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Algebraic polymorphisms

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Dedicated to the memory of our colleague and friend William Parry

Abstract. In this paper we consider a special class of polymorphisms with invariant measure, the algebraic polymorphisms of compact groups. A general polymorphism is—by definition—a many-valued map with invariant measure, and the conjugate operator of a polymorphism is a Markov operator (i.e. a positive operator on L^2 of norm 1 which preserves the constants). In the algebraic case a polymorphism is a correspondence in the sense of algebraic geometry, but here we investigate it from a dynamical point of view. The most important examples are the algebraic polymorphisms of a torus, where we introduce a parametrization of the semigroup of toral polymorphisms in terms of rational matrices and describe the spectra of the corresponding Markov operators. A toral polymorphism is an automorphism of $\mathbb{T}^m$ if and only if the associated rational matrix lies in $\mathrm{GL}(m, \mathbb{Z})$. We characterize toral polymorphisms which are factors of toral automorphisms.

1. *Algebraic polymorphisms*

Definition 1.1. Let G be a compact group with Borel field $\mathcal{B}_G$, normalized Haar measure λ_G and identity element $1 = 1_G$. A closed subgroup $\mathsf{P} \subset G \times G$ is an (*algebraic*) *correspondence of* G if $\pi_1(\mathsf{P}) = \pi_2(\mathsf{P}) = G$, where $\pi_i : G \times G \longrightarrow G$, $i = 1, 2$, are the coordinate projections (which are obviously group homomorphisms).

Every correspondence $\mathsf{P} \subset G \times G$ defines a map Π_{P} from G to the set of all non-empty closed subsets of G by

$$\Pi_{\mathsf{P}}(x) = \{y \mid (x, y) \in \mathsf{P}\}, \tag{1.1}$$

for every $x \in G$. Clearly, π_i sends the Haar measure on P to the Haar measure on G; in the terminology of [**1**], the correspondence P defines an (*algebraic*) *polymorphism* of G (more exactly, P determines a *polymorphism of the measure space* $(G, \mathcal{B}_G, \lambda_G)$ *to itself*)†.

The correspondence $\mathsf{P} \subset G \times G$ and the polymorphism Π_{P} obviously determine each other.

Algebraic polymorphisms from one compact group to another are defined similarly.

A correspondence $\mathsf{P}' \subset G' \times G'$ is a *factor* of a correspondence $\mathsf{P} \subset G \times G$ (and the polymorphism $\Pi_{\mathsf{P}'}$ is a factor of Π_{P}) if there exists a surjective group homomorphism $\phi \colon G \longrightarrow G'$ with $(\phi \times \phi)(\mathsf{P}) = \mathsf{P}'$. If ϕ can be chosen to be a group isomorphism then P and P' (respectively Π_{P} and $\Pi_{\mathsf{P}'}$) are *isomorphic*.

This notion of factors is consistent with the terminology in [**1**]: if Π is a measure-preserving polymorphism of a probability space $(X, \mathcal{S}, \mu)$ determined by a self-coupling ν of μ, and if $\mathcal{T} \subset \mathcal{S}$ is a sub-sigma-algebra, then the factor polymorphism $\Pi_{\mathcal{T}}$ of $(X, \mathcal{T})$ is determined by the restriction of ν to the sigma-algebra $\mathcal{T} \otimes \mathcal{T} \subset \mathcal{S} \otimes \mathcal{S}$.

Let $\mathsf{P} \subset G \times G$ be a correspondence (since we only consider algebraic correspondences and polymorphisms we drop the term *algebraic* from now on). The subgroup

$$\mathsf{P}^* = \{(y, x) \mid (x, y) \in \mathsf{P}\} \tag{1.2}$$

corresponds to the *conjugate* (or *inverse*) polymorphism of Π_{P}. If $\mathsf{P}_1, \mathsf{P}_2$ are two correspondences of G, their *product* $\mathsf{P}_1 \star \mathsf{P}_2$ is the correspondence

$$\mathsf{P}_1 \star \mathsf{P}_2 = \{(x, z) \in G \times G \mid (x, y) \in \mathsf{P}_2 \ \text{ and } \ (y, z) \in \mathsf{P}_1 \ \text{ for at least one } y \in G\}. \tag{1.3}$$

Clearly,

$$\Pi_{\mathsf{P}_1 \star \mathsf{P}_2}(x) = \Pi_{\mathsf{P}_1} \circ \Pi_{\mathsf{P}_2}(x) = \bigcup_{y \in \Pi_{\mathsf{P}_2}(x)} \Pi_{\mathsf{P}_1}(y),$$

for every $x \in G$. With respect to the composition (1.3) the set of all correspondences (or, equivalently, the set of all polymorphisms) of G is a semigroup, denoted by $\mathcal{P}(G)$, with involution $\mathsf{P} \mapsto \mathsf{P}^*$, identity element $P_1 = \{(g, g), g \in G\}$ and zero element $P_0 = G \times G$.

For later use we introduce also the higher powers P^n of P, $n \geq 2$, defined recursively by

$$\mathsf{P}^n = \mathsf{P}^{n-1} \star \mathsf{P}. \tag{1.4}$$

If $\mathsf{P} \subset G \times G$ is a correspondence such that the group homomorphisms $\pi_i \colon \mathsf{P} \longrightarrow G$, $i = 1, 2$, are injections, then P is (the graph of) an automorphism of G, and the conjugate correspondence yields the inverse automorphism. If π_2 is an injection then P is (the graph of) an *endomorphism* (i.e. of a surjective group homomorphism), and if π_1 is an injection then P is (the graph of) an *exomorphism* (i.e. P^* is the graph of an endomorphism). *The group of automorphisms as well as semigroups of endo- and exomorphisms are sub-semigroups of the semigroup of* $\mathcal{P}(G)$ *of correspondences of* G.

