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Abstract

In this article, we further the study of higher K-theory of differential graded (dg) categories via universal invariants, initiated in [G. Tabuada, *Higher K-theory via universal invariants*, Duke Math. J. **145** (2008), 121–206]. Our main result is the co-representability of non-connective K-theory by the base ring in the 'universal localizing motivator'. As an application, we obtain for free higher Chern characters, respectively higher trace maps, from non-connective K-theory to cyclic homology, respectively to topological Hochschild homology.

Introduction

Dg categories

A *differential graded (dg) category*, over a commutative base ring k, is a category enriched over cochain complexes of k-modules (morphisms sets are such complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Dg categories enhance and solve many of the technical problems inherent to triangulated categories; see Keller's address at the International Congress of Mathematics (ICM) at Madrid 2006 [Kel06]. In *non-commutative algebraic geometry* in the sense of Bondal, Drinfeld, Kapranov, Kontsevich, Toën, Van den Bergh [BK90, BV03, Dri, Dri04, Kon05, Kon09, Toe07], they are considered as dg-enhancements of derived categories of quasi-coherent sheaves on a hypothetic non-commutative space.

Additive and localizing invariants

All the classical (functorial) invariants, such as algebraic K-theory, cyclic homology (and its variants), and even topological cyclic homology (see [Tab10, § 10]) extend naturally from k-algebras to dg categories. In a 'motivic spirit', in order to study *all* these classical invariants simultaneously, the notions of *additive* and *localizing* invariant have been introduced in [Tab08]. These notions, that we now recall, make use of the language of Grothendieck derivators [Gro], a formalism which allows us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using model categories; consult Appendix A. For this introduction, it is sufficient to think of (triangulated) derivators as (triangulated) categories which have all the good properties of the homotopy category of a (stable) model category.

Let $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ be a morphism of derivators, from the pointed derivator associated to the Morita model structure (see § 2.5), to a strong triangulated derivator (in practice, $\mathbb{D}$ will

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be the derivator associated to a stable model category $\mathcal{M}$, and E will come from a functor $\mathsf{dgcat} \to \mathcal{M}$ which sends derived Morita equivalences to weak equivalences in $\mathcal{M}$). We say that E is an *additive invariant* if it preserves filtered homotopy colimits as well as the terminal object and sends *split exact sequences* (see 3.1) to direct sums:

$$\mathcal{A} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{B} \underset{P}{\overset{S}{\leftrightarrows}} \mathcal{C} \;\mapsto [E(I) E(S)] : E(\mathcal{A}) \oplus E(\mathcal{C}) \xrightarrow{\sim} E(\mathcal{B}).$$

We say that E is a *localizing invariant* if it satisfies the same conditions as an additive invariant and sends *exact sequences* (see 3.1) to distinguished triangles

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C} \mapsto E(\mathcal{A}) \xrightarrow{E(I)} E(\mathcal{B}) \xrightarrow{E(P)} E(\mathcal{C}) \longrightarrow E(\mathcal{A})[1].$$

In [Tab08], the second named author constructed the *universal additive* and *localizing* invariants:

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}, \quad \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}.$$

Roughly, every additive (respectively localizing) invariant $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathbb{D}$ factors uniquely through $\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}$ (respectively $\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}$); see Theorem 7.13. For instance, Waldhausen's K-theory of dg categories defines a functor from dgcat to the model category of spectra Spt which sends derived Morita equivalences to stable homotopy equivalences (see 2.29). Hence it gives rise to a morphism of derivators $K : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$. As K-theory preserves filtered (homotopy) colimits, it follows from Waldhausen's additivity theorem that K is an additive invariant. Hence there is a unique homotopy colimit preserving morphism of triangulated derivators $K_{\mathsf{add}} : \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}} \to \mathsf{HO}(\mathsf{Spt})$ such that $K = K_{\mathsf{add}} \circ \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}$. Notice that a localizing invariant is also an additive invariant, but the converse does not hold. Because of this universality property, which is a reminiscence of motives, $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ is called the *additive motivator* and $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ the *localizing motivator*. Recall that they are both triangulated derivators. Before going further, note that any triangulated derivator is canonically enriched over spectra; see § A.3. The spectra of morphisms in a triangulated derivator $\mathbb{D}$ will be denoted by $\mathbb{R}\mathsf{Hom}(-,-)$. The main result from [Tab08] is the following.

Theorem [Tab08, Theorem 15.10]. *For every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$K(\mathcal{A}) \simeq \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}(\mathcal{A})),$$

where $\underline{k}$ is the dg category with a single object and k as the dg algebra of endomorphisms, and $K(\mathcal{A})$ is Waldhausen's K-theory spectrum of $\mathcal{A}$.

However, from the 'motivic' point of view, this co-representability result is not completely satisfactory, in the sense that all the classical invariants (except Waldhausen's K-theory) are localizing. Therefore, the base category $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}(e)$ of the localizing motivator is morally what we would like to consider as the triangulated category of *non-commutative motives*. From this point of view, the importance of the computation of the (spectra of) morphisms between two objects in the localizing motivator is by now clear. Hence, a fundamental problem of the theory of non-commutative motives is to compute the localizing invariants (co-)represented by dg categories in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$. In particular, we would like a co-representability theorem analogous to the preceding one, with $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ replaced by $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ and $K(\mathcal{A})$ replaced by the non-connective K-theory spectrum $\mathbb{K}(\mathcal{A})$ (see Notation 2.29). The main result of this paper is a proof of the co-representability of non-connective K-theory by the base ring k.

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THEOREM (Theorem 7.16). *For every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{K}(\mathcal{A}) \simeq \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})).$$

In particular, we obtain isomorphisms of abelian groups

$$\mathbb{K}_n(\mathcal{A}) \simeq \mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})[n], \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})), \quad n \in \mathbb{Z}.$$

The above theorem asserts that non-connective K-theory is 'non-commutative motivic cohomology'. Non-connective K-theory goes back to the work of Bass [Bas68], Karoubi [Kar70], Pedersen and Weibel [Ped84, PW89], Thomason and Trobaugh [TT90], Schlichting [Sch06], and has been the source of many deep results 'amalgamated' with somewhat *ad hoc* constructions. The above co-representability theorem radically changes this state of affairs by offering, to the best of the authors' knowledge, the first conceptual characterization of non-connective K-theory in the setting of dg categories. It provides a completely new understanding of non-connective K-theory as the universal construction, with values in a stable context, which preserves filtered homotopy colimits and satisfies the localization property. More precisely, the co-representability theorem above should be understood as the construction of the universal Chern character from non-connective K-theory: it follows from this result and from the Yoneda lemma that, for any localizing invariant $E : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$ with values in the triangulated derivator of spectra, the data of natural maps $\mathbb{K}(\mathcal{A}) \to E(\mathcal{A})$ are equivalent to the datum of a single class in the stable homotopy group $E_0(k) = \pi_0(E(\underline{k}))$; see Theorem 8.1. This is illustrated as follows.

Applications

Let $\mathbb{K}_n$, HC_j and THH_j be respectively the nth algebraic K-theory group functor, the jth cyclic homology group functor, and the jth topological Hochschild homology group functor; see § 8. As an application of the above co-representability theorem we obtain the following.

