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ON THE CAPACITY FUNCTIONAL OF EXCURSION SETS OF GAUSSIAN RANDOM FIELDS ON $\mathbb{R}^2$

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Abstract

When a random field $(X_t,\ t \in \mathbb{R}^2)$ is thresholded on a given level u, the excursion set is given by its indicator $\mathbf{1}_{[u,\infty)}(X_t)$. The purpose of this work is to study functionals (as established in stochastic geometry) of these random excursion sets as, e.g. the capacity functional as well as the second moment measure of the boundary length. It extends results obtained for the one-dimensional case to the two-dimensional case, with tools borrowed from crossings theory, in particular, Rice methods, and from integral and stochastic geometry.

Keywords: Capacity functional; crossings; excursion set; Gaussian field; growing circle method; Rice formula; second moment measure; sweeping line method; stereology; stochastic geometry

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1. Introduction

Let $\mathbb{R}^2$ denote the two-dimensional Euclidean plane with the origin 0, the inner product $\langle \cdot, \cdot \rangle$, the norm $\| \cdot \|$, and the unit sphere $\mathcal{S}^1 = \{v \in \mathbb{R}^2 \colon \|v\| = 1\}$. We will refer to the elements of $\mathbb{R}^2$ both as points and as vectors. The Borel σ-algebra is denoted $\mathcal{R}_2$.

Let X be a stationary random field taking values in $\mathbb{R}$, with continuously differentiable paths (C^1 paths, for short). It will be described by

$$X = (X_x,\ x \in \mathbb{R}^2) \quad \text{or} \quad (X_{sv},\ s \in [0,\infty),\ v \in \mathcal{S}^1).$$

We denote by r its correlation function and by f_{X_0} its one-dimensional marginal density function, which is a standard normal density function.

Denote by A_u the excursion set of the random field X over a threshold $u \in \mathbb{R}$, i.e.

$$A_u = \{x \in \mathbb{R}^2 \colon X_x \geq u\} = \{sv \colon X_{sv} \geq u,\ s \in [0,\infty),\ v \in \mathcal{S}^1\}. \tag{1}$$

Since X is a random field with C^1 paths, then, for all $u \in \mathbb{R}$, the set A_u is a random closed set (see [14, Section 5.2]) and the topological closure of the complement, denoted by $\mathrm{cl}(A_u^{\mathrm{c}})$, is also a random closed set (see [18, p. 19 and Theorem 12.2.6.(b)]). The distribution of a random closed set is fully characterized by its capacity functional T (see [12], or [14], [18]), which for A_u is defined by

$$T(K) = \mathbb{P}(A_u \cap K \neq \varnothing) \quad \text{for all compact subsets } K \subset \mathbb{R}^2. \tag{2}$$

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Since $T(K) = \mathbb{P}(\sup\{X_x;\ x \in K\} \geq u)$ the results for the distribution of the supremum of X over a set K (see, e.g. [2], [4], and [15]) can be applied to the capacity functional.

Often it is too complicated to describe the capacity functional completely. Therefore, one usually restricts the family of sets K considered in (2) to certain parametric families of sets, e.g. circles with varying radii or linear segments with a fixed direction and varying lengths, yielding the spherical or linear contact distributions, respectively (see [6]). Thus, at least partial information about the distribution of the random set is available. This approach is also used in spatial statistics.

In this paper we choose $k \geq 2$ directions given by unit vectors $v_1, \dots, v_k \in \mathcal{S}^1$, and denote by $[0, l_i v_i] = \{s v_i : 0 \leq s \leq l_i\}$ the linear segment with one endpoint in the origin 0, length $l_i > 0$, and direction v_i. We consider the sets

$$K = \bigcup_{i=1}^{k} [0, l_i v_i] \quad \text{with } l_i > 0,\ i = 1, \dots, k. \tag{3}$$

By $L_i = \sup\{l : [0, l v_i] \subset A_u^{\mathrm{c}}\}$ we denote the random distance—the visibility—in direction v_i from the origin 0 to the next point of the boundary ∂A_u, if $0 \in A_u^{\mathrm{c}}$; otherwise, $L_i = 0$. The joint survival function of the visibilities can now be related to the capacity functional by either of the following:

$$\mathbb{P}(L_1 > l_1, \dots, L_k > l_k) = \mathbb{P}(K \subset A_u^{\mathrm{c}}) = 1 - T(K), \tag{4a}$$

$$T(K) = 1 - \mathbb{P}\Big(X_0 < u,\ \sup_{s \in [0, l_i]} X_{s v_i} < u,\ i = 1, \dots, k\Big). \tag{4b}$$

The event in the last expression means that $0 \in A_u^{\mathrm{c}}$ and that there is no up-crossing of the random field X on the segments of K.

Besides the capacity functional of a random set, moment measures of some random measures which are induced by this set are of interest; see [6] and [18].

In [1], [2], [4], and [21] the geometry of excursion sets is studied thoroughly, in particular in [2] with explicit results for the Lipschitz-Killing curvatures (intrinsic volumes) of the excursion sets (see also [3]). In this paper we consider the capacity functional of the excursion set for families of sets K which consist of two or more linear segments, originating from a common point. This can also be interpreted as the joint distribution of the visibility in different directions from a certain point to the boundary of the excursion set. On the other hand, it can be seen as an approximation of the capacity functional of the excursion set for classes of convex polygons.

To study $T(K)$, we extend results obtained for the one-dimensional case (see, e.g. [8]) to the two-dimensional case and borrow tools from the literature on level crossings (see [2], [7], and [9]), in particular, Rice-type methods (see [4], [13], [17], and [21]). We also extend an approach given in [13], that we call the 'sweeping line' method, into a 'growing circle' method. It will be developed in Section 2.

Furthermore, via our approach, we study the second moment measure of the boundary length measure of the excursion set, provided that the boundary is smooth enough. If the boundary ∂A_u is Hausdorff-rectifiable then with the help of the one-dimensional Hausdorff-measure $\mathcal{H}^1$, we define the random measure $\mathcal{L}$ on $[\mathbb{R}^2, \mathcal{R}_2]$ by

$$\mathcal{L}(B) = \mathcal{H}^1(\partial A_u \cap B) \quad \text{for all } B \in \mathcal{R}_2.$$

Then the first moment measure, also termed the intensity measure of the random length measure, is given by

$$\mu^{(1)}(B) = \mathbb{E}[\mathcal{L}(B)] = \mathbb{E}[\mathcal{H}^1(\partial A_u \cap B)] \quad \text{for all } B \in \mathcal{R}_2,$$

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and the second moment measure by

$$\mu^{(2)}(B_1 \times B_2) = \mathbb{E}[\mathcal{L}(B_1)\mathcal{L}(B_2)] = \mathbb{E}[\mathcal{H}^1(\partial A_u \cap B_1)\mathcal{H}^1(\partial A_u \cap B_2)] \quad \text{for all } B_1, B_2 \in \mathcal{R}_2.$$

The stationarity of X, and, thus, also of A_u, yields that the intensity measure is a multiple of the Lebesgue measure λ_2 on $[\mathbb{R}^2, \mathcal{R}_2]$, i.e. $\mu^{(1)} = L_A \lambda_2$ with a positive constant L_A which is the mean length of ∂A_u per unit area; see [6] and [18].