We note in passing that the product of the algebraic polymorphisms is a special case of the general notion of the product of measure-preserving polymorphisms in [**1**].

† In general, a measure-preserving polymorphism Π of a probability space $(X, \mathcal{S}, \mu)$ is determined by a probability measure ν on $X \times X$ with $\pi_{i*}\nu = \mu$ for $i = 1, 2$, i.e. by a *coupling* of μ with itself.

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Definition 1.2. For the following definitions we fix a correspondence P of a compact group G. We write $\mathcal{B}_{\mathsf{P}}$ and λ_{P} for the Borel field and the Haar measure of P.

(1) *Algebraic factor polymorphisms.* Let $H \subset G$ be a closed subgroup, and let $\mathsf{P}_H = \mathsf{P}/(H \times H) \subset (G/H \times G/H)$ be the associated *factor correspondence*. The subgroup $H \subset G$ is *invariant, co-invariant* or *doubly invariant* under the polymorphism Π_{P} if P_H is an endomorphism, exomorphism or an automorphism, respectively. Examples will be given in §3.

(2) *The Markov operator.* Put $\mathcal{B}_{\mathsf{P}}^{(i)} = \pi_i^{-1}(\mathcal{B}_G) \subset \mathcal{B}_{\mathsf{P}}$, $i = 1, 2$, and let $F_i \subset L^2(G, \mathcal{B}_G, \lambda_G)$ be the subspace of functions measurable with respect to $\mathcal{B}_{\mathsf{P}}^{(i)}$, $i = 1, 2$. Let Pr_i be the orthogonal projection in $L^2(G, \mathcal{B}_G, \lambda_G)$ onto F_i, $i = 1, 2$. We define the Markov operator

$$V_{\mathsf{P}} \colon L^2(G, \mathcal{B}_G, \lambda_G) \longrightarrow L^2(G, \mathcal{B}_G, \lambda_G)$$

as follows. If $f \in L^2(G, \mathcal{B}_G, \lambda_G)$, we define $h \in L^2(\mathsf{P}, \mathcal{B}_{\mathsf{P}}, \lambda_{\mathsf{P}})$ by

$$h(x, y) = f(x),$$

for every $(x, y) \in \mathsf{P}$ and set

$$V_{\mathsf{P}} f = E_{\lambda_{\mathsf{P}}}(h | \mathcal{B}_{\mathsf{P}}^{(2)}), \tag{1.5}$$

where $E_{\lambda_{\mathsf{P}}}(\cdot|\cdot)$ stands for conditional expectation with respect to λ_{P}. Then

$$V_{\mathsf{P}} = \mathrm{Pr}_2 \cdot \mathrm{Pr}_1, \quad V_{\mathsf{P}}^* = \mathrm{Pr}_1 \cdot \mathrm{Pr}_2, \tag{1.6}$$

and

$$V_{\mathsf{P}^*} = V_{\mathsf{P}}^* \tag{1.7}$$

(cf. (1.2)). Note that V_{P} preserves positivity and has norm 1.

(3) *The Markov process* X_{P}. The closed, shift-invariant subgroup

$$X_{\mathsf{P}} = \{(x_n) \in G^{\mathbb{Z}} \mid (x_n, x_{n+1}) \in \mathsf{P} \text{ for every } n \in \mathbb{Z}\} \tag{1.8}$$

is the *Markov process* of P, and the corresponding *Markov shift* $\sigma_{\mathsf{P}} \colon X_{\mathsf{P}} \longrightarrow X_{\mathsf{P}}$ is defined by $(\sigma_{\mathsf{P}} x)_n = x_{n+1}$ for every $x = (x_n) \in X_{\mathsf{P}}$. Note that σ_{P} is an automorphism of the compact group X_{P} which preserves the normalized Haar measure $\lambda_{X_{\mathsf{P}}}$ of X_{P}, and that the Markov shift $\sigma_{\mathsf{P}^*} \colon X_{\mathsf{P}^*} \longrightarrow X_{\mathsf{P}^*}$ corresponding to P^* is the *time reversal* of σ_{P}.

Motivated by considering the various tail sigma-algebras (past, future and two-sided) of the Markov process X_{P}, we call the polymorphism Π_{P} *right* (*left* or *totally*) *non-deterministic* if there is no closed invariant (co-invariant or doubly invariant, respectively) proper subgroup $H \subset G$ (cf. Theorem 2.4).

(4) *Ergodicity.* The polymorphism Π_{P} is *ergodic* if the constants are the only V_{P}-invariant functions.

PROPOSITION 1.3. *Let G be a compact group and $\mathsf{P} \subset G \times G$ a correspondence. Then there exist closed normal subgroups $K_{\mathsf{P}}^{(i)} \subset G$, $i = 1, 2$, and a continuous group isomorphism $\eta_{\mathsf{P}} \colon G/K_{\mathsf{P}}^{(1)} \longrightarrow G/K_{\mathsf{P}}^{(2)}$ such that*

$$\mathsf{P} = \{(g_1, g_2) \in G \times G \mid \eta_{\mathsf{P}}(g_1 K_{\mathsf{P}}^{(1)}) = g_2 K_{\mathsf{P}}^{(2)}\}. \tag{1.9}$$

