$a(\lambda_n v_n)\partial_x v_n \to a(\lambda v)\partial_x v \quad \text{in } D'(\mathbb{R}\times[0,T]).$$

Thus, v solves

$$v_t + v_{xxx} - v_{xx} + a(\lambda v)v_x + bv = 0 \quad \text{in } D'([0,T]\times\mathbb{R}),$$

and from (3.24) and (3.28)–(3.30), it follows that

$$\|v\|_{L^2(0,T;L^2(\mathbb{R}))} = 1. \tag{3.31}$$

Claim 3.9. *Let $0 < t_1 < t_2 < T$. Then, there exist $(t_1', t_2') \subset (t_1,t_2)$ such that $v \in L^\infty(t_1',t_2';H^1(\mathbb{R}))$.*

Proof. Let w_n be a solution of

$$\partial_t w_n - \partial_x^2 w_n + \partial_x^3 w_n + a_n(\lambda_n w_n)\partial_x w_n = 0 \quad \text{in } \mathbb{R}\times(0,T),$$
$$w_n(x,0) = v_n(x,0) \quad \text{in } \mathbb{R},$$

where $a_n \in C_0^\infty(\mathbb{R})$ satisfies (2.42) and (2.43). Proceeding as in the proof of theorem 2.14, we have that

$$w_n - v_n \to 0 \quad \text{in } C([0,T];H^{-1}_{\mathrm{loc}}(\mathbb{R})) \quad \text{and} \quad \|w_n\|_{L^2(0,T;H^1(\mathbb{R}))} \leqslant C. \tag{3.32}$$

Consider $\tau_n \in (t_1, \frac{1}{2}(t_1+t_2))$ such that

$$\tau_n \to \tau \quad \text{and} \quad \|w_n(\tau_n)\|_{L^2(0,T;H^1(\mathbb{R}))} \leqslant C.$$

Hence, by theorem 2.19,

$$\|w_n(\tau_n + \cdot)\|_{L^2(0,\varepsilon;H^1(\mathbb{R}))} \leqslant C \tag{3.33}$$

for any $\varepsilon \leqslant T$. On the other hand, note that (3.32) implies that

$$w_n(\tau_n+\cdot) \to v(\tau+\cdot) \quad \text{in } C([0,\varepsilon];H^{-1}_{\mathrm{loc}}(\mathbb{R})) \tag{3.34}$$

for $\varepsilon < \frac{1}{2}(t_2-t_1)$. Thus by (3.33) and (3.34), $v \in L^\infty(\tau,\tau+\varepsilon;H^1(\mathbb{R}))$. □

Applying the claim above and lemma 3.6, we deduce that $v = 0$ in $\mathbb{R}\times(t_1',t_2')$, where $(t_1',t_2') \subset (t_1,t_2)$. As t_2 can take an arbitrary value close to t_1, by continuity of v in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ we obtain that $v \equiv 0$, which contradicts (3.31). □

Returning to the proof of theorem 3.7, note that (3.19) implies

$$\|u(T)\|^2_{L^2(\mathbb{R})} + 2\lambda_0 \int_0^T \int_{\mathbb{R}\setminus(\alpha,\beta)} |u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant \|u_0\|^2_{L^2(\mathbb{R})}.$$

Consequently,

$$\frac{1}{2\lambda_0}\|u(T)\|^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} |u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant \frac{1}{2\lambda_0}\|u_0\|^2_{L^2(\mathbb{R})} + \int_0^T \int_\alpha^\beta |u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Then, by claim 3.8 and the monotonicity of $\|u(\cdot,t)\|^2_{L^2(\mathbb{R})}$, the following estimate holds:

$$\begin{aligned}\left(\frac{1}{2\lambda_0} + T\right)&\|u(T)\|^2_{L^2(\mathbb{R})} \\ &\leqslant \frac{1}{2\lambda_0}\|u_0\|^2_{L^2(\mathbb{R})} + C(r,T)\left(\|u_x\|^2_{L^2(0,T;L^2(\mathbb{R}))} + \int_0^T \int_{\mathbb{R}} b(x)|u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t\right),\end{aligned}$$

and by (3.19) we get

$$\left(\frac{1}{2\lambda_0} + T + \frac{C(r,T)}{2}\right)\|u(T)\|^2_{L^2(\mathbb{R})} \leqslant \left(\frac{1}{2\lambda_0} + \frac{C(r,T)}{2}\right)\|u_0\|^2_{L^2(\mathbb{R})},$$

i.e.

$$\|u(T)\|^2_{L^2(\mathbb{R})} \leqslant \gamma\|u_0\|^2_{L^2(\mathbb{R})}, \quad \text{with } 0 < \gamma < 1.$$

Consequently,

$$\|u(kT)\|^2_{L^2(\mathbb{R})} \leqslant \gamma^k\|u_0\|^2_{L^2(\mathbb{R})}, \quad \forall k \geqslant 0.$$

Moreover, for any $t \geqslant 0$, there exist $k > 0$ such that $kT \leqslant t < (k+1)T$. Thus,

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R})} \leqslant \|u(kT)\|^2_{L^2(\mathbb{R})} &\leqslant \gamma^k\|u_0\|^2_{L^2(\mathbb{R})} \\ &\leqslant \gamma^{t/T}\gamma^{-1}\|u_0\|^2_{L^2(\mathbb{R})} \\ &\leqslant \gamma^{-1}\|u_0\|^2_{L^2(\mathbb{R})}\mathrm{e}^{-\eta t},\end{aligned}$$

where $\eta = -(\ln\gamma)/T > 0$. □

The next result asserts that (1.1) is globally uniformly exponentially stable in $L^2(\mathbb{R})$. It means that the constant η in proposition 3.7 is independent of r when $\|u_0\|_{L^2(\mathbb{R})} \leqslant r$.

THEOREM 3.10. *Let a be a $C^2(\mathbb{R})$-function satisfying (1.3), with $1 \leqslant p < 5$ and let b satisfy (1.5). Then, (1.1) is globally uniformly exponentially stable in $L^2(\mathbb{R})$, i.e. there exist a positive constant η and a non-negative continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that, for any $u_0 \in L^2(\mathbb{R})$ with $\|u_0\|_{L^2(\mathbb{R})} < r$ and any weak solution u of (1.1),*

$$\|u(t)\|_{L^2(\mathbb{R})} \leqslant \alpha(\|u_0\|_{L^2(\mathbb{R})})\mathrm{e}^{-\eta t}, \quad t \geqslant 0. \tag{3.35}$$

Theorem 3.10 is a direct consequence of theorem 3.7, as the decay η can be taken as the decay for $r = 1$ (the decay rate is given by the behaviour of the solutions in a neighbourhood of the origin, since all trajectories enter into this neighbourhood). The estimate (3.35) holds for all $t \geqslant 0$.

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