Furthermore, stationarity allows the following implicit definition of the reduced second moment measure κ on $[\mathbb{R}^2, \mathcal{R}_2]$:

$$\mu^{(2)}(B_1 \times B_2) = L_A^2 \int\int \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(x+h)\kappa(\mathrm{d}h)\lambda_2(\mathrm{d}x). \tag{5}$$

The value $L_A\kappa(B)$ is the mean length of ∂A_u within $B \in \mathcal{R}_2$, given that the origin is located at the 'typical point' of the boundary with respect to the length measure and the corresponding Palm distribution; see [6] and [18].

Note that this second moment measure for the length of the boundary has been studied in [4, Theorems 6.8 and 6.9] using the co-area formula. Here we present an alternative approach, based on stereology, to provide another expression for the second moment measure. Since this second moment measure can be determined from intersections of ∂A_u with pairs of lines and from the observation of pairs of intersection points (see [20]), our method of counting crossings of the random field X on linear segments, developed in Section 2, can be applied to the estimation of the second moment measure. This will be done in Section 3.

The contact distribution functions as well as the intensity L_A and the (reduced) second moment measure yield established tools for model adaption and goodness-of-fit tests; see [6].

Finally, note, as in [8], that we will not tackle the numerical part, which is a subject in itself, as attested by the literature dedicated to this approach in recent years; see [5] and [13] or [4, Chapter 5] and [17].

From now on, let us assume that X is Gaussian, with mean 0 and variance 1.

2. A sweeping line and growing circle methods for an algorithmic computation of the capacity functional

Sweeping line methods are well established in geometry, e.g. for the definition of the Euler–Poincaré characteristic of a set, in image analysis (for both, see [19]), in computational geometry (see [16]), and in probability (see, e.g. [13]). We will apply it together with Gaussian regression and discretization to set an algorithmic computation of the capacity functional for a pair of segments. Then we will modify the method in order to calculate the capacity functional for a bundle of segments, now using circles with growing radii.

Suppose that $C \subset \mathbb{R}^2$ is a compact convex set with $0 \in C$. For $s > 0$, we denote by $s\partial C = \{sx : x \in \partial C\}$ a homothet of the boundary of C, and we consider the family $(s\partial C,\ s > 0)$ as a sweeping contour, determined by C. In this paper we will only use $C = \{x \in \mathbb{R}^2 : \|x\| = 1\}$, the boundary of the unit circle around the origin.

2.1. The capacity functional for a bundle of two line segments

Consider K defined in (3) with $k = 2$, so that $K = [0, l_1 v_1] \cup [0, l_2 v_2]$ with $v_1 \neq v_2$.

Now we specify the sweeping line method for these sets K using an appropriate parametrization for the bundle of two line segments.

We also introduce the $\mathcal{C}^1$-diffeomorphism ρ (except in a finite number of points where it might only be $\mathcal{C}^0$) defined by

$$\rho\colon [0,\ l_1+l_2] \to K, \qquad \theta \mapsto \begin{cases} (l_1-\theta)v_1 & \text{if } 0 \le \theta \le l_1, \\ (\theta-l_1)v_2 & \text{if } l_1 \le \theta \le l_1+l_2. \end{cases} \tag{6}$$

We have, via (4a) and (4b), and noting that $\mathbb{P}(X_0 = u) = 0$,

$$\mathbb{P}[L_1 > l_1, L_2 > l_2] = 1 - \mathbb{P}\Big[\sup_{s\in K} X_s > u\Big] = 1 - \mathbb{P}\Big[\sup_{\theta\in[0,l_1+l_2]} Y_\theta > u\Big],$$

where the process $Y = (Y_\theta,\ 0 \le \theta \le l_1+l_2)$ is defined by

$$Y_\theta = X(\rho(\theta)). \tag{7}$$

Let $Y'_\theta = \partial_\theta Y_\theta$ denote the derivative of Y_θ with respect to the parameter θ. Let (e_1, e_2) be an orthonormal basis in $\mathbb{R}^2$. The idea is to introduce a sweeping line parallel to the $(0e_1)$ axis, and to translate it along the $(0e_2)$ axis until meeting a u-crossing by X_s, $s \in K$.

Here we choose the $(0e_2)$ axis in such a way that the vectors v_1 and v_2 become symmetric to the $(0e_2)$ axis and define

$$\widetilde{\varphi} = \angle(v_2, 0e_2) \in \Big(0; \frac{\pi}{2}\Big], \qquad v_1 = (-\sin\widetilde{\varphi}, \cos\widetilde{\varphi}), \qquad v_2 = (\sin\widetilde{\varphi}, \cos\widetilde{\varphi}). \tag{8}$$

We then start with the sweeping line method to express the capacity functional for a bundle of two line segments.

Noting that $X_0 \neq u$ almost surely, we have the following result.

Theorem 1. *Let X be a stationary Gaussian random field, mean 0 and variance 1, with C^1 paths, and Y be defined as in (7). Furthermore, let $K = [0, l_1v_1] \cup [0, l_2v_2]$ and $\widetilde{\varphi}$ be as in (8). The capacity functional T of A_u is given for K as follows.*

If $l_1 \le l_2$ then

$$\begin{aligned} T(K) = f_{X_0}(u) \int_{[0;l_1]} &(\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[\theta;2l_1-\theta]\}} \mid Y_\theta = u] \\ &- \mathbb{E}[|Y'_{2l_1-\theta}| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[\theta;2l_1-\theta]\}} \mid Y_{2l_1-\theta} = u])\,\mathrm{d}\theta \\ + f_{X_0}(u) &\int_{[2l_1;l_1+l_2]} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[0;\theta]\}} \mid Y_\theta = u]\,\mathrm{d}\theta. \end{aligned} \tag{9}$$