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Proof. We set $K_{\mathsf{P}}^{(1)} = \{g \in G \mid (g, 1) \in \mathsf{P}\}$, $K_{\mathsf{P}}^{(2)} = \{g \in G \mid (1, g) \in \mathsf{P}\}$ and observe that $K_{\mathsf{P}}^{(1)}$ and $K_{\mathsf{P}}^{(2)}$ are normal subgroups of G, since $\pi_1(\mathsf{P}) = \pi_2(\mathsf{P}) = G$. Since $\mathsf{P} = \{(g_1 p_1, g_2 p_2) \mid (g_1, g_2) \in \mathsf{P},\ p_1 \in K_{\mathsf{P}}^{(1)},\ p_2 \in K_{\mathsf{P}}^{(2)}\}$, we may view P as a subset $\bar{\mathsf{P}} \subset G/K_{\mathsf{P}}^{(1)} \times G/K_{\mathsf{P}}^{(2)}$, and the definition of the groups $K_{\mathsf{P}}^{(i)}$ implies that $\bar{\mathsf{P}}$ is the graph of a continuous group isomorphism $\eta_{\mathsf{P}} \colon G/K_{\mathsf{P}}^{(1)} \longrightarrow G/K_{\mathsf{P}}^{(2)}$. □

Remark 1.4. The triples $(K_{\mathsf{P}}^{(1)}, K_{\mathsf{P}}^{(2)}, \eta_{\mathsf{P}})$, where $K_{\mathsf{P}}^{(i)}$, $i = 1, 2$, are subgroups of G and $\eta_{\mathsf{P}} \colon G/K_{\mathsf{P}}^{(1)} \longrightarrow G/K_{\mathsf{P}}^{(2)}$ is a group isomorphism, form a parametrization of the algebraic polymorphisms of G.

Definition 1.5. A correspondence $\mathsf{P} \subset G \times G$ is *finite-to-one* (and defines a *polymorphism of discrete type*) if the groups $K_{\mathsf{P}}^{(i)}$ in (1.9) are both finite.

The finite-to-one correspondences of G form a subsemigroup $\mathcal{P}_f(G) \subset \mathcal{P}(G)$ of the semigroup of all correspondences of G.

For the notation in the following characterization of (co-)invariance we again refer to (1.9).

THEOREM 1.6. *Let $\mathsf{P} \subset G \times G$ be a correspondence and $H \subset G$ a closed normal subgroup:*

(1) *H is invariant under the polymorphism Π_{P} if and only if $\eta_{\mathsf{P}}(HK_{\mathsf{P}}^{(1)}) \subset H$;*

(2) *H is co-invariant under Π_{P} if and only if $\eta_{\mathsf{P}}^{-1}(HK_{\mathsf{P}}^{(2)}) \subset H$;*

(3) *H is bi-invariant under Π_{P} if and only if $K_{\mathsf{P}}^{(1)} \subset H$ and $\eta_{\mathsf{P}}(H) = H$ (in which case we also have that $K_{\mathsf{P}}^{(2)} \subset H$).*

Proof. Clearly, $K_{\mathsf{P}}^{(2)} \subset \eta_{\mathsf{P}}(HK_{\mathsf{P}}^{(1)})$. If $\eta_{\mathsf{P}}(HK_{\mathsf{P}}^{(1)}) \subset H$ then invariance follows from Definition 1.2(1). Conversely, if $K_{\mathsf{P}}^{(2)} \subset \eta_{\mathsf{P}}(HK_{\mathsf{P}}^{(1)}) \subset H$, then P_H is the graph of a group endomorphism.

The other assertions are proved similarly. □

COROLLARY 1.7. *Let $\mathsf{P} \subset G \times G$ be a correspondence and let $H \subset G$ be a closed normal subgroup. We denote by $K_{\mathsf{P}^n}^{(i)}$, $i = 1, 2$, the closed normal subgroups of G associated with the correspondence P^n, $n \geq 2$, in (1.4) by (1.9). The sequences of subgroups $(K_{\mathsf{P}^n}^{(i)}, n \geq 1)$ are non-decreasing and have the following property:*

(1) *H is invariant under Π_{P} if and only if it contains $\bigcup_{n \geq 1} K_{\mathsf{P}^n}^{(2)}$;*

(2) *H is co-invariant under Π_{P} if and only if it contains $\bigcup_{n \geq 1} K_{\mathsf{P}^n}^{(1)}$.*

Proof. If a closed normal subgroup $H \subset G$ is invariant under Π_{P} then Theorem 1.6(1) shows that $K_{\mathsf{P}^2}^{(2)} = \eta_{\mathsf{P}}(K_{\mathsf{P}}^{(1)} K_{\mathsf{P}}^{(2)}) \subset \eta_{\mathsf{P}}(K_{\mathsf{P}}^{(1)} H) \subset H$, so hence $\eta_{\mathsf{P}}(K_{\mathsf{P}^2}^{(2)}) \subset H$ and, by induction, $\eta_{\mathsf{P}}(K_{\mathsf{P}^n}^{(2)}) \subset H$ for every $n \geq 1$.

Conversely, if $H \supset \bigcup_{n \geq 1} K_{\mathsf{P}^n}^{(2)}$, then it is obviously invariant.

The proof of the second assertion is analogous. □

If the group G is abelian, the characterization of ergodicity of a polymorphism of G is completely analogous to that of ergodicity of an automorphism of G.

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THEOREM 1.8. *Let $\mathsf{P} \subset G \times G$ be a correspondence of a compact abelian group G with Markov operator V_{P} (cf. (1.5)). Then Π_{P} is non-ergodic if and only if there exist a non-trivial character χ of G and an integer $n \geq 1$ with $V_{\mathsf{P}}^n \chi = \chi$.*

Proof. If χ is a non-trivial character of G then the restriction to P of $h = \chi \circ \pi_1$ is a non-trivial character on P, and $V_{\mathsf{P}} \chi = E_{\lambda_{\mathsf{P}}}(h | \mathcal{B}_{\mathsf{P}}^{(2)})$ is either equal to zero or a non-trivial character of G (depending on whether h is constant on $\{1_G\} \times K_{\mathsf{P}}^{(2)}$ or not). Fourier expansion completes the proof of the theorem. □

2. *Toral polymorphisms*

Compact groups do not have dynamically interesting polymorphisms unless they have large abelian quotients. For this reason we focus our attention in this section on compact *abelian* groups, and in particular on finite-dimensional tori.