THEOREM (Theorem 8.4). *We have the following canonical morphisms of abelian groups:*

(i) *higher Chern characters*

$$\mathrm{ch}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow HC_{n+2r}(\mathcal{A}), \quad n \in \mathbb{Z}, r \geqslant 0,$$

such that $\mathrm{ch}_{0,r}$ *sends* $1 \in \mathbb{K}_0(k)$ *to a generator of the* k*-module of rank one* $HC_{2r}(k)$*;*

(ii) *when* $k = \mathbb{Z}$*, higher trace maps*

$$\mathrm{tr}_n : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_n(\mathcal{A}), \quad n \in \mathbb{Z},$$

such that tr_0 *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to* $1 \in THH_0(\mathbb{Z})$*, and*

$$\mathrm{tr}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r-1}(\mathcal{A}), \quad n \in \mathbb{Z}, r \geqslant 1,$$

such that $\mathrm{tr}_{0,r}$ *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to a generator in the cyclic group* $THH_{2r-1}(\mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z}$*;*

(iii) *when* $k = \mathbb{Z}/p\mathbb{Z}$*, with* p *a prime number, higher trace maps*

$$\mathrm{tr}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r}(\mathcal{A}), \quad n, r \in \mathbb{Z},$$

such that $\mathrm{tr}_{0,r}$ *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to a generator in the cyclic group* $THH_0(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

Before explaining the main ideas behind the proof of the co-representability theorem, we would like to emphasize that, as condition (loc) is much more subtle than condition (add), the tools and arguments used in the proof of co-representability result in the additive motivator, *are not available* when we work with the localizing motivator.

Strategy of the proof

The proof is based on a rather 'orthogonal' construction of the localizing motivator $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$. A crucial ingredient is the observation that we should not start with the notion of derived Morita equivalence, but with the weaker notion of quasi-equiconic dg functor (these are the dg functors which are homotopically fully-faithful and essentially surjective after pre-triangulated completion): in other words, this means that we should work first with pre-triangulated dg categories which are not necessarily idempotent complete; see Proposition 2.19. The reason for this is that negative K-theory is, in particular, an obstruction for the (triangulated) quotients to be Karoubian (see [Sch06]), which is a difficult aspect to see when we work with the notion of derived Morita equivalence. The other difficulty consists in understanding the localizing invariants associated to additive ones: Schlichting's construction of the non-connective K-theory spectrum is general enough to be applied to a wider class of functors than connective K-theory, and we would like to use it to understand as explicitly as possible $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ as a left Bousfield localization of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$. One crucial step in Schlichting's construction is the study of a flasque resolution (of dg categories), obtained as a completion by countable sums. A technical problem with this completion functor is the fact that it doesn't preserves filtered (homotopy) colimits. Our strategy consists in skirting this technical problem by considering invariants of dg categories which only preserve α-filtered homotopy colimits, for a big enough cardinal α. Through a careful analysis of Schlichting's construction, we prove a co-representability result in this wider setting, and then use the nice behaviour of non-connective K-theory to solve the problem with filtered homotopy colimits at the end.

In particular, we shall work with quite a few different kinds of K-theories. Given a small dg category $\mathcal{A}$, there are two ways to complete it. First, one can consider its pre-triangulated envelope; see Proposition 2.19. Waldhausen's K-theory of the pre-triangulated envelope of $\mathcal{A}$ will be denoted by $\underline{K}(\mathcal{A})$; see Notation 2.27. One can also consider the Morita envelope of $\mathcal{A}$, that roughly speaking is the pseudo-abelian completion of the pre-triangulated envelope of $\mathcal{A}$; see Proposition 2.25. We will write $K(\mathcal{A})$ for the Waldhausen K-theory of the Morita envelope of $\mathcal{A}$; see Notation 2.28. This corresponds to what we usually call K-theory of $\mathcal{A}$; see the example below. We thus have a natural comparison map $\underline{K}(\mathcal{A}) \to K(\mathcal{A})$ which induces isomorphisms $\underline{K}_i(\mathcal{A}) \simeq K_i(\mathcal{A})$ for $i > 0$, and a monomorphism $\underline{K}_0(\mathcal{A}) \subset K_0(\mathcal{A})$. Of course, we shall deal also with non-connective K-theory $\mathbb{K}(\mathcal{A})$ of $\mathcal{A}$; see Notation 2.29. By construction, we have a comparison map $K(\mathcal{A}) \to \mathbb{K}(\mathcal{A})$ which induces isomorphisms $K_i(\mathcal{A}) \simeq \mathbb{K}_i(\mathcal{A})$ for $i \geqslant 0$. See § 2.6 for further details.

Example. Any k-algebra A can be seen as a dg category with one object whose endomorphisms are given by A (seen as a complex of k-modules concentrated in degree zero). Then $K(A)$ (respectively $\underline{K}(A)$) is (equivalent to) the spectrum (seen as an infinite loop space) obtained as the loop space of Quillen's Q-construction applied to the exact category of projective right A-modules of finite type (respectively to the exact category of free A-modules of finite type). In particular $K(A)$ agrees with what we usually call the K-theory of A, which is not the case of $\underline{K}(A)$ in general. Finally, the negative (stable) homotopy groups of $\mathbb{K}(A)$ are (isomorphic to) the usual negative K-groups of A (as considered by Bass, Karoubi, Weibel, etc.).

Here is a more detailed account on the contents of the paper. In the first two sections we recall some basic results and constructions concerning dg categories. Let $\mathsf{Trdgcat}$ be the derivator associated with the quasi-equiconic model structure, α a regular cardinal and $\mathcal{A}$ a (fixed) dg category. In § 3, we construct the *universal α-additive invariant*

$$\underline{\mathcal{U}}_\alpha^{\mathsf{add}} : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$$

(see Theorem 3.2) (i.e. $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ sends quasi-equiconic dg functors to isomorphisms, preserves the terminal object as well as α-filtering homotopy colimits, satisfies condition add), and is universal for these properties). We then prove the identification of spectra

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A})$$

(see Theorem 3.9). In § 5, we introduce the class of *strict exact sequences* (see 5.1):

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A},$$

where $\mathcal{B}$ is a pre-triangulated dg category, $\mathcal{A}$ is a *thick* dg subcategory of $\mathcal{B}$, and $\mathcal{B}/\mathcal{A}$ is the Drinfeld dg quotient [Dri04] of $\mathcal{B}$ by $\mathcal{A}$. We localize the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$ to force the strict exact sequences to become distinguished triangles, and obtain a homotopy colimit preserving morphism of triangulated derivators $\gamma_! : \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$. Using Theorem 3.9 and Waldhausen's fibration theorem we obtain

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A})$$

(see Theorem 5.6). Then, we localize the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ so that the derived Morita equivalences become invertible, which leads to a new homotopy colimit preserving morphism of triangulated derivators $l_! : \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$. The derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$ is the analog of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ (in the case $\alpha = \aleph_0$, we have by definition $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} = \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$). In § 6, we adapt Schlichting's construction of non-connective K-theory of Frobenius pairs to the setting of dg categories (see Propositions 6.5 and 6.6). This is used in § 7 to construct a morphism of derivators $V_l(-) : \mathsf{Trdgcat} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ for which, if α is big enough, we have

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A})$$

(see Proposition 7.6). A key technical point is the fact that $V_l(-)$ is an α-localizing invariant; see Proposition 7.8. This allows us to prove the identification (still for α big enough)