If $l_1 \ge l_2$ then

$$\begin{aligned} T(K) = f_{X_0}(u) &\int_{[0;l_1-l_2]} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[\theta;l_1+l_2]\}} \mid Y_\theta = u]\,\mathrm{d}\theta \\ + f_{X_0}(u) &\int_{[l_1-l_2;l_1]} (\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[\theta;2l_1-\theta]\}} \mid Y_\theta = u] \\ &- \mathbb{E}[|Y'_{2l_1-\theta}| \mathbf{1}_{\{Y_\eta \le u \textit{ for all } \eta\in[\theta;2l_1-\theta]\}} \mid Y_{2l_1-\theta} = u])\,\mathrm{d}\theta. \end{aligned} \tag{10}$$

Proof. As already mentioned, we introduce a sweeping line parallel to the $(0e_1)$ axis and translate it along the $(0e_2)$ axis until meeting a u-crossing by X_s, $s \in K$. Setting

$$\Gamma_t = \{s = (s_1, s_2) \in K : s_2 \le t_2\}, \qquad t = (t_1, t_2) \in \mathbb{R}^2,$$

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where the parameter t_2 indicates the position of that sweeping line, we can write

$$\mathbb{P}[L_1 > l_1, L_2 > l_2] = 1 - \mathbb{E}[\#\{\theta \in [0, l_1 + l_2], Y_\theta = u,\ X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\}],$$

where $\#\{\theta \in [0, l_1 + l_2], Y_\theta = u,\ X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\} = 1$ if there is a (first) crossing by X on K, and 0 otherwise.

So, using the Rice formula (f_{Y_θ} denoting the density function of Y_θ), then the stationarity of X, we obtain

$$\begin{aligned}\mathbb{P}[L_1 > l_1, L_2 > l_2] &= 1 - \int_0^{l_1+l_2} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\}} \mid Y_\theta = u] f_{Y_\theta}(u)\, \mathrm{d}\theta \\ &= 1 - f_{X_0}(u) \int_0^{l_1+l_2} \mathbb{E}[|Y'_\theta|\, \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\}} \mid Y_\theta = u]\, \mathrm{d}\theta. \qquad (11)\end{aligned}$$

Note that this type of integral can be numerically evaluated as in [13].

Let us go further in the study of the integral appearing in (11), reducing the problem to a one-dimensional parameter set.

If $l_1 \le l_2$ then

$$\begin{aligned}&\int_{[0;l_1+l_2]} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\}} \mid Y_\theta = u]\, \mathrm{d}\theta \\ &\quad = \int_{[0;l_1]} (\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [\theta; 2l_1-\theta]\}} \mid Y_\theta = u] \\ &\qquad\qquad - \mathbb{E}[|Y'_{2l_1-\theta}| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [\theta; 2l_1-\theta]\}} \mid Y_{2l_1-\theta} = u])\, \mathrm{d}\theta \\ &\qquad + \int_{[2l_1;l_1+l_2]} \mathbb{E}[|Y'_\theta|\, \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [0;\theta]\}} \mid Y_\theta = u]\, \mathrm{d}\theta.\end{aligned}$$

If $l_1 \ge l_2$ then

$$\begin{aligned}&\int_{[0;l_1+l_2]} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Gamma_{\rho(\theta)}\}} \mid Y_\theta = u]\, \mathrm{d}\theta \\ &\quad = \int_{[0;l_1-l_2]} \mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [\theta; l_1+l_2]\}} \mid Y_\theta = u]\, \mathrm{d}\theta \\ &\qquad + \int_{[l_1-l_2;l_1]} (\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [\theta; 2l_1-\theta]\}} \mid Y_\theta = u] \\ &\qquad\qquad - \mathbb{E}[|Y'_{2l_1-\theta}| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in [\theta; 2l_1-\theta]\}} \mid Y_{2l_1-\theta} = u])\, \mathrm{d}\theta.\end{aligned}$$

Hence, the result follows. □

Let $I(\theta)$ denote the following interval (as it appears in the indicator functions of (9) and (10)), i.e.

$$I(\theta) = \begin{cases} [\theta, 2l_1 - \theta] & \text{for } l_1 \le l_2,\ 0 \le \theta \le l_1, \\ [0, \theta] & \text{for } l_1 \le l_2,\ 2l_1 \le \theta \le l_1 + l_2, \\ [0, l_1 + l_2] & \text{for } l_1 > l_2,\ 0 \le \theta \le l_1 - l_2, \\ [\theta, 2l_1 - \theta] & \text{for } l_1 > l_2,\ l_1 - l_2 < \theta \le l_1. \end{cases} \qquad (12)$$

The integrands appearing in Theorem 1 as conditional expectations of the form

$$\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in I(\theta)\}} \mid Y_\theta = u]$$

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will now be treated via an approximation by discretization. We will use a standard method when working with Gaussian vectors; namely, the Gaussian regression (see, e.g. [11]). This may allow us to handle numerically the computation of the conditional expectations.

Before stating the main result, let us introduce some further notation.

Let ∂v_i denote the directional derivative with respect to v_i, for $i = 1, 2$, which corresponds to

$$\partial v_1 X_{lv_1} = \lim_{h\to 0} \frac{1}{h}(X_{(l+h)v_1} - X_{lv_1}) = -\sin\widetilde{\varphi}\,\partial_{10} X_{-l\sin\widetilde{\varphi},\, l\cos\widetilde{\varphi}} + \cos\widetilde{\varphi}\,\partial_{01} X_{-l\sin\widetilde{\varphi},\, l\cos\widetilde{\varphi}}$$

and

$$\partial v_2 X_{lv_2} = \sin\widetilde{\varphi}\,\partial_{10} X_{l\sin\widetilde{\varphi}, l\cos\widetilde{\varphi}} + \cos\widetilde{\varphi}\,\partial_{01} X_{l\sin\widetilde{\varphi},\, l\cos\widetilde{\varphi}},$$

where ∂_{ij} denotes the partial derivative of order $i+j$ with the ith partial derivative in direction e_1, and the jth partial derivative in direction e_2.

Recall that the covariances between the random field X and its partial derivatives, when existing, are given, for $s, t, h_1, h_2 \in \mathbb{R}^2$, by (see [10])

$$\mathbb{E}[\partial_{jk} X_{s+h_1, t+h_2} \partial_{lm} X_{s,t}] = (-1)^{l+m} \partial_{j+l, k+m} r(h_1, h_2) \tag{13}$$

for all $0 \le j+k \le 2,\ 0 \le l+m \le 2$.