Let $m \geq 1$, and let $\mathcal{P}_f(\mathbb{T}^m)$ be the semigroup of all finite-to-one correspondences of $\mathbb{T}^m$. We denote by $\mathcal{L}$ the semigroup of all finite index subgroups of $\mathbb{Z}^m$ with respect to the addition $L_1 + L_2 = \{u + v \mid u \in L_1, v \in L_2\}$. For every $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$ and $x = (x_1, \ldots, x_m) \in \mathbb{T}^m$ we write

$$\chi_{\mathbf{n}}(x) = e^{2\pi i \sum_{j=1}^m n_j x_j} \tag{2.1}$$

for the value of the corresponding character $\chi_{\mathbf{n}}$ of $\mathbb{T}^m$ at x. The annihilator of a subgroup $F \subset \mathbb{T}^m$ (or $F' \subset \mathbb{T}^{2m}$) is denoted by $F^\perp$ (respectively $F'^\perp$).

For $Q \in \mathrm{GL}(m, \mathbb{Q})$ we put

$$\Lambda_Q = \mathbb{Z}^m \cap Q\mathbb{Z}^m \in \mathcal{L}. \tag{2.2}$$

Finally, we introduce the semigroup

$$\mathcal{M} = \{(Q, \Lambda) \mid Q \in \mathrm{GL}(m, \mathbb{Q}), \Lambda \in \mathcal{L}, \Lambda \subset \Lambda_Q\}, \tag{2.3}$$

with composition

$$(Q, \Lambda) \cdot (Q', \Lambda') = (QQ', \Lambda + Q\Lambda'). \tag{2.4}$$

PROPOSITION 2.1. *The semigroup $\mathcal{P}_f(\mathbb{T}^m)$ is isomorphic to the semigroup $\mathcal{M}$ in (2.3), where the isomorphism $\theta : \mathcal{M} \longrightarrow \mathcal{P}_f(\mathbb{T}^m)$ is given by*

$$\theta(Q, \Lambda)^\perp = \{(Q^{-1}\mathbf{n}, \mathbf{n}) \mid \mathbf{n} \in \Lambda\}, \tag{2.5}$$

for every $(Q, \Lambda) \in \mathcal{M}$.

A correspondence $\mathsf{P} \in \mathcal{P}_f(\mathbb{T}^m)$ is connected if and only if

$$\mathsf{P} = \mathsf{P}_Q = \theta(Q, \Lambda_Q), \tag{2.6}$$

for some $Q \in \mathrm{GL}(m, \mathbb{Q})$ (cf. (2.2)). Finally, if $\mathsf{P} = \theta(Q, \Lambda) \in \mathcal{P}_f(\mathbb{T}^m)$, then $\mathsf{P}^* = \theta(Q^{-1}, Q^{-1}\Lambda)$.

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Proof. For every $Q \in \mathrm{GL}(m, \mathbb{Q})$ and $\Lambda \subset \Lambda_Q$, $\theta(Q, \Lambda) \subset \mathbb{T}^m \times \mathbb{T}^m$ is obviously an element of $\mathcal{P}_f(\mathbb{T}^m)$, and (1.9) guarantees that every $\mathsf{P} \in \mathcal{P}_f(\mathbb{T}^m)$ is obtained in this manner.

If $\Lambda \subsetneq \Lambda_Q \subset \mathbb{Z}^m$, then $\mathsf{P} = \theta(Q, \Lambda)$ contains $\mathsf{P}_Q = \theta(Q, \Lambda_Q)$ as a finite index subgroup and is therefore not connected. In order to prove the converse we set

$$W_Q = \{(Q^{-1}\mathbf{n}, \mathbf{n}) \mid \mathbf{n} \in \Lambda_Q\}.$$

The dual group of P_Q is of the form $(\mathbb{Z}^m \times \mathbb{Z}^m)/W_Q$. If P_Q is not connected, then there exist an element $(\mathbf{m}, \mathbf{n}) \in (\mathbb{Z}^m \times \mathbb{Z}^m) \smallsetminus W_Q$ and an $l > 1$ with $(l\mathbf{m}, l\mathbf{n}) \in W_Q$. Hence $(\mathbf{m}, \mathbf{n}) = (Q^{-1}\mathbf{k}, \mathbf{k})$ for some $\mathbf{k} \in \mathbb{Z}^m \cap Q\mathbb{Z}^m = \Lambda_Q$, and $(\mathbf{m}, \mathbf{n}) \in W_Q$. This contradiction proves that P_Q is connected.

The last assertion is obvious. □

Remark 2.2. Proposition 2.1 shows that connected finite-to-one correspondences are in one-to-one correspondence with the elements of $\mathrm{GL}(m, \mathbb{Q})$.

For every $n \geq 1$ we define P^n and $K^{(i)}_{\mathsf{P}^n}$ as in Corollary 1.7.

THEOREM 2.3. *Let $Q \in \mathrm{GL}(m, \mathbb{R})$ and $\mathsf{P} = \xi(Q, \Lambda) \in \mathcal{P}_f(\mathbb{T}^m)$, where $\Lambda \subset \Lambda_Q = \mathbb{Z}^m \cap Q\mathbb{Z}^m$ is a finite index subgroup (cf. (2.2) and (2.5)).*

(1) *The following conditions are equivalent.*
  - (a) Π_{P} *is right non-deterministic.*
  - (b) $\Xi^+_{\mathsf{P}} = \{\mathbf{n} \in \Lambda \mid Q^k\mathbf{n} \in \mathbb{Z}^m$ *for every* $k \leq 0\} = \{\mathbf{0}\}$.
  - (c) $\bigcup_{n \geq 1} K^{(2)}_{\mathsf{P}^n}$ *is dense in* $\mathbb{T}^m$.