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A})$$

(see Theorem 7.12). In Proposition 7.15, we prove that the localizing motivator $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}$ can be obtained from $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}$ by localizing it with respect to the morphisms of shape

$$\mathrm{hocolim}_j\, \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(D_j) \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}),$$

where $\mathcal{A}$ is a α-small dg cell (see § 2.1), J is a α-small filtered category, and $D : J \to \mathsf{dgcat}$ is a functor such that $\mathrm{hocolim}_j\, D_j \simeq \mathcal{A}$. Let us now sum up the steps of the construction of the universal localizing motivator.

$$\begin{array}{ccccccc}
\mathsf{Trdgcat} & & \longrightarrow & & & & \mathsf{HO}(\mathsf{dgcat}) \\
\downarrow {\scriptstyle \underline{\mathcal{U}}_\alpha^{\mathsf{add}}} & & & & & & \downarrow {\scriptstyle \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}} \\
\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} & \xrightarrow{\gamma_!} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} & \xrightarrow{l_!} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} & \longrightarrow & \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}}
\end{array}$$

Finally, using Theorem 7.12 and the fact that negative K-theory preserves filtered colimits, we prove the co-representability theorem. The last section is devoted to the construction of Chern characters and trace maps. We have also included an Appendix A containing several results on Grothendieck derivators, used throughout the article and which are also of independent interest.

1. Preliminaries

1.1 Notations

Throughout the article we will work over a fixed commutative base ring k. Let $\mathcal{C}(k)$ be the category of complexes of k-modules. We will use cohomological notation, i.e. the differential increases the degree. The tensor product of complexes will be denoted by $\otimes$. The category $\mathcal{C}(k)$ is a symmetric monoidal model category, where one uses the projective model structure for which the fibrations are the degreewise surjections, while the weak equivalences are the quasi-isomorphisms. Let sSet be the category of simplicial sets and Spt the category of spectra; see [BF78] for details. Given a Quillen model category $\mathcal{M}$ [Qui67], we will denote by $\mathsf{Ho}(\mathcal{M})$ its homotopy category.

We will use freely the language and basic results of the theory of Grothendieck derivators. A short reminder of our favourite tools and notations can be found in the Appendix A. If $\mathcal{M}$ is a model category, we will denote by $\mathsf{HO}(\mathcal{M})$ its associated derivator; see § A.2. We will write e for the terminal category, so that for a derivator $\mathbb{D}$, $\mathbb{D}(e)$ will be its underlying category (for instance, if $\mathbb{D} = \mathsf{HO}(\mathcal{M})$, $\mathbb{D}(e)$ is the usual homotopy category of the model category $\mathcal{M}$). Every triangulated derivator $\mathbb{D}$ is canonically enriched over spectra; see § A.3. We will denote by $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y)$ the spectrum of maps from X to Y in $\mathbb{D}$. If there is not any ambiguity in doing so, we will simplify the notations by writing $\mathbb{R}\mathsf{Hom}(X, Y) = \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y)$.

Throughout the article the adjunctions will be displayed vertically with the left (respectively right) adjoint on the left-hand (respectively right-hand) side.

1.2 Triangulated categories

We assume the reader is familiar with the basic notions concerning triangulated categories. The unfamiliar reader is invited to consult Neeman's book [Nee01], for instance.

Notation 1.1. We will denote by Tri the category of triangulated categories: the objects are the triangulated categories, and the maps are the exact functors.

Definition 1.2 [Nee01, 4.2.7]. Let $\mathcal{T}$ be a triangulated category admitting arbitrary coproducts. An object X in $\mathcal{T}$ is called *compact* if for each family $\{Y_i\}_{i \in I}$ of objects in $\mathcal{T}$, the canonical morphism

$$\bigoplus_{i \in I} \mathsf{Hom}_{\mathcal{T}}(X, Y_i) \longrightarrow \mathsf{Hom}_{\mathcal{T}}\left(X, \bigoplus_{i \in I} Y_i\right)$$

is invertible. Given a triangulated category $\mathcal{T}$ admitting arbitrary coproducts, we will denote by $\mathcal{T}_c$ its full subcategory of compact objects (note that $\mathcal{T}_c$ is a thick triangulated subcategory of $\mathcal{T}$).

Definition 1.3 [Nee01, § 2]. A sequence of triangulated categories

$$\mathcal{R} \xrightarrow{I} \mathcal{S} \xrightarrow{P} \mathcal{T}$$

is called *exact* if the composition $P \circ I$ is zero, while the functor $\mathcal{R} \xrightarrow{I} \mathcal{S}$ is fully-faithful and the induced functor from the Verdier quotient $\mathcal{S}/\mathcal{R}$ to $\mathcal{T}$ is *cofinal*, i.e. it is fully-faithful and every object in $\mathcal{T}$ is a direct summand of an object of $\mathcal{S}/\mathcal{R}$. An exact sequence of triangulated categories as above is called *split exact*, if there exist triangulated functors

$$\mathcal{S} \xrightarrow{R} \mathcal{R} \quad \text{and} \quad \mathcal{T} \xrightarrow{S} \mathcal{S},$$

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with R right adjoint to I, S right adjoint to P and $P \circ S = \mathrm{Id}_{\mathcal{T}}$ and $R \circ I = \mathrm{Id}_{\mathcal{R}}$ via the adjunction morphisms.

Notation 1.4. Given a triangulated category $\mathcal{T}$, we will denote by $\widetilde{\mathcal{T}}$ its *idempotent completion*; see [BS01] for details. This construction is functorial. Notice also that the idempotent completion of a split exact sequence is split exact.

2. Homotopy theories of dg categories

In this section, we will review the three homotopy theories of dg categories developed in [Tab07, §§ 1–2]. This will give us the opportunity to fix some notation, which will be used throughout the article. For a survey article on dg categories, consult Keller's ICM address [Kel06].

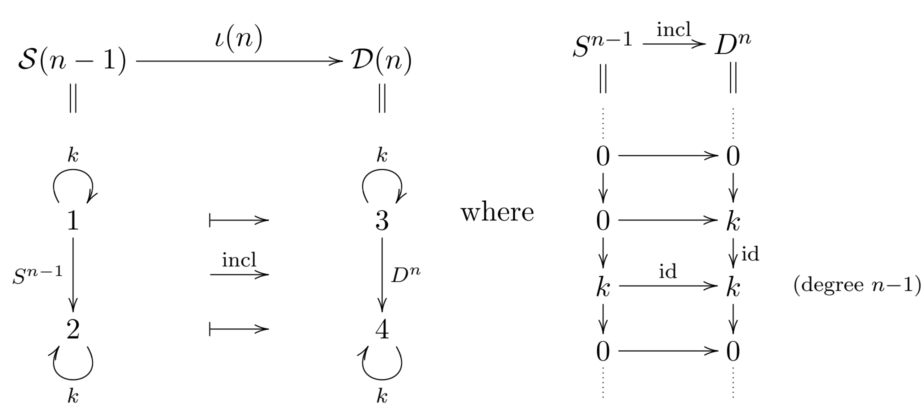
DEFINITION 2.1. A *small dg category* $\mathcal{A}$ is a $\mathcal{C}(k)$-enriched category [Bor94, Definition 6.2.1]. Recall that this consists of the following data: a small set of objects $\mathrm{obj}(\mathcal{A})$ (usually denoted by $\mathcal{A}$ itself); for each pair of objects (x, y) in $\mathcal{A}$, a complex of k-modules $\mathcal{A}(x, y)$; for each triple of objects (x, y, z) in $\mathcal{A}$, a composition morphism $\mathcal{A}(y, z) \otimes \mathcal{A}(x, y) \to \mathcal{A}(x, z)$ in $\mathcal{C}(k)$ satisfying the usual associativity condition; for each object x in $\mathcal{A}$, a morphism $k \to \mathcal{A}(x, x)$ in $\mathcal{C}(k)$, satisfying the usual unit condition with respect to the above composition.