Theorem 2. *Let X be a stationary Gaussian random field, mean 0 and variance 1, with C^1 paths and a twice differentiable correlation function r. Furthermore, for all $m \in \mathbb{N}$, let $\eta_1, \dots, \eta_m$ be equidistant points, partitioning $I(\theta)$ (defined in (12)), into $m-1$ intervals (where η_1 and η_m coincide with the left and right boundary of $I(\theta)$, respectively). Then we have*

$$\begin{aligned}\mathbb{E}[|Y'_\theta| \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta \in I(\theta)\}} \mid Y_\theta = u] \\ = \lim_{m\to\infty} \int_{\mathbb{R}} |y| F_{\xi^{(m)}}(u(1 - a(\eta_i;\theta)) - y\, b(\eta_i;\theta);\ i = 1, \dots, m) f_{Y'_\theta}(y)\, \mathrm{d}y,\end{aligned} \tag{14}$$

where the density $f_{Y'_\theta}$ of Y'_θ is Gaussian with mean 0 and variance given by

$$\begin{aligned}\mathbb{E}[{Y'_\theta}^2] &= (-\partial_{20} r(0,0)\sin^2\widetilde{\varphi} - \partial_{02} r(0,0)\cos^2\widetilde{\varphi} + 2\partial_{11} r(0,0)\sin\widetilde{\varphi}\cos\widetilde{\varphi})\mathbf{1}_{\{0\le\theta<l_1\}} \\ &\quad - (\partial_{20} r(0,0)\sin^2\widetilde{\varphi} + \partial_{02} r(0,0)\cos^2\widetilde{\varphi} + 2\partial_{11} r(0,0)\sin\widetilde{\varphi}\cos\widetilde{\varphi})\mathbf{1}_{\{l_1<\theta\le l_1+l_2\}},\end{aligned}$$

with $\widetilde{\varphi}$ being defined in (8), and where $F_{\xi^{(m)}}$ is the cumulative distribution function of the Gaussian vector $\xi^{(m)} = (\xi_i,\ i = 1, \dots, m)\colon \mathcal{N}(0, \Sigma_m)$. The covariance matrix Σ_m is given by

$$\operatorname{var}(\xi_i) = 1 - a^2(\eta_i, \theta) - b^2(\eta_i, \theta)$$

and, for η_i, $i = 1, \dots, m$ pairwise different,

$$\operatorname{cov}(\xi_i, \xi_j) = a(\eta_i, \eta_j) - a(\eta_i, \theta) a(\eta_j, \theta) - b(\eta_i, \theta) b(\eta_j, \theta)\mathbb{E}[{Y'_\theta}^2],$$

the coefficients $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ being defined below in (18) and (19), respectively.

Looking for the rate of convergence to the limit in (14) might be of interest for applications and is an area of investigation for future research. Nevertheless, we can already deduce from this theorem an approximation that is quite useful for a numerical evaluation of the capacity functional.

Corollary 1. *The capacity functional $T(K)$ given in Theorem 1 can be numerically evaluated by approximating, for large m, its integrands as*

$$\mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \mid Y_\theta = u] \approx \int_{\mathbb{R}} |y| F_{\xi^{(m)}}(u(1-a(\eta_i;\theta)) - yb(\eta_i;\theta);\ i=1,\dots,m) f_{Y'_\theta}(y)\,\mathrm{d}y. \tag{15}$$

The proof of Theorem 2 is based on the following lemma.

Lemma 1. *Under the assumptions of Theorem 2, we have*

$$\mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \mid Y_\theta = u] = \lim_{m\to\infty} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} \mid Y_\theta = u].$$

Proof. Let $D_i^{(m)} = \{C_u(I_i^{(m)}) \ge 2\}$ denote the event that the number of crossings in the interval $I_i^{(m)}$, $i=1,\dots,m-1$, is greater than or equal to 2, where $I_i^{(m)}$ is the ith open interval of the equidistant partition of $I(\theta)$ into $m-1$ intervals by $\eta_1,\dots,\eta_m$.

Noting that

$$\mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} - \mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} \le \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \le \mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}},$$

we can write

$$\begin{aligned}\mathbb{E}[|Y'_\theta|(\mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} - \mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}}) \mid Y_\theta = u] \\ \le \mathbb{E}[|Y'_\theta|\, \mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \mid Y_\theta = u] \\ \le \mathbb{E}[|Y'_\theta|\, \mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} \mid Y_\theta = u].\end{aligned} \tag{16}$$

Moreover, since for all $m\in\mathbb{N}$, $|Y'_\theta|\mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} \le |Y'_\theta|$, and $|Y'_\theta|$ is integrable with respect to the conditional distribution given $Y_\theta = u$ (the number of crossings in $I(\theta)$ having finite mean), then, using the theorem of dominated convergence, we obtain

$$\lim_{m\to\infty} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} \mid Y_\theta = u] = \mathbb{E}[|Y'_\theta| \lim_{m\to\infty} \mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} \mid Y_\theta = u].$$

Since $\lim_{m\to\infty} \mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} = 0$ for almost all paths of Y, we deduce that

$$\lim_{m\to\infty} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\bigcup_{i=1}^{m-1} D_i^{(m)}} \mid Y_\theta = u] = 0. \tag{17}$$

Combining (16) and (17), we conclude that

$$\mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \mid Y_\theta = u] = \lim_{m\to\infty} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} \mid Y_\theta = u]. \qquad \square$$

Proof of Theorem 2. Regressing the random vector $Y_\eta^{(m)} = (Y_{\eta_1},\dots,Y_{\eta_m})$, $m\ge 1$, on Y_θ and Y'_θ, which are independent at fixed θ (see, e.g. [7]), we obtain

$$Y_\eta^{(m)} = \delta^{(m)}\, \xi^{(m)} + a^{(m)} Y_\theta + b^{(m)} Y'_\theta,$$

where the deterministic vectors

$$\delta^{(m)} = (\delta(\eta_1,\theta),\dots,\delta(\eta_m,\theta)), \qquad a^{(m)} = (a(\eta_1,\theta),\dots,a(\eta_m,\theta)),$$

and

$$b^{(m)} = (b(\eta_1,\theta),\dots,b(\eta_m,\theta))$$

have their components defined respectively by

$$\delta(\alpha,\theta) = \mathbf{1}_{\{\alpha\ne\theta\}}, \qquad a(\theta,\theta) = 1, \qquad b(\theta,\theta) = 0,$$