(2) *The following conditions are equivalent.*
  - (a) Π_{P} *is left non-deterministic.*
  - (b) $\Xi^-_{\mathsf{P}} = \{\mathbf{n} \in \Lambda \mid Q^k\mathbf{n} \in \mathbb{Z}^m$ *for every* $k \geq 0\} = \{\mathbf{0}\}$.
  - (c) $\bigcup_{n \geq 1} K^{(1)}_{\mathsf{P}^n}$ *is dense in* $\mathbb{T}^m$.

(3) *The following conditions are equivalent.*
  - (a) Π_{P} *is totally non-deterministic.*
  - (b) $\Xi^+_{\mathsf{P}} \cap \Xi^-_{\mathsf{P}} = \{\mathbf{n} \in \Lambda \mid Q^k\mathbf{n} \in \mathbb{Z}^m$ *for every* $k \in \mathbb{Z}\} = \{\mathbf{0}\}$.
  - (c) *Both* $\bigcup_{n \geq 1} K^{(1)}_{\mathsf{P}^n}$ *and* $\bigcup_{n \geq 1} K^{(2)}_{\mathsf{P}^n}$ *are dense in* $\mathbb{T}^m$.

Proof. In order to prove (1) we note that Ξ^+_{P} is a group and that $Q^{-1}\Xi^+_{\mathsf{P}} \subset \Xi^+_{\mathsf{P}}$. We set $H = (\Xi^+_{\mathsf{P}})^{\perp} \subset \mathbb{T}^m$. Then the correspondence $\mathsf{P}_H = \mathsf{P}/(H \times H)$ is the graph of a continuous surjective homomorphism of the group $Y = \mathbb{T}^m/H^{\perp}$ to itself. The converse is proved by reversing this argument.

If the group $H = \bigcup_{n \geq 1} K^{(2)}_{\mathsf{P}^n}$ is trivial, then P is the graph of an endomorphism. If H is not dense in $\mathbb{T}^m$, then its closure $\bar{H}$ is non-trivial and is the smallest proper invariant subgroup of Π_{P} (cf. Corollary 1.7).

The assertions (2) and (3) are proved in exactly the same manner. □

The property of being left, right or totally non-deterministic can also be expressed in terms of the Markov group X_{P} in (1.8).

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THEOREM 2.4. *Under the hypotheses of Theorem 2.3 the polymorphism Π_{P} is right non-deterministic if and only if the remote past of the process X_{P} is trivial*†.

Similarly, Π_{P} is left non-deterministic if and only if the remote future of X_{P} is trivial.

Proof. We denote by $\lambda_{X_{\mathsf{P}}}$ the Haar measure of the compact abelian group X_{P}. For every $n \geq 1$ we define the subgroups $K^{(1)}_{\mathsf{P}^n}$ and $K^{(2)}_{\mathsf{P}^n}$ as in Corollary 1.7.

For the proof of the theorem it is enough to notice that the conditional measure $\lambda_{X_{\mathsf{P}}}(\cdot \mid x_n = t)$ for fixed $t \in \mathbb{T}^m$ and $n < 0$ is the uniform measure on a coset of the group $K^{(2)}_{\mathsf{P}^n}$. According to Theorem 1.6(1), the polymorphism Π_{P} is right non-deterministic if and only if $\bigcup_{n\geq 1} K^{(2)}_{\mathsf{P}^n}$ is dense in $\mathbb{T}^m$, in which case the conditional measures $\lambda_{X_{\mathsf{P}}}(\cdot \mid x_{-n} = t)$ converge to $\lambda_{\mathbb{T}^m}$ as $m \to \infty$. □

THEOREM 2.5. *Let $Q \in \mathrm{GL}(m, \mathbb{R})$ and $\mathsf{P} = \xi(Q, \Lambda) \in \mathcal{P}_f(\mathbb{T}^m)$, where $\Lambda \subset \Lambda_Q = \mathbb{Z}^m \cap Q\mathbb{Z}^m$ is a finite index subgroup (cf. (2.5)). Then Π_{P} is non-ergodic if and only if Q has a non-trivial root of unity as an eigenvalue.*

Furthermore, if Π_{P} is left, right or totally non-deterministic, then it is ergodic.

Proof. By definition, $\mathsf{P}^{\perp} = \{(Q^{-1}\mathbf{n}, \mathbf{n}) \mid \mathbf{n} \in \Lambda\}$. A direct calculation shows that, for every $\mathbf{n} \in \mathbb{Z}^m$,

$$V_{\mathsf{P}}(\chi_{\mathbf{n}}) = \begin{cases} \chi_{Q^{-1}\mathbf{n}} & \text{if } \mathbf{n} \in \Lambda, \\ 0 & \text{otherwise}, \end{cases} \qquad V^*_{\mathsf{P}}(\chi_{\mathbf{n}}) = \begin{cases} \chi_{Q\mathbf{n}} & \text{if } \mathbf{n} \in Q^{-1}\Lambda, \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

(cf. (2.1)). The existence of an $\mathbf{n} \in \Lambda$ with $V^k_{\mathsf{P}}\chi_{\mathbf{n}} = \chi_{\mathbf{n}}$ for some $k \geq 1$ is obviously equivalent to Q having a root of unity as an eigenvalue.

The last assertion is obvious. □

We turn to the spectral properties of the Markov operator V_{P} associated with a correspondence $\mathsf{P} \in \mathcal{P}_f(\mathbb{T}^m)$.