DEFINITION 2.2. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{C}(k)$-functor [Bor94, Definition 6.2.3]. Recall that this consists of the following data: a map of sets $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$; for each pair of objects (x, y) in $\mathcal{A}$, a morphism $F(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ in $\mathcal{C}(k)$ satisfying the usual unit and associativity conditions.

Notation 2.3. We denote by dgcat the category of small dg categories.

2.1 Dg cells

DEFINITION 2.4. Let $\underline{k}$ be the dg category with one object $*$ such that $\underline{k}(*, *) := k$. For $n \in \mathbb{Z}$, let S^n be the complex $k[n]$ (with k concentrated in degree n) and let D^n be the mapping cone on the identity of S^{n-1}. We denote by $\mathcal{S}(n)$ the dg category with two objects, 1 and 2, such that $\mathcal{S}(n)(1, 1) = k$, $\mathcal{S}(n)(2, 2) = k$, $\mathcal{S}(n)(2, 1) = 0$, $\mathcal{S}(n)(1, 2) = S^n$ and composition given by multiplication. Let $\mathcal{D}(n)$ be the dg category with two objects, 3 and 4, such that $\mathcal{D}(n)(3, 3) = k$, $\mathcal{D}(n)(4, 4) = k$, $\mathcal{D}(n)(4, 3) = 0$, $\mathcal{D}(n)(3, 4) = D^n$ and with composition given by multiplication. Finally, let $\iota(n) : \mathcal{S}(n-1) \to \mathcal{D}(n)$ be the dg functor that sends 1 to 3, 2 to 4 and S^{n-1} to D^n by the identity on k in degree $n-1$.



Notation 2.5. We denote by I the set consisting of the dg functors $\{\iota(n)\}_{n\in\mathbb{Z}}$ and the dg functor $\emptyset \to \underline{k}$ (where the empty dg category $\emptyset$ is the initial one).

Definition 2.6 [Hir03, Definition 10.4.1]. Let α be an infinite regular cardinal; see [Hir03, Definition 10.1.10]. A dg category $\mathcal{A}$ is called *α-small* if for every regular cardinal $\lambda \geqslant \alpha$ and every λ-sequence of dg categories $\{\mathcal{B}_\beta\}_{\beta<\lambda}$, the following induced map of sets is invertible

$$\mathrm{colim}_{\beta<\lambda}\, \mathsf{Hom}_{\mathsf{dgcat}}(\mathcal{A}, \mathcal{B}_\beta) \longrightarrow \mathsf{Hom}_{\mathsf{dgcat}}(\mathcal{A}, \mathrm{colim}_{\beta<\lambda}\, \mathcal{B}_\beta).$$

Definition 2.7. A dg category $\mathcal{A}$ is a *dg cell* if the (unique) map $\emptyset \to \mathcal{A}$ is a transfinite composition of pushouts of elements of I. If $\mathcal{A}$ is moreover α-small, we say that $\mathcal{A}$ is a *α-small dg cell*.

2.2 Dg modules

Let $\mathcal{A}$ be a small dg category. The *opposite dg category* $\mathcal{A}^{\mathsf{op}}$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and complexes of morphisms given by $\mathcal{A}^{\mathsf{op}}(x, y) = \mathcal{A}(y, x)$. A *right dg $\mathcal{A}$-module* (or simply an $\mathcal{A}$-module) is a dg functor $M : \mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathsf{dg}}(k)$ of complexes of k-modules.

Notation 2.8. We denote by $\mathcal{C}(\mathcal{A})$ (respectively by $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$) the category (respectively dg category) of $\mathcal{A}$-modules. Notice that $\mathcal{C}(\mathcal{A})$ is endowed with the projective Quillen model structure (see [Kel06, Theorem 3.2]), whose weak equivalences and fibrations are defined objectwise. We denote by $\mathcal{D}(\mathcal{A})$ the *derived category of $\mathcal{A}$*, i.e. the homotopy category $\mathsf{Ho}(\mathcal{C}(\mathcal{A}))$ or equivalently the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Notice that $\mathcal{D}(\mathcal{A})$ is a triangulated category.

Definition 2.9. The *Yoneda dg functor*

$$\underline{h} : \mathcal{A} \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathcal{A}) \quad x \mapsto \mathcal{A}(-, x) =: \underline{h}(x)$$

sends an object x to the $\mathcal{A}$-module $\mathcal{A}(-, x)$ *represented by x*.

Notation 2.10. Let $\mathsf{tri}(\mathcal{A})$ be the smallest triangulated subcategory of $\mathcal{D}(\mathcal{A})$ (see Notation 2.8) which contains the $\mathcal{A}$-modules $\underline{h}(x)$, $x \in \mathcal{A}$. We denote by $\mathsf{perf}(\mathcal{A})$ the smallest thick triangulated subcategory of $\mathcal{D}(\mathcal{A})$ (i.e. stable under direct factors) which contains the $\mathcal{A}$-modules $\underline{h}(x)$, $x \in \mathcal{A}$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to a *restriction and extension of scalars* adjunctions.

$$\begin{array}{ccc} \mathcal{C}(\mathcal{B}) & \qquad & \mathcal{D}(\mathcal{B}) \\ {\scriptstyle F_!}\uparrow\ \downarrow{\scriptstyle F^*} & & {\scriptstyle \mathbb{L}F_!}\uparrow\ \downarrow{\scriptstyle F^*} \\ \mathcal{C}(\mathcal{A}) & & \mathcal{D}(\mathcal{A}) \end{array}$$

Remark 2.11. Notice that the derived extension of scalars functor $\mathbb{L}F_!$ restricts to the triangulated functors $\mathbb{L}F_! : \mathsf{tri}(\mathcal{A}) \to \mathsf{tri}(\mathcal{B})$ and $\mathbb{L}F_! : \mathsf{perf}(\mathcal{A}) \to \mathsf{perf}(\mathcal{B})$.

2.3 Quasi-equivalences

Let $\mathcal{A}$ be a dg category. The category $\mathsf{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{H}^0(\mathcal{A})(x, y) = \mathrm{H}^0(\mathcal{A}(x, y))$, where H^0 denotes the 0th cohomology group. This construction is functorial and so we obtain a well-defined functor with values in the category of small categories

$$\mathsf{H}^0(-) : \mathsf{dgcat} \longrightarrow \mathrm{Cat}, \quad \mathcal{A} \mapsto \mathsf{H}^0(\mathcal{A}).$$

DEFINITION 2.12. A dg functor $F:\mathcal{A}\to\mathcal{B}$ is a *quasi-equivalence* if for all objects $x,y\in\mathcal{A}$, the morphism $F(x,y):\mathcal{A}(x,y)\to\mathcal{B}(Fx,Fy)$ in $\mathcal{C}(k)$ is a quasi-isomorphism and the induced functor $\mathsf{H}^0(F):\mathsf{H}^0(\mathcal{A})\to\mathsf{H}^0(\mathcal{B})$ is an equivalence of categories.