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and, for $\alpha \neq \theta$,

$$a(\alpha,\theta) = \mathbb{E}[Y_\alpha Y_\theta]$$
$$= \begin{cases} r((\theta-\alpha)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \text{if } 0 \le \theta, \alpha \le l_1, \\ r((\theta-\alpha)\sin\widetilde{\varphi}, (\theta-\alpha)\cos\widetilde{\varphi}) & \text{if } \theta, \alpha \ge l_1, \\ r((2l_1-\alpha-\theta)\sin\widetilde{\varphi}, (\theta-\alpha)\cos\widetilde{\varphi}) & \text{if } 0 \le \theta \le l_1 \le \alpha \le l_1+l_2, \\ r((2l_1-\alpha-\theta)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \text{if } 0 \le \alpha \le l_1 \le \theta \le l_1+l_2, \end{cases} \tag{18}$$

$$b(\alpha,\theta)$$
$$= \mathbb{E}[Y_\alpha Y'_\theta]$$
$$= \mathbb{E}[Y_\alpha \partial_{v_1} Y_\theta]\mathbf{1}_{\{\theta\in[0,l_1]\}} + \mathbb{E}[Y_\alpha \partial_{v_2} Y_\theta]\mathbf{1}_{\{\theta\in(l_1,l_1+l_2]\}}$$
$$= \begin{cases} -\sin\widetilde{\varphi}\partial_{10} r((\theta-\alpha)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \\ \quad +\cos\widetilde{\varphi}\partial_{01} r((\theta-\alpha)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \text{if } 0 \le \theta,\ \alpha \le l_1, \\ \sin\widetilde{\varphi}\ \partial_{10} r((\alpha-\theta)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \\ \quad +\cos\widetilde{\varphi}\partial_{01} r((\alpha-\theta)\sin\widetilde{\varphi}, (\alpha-\theta)\cos\widetilde{\varphi}) & \text{if } \theta,\ \alpha \ge l_1, \\ \sin\widetilde{\varphi}\partial_{10} r((\alpha-\theta)\sin\widetilde{\varphi}, (\alpha+\theta-2l_1)\cos\widetilde{\varphi}) & \\ \quad -\cos\widetilde{\varphi}\partial_{01} r((\alpha-\theta)\sin\widetilde{\varphi}, (\alpha+\theta-2l_1)\cos\widetilde{\varphi}) & \text{if } 0 \le \theta \le l_1 \le \alpha \le l_1+l_2, \\ \sin\widetilde{\varphi}\partial_{10} r((\theta-\alpha)\sin\widetilde{\varphi}, (\alpha+\theta-2l_1)\cos\widetilde{\varphi}) & \\ \quad +\cos\widetilde{\varphi}\partial_{01} r((\theta-\alpha)\sin\widetilde{\varphi}, (\alpha+\theta-2l_1)\cos\widetilde{\varphi}) & \text{if } 0 \le \alpha \le l_1 \le \theta \le l_1+l_2, \end{cases} \tag{19}$$

and where the random vector $\xi^{(m)} = (\xi_1, \dots, \xi_m)$ is independent of (Y_θ, Y'_θ), Gaussian ($F_{\xi^{(m)}}$ denoting its cumulative distribution function), mean 0, covariance matrix Σ_m with

$$\operatorname{var}(\xi_i) = 1 - a^2(\eta_i,\theta) - b^2(\eta_i,\theta), \qquad i = 1, \dots, m,$$

and, for $\eta_1, \dots, \eta_m$ pairwise different,

$$\operatorname{cov}(\xi_i,\xi_j) = \mathbb{E}[\xi_i\xi_j] = a(\eta_i,\eta_j) - a(\eta_i,\theta)a(\eta_j,\theta) - b(\eta_i,\theta)b(\eta_j,\theta)\mathbb{E}[{Y'_\theta}^2]$$

since $\mathbb{E}[Y_\theta^2] = \operatorname{var}(X_{\rho(\theta)}) = 1$. On the one hand, using (13), we obtain, if $0 \le \theta < l_1$,

$$\mathbb{E}[{Y'_\theta}^2] = \mathbb{E}[(\partial_{v_1} X_{(l_1-\theta)v_1})^2]$$
$$= \mathbb{E}[(-\sin\widetilde{\varphi}\ \partial_{10} X_{-(l_1-\theta)\sin\widetilde{\varphi},\ (l_1-\theta)\cos\widetilde{\varphi}} + \cos\widetilde{\varphi}\partial_{01} X_{-(l_1-\theta)\sin\widetilde{\varphi},\ (l_1-\theta)\cos\widetilde{\varphi}})^2]$$
$$= -\partial_{20} r(0,0)\sin^2\widetilde{\varphi} - \partial_{02} r(0,0)\cos^2\widetilde{\varphi} + 2\partial_{11} r(0,0)\sin\widetilde{\varphi}\cos\widetilde{\varphi},$$

and, on the other hand, if $l_1 < \theta \le l_1 + l_2$,

$$\mathbb{E}[{Y'_\theta}^2] = -\partial_{20} r(0,0)\sin^2\widetilde{\varphi} - \partial_{02} r(0,0)\cos^2\widetilde{\varphi} - 2\partial_{11} r(0,0)\sin\widetilde{\varphi}\cos\widetilde{\varphi}.$$

Therefore, using this Gaussian regression for any vector $Y_\eta^{(m)}$ of any size m, and the independence of $(Y_\theta, Y'_\theta, \xi)$, we can write, for the interval $I(\theta)$, $\xi = (\xi_\eta)$ denoting the Gaussian process defined by its finite-dimensional distributions of $\xi^{(m)}$,

$$\mathbb{E}[|Y'_\theta|\mathbf{1}_{\{Y_\eta \le u \text{ for all } \eta\in I(\theta)\}} \mid Y_\theta = u]$$
$$= \mathbb{E}[|Y'_\theta|\mathbf{1}_{\{b(\eta,\theta)Y'_\theta \le u(1-a(\eta,\theta)) - \delta(\eta,\theta)\xi_\eta \text{ for all } \eta\in I(\theta)\}}]$$
$$= \mathbb{E}[\mathbb{E}[|Y'_\theta|\mathbf{1}_{\{b(\eta,\theta)Y'_\theta \le u(1-a(\eta,\theta)) - \delta(\eta,\theta)\xi_\eta \text{ for all } \eta\in I(\theta)\}}] \mid \xi].$$