THEOREM 2.6. *Let $m \geq 1$, $\mathsf{P} \in \mathcal{P}_f(\mathbb{T}^m)$, and let $\mathrm{Sp}(V_{\mathsf{P}}) \subset \mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the spectrum of the linear operator $V_{\mathsf{P}}\colon L^2_0(\mathbb{T}^m, \mathcal{B}^m_{\mathbb{T}}, \lambda_{\mathbb{T}^m}) \longrightarrow L^2_0(\mathbb{T}^m, \mathcal{B}^m_{\mathbb{T}}, \lambda_{\mathbb{T}^m})$ in (1.5), where $L^2_0(\mathbb{T}^m, \mathcal{B}^m_{\mathbb{T}}, \lambda_{\mathbb{T}^m}) = \{f \in L^2(\mathbb{T}^m, \mathcal{B}^m_{\mathbb{T}}, \lambda_{\mathbb{T}^m}) \mid \int f\, d\lambda_{\mathbb{T}^m} = 0\}$ is the orthocomplement of the constants:*

(1) $\mathrm{Sp}(V_{\mathsf{P}}) = \mathrm{Sp}(V^*_{\mathsf{P}}) = \{0\}$ *if and only if* P *is totally non-deterministic;*
(2) $\mathrm{Sp}(V_{\mathsf{P}}) = \mathrm{Sp}(V^*_{\mathsf{P}}) = \mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$ *if and only if* $\Xi^+_{\mathsf{P}} = \Xi^-_{\mathsf{P}} = \Lambda$*;*
(3) $\mathrm{Sp}(V_{\mathsf{P}}) = \mathrm{Sp}(V^*_{\mathsf{P}}) \subset \mathbb{S} \cup \{0\}$ *if and only if* $\Xi^+_{\mathsf{P}} = \Xi^-_{\mathsf{P}} \subsetneq \Lambda$*;*
(4) *if* $\Xi^-_{\mathsf{P}} \smallsetminus \Xi^+_{\mathsf{P}} \neq \varnothing$ *then* $\mathrm{Sp}(V_{\mathsf{P}}) = \mathbb{D}$*;*
(5) *if* $\Xi^+_{\mathsf{P}} \smallsetminus \Xi^-_{\mathsf{P}} \neq \varnothing$ *then* $\mathrm{Sp}(V^*_{\mathsf{P}}) = \mathbb{D}$.

Proof. We choose $(Q, \Lambda) \in \mathcal{M}$ with $\xi(Q, \Lambda) = \mathsf{P}$ (cf. (2.5)). By definition of $\mathcal{M}$, we have $\Lambda \subset \Lambda \cap Q\Lambda$.

† The *remote past* of the process $X_{\mathsf{P}} \subset (\mathbb{T}^m)^{\mathbb{Z}}$ is the intersection $\mathcal{A}_{-\infty} = \bigcap_{n\in\mathbb{Z}} \mathcal{A}^-_n$, where $\mathcal{A}^-_n$ is the sigma-algebra generated by the coordinates of the process X_{P} with index less than or equal to n. The *remote future* of $X_{\mathsf{P}} \subset (\mathbb{T}^m)^{\mathbb{Z}}$ is the sigma-algebra $\mathcal{A}_{\infty} = \bigcap_{n\in\mathbb{Z}} \mathcal{A}^+_n$, where $\mathcal{A}^+_n$ is generated by the coordinates with index greater than or equal to n of X_{P}.

If P is totally non-deterministic then there exist, for every non-zero $\mathbf{n} \in \Lambda$, a smallest positive integer $k^+(\mathbf{n})$ and a largest negative integer $k^-(\mathbf{n})$ such that $Q^{k^\pm(\mathbf{n})}\mathbf{n} \notin \Lambda$. If we set

$$\mathcal{O}_Q(\mathbf{n}) = \{Q^{k^-(\mathbf{n})+1}\mathbf{n}, \dots, Q^{k^+(\mathbf{n})-1}\mathbf{n}\},$$

then the restriction of V_P^* to the linear span $\langle \mathcal{O}_Q(\mathbf{n})\rangle$ of $\mathcal{O}_Q(\mathbf{n})$ in $L^2(\mathbb{T}^m, \mathcal{B}_{\mathbb{T}^m}, \lambda_{\mathbb{T}^m})$ is unitarily equivalent to a matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which has spectrum $\{0\}$. By taking the direct sum of the subspaces $\langle \mathcal{O}_Q(\mathbf{n})\rangle$, $\mathbf{n} \in \mathbb{Z}^n$, in $L^2(\mathbb{T}^m, \mathcal{B}_{\mathbb{T}^m}, \lambda_{\mathbb{T}^m})$ we see that $\mathrm{Sp}(V_\mathsf{P}^*) = \{0\}$, and that the same is true for $\mathrm{Sp}(V_\mathsf{P})$. This proves (1).

The assertion (2) is obvious, since the condition given there is equivalent to Q being an element of $\mathrm{GL}(m, \mathbb{Z})$.

In order to prove (3) we set $\Xi = \Xi_\mathsf{P}^+ = \Xi_\mathsf{P}^-$, $S = \Xi^\perp$, $Y = \widehat{\Xi} = \mathbb{T}^m/S$, and we observe that the restriction of Q to Ξ is a group automorphism. Hence the restrictions of V_P and V_P^* to the closed linear span of $\{\chi_\mathbf{n} \mid \mathbf{n} \in \Xi\}$ in $L^2(\mathbb{T}^m, \mathcal{B}_{\mathbb{T}^m}, \lambda_{\mathbb{T}^m})$ are unitary.