THEOREM 2.13 [Tab07, Theorem 1.8]. *The category* dgcat *carries a cofibrantly generated Quillen model structure whose weak equivalences are the quasi-equivalences. The set of generating cofibrations is the set I of Notation 2.5.*

We denote by Hqe the homotopy category hence obtained. Given small dg categories $\mathcal{A}$ and $\mathcal{B}$, its *tensor product* $\mathcal{A}\otimes\mathcal{B}$ is defined as follows: the set of objects of $\mathcal{A}\otimes\mathcal{B}$ is $\mathrm{obj}(\mathcal{A})\times\mathrm{obj}(\mathcal{B})$ and for two objects (x,y) and (x',y') in $\mathcal{A}\otimes\mathcal{B}$, we define $(\mathcal{A}\otimes\mathcal{B})((x,y),(x',y')):=\mathcal{A}(x,x')\otimes\mathcal{B}(y,y')$. The tensor product of dg categories defines a symmetric monoidal structure on dgcat (with unit the dg category $\underline{k}$ of Definition 2.4), which is easily seen to be closed. However, the model structure of Theorem 2.13 endowed with this symmetric monoidal structure is *not* a symmetric monoidal model category, as the tensor product of two cofibrant objects in dgcat is not cofibrant in general. Nevertheless, the bifunctor $-\otimes-$ can be derived into a bifunctor

$$-\otimes^{\mathbb{L}}-:\mathsf{Hqe}\times\mathsf{Hqe}\longrightarrow\mathsf{Hqe},\quad(\mathcal{A},\mathcal{B})\mapsto Q(\mathcal{A})\otimes\mathcal{B}=\mathcal{A}\otimes^{\mathbb{L}}\mathcal{B},$$

where $Q(A)$ is a cofibrant resolution of $\mathcal{A}$.

THEOREM 2.14 [Toe07, Theorem 6.1]. *The derived tensor monoidal structure $-\otimes^{\mathbb{L}}-$ on* Hqe *admits an internal Hom-functor (denoted by $\mathbb{R}\underline{\mathsf{Hom}}(-,-)$ in [Toe07])*

$$\mathsf{rep}(-,-):\mathsf{Hqe}^{\mathsf{op}}\times\mathsf{Hqe}\longrightarrow\mathsf{Hqe}.$$

The category $\mathsf{H}^0(\mathsf{rep}(\mathcal{A},\mathcal{B}))$ is the full subcategory of $\mathcal{D}(\mathcal{A}^{\mathsf{op}}\otimes^{\mathbb{L}}\mathcal{B})$ spanned by bimodules X such that, for any object a of $\mathcal{A}$, $X(a)$ belongs to the essential image of the Yoneda embedding $\mathsf{H}^0(\mathcal{B})\to\mathcal{D}(\mathcal{B})$.

2.4 Quasi-equiconic dg functors

DEFINITION 2.15. A dg functor $F:\mathcal{A}\to\mathcal{B}$ is *quasi-equiconic* if the induced triangulated functor $\mathbb{L}F_!:\mathsf{tri}(\mathcal{A})\to\mathsf{tri}(\mathcal{B})$ (see Remark 2.11) is an equivalence of categories.

THEOREM 2.16 [Tab07, Theorem 2.2]. *The category* dgcat *carries a cofibrantly generated Quillen model structure (called the* quasi-equiconic model structure*) whose weak equivalences are the quasi-equiconic dg functors. The cofibrations are the same as those of Theorem 2.13.*

Notation 2.17. We denote by Hec the homotopy category hence obtained.

PROPOSITION 2.18 [Tab07, Proposition 2.14]. *The Quillen model structure of Theorem 2.16 is a left Bousfield localization of the Quillen model structure of Theorem 2.13.*

Notice that we have a well-defined functor (see Notations 1.1 and 2.10):

$$\mathsf{tri}:\mathsf{Hec}\longrightarrow\mathsf{Tri},\quad\mathcal{A}\mapsto\mathsf{tri}(\mathcal{A}).$$

PROPOSITION 2.19 [Tab07, Proposition 2.10]. *The fibrant dg categories, which respect to the quasi-equiconic model structure (called the* pre-triangulated dg categories*), are the dg categories $\mathcal{A}$ for which the image of the Yoneda embedding $\mathsf{H}^0(\mathcal{A})\hookrightarrow\mathcal{D}(\mathcal{A})$ (Definition 2.9) is stable under (co)suspensions and cones. Equivalently, these are the dg categories $\mathcal{A}$ for which we have an equivalence $\mathsf{H}^0(\mathcal{A})\xrightarrow{\sim}\mathsf{tri}(\mathcal{A})$ of triangulated categories.*

Proposition 2.20 [Tab07, Remark 2.8]. *The monoidal structure* $-\otimes^{\mathbb{L}}-$ *on* Hqe *descends to* Hec *and the internal Hom-functor* $\mathsf{rep}(-,-)$ *(see Theorem 2.14) can be naturally derived:*

$$\mathsf{rep}_{\mathrm{tr}}(-,-) : \mathsf{Hec}^{\mathsf{op}} \times \mathsf{Hec} \longrightarrow \mathsf{Hec}, \quad (\mathcal{A}, \mathcal{B}) \mapsto \mathsf{rep}(\mathcal{A}, \mathcal{B}_f) = \mathsf{rep}_{\mathrm{tr}}(\mathcal{A}, \mathcal{B}),$$

where $\mathcal{B}_f$ *is a pre-triangulated resolution of* $\mathcal{B}$ *as in Proposition 2.19. The category* $\mathsf{H}^0(\mathsf{rep}_{\mathrm{tr}}(\mathcal{A}, \mathcal{B}))$ *is the full subcategory of* $\mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ *spanned by bimodules* X *such that, for any object* a *of* $\mathcal{A}$, $X(a)$ *belongs to the essential image of the full inclusion* $\mathsf{tri}(\mathcal{B}) \to \mathcal{D}(\mathcal{B})$.

2.5 Derived Morita equivalences

Definition 2.21. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is a *derived Morita equivalence* if the induced triangulated functor $\mathbb{L}F_! : \mathsf{perf}(\mathcal{A}) \to \mathsf{perf}(\mathcal{B})$ (see Remark 2.11) is an equivalence of categories.

Since the triangulated categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ (see Notation 2.8) are compactly generated (see [Nee01, § 8.1]) and the triangulated functor

$$\mathbb{L}F_! : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B}) \tag{1}$$

preserves arbitrary sums, we conclude that a dg functor $F : \mathcal{A} \to \mathcal{B}$ is a derived Morita equivalence if and only if the triangulated functor (1) is an equivalence.

Theorem 2.22 [Tab07, Theorem 2.27]. *The category* dgcat *carries a cofibrantly generated model structure (called the* Morita model structure*), whose weak equivalences are the derived Morita equivalences. The cofibrations are the same as those of Theorem 2.16.*

Notation 2.23. We denote by Hmo the homotopy category hence obtained.