To compute this last expression, we proceed by discretization, working on vectors. We have, for a given vector $(\eta_1, \dots, \eta_m)$,

$$\begin{aligned}
&\mathbb{E}[|Y'_\theta|\,\mathbf{1}_{\{Y_{\eta_1}\le u,\dots,Y_{\eta_m}\le u\}} \mid Y_\theta = u] \\
&\quad= \int_{\mathbb{R}^m} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\{b(\eta_i;\theta)Y'_\theta\le u(1-a(\eta_i;\theta))-z_i;\ i=1,\dots,m\}} \mid \xi^{(m)} = z] f_{\xi^{(m)}}(z)\,\mathrm{d}z \\
&\quad= \int_{\mathbb{R}^m} \mathbb{E}[|Y'_\theta|\mathbf{1}_{\{b(\eta_i;\theta)Y'_\theta\le u(1-a(\eta_i;\theta))-z_i;\ i=1,\dots,m\}}] f_{\xi^{(m)}}(z)\,\mathrm{d}z \\
&\quad= \int_{\mathbb{R}} |y| \int_{\mathbb{R}^m} \mathbf{1}_{\{z_i\le(1-a(\eta_i;\theta))u-b(\eta_i;\theta)y;\ i=1,\dots,m\}} f_{\xi^{(m)}}(z)\,\mathrm{d}z f_{Y'_\theta}(y)\,\mathrm{d}y \\
&\quad= \int_{\mathbb{R}} |y|\mathbb{P}[\xi_i \le u(1-a(\eta_i;\theta)) - yb(\eta_i;\theta);\ i = 1,\dots,m] f_{Y'_\theta}(y)\,\mathrm{d}y \\
&\quad= \int_{\mathbb{R}} |y| F_{\xi^{(m)}}(u(1-a(\eta_i;\theta)) - yb(\eta_i;\theta);\ i = 1,\dots,m) f_{Y'_\theta}(y)\,\mathrm{d}y
\end{aligned}$$

using the independence of ξ and Y'_θ in the second equality.

Taking the limit as $m \to \infty$ in the previous equations and applying Lemma 1 provides the result (14). □

Example 1. Let us consider a stationary and isotropic Gaussian random field X, with correlation function r defined on $\mathbb{R}^2$ by $r(x) = \mathrm{e}^{-\|x\|^2/2}$. Then, for $x = (x_1, x_2)$, we have

$$\partial_{10} r(x) = -x_1 r(x), \qquad \partial_{01} r(x) = -x_2 r(x), \qquad \partial_{11} r(x) = -x_2 \partial_{10} r(x) = -x_1 \partial_{01} r(x),$$
$$\partial_{20} r(x) = (x_1^2 - 1) r(x), \qquad \partial_{02} r(x) = (x_2^2 - 1) r(x);$$

hence, the variance of Y'_θ becomes

$$\mathbb{E}[{Y'_\theta}^2] = 1 \quad \text{for all } \theta \in [0, l_1 + l_2],$$

and the coefficients $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy

$$a(\alpha,\theta) = a(\theta,\alpha) = \begin{cases} \exp\{-\frac{1}{2}(\alpha-\theta)^2\} & \text{if } 0 \le \theta,\ \alpha \le l_1 \text{ or if } \theta,\ \alpha \ge l_1, \\ \exp\{-\frac{1}{2}(\alpha-\theta)^2 - 4|(l_1-\alpha)(l_1-\theta)|\sin^2\widetilde{\varphi}\} & \text{if } 0 \le \theta \le l_1 \le \alpha \le l_1 + l_2, \text{ or if } 0 \le \alpha \le l_1 \le \theta \le l_1 + l_2, \end{cases}$$

$$b(\alpha,\theta) = \begin{cases} (\theta-\alpha)\exp\{-\frac{1}{2}(\alpha-\theta)^2\} & \text{if } 0 \le \theta, \alpha \le l_1, \text{ or if } \theta, \alpha \ge l_1, \\ (\theta-\alpha+2(\alpha-l_1)\cos^2\widetilde{\varphi}) \times \exp\{-\frac{1}{2}[(\alpha-\theta)^2 + 4(\alpha-l_1) \times (2\alpha - l_1 - \theta)\cos^2\widetilde{\varphi}]\} & \text{if } 0 \le \theta \le l_1 \le \alpha \le l_1 + l_2, \\ -(\theta-\alpha+2(\alpha-l_1)\cos^2\widetilde{\varphi}) \times \exp\{-\frac{1}{2}[(\alpha-\theta)^2 + 4(\alpha-l_1) \times (2\alpha - l_1 - \theta)\cos^2\widetilde{\varphi}]\} & \text{if } 0 \le \alpha \le l_1 \le \theta \le l_1 + l_2. \end{cases}$$

Therefore, (15) can be computed numerically when replacing $f_{Y'_\theta}$ by a standard normal density function and $\xi^{(m)} = (\xi_1, \dots, \xi_m)$ by a Gaussian $\mathcal{N}(0, \Sigma_m)$ with the covariance matrix Σ_m

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given by

$$\mathrm{var}(\xi_i) = 1 - a^2(\eta_i, \theta) - b^2(\eta_i, \theta), \qquad i = 1, \dots, m,$$

and, for $1 \le i \ne j \le m$, for $\eta_i \ne \eta_j$,

$$\mathrm{cov}(\xi_i, \xi_j) = a(\eta_i, \eta_j) - a(\eta_i, \theta)a(\eta_j, \theta) - b(\eta_i, \theta)b(\eta_j, \theta).$$

2.2. Joint distribution for k line-segments via a growing circle

We can extend to k segments what has been previously developed for two segments, considering a growing circle of radius $t > 0$, with center in 0, under the same assumptions on X. Let $v_1, \dots, v_k \in \mathcal{S}^1$, denoting k directions, and φ_j be the angle between (oe_1) and (ov_j), i.e.

$$\varphi_j = \angle(oe_1, ov_j), \qquad j = 1, \dots, k.$$

Then $X_{tv_j} = X_{t\cos\varphi_j, t\sin\varphi_j}$.

For $l_1, \dots, l_k > 0$, we define the union of segments $K = \bigcup_{i=1}^k [0, l_i v_i]$. The method consists of introducing a circle and making it grow with t until meeting a u-crossing by X_s for $s \in K$.

Setting $\Theta_t = \{s = (s_1, \dots, s_k) \in K \colon \sum_{i=1}^k s_i^2 \le t^2\}$, we can write (analogously to (11), using the Rice formula)

$$\mathbb{P}[L_1 > l_1, \dots, L_k > l_k] = 1 - \sum_{i=1}^k \int_0^{l_i} \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Theta_t\}} \mid X_{tv_i} = u] f_{X_{tv_i}}(u)\, \mathrm{d}t. \tag{20}$$

Now let us compute the conditional expectation, denoted by $\mathbb{E}_i(t)$, appearing as an integrand in (20). We can write, for fixed i and $t \le l_i$,

$$\begin{aligned}\mathbb{E}_i(t) &= \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{X_s \le u \text{ for all } s \in \Theta_t\}} \mid X_{tv_i} = u] \\ &= \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{X_{hv_i} \le u \text{ for all } h \le t\}} \mathbf{1}_{\{X_{hv_j} \le u \text{ for all } h \le \min(l_j, t),\ j \ne i\}} \mid X_{tv_i} = u] \\ &= \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{X_{hv_j} \le u \text{ for all } h \le min(l_j, t),\ j = 1, \dots, k\}} \mid X_{tv_i} = u]\end{aligned} \tag{21}$$

since, for $j = i$, $\min(l_i, t) = t$.