For $\mathbf{n} \notin \Xi$ there exist a smallest non-negative integer $k^+(\mathbf{n})$ and a largest non-positive integer $k^-(\mathbf{n})$ such that $Q^{k^\pm(\mathbf{n})}\mathbf{n} \notin \mathbb{Z}^n$, and by combining the preceding paragraph with the argument in the proof of (1) we obtain (3).

If $\Xi_\mathsf{P}^- \smallsetminus \Xi_\mathsf{P}^+ \neq \varnothing$ there exists an $\mathbf{n} \in \Lambda$ with $Q^k\mathbf{n} \in \Lambda$ for every $k \geq 0$, but $Q^{-1}\mathbf{n} \notin \Lambda$. The restriction W of V_P to the closed linear span H of $\{\chi_{Q^k\mathbf{n}} \mid k \geq 0\}$ has a non-zero kernel, since $V_\mathsf{P}\chi_\mathbf{n} = 0$. Furthermore, if $\gamma \in \mathbb{C}$, $|\gamma| < 1$, and if $v_\gamma = \sum_{k\leq 0} \gamma^k \chi_{Q^k\mathbf{n}} \in H$, then $V_\mathsf{P} v_\gamma = \gamma v_\gamma$, i.e. v_γ is an eigenvector of V_P with eigenvalue γ. This proves that $\mathrm{Sp}(V_\mathsf{P}) \supset \mathrm{Sp}(W) = \mathbb{D}$.

The same argument shows that $\mathrm{Sp}(V_\mathsf{P}^*) \supset \mathrm{Sp}(W) = \mathbb{D}$ if $\Xi_\mathsf{P}^+ \smallsetminus \Xi_\mathsf{P}^- \neq \varnothing$, and the remaining implications are immediate consequences of what has already been shown. □

3. *Factors of toral automorphisms and other examples*

If A, B are endomorphisms of $\mathbb{T}^m$, then

$$\mathsf{P}(A, B) = \{(x, y) \in \mathbb{T}^m \times \mathbb{T}^m \mid Ax = By\} \tag{3.1}$$

is a finite-to-one correspondence, and every $\mathsf{P} \in \mathcal{P}_f(\mathbb{T}^m)$ is of this form. Note that $\mathsf{P}(A, B) = \mathsf{P}(CA, CB)$ for every $C \in \mathrm{GL}(m, \mathbb{Z})$, and that $\mathsf{P}(AC', BC')$ and $\mathsf{P}(A, B)$ are isomorphic if $C' \in \mathrm{GL}(m, \mathbb{Z})$.

The subgroups $K^{(1)}_{\mathsf{P}(A,B)}$ and $K^{(2)}_{\mathsf{P}(A,B)}$ and the isomorphism $\eta_{\mathsf{P}(A,B)} \colon \mathbb{T}^m/K^{(1)}_{\mathsf{P}(A,B)} \longrightarrow \mathbb{T}^m/K^{(2)}_{\mathsf{P}(A,B)}$ associated with $\mathsf{P}(A, B)$ by (1.9) are given by $K^{(1)}_{\mathsf{P}(A,B)} = (A^\top\mathbb{Z}^m)^\perp$, $K^{(2)}_{\mathsf{P}(A,B)} = (B^\top\mathbb{Z}^m)^\perp$ and $\eta_{\mathsf{P}(A,B)} = -B^{-1}A$, respectively, where $^\top$ denotes transpose.

Ergodicity of $\mathsf{P}(A, B)$ is thus equivalent to the assumption that $B^{-1}A \in \mathrm{GL}(m, \mathbb{Q})$ has no eigenvalues which are roots of unity (cf. Theorem 1.8).

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In order to characterize the connectedness of $\mathsf{P}(A, B)$ we set $Q = \widehat{\eta_{\mathsf{P}}^{-1}} = B^\top (A^\top)^{-1}$. According to Proposition 2.1, $\mathsf{P}(A, B)$ is connected if and only if

$$B^\top \mathbb{Z}^m = \mathbb{Z}^m \cap B^\top (A^\top)^{-1} \mathbb{Z}^m = \Lambda_Q,$$

i.e. if and only if

$$\mathbb{Z}^m = (A^\top)^{-1} \mathbb{Z}^m \cap (B^\top)^{-1} \mathbb{Z}^m. \tag{3.2}$$

Example 3.1. Let $m = 1$, $k, l \in \mathbb{N}$ and $\mathsf{P} = \{(u, v) \mid u, v \in \mathbb{T}, ku = lv\}$. We denote by s the highest common factor of k, l and set $k = k's$, $l = l's$. Then $\mathsf{P} = \theta(Q, \Lambda)$, where $\Lambda = l\mathbb{Z}$ and Q is multiplication by $-(k'/l')$ (cf. (2.5)).

If k, l are coprime (i.e. if $s = 1$) then $\mathsf{P} = \mathsf{P}_Q$ is connected by (3.2).

If $s > 1$ then ϕ is the quotient map from $\mathbb{Z}$ to $F = \mathbb{Z}/s\mathbb{Z}$, P is disconnected and $K_{\mathsf{P}}^{(1)} \cap K_{\mathsf{P}}^{(2)} = \{x \in \mathbb{T} \mid sx = 0 \ (\mathrm{mod}\ 1)\})$.

Finally, if $|k/l| \neq 1$ then Π_{P} is ergodic. If k, l are coprime and $|k| > 1$, $|l| > 1$, then Π_{P} is totally non-deterministic.

Examples 3.2. (Factors of polymorphisms) (1) Consider the correspondence $\mathsf{P} = \{(u, v) \mid u, v \in \mathbb{T}, 3u = 2v\}$ (cf. Example 3.1(1)), and let $H = \{0, 1/5, 2/5, 3/5, 4/5\} \subset \mathbb{T}$. Then P is the annihilator of $\{(3k, -2k) \mid k \in \mathbb{Z}\} \subset \mathbb{Z}^2$ and P_H is the annihilator of $\{(15k, -10k) \mid k \in \mathbb{Z}\}$. Note that P and P_H are isomorphic.