Proposition 2.24 [Tab07, Proposition 2.35]. *The Quillen model structure of Theorem 2.22 is a left Bousfield localization of the Quillen model structure of Theorem 2.16.*

Notice that we have a well-defined functor (see Notations 1.1 and 2.10):

$$\mathsf{perf} : \mathsf{Hmo} \longrightarrow \mathsf{Tri} \quad \mathcal{A} \mapsto \mathsf{perf}(\mathcal{A}).$$

Proposition 2.25 [Tab07, Proposition 2.34]. *The fibrant dg categories, which respect to the Morita model structure (called the* Morita fibrant dg categories*), are the dg categories* $\mathcal{A}$ *for which the image of the Yoneda embedding* $\mathsf{H}^0(\mathcal{A}) \hookrightarrow \mathcal{D}(\mathcal{A})$ *(Definition 2.9) is stable under (co)suspensions, cones, and direct factors. Equivalently, these are the dg categories* $\mathcal{A}$ *for which we have an equivalence* $\mathsf{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathsf{perf}(\mathcal{A})$ *of (idempotent complete) triangulated categories.*

Proposition 2.26 [Tab07, Remark 2.40]. *The monoidal structure* $-\otimes^{\mathbb{L}}-$ *on* Hec *descends to* Hmo *and the internal Hom-functor* $\mathsf{rep}_{\mathrm{tr}}(-,-)$ *(see Proposition 2.20) can be naturally derived:*

$$\mathsf{rep}_{\mathrm{mor}}(-,-) : \mathsf{Hmo}^{\mathsf{op}} \times \mathsf{Hmo} \longrightarrow \mathsf{Hmo} \quad (\mathcal{A}, \mathcal{B}) \mapsto \mathsf{rep}_{\mathrm{tr}}(\mathcal{A}, \mathcal{B}_f) = \mathsf{rep}_{\mathrm{mor}}(\mathcal{A}, \mathcal{B}),$$

where $\mathcal{B}_f$ *is a Morita fibrant resolution of* $\mathcal{B}$ *as in Proposition 2.25. The category* $\mathsf{H}^0(\mathsf{rep}_{\mathrm{mor}}(\mathcal{A}, \mathcal{B}))$ *is the full subcategory of* $\mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ *spanned by bimodules* X *such that, for any object* a *of* $\mathcal{A}$, $X(a)$ *belongs to the essential image of the full inclusion* $\mathsf{perf}(\mathcal{B}) \to \mathcal{D}(\mathcal{B})$.

2.6 *K*-theories

Notation 2.27. ($\underline{K}$-theory) Let $\mathcal{A}$ be a small dg category. We denote by $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ (see Notation 2.8) whose objects are the $\mathcal{A}$-modules which become isomorphic in $\mathcal{D}(\mathcal{A})$ to elements of $\mathsf{tri}(\mathcal{A})$ (see Notation 2.10), and which are, moreover, cofibrant. The category $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$, endowed with the weak equivalences and cofibrations of the Quillen model

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structure on $\mathcal{C}(\mathcal{A})$, is a Waldhausen category; see [DS04, § 3] or [Cis10]. We denote by $\underline{K}(\mathcal{A})$ the Waldhausen K-theory spectrum [Wal85] of $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$. Given a dg functor $F : \mathcal{A} \to \mathcal{B}$, the extension of scalars left Quillen functor $F_! : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ preserves weak equivalences, cofibrations, and pushouts. Therefore, it restricts to an exact functor $F_! : \mathsf{tri}^{\mathcal{W}}(\mathcal{A}) \to \mathsf{tri}^{\mathcal{W}}(\mathcal{B})$ between Waldhausen categories. Moreover, if F is a quasi-equiconic dg functor, the Waldhausen functor $F_!$ is an equivalence. We obtain then a well-defined functor with values in the homotopy category of spectra:

$$\mathsf{Hec} \longrightarrow \mathsf{Ho}(\mathsf{Spt}) \quad \mathcal{A} \mapsto \underline{K}(\mathcal{A}).$$

Notation 2.28. (K-theory) Let $\mathcal{A}$ be a small dg category. We denote by $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$ the full subcategory of $\mathcal{C}(\mathcal{A})$ (see Notation 2.8) whose objects are the $\mathcal{A}$-modules which become isomorphic in $\mathcal{D}(\mathcal{A})$ to elements of $\mathsf{perf}(\mathcal{A})$ (see Notation 2.10), and which are, moreover, cofibrant. The category $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$, endowed with the weak equivalences and cofibrations of the Quillen model structure on $\mathcal{C}(\mathcal{A})$, is a Waldhausen category; see [DS04, § 3] or [Cis10]. We denote by $K(\mathcal{A})$ the Waldhausen K-theory spectrum [Wal85] of $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$. Given a dg functor $F : \mathcal{A} \to \mathcal{B}$, the extension of scalars left Quillen functor $F_! : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ preserves weak equivalences, cofibrations, and pushouts. Therefore, it restricts to an exact functor $F_! : \mathsf{perf}^{\mathcal{W}}(\mathcal{A}) \to \mathsf{perf}^{\mathcal{W}}(\mathcal{B})$ between Waldhausen categories. Moreover, if F is a derived Morita equivalence, the Waldhausen functor $F_!$ is an equivalence. We obtain then a well-defined functor with values in the homotopy category of spectra:

$$\mathsf{Hmo} \longrightarrow \mathsf{Ho}(\mathsf{Spt}) \quad \mathcal{A} \mapsto K(\mathcal{A}).$$

Notation 2.29. ($\mathbb{K}$-theory) Let $\mathcal{A}$ be a small dg category. We denote by $\mathbb{K}(\mathcal{A})$ the non-connective K-theory spectrum of $\mathsf{perf}^{\mathcal{W}}(\mathcal{A})$; see [Sch06, § 12.1] (one can also consider Proposition 7.5 below as a definition). This construction induces a functor with values in the homotopy category of spectra:

$$\mathsf{Hmo} \longrightarrow \mathsf{Ho}(\mathsf{Spt}), \quad \mathcal{A} \mapsto \mathbb{K}(\mathcal{A}).$$

3. Additive invariants

In this section, we will recall the main theorems of [Tab08], using the quasi-equiconic model structure (see Theorem 2.16) instead of the Morita model structure (see Theorem 2.22). Moreover, and in contrast with [Tab08], we will work with a fixed infinite regular cardinal α (see [Hir03, Definition 10.1.10]) since we want to study additive invariants which preserve only α-filtered homotopy colimits of dg categories.

DEFINITION 3.1. A sequence in Hec, respectively in Hmo, (see Notations 2.17 and 2.23)

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C}$$

is called *exact* if the induced sequence of triangulated categories on the left, respectively on the right,

$$\mathsf{tri}(\mathcal{A}) \longrightarrow \mathsf{tri}(\mathcal{B}) \longrightarrow \mathsf{tri}(\mathcal{C}) \quad \mathsf{perf}(\mathcal{A}) \longrightarrow \mathsf{perf}(\mathcal{B}) \longrightarrow \mathsf{perf}(\mathcal{C})$$

is exact (1.3). A *split exact sequence* in Hec, respectively in Hmo, is an exact sequence in Hec, respectively in Hmo, which is equivalent to one of the form

$$\mathcal{A} \underset{I}{\overset{R}{\rightleftarrows}} \mathcal{B} \underset{P}{\overset{S}{\rightleftarrows}} \mathcal{C},$$

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where I, P, R and S are dg functors, with R right adjoint to I, S right adjoint to P and $P \circ S = \mathrm{Id}_{\mathcal{C}}$ and $R \circ I = \mathrm{Id}_{\mathcal{A}}$ via the adjunction morphisms.

Let us denote by $\mathsf{Trdgcat}$ the derivator associated with the quasi-equiconic model structure (see Theorem 2.16) on dgcat. In particular, we have $\mathsf{Trdgcat}(e) = \mathsf{Hec}$.

Theorem 3.2. *There exists a morphism of derivators*

$$\underline{\mathcal{U}}_\alpha^{\mathsf{add}} : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}},$$

with values in a strong triangulate derivator (A.1), which:

(α-flt) *commutes with α-filtered homotopy colimits;*

($\underline{\mathrm{p}}$) *sends the terminal object in $\mathsf{Trdgcat}$ to the terminal object in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$; and*

($\underline{\mathrm{add}}$) *sends the split exact sequences in Hec (3.1) to direct sums*

$$\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) \oplus \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{C}) \xrightarrow{\sim} \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}).$$

Moreover, $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ is universal with respect to these properties, i.e. for every strong triangulated derivator $\mathbb{D}$, we have an equivalence of categories

$$(\underline{\mathcal{U}}_\alpha^{\mathsf{add}})^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_\alpha^{\mathsf{add}}(\mathsf{Trdgcat}, \mathbb{D}),$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{Trdgcat}, \mathbb{D})$ of morphisms of derivators which verify the above three conditions.

Notation 3.3. The objects of the category $\underline{\mathsf{Hom}}_\alpha^{\mathsf{add}}(\mathsf{Trdgcat}, \mathbb{D})$ will be called *α-additive invariants* and $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ the *universal α-additive invariant*.

The construction on $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$ is analogous to the construction of the additive motivator $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ (see [Tab08, Definition 15.1]) and so we omit it. The only difference is that we start with the quasi-equiconic model structure instead of the Morita model structure and we consider α-filtered homotopy colimits instead of filtered homotopy colimits. Let us now state a natural variation on a result due to Toën and Vaquié [TV07, Proposition 2.2], which is verified by all three model structures on dgcat (see §§ 2.3–2.5).

Notation 3.4. Given a Quillen model category $\mathcal{M}$, we denote by $\mathsf{Map}(-,-)$ its homotopy function complex [Hir03, Definition 17.4.1].

Definition 3.5. Let α be a infinite regular cardinal; see [Hir03, Definition 10.1.10]. An object X in $\mathcal{M}$ is said to be *homotopically α-small* if, for any α-filtered direct system $\{Y_j\}_{j\in J}$ in $\mathcal{M}$, the following induced map is an isomorphism in $\mathsf{Ho}(\mathsf{sSet})$:

$$\mathrm{hocolim}_j\, \mathsf{Map}(X, Y_j) \longrightarrow \mathsf{Map}(X, \mathrm{hocolim}_j\, Y_j).$$

Proposition 3.6. *Let $\mathcal{M}$ be a cellular Quillen model category, with I a set of generating cofibration. If the (co)domains of the elements of I are cofibrant, α-small and homotopically α-small, then the following hold.*

(i) *An α-filtered colimit of trivial fibration is a trivial fibration.*

(ii) *For any α-filtered direct system $\{X_j\}_{j\in J}$ in $\mathcal{M}$, the natural morphism*

$$\mathrm{hocolim}_j\, X_j \longrightarrow \mathrm{colim}_j\, X_j$$

is an isomorphism in $\mathsf{Ho}(\mathcal{M})$.

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(iii) *Any object X in $\mathcal{M}$ is equivalent to a α-filtered colimit of α-small I-cell objects.*

(iv) *An object X in $\mathcal{M}$ is homotopically α-small presented if and only if it is equivalent to a retraction in $\mathsf{Ho}(\mathcal{M})$ of a α-small I-cell object.*

The proof of Proposition 3.6 is similar to the proof of [TV07, Proposition 2.2] and so we omit it.

PROPOSITION 3.7. *The set of objects $\{\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B})[n] \mid n \in \mathbb{Z}\}$, with $\mathcal{B}$ a α-small dg cell (2.7), forms a set of compact generators (see [Nee01, § 8.1]) of the triangulated category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}(e)$.*

Proof. As in [Tab08, § 15], the morphism $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ is given by the composition

$$\mathsf{Trdgcat} \xrightarrow{\Phi\circ\mathbb{R}\underline{h}} \mathsf{L}_{\Sigma,\underline{\mathsf{p}}}\mathsf{Hot}_{\mathsf{dgcat}_\alpha} \xrightarrow{\varphi\circ\psi} \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}.$$

The objects $(\Phi \circ \mathbb{R}\underline{h})(\mathcal{B})$, with $\mathcal{B}$ a α-small dg cell, form a set of homotopically finitely presented generators and $\varphi \circ \psi$ corresponds to the operations of stabilization and left Bousfield localization with respect to a set of morphisms whose (co)domains are homotopically finitely presented. Therefore [Tab08, Lemma 8.2] and [Tab08, Lemma 7.1] allow us to conclude the proof. □

Remark 3.8. Notice that, for a triangulated derivator $\mathbb{D}$ (A.1), and for an object X of $\mathbb{D}(e)$, the spectrum of maps functor $\mathbb{R}\mathsf{Hom}(X, -)$ preserves small sums if and only if X is compact in the triangulated category $\mathbb{D}(e)$ (which means that the functors $\mathsf{Hom}_{\mathbb{D}(e)}(X[n], -)$ preserve small sums for any integer n). Moreover, as the spectrum of maps functor $\mathbb{R}\mathsf{Hom}(X, -)$ preserves finite homotopy (co-)limits for any object X, the spectrum of maps functor $\mathbb{R}\mathsf{Hom}(X, -)$ preserves small sums if and only if it preserves small homotopy colimits. Hence X is compact in $\mathbb{D}$ (in the sense that $\mathbb{R}\mathsf{Hom}(X, -)$ preserves filtered homotopy colimits) if and only if it is compact in the triangulated category $\mathbb{D}(e)$, which is also equivalent to the property that $\mathbb{R}\mathsf{Hom}(X, -)$ preserves arbitrary small homotopy colimits. We will use freely this last characterization in the following of this article. In particular, Proposition 3.7 can be restated by saying that $\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}), -)$ preserves homotopy colimits for any α-small dg cell $\mathcal{B}$.

THEOREM 3.9. *For any dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{B}$ a α-small dg cell (2.7), we have a natural isomorphism in the stable homotopy category of spectra (see Proposition 2.20 and Notation 2.27)*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, \mathcal{A})).$$

The proof of Theorem 3.9 is similar to the one of [Tab08, Theorem 15.10] and so we omit it.

4. Idempotent completion

Notation 4.1. We denote by

$$(-)^\wedge : \mathsf{dgcat} \longrightarrow \mathsf{dgcat} \quad \mathcal{A} \mapsto \mathcal{A}^\wedge$$

the Morita fibrant resolution functor (see Proposition 2.25), obtained by the small object argument [Hir03, § 10.5.14], using the generating trivial cofibrations (see [Tab07, § 2.5]) of the Morita model structure. Thanks to [Tab07, Proposition 2.27], the functor $(-)^\wedge$ preserves quasi-equiconic dg functors and so it gives rise to a morphism of derivators

$$(-)^\wedge : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat}.$$

Notice that by construction, we have also a 2-morphism $\mathrm{Id} \Rightarrow (-)^\wedge$ of derivators.

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Proposition 4.2. *The morphism* $(-)^\wedge$ *of derivators preserves:*

(i) *α-filtered homotopy colimits;*

(ii) *the terminal object;*

(iii) *split exact sequences (3.1).*

Proof. Condition (i) follows from Proposition 3.6(ii) and the fact that the (co)domains of the generating trivial cofibrations of the Morita model structure are α-small. Condition (ii) is clear. In what concerns condition (iii), notice that we have a (up to isomorphism) commutative diagram.

$$\begin{array}{ccc} \mathsf{Hec} & \xrightarrow{(-)^\wedge} & \mathsf{Hec} \\ {\scriptstyle \mathsf{tri}}\downarrow & & \downarrow{\scriptstyle \mathsf{tri}} \\ \mathsf{Tri} & \xrightarrow[\widetilde{(-)}]{} & \mathsf{Tri} \end{array}$$

Since the idempotent completion functor $\widetilde{(-)}$ (see Notation 1.4) preserves split exact sequences, the proof is finished. □

By Proposition 4.2, the composition

$$\mathsf{Trdgcat} \xrightarrow{(-)^\wedge} \mathsf{Trdgcat} \xrightarrow{\underline{\mathcal{U}}^{\mathsf{add}}_\alpha} \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$$

is a α-additive invariant (3.3). Therefore, by Theorem 3.2 we obtain an induced morphism of derivators and a 2-morphism,

$$(-)^\wedge : \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha, \quad \mathrm{Id} \Rightarrow (-)^\wedge,$$

such that $\underline{\mathcal{U}}^{\mathsf{add}}_\alpha(\mathcal{A})^\wedge \simeq \underline{\mathcal{U}}^{\mathsf{add}}_\alpha(\mathcal{A}^\wedge)$ for every dg category $\mathcal{A}$.

5. Localizing invariants

In this section, we will work with a fixed infinite regular cardinal α [Hir03, Definition 10.1.10].

5.1 Waldhausen exact sequences

Definition 5.1. Let $\mathcal{B}$ be a pre-triangulated dg category, see Proposition 2.19. A *thick* dg subcategory of $\mathcal{B}$ is a full dg subcategory $\mathcal{A}$ of $\mathcal{B}$ such that the induced functor $\mathsf{H}^0(\mathcal{A}) \to \mathsf{H}^0(\mathcal{B})$ turns $\mathsf{H}^0(\mathcal{A})$ into a thick subcategory of $\mathsf{H}^0(\mathcal{B})$ (which means that $\mathcal{A}$ is pre-triangulated and that any object of $\mathsf{H}^0(\mathcal{B})$ which is a direct factor of an object of $\mathsf{H}^0(\mathcal{A})$ is an object of $\mathsf{H}^0(\mathcal{A})$). A *strict exact sequence* is a diagram of shape

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A},$$

in which $\mathcal{B}$ is a pre-triangulated dg category, $\mathcal{A}$ is a thick dg subcategory of $\mathcal{B}$, and $\mathcal{B}/\mathcal{A}$ is the Drinfeld dg quotient [Dri04] of $\mathcal{B}$ by $\mathcal{A}$. Note that any split short exact sequence (3.1) is a strict exact sequence.

Notation 5.2. We fix once and for all a *set* $\mathcal{E}$ of representatives of homotopy α-small thick inclusions, by which we mean that $\mathcal{E}$ is a set of full inclusions of dg categories $\mathcal{A} \to \mathcal{B}$ with the following properties.

(i) For any inclusion $\mathcal{A} \to \mathcal{B}$ in $\mathcal{E}$, the dg category $\mathcal{B}$ is pre-triangulated, and $\mathcal{A}$ is thick in $\mathcal{B}$.

(ii) For any inclusion $\mathcal{A} \to \mathcal{B}$ in $\mathcal{E}$, there exists an α-small dg cell $\mathcal{B}_0$ (2.1) and a quasi-equiconic dg functor $\mathcal{B}_0 \to \mathcal{B}$ (2.15).

(iii) For any α-small dg cell $\mathcal{B}_0$, there exists a quasi-equiconic dg functor $\mathcal{B}_0 \to \mathcal{B}$, with $\mathcal{B}$ cofibrant and pre-triangulated, such that any inclusion $\mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ thick in $\mathcal{B}$ is in $\mathcal{E}$.

Proposition 5.3. *Let $\mathcal{B}$ be a pre-triangulated dg category, and $\mathcal{A}$ a thick dg subcategory of $\mathcal{B}$ (5.1). Then there exists an α-filtered direct system $\{\mathcal{A}_j \xrightarrow{\epsilon_j} \mathcal{B}_j\}_{j\in J}$ of dg functors, such that for any index j, we have a commutative diagram of dg categories of shape*

$$\begin{array}{ccc} \mathcal{A}'_j & \longrightarrow & \mathcal{B}'_j \\ \downarrow & & \downarrow \\ \mathcal{A}_j & \longrightarrow & \mathcal{B}_j \end{array}$$

in which the vertical maps are quasi-equivalences (2.12), the map $\mathcal{A}'_j \to \mathcal{B}'_j$ belongs to $\mathcal{E}$ (see Notation 5.2), and moreover there exists an isomorphism in the homotopy category of arrows between dg categories (with respect to the quasi-equiconic model structure of Theorem 2.16) of shape

$$\mathrm{hocolim}_j\{\mathcal{A}_j \xrightarrow{\epsilon_j} \mathcal{B}_j\} \xrightarrow{\sim} (\mathcal{A} \longrightarrow \mathcal{B}).$$

Proof. By Proposition 3.6, there exists an α-filtered direct system of α-small dg cells $\{\mathcal{B}''_j\}_{j\in J}$ in Hec, such that

$$\mathrm{hocolim}_j\, \mathcal{B}''_j \xrightarrow{\sim} \mathcal{B}.$$

By taking a termwise fibrant replacement $\{\mathcal{B}_j\}_{j\in J}$ of the diagram $\{\mathcal{B}''_j\}_{j\in J}$, with respect to the quasi-equiconic model category structure (see Proposition 2.19), we obtain a α-filtered diagram of pre-triangulated dg categories, and so an isomorphism in Hec of shape

$$\mathrm{hocolim}_j\, \mathcal{B}_j \xrightarrow{\sim} \mathcal{B}.$$

Using the following fiber products

$$\begin{array}{ccc} \mathcal{A}_j & \xrightarrow{\epsilon_j} & \mathcal{B}_j \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

we construct an α-filtered direct system $\{\mathcal{A}_j \xrightarrow{\epsilon_j} \mathcal{B}_j\}_{j\in J}$ such that the map

$$\mathrm{hocolim}_j\, \mathcal{A}_j \xrightarrow{\sim} \mathcal{A}$$

is an isomorphism in Hec. To prove this last assertion, we consider the commutative diagram of triangulated categories

$$\begin{array}{ccccc} \mathrm{colim}\, \mathsf{H}^0(\mathcal{A}_j) & = & \mathsf{H}^0(\mathrm{hocolim}\, \mathcal{A}_j) & \longrightarrow & \mathsf{H}^0(\mathcal{A}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{colim}\, \mathsf{H}^0(\mathcal{B}_j) & = & \mathsf{H}^0(\mathrm{hocolim}\, \mathcal{B}_j) & \longrightarrow & \mathsf{H}^0(\mathcal