Once again, we proceed by standard Gaussian regression, regressing X_{hv_j} on $(X_{tv_i}, \partial_{v_i} X_{tv_i})$ at given h, i, and t, for any $j = 1, \dots, k$. So we consider

$$X_{hv_j} = Z_{h,j} + \alpha_h^j X_{tv_i} + \beta_h^j\, \partial_{v_i} X_{tv_i} \tag{22}$$

with $\alpha_h^j = r(tv_i - hv_j)$, $\beta_h^j = \cos\varphi_i\ \partial_{10} r(tv_i - hv_j) + \sin\varphi_i\ \partial_{01} r(tv_i - hv_j)$, and $Z_{h,j}$: independent of $(X_{tv_i}, \partial_{v_i} X_{tv_i})$, Gaussian, mean 0, $\mathrm{var}(Z_{h,j}) = 1 - (\alpha_h^j)^2 - (\beta_h^j)^2$, and

$$\mathbb{E}[Z_{h,j} Z_{l,n}] = \mathbb{E}[X_{hv_j} X_{lv_n}] - \alpha_h^j \alpha_l^n - \beta_h^j \beta_l^n = r(hv_j - lv_n) - \alpha_h^j \alpha_l^n - \beta_h^j \beta_l^n.$$

Note that we took $Z_{h,j} = Z_{h,j}^{i,t}$, $\alpha_h^j = \alpha_h^{i,j}$, and $\beta_h^j = \beta_h^{i,j}$ to simplify the notation when working at given i and t.

The conditional expectation (21) can be written as

$$\begin{aligned}\mathbb{E}_i(t) &= \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{Z_{h,j} + \alpha_h^j X_{tv_i} + \beta_h^j \partial_{v_i} X_{tv_i} \le u \text{ for all } h \le \min(l_j, t),\ j = 1, \dots, k\}} \mid X_{tv_i} = u] \\ &= \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{Z_{h,j} + \beta_h^j \partial_{v_i} X_{tv_i} \le u(1 - \alpha_h^j) \text{ for all } h \le \min(l_j, t),\ j = 1, \dots, k\}}] \\ &= \mathbb{E}[\mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{Z_{h,j} + \beta_h^j \partial_{v_i} X_{tv_i} \le u(1 - \alpha_h^j) \text{ for all } h \le \min(l_j, t),\ j = 1, \dots, k\}}] \mid (Z_{h,j})_{h \le t,\ 1 \le j \le k}]\end{aligned}$$

using the independence of $(X_{tv_i}, \partial_{v_i} X_{tv_i}, (Z_{h,i}))$.

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Now we can evaluate $\mathbb{E}_i(t)$ via discretization and using once again the above mentioned independence. We discretize equidistantly the interval $[0, \max_{1\le i\le k} l_i]$ as $[0, h_1] \cup (\bigcup_{i=1}^{n-1}(h_i, h_{i+1}])$ with $h_n = \max_{1\le i\le k} l_i$ and introduce the corresponding Gaussian vector $Z^{(n)} = (Z_{h_m,j};\, 1 \le m \le n,\ 1 \le j \le k)$ with density function $f_{Z^{(n)}}$ and cumulative distribution function $F_{Z^{(n)}}$. Note that we apply the same discretization in any direction v_i, $i = 1, \ldots, k$.

Then Lemma 1 can be applied to the k segments, substituting $I(\theta)$ by $[0, l_i v_i]$, and $\eta_1, \ldots, \eta_m$ by $0, h_1 v_i, \ldots, h_{m_i} v_i$, with $h_{m_i} \le l_i < h_{m_i+1}$ for $i = 1, \ldots, k$. We obtain

$$\begin{aligned}\mathbb{E}_i(t) &= \lim_{n\to\infty} \int_{\mathbb{R}^{n\times k}} \mathbb{E}[|\partial_{v_i} X_{tv_i}| \mathbf{1}_{\{\beta^j_{h_m}\partial_{v_i} X_{tv_i} \le u(1-\alpha^j_{h_m}) - z_{h_m,j} \text{ for all } h_m \le \min(l_j,t),\ j=1,\ldots,k\}}] \\ &\qquad\qquad \times f_{Z^{(n)}}(z)\,\mathrm{d}z \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} |y| \bigg(\int_{\mathbb{R}^{n\times k}} \mathbf{1}_{\{z_{h_m,j} \le u(1-\alpha^j_{h_m}) - y\beta^j_{h_m} \text{ for all } h_m \le \min(l_j,t),\ j=1,\ldots,k\}} f_{Z^{(n)}}(z)\,\mathrm{d}z \bigg) \\ &\qquad\qquad \times f_{\partial_{v_i} X_{tv_i}}(y)\,\mathrm{d}y \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} |y| F_{Z^{(n)}}(w(y,u,\alpha,\beta,t)) f_{\partial_{v_i} X_{tv_i}}(y)\,\mathrm{d}y,\end{aligned}$$

where $f_{\partial_{v_i} X_{tv_i}}$ denotes the density function of $\partial_{v_i} X_{tv_i}$, and

$$w(y,u,\alpha,\beta,t)$$

is an $(n\times k)$ matrix having components $(w_{mj};\, 1 \le m \le n,\ 1 \le j \le k)$ given by

$$w_{mj} = \begin{cases} u(1-\alpha^j_{h_m}) - y\beta^j_{h_m} & \text{if } h_m \le \min(l_j, t), \\ +\infty & \text{otherwise.}\end{cases} \tag{23}$$

We conclude with the following result.

Theorem 3. *Let X be a stationary Gaussian random field, mean 0 and variance 1, with C^1 paths and a twice differentiable correlation function r. Then*

$$\begin{aligned}&\mathbb{P}[L_1 > l_1, \ldots, L_k > l_k] \\ &\qquad = 1 - \lim_{n\to\infty} \sum_{i=1}^k \int_0^{l_i} \bigg(\int_{\mathbb{R}} |y| F_{Z^{(n)}}(w(y,u,\alpha,\beta,t)) f_{\partial_{v_i} X_{tv_i}}(y)\,\mathrm{d}y \bigg) f_{X_{tv_i}}(u)\,\mathrm{d}t,\end{aligned}$$

where w is defined in (23).

Note that we can deduce from this result a way to evaluate numerically the joint distribution $\mathbb{P}[L_1 > l_1, \ldots, L_k > l_k]$, as carried out in Corollary 1.

3. The second moment measure

Now we describe a method to determine the second moment measure of the length measure of the boundary ∂A_u, as defined in (5). It is based on the classical Crofton formula of integral geometry which is widely used in stereology. It allows the determination of the length of a planar curve by an integral of the number of intersection points of the curve with 'test' lines, and the integration goes over all lines of the plane with respect to a motion invariant measure on the set of lines. Note that this second moment measure has been studied in [4, Theorem 6.9 and the associated comment p. 181], using another approach, namely the co-area formula.

Denote by G the set of all lines in the plane. The σ-algebra $\mathcal{G}$ on G is induced by an appropriate parametrization and the Borel σ-algebra on the parameter space. Furthermore, dg denotes the element of the measure on $(G, \mathcal{G})$ which is invariant under translation and rotation of the plane, and normalized such that $\int \mathbf{1}_{\{g\cap A\neq\varnothing\}}\,\mathrm{d}g = 2\pi$ for the unit circle $A \subset \mathbb{R}^2$.

Let $C(g \cap B)$ denote the number of crossings of u by X on the line g within a set $B \subset \mathbb{R}^2$.

Theorem 4. *Let X be a stationary Gaussian random field, mean 0 and variance 1, with C^1 paths. Assume ∂A_u to be smooth (in the sense that it can be parametrized by a C^1 mapping). Then, for bounded Borel sets $B_1, B_2 \subset \mathbb{R}^2$, for which $g_1 \cap B_1$ and $g_2 \cap B_2$ consist of finitely many line segments for all pairs (g_1, g_2) of lines, we have*

$$\mu^{(2)}(B_1 \times B_2) = \frac{1}{4}\int\int \mathbb{E}[C(g_1 \cap B_1)C(g_2 \cap B_2)]\,\mathrm{d}g_1\mathrm{d}g_2. \tag{24}$$

For $g_1 \neq g_2$ and not parallel, denote $p \in \mathbb{R}^2$ such that $\{p\} = g_1 \cap g_2$, and consider $v_1, v_2 \in \mathcal{S}^1$ with $v_1 \neq v_2$ such that $g_1 = \mathbb{R}v_1 + p$, $g_2 = \mathbb{R}v_2 + p$. Then the expectation appearing as the integrand in (24) is given by

$$\mathbb{E}[C(g_1 \cap B_1)C(g_2 \cap B_2)] = \int\int \mathbb{E}[|\partial_{v_1}X_{sv_1} \cdot \partial_{v_2}X_{tv_2}| \mid X_{sv_1} = X_{tv_2} = u] \times f_{X_{sv_1},X_{tv_2}}(u,u)\mathbf{1}_{B_1-p}(sv_1)\mathbf{1}_{B_2-p}(tv_2)\,\mathrm{d}s\mathrm{d}t,$$

where $f_{X_{sv_1},X_{tv_2}}$ denotes the density function of (X_{sv_1}, X_{tv_2}).

Comments. (i) The product $\partial_{v_1}X_{sv_1} \cdot \partial_{v_2}X_{tv_2}$ may again be treated, using Gaussian regression given in (22), but it will not provide as simple a covariance matrix to the one of $(\partial_{v_1}X_{sv_1}, \partial_{v_2}X_{tv_2})$ that we computed using (13).

(ii) Sufficient conditions can be given on X and u for ∂A_u to be smooth. We refer the reader to [4, Section 6.2.2] or [2, Section 6.2].

Proof of Theorem 4. The proof is based on two main steps.

(i) We apply the second-order stereology for planar fibre processes proposed in [20]. Applying [20, Theorem 3.1] for ∂A_u yields

$$\begin{aligned}\mu^{(2)}(B_1 \times B_2) &= \frac{1}{4}\,\mathbb{E}\bigg(\int\int \sum_{y\in\partial A_u\cap g_1}\ \sum_{z\in\partial A_u\cap g_2} \mathbf{1}_{B_1\times B_2}(y,z)\,\mathrm{d}g_1\mathrm{d}g_2\bigg)\\ &= \frac{1}{4}\int\int \mathbb{E}[C(g_1 \cap B_1)C(g_2 \cap B_2)]\,\mathrm{d}g_1\mathrm{d}g_2.\end{aligned}$$

Note that integrating on the restricted domain $\{g_1 = g_2\} \cup \{g_1 \parallel g_2\}$ would give 0 for the double integral and, therefore, we consider integration only on $\{g_1 \neq g_2\} \cap \{g_1$ not parallel to $g_2\}$.

(ii) We use the approach developed in Theorem 1. According to the assumption on the B_is, we can write $g_i \cap B_i = \bigcup_{j=1}^{n_i} I_{ij}$ for $i = 1, 2$, and $n_i \in \mathbb{N}$, where the I_{ij} are pairwise disjoint intervals. Then, we obtain

$$\mathbb{E}[C(g_1 \cap B_1)C(g_2 \cap B_2)] = \sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\mathbb{E}[C(I_{1j})C(I_{2k})].$$

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Let us compute each term of the double sum. For fixed j, k, first shift and rotate g_1, g_2, B_1, B_2 such that the lines have a representation $g_i = \mathbb{R}v_i$, $i = 1, 2$, with v_1, v_2 as in (8). Let $\widetilde{B}_i$ and $\widetilde{I}_{1j}, \widetilde{I}_{2k}$ denote the adequate transformations of B_i and I_{1j}, I_{2k}, respectively. Then, using the diffeomorphism ρ analogous to (6), which may also be applied if the intervals do not intersect, and applying a Rice-type formula for the second moment (see [4, Equation (6.28)]), provides

$$\mathbb{E}[C(I_{1j})C(I_{2k})] = \int_{\widetilde{I}_{1j}\times\widetilde{I}_{2k}} \mathbb{E}[|\partial_{v_1}Y_{\theta_1}\cdot\partial_{v_2}Y_{\theta_2}| \mid Y_{\theta_1} = Y_{\theta_2} = u] \times f_{Y_{\theta_1},Y_{\theta_2}}(u,u)\mathbf{1}_{\{\widetilde{B}_1\times\widetilde{B}_2\}}(\rho(\theta_1),\rho(\theta_2))\,\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Note that the rotation has been introduced only to apply (6); what does matter is the shift by p, the intersection point of g_1 and g_2.

Combining those results provides the theorem. □

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