(2) Let

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $\mathsf{P} = \mathsf{P}_Q = \theta(Q, \mathbb{Z}^2)$. Put

$$H = \left\{ \mathbf{0}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right\} \subset \mathbb{T}^2,$$

and set $\mathsf{P}' = \mathsf{P}_H$. We identify $\mathbb{T}^2/H$ with $\mathbb{T}^2$ by the map

$$\phi \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 2s \\ t \end{pmatrix},$$

and view P' as a correspondence of $\mathbb{T}^2$. Then P' is isomorphic to the polymorphism of $\mathsf{P}(A, B)$ of $\mathbb{T}^2$ with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Since $A, B \notin \mathrm{GL}(2, \mathbb{Z})$, P' is not the graph of an automorphism.

Example 3.2(2) shows that a toral automorphism P may have a proper polymorphism as a factor (*proper* means that the groups $K_{\mathsf{P}}^{(1)}$, $K_{\mathsf{P}}^{(2)}$ in (1.9) are not both trivial). However, the following theorem shows that factors of automorphisms always have a non-trivial doubly invariant subgroup.

THEOREM 3.3. *Let $\mathsf{P}(A) \in \mathcal{P}_f(\mathbb{T}^m)$ be the graph of a toral automorphism $A \in \mathrm{GL}(m, \mathbb{Z})$. For every finite subgroup $H \subset \mathbb{T}^m$ there exists a finite doubly invariant subgroup $H' \subset \mathbb{T}^m$ containing H. In particular, $\mathsf{P}_{H'}$ is the graph of an automorphism of $\mathbb{T}^m/H'$.*

In other words, if a polymorphism is a factor of a toral automorphism then it has a further factor which is again an automorphism.

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Proof. Since H is finite, there exists a $q \geq 1$ such that $H \subset H' = \{\mathbf{t} \in \mathbb{T}^m \mid q\mathbf{t} = \mathbf{0}\}$. As one can check easily, $\bigcup_{n\geq 1} K^{(i)}_{\mathsf{P}^n} \subset H'$ for $i = 1, 2$. Theorem 2.3 shows that H' is invariant under P_H, which proves our claim. □

Theorem 3.3 allows us to say a little more about the structure of factors of toral automorphisms.

COROLLARY 3.4. *Let $\mathsf{P}(A) \in \mathcal{P}_f(\mathbb{T}^m)$ be the graph of a toral automorphism $A \in \mathrm{GL}(m, \mathbb{Z})$, $H \subset \mathbb{T}^m$ a finite subgroup and $H' \subset \mathbb{T}^m$ a finite doubly invariant subgroup containing H.*

If we identify both $\mathbb{T}^m/H$ and $\mathbb{T}^m/H'$ with $\mathbb{T}^m$, then the correspondence $\mathsf{P}'' = \mathsf{P}_{H'}$ is the graph of a toral automorphism A'' (i.e. $\mathsf{P}'' = \mathsf{P}(A'')$), and the correspondence $\mathsf{P}' = \mathsf{P}_H$ has the graph of the automorphism $A'' \in \mathrm{GL}(m, \mathbb{Z})$ as a factor with kernel $(H/H') \times (H/H')$.

Proof. The identifications of $\mathbb{T}^m/H$ and $\mathbb{T}^m/H'$ with $\mathbb{T}^m$ yield finite-to-one equivariant homomorphisms

$$\mathbb{T}^m \longrightarrow \mathbb{T}^m/H \longrightarrow \mathbb{T}^m/H',$$

where the automorphisms A and A'' act on the first and third tori and the polymorphism $\Pi_{\mathsf{P}'}$ on the second. □

Remarks 3.5. (1) The automorphism $A'' \in \mathrm{GL}(m, \mathbb{Z})$ in Corollary 3.4 is obviously conjugate to A in $\mathrm{GL}(m, \mathbb{Q})$, but not necessarily in $\mathrm{GL}(m, \mathbb{Z})$.

Conversely, if $\mathsf{P} = \mathsf{P}(A)$ is the graph if some $A \in \mathrm{GL}(m, \mathbb{Z})$, and if $A'' \in \mathrm{GL}(m, \mathbb{Z})$ is conjugate to A in $\mathrm{GL}(m, \mathbb{Q})$, then the graph $\mathsf{P}(A'')$ is isomorphic to $\mathsf{P}(A)_{H'}$ for some finite subgroup $H' \subset \mathbb{T}^m$.

(2) There is a minimal choice of the subgroup $H' \subset \mathbb{T}^m$ in Theorem 3.3: the subgroup generated by $\bigcup_{n\in\mathbb{Z}} A^k H$ (which we know to be finite from the proof of Theorem 3.3).

(3) Corollary 3.4 shows that a polymorphism $\Pi_{\mathsf{P}'}$ is a factor of an automorphism $A \in \mathrm{GL}(m, \mathbb{Z})$ of $\mathbb{T}^m$ if and only if there exist an $A'' \in \mathrm{GL}(m, \mathbb{Z})$ which is conjugate to A in $\mathrm{GL}(m, \mathbb{Q})$ and finite groups $H \subset H' \subset \mathbb{T}^m$ such that P' is a skew product over the (the graph of) automorphism A'' with fibre (H/H'). Note, however, that P' is connected and is therefore a non-trivial H/H'-bundle over the base $\mathbb{T}^m$ on which A'' acts.

(4) In [**1**] it is shown that every polymorphism is a factor of an automorphism with respect to some invariant partition (i.e. invariant sub-sigma-algebra), but Theorem 3.3 shows that this is not true in the algebraic category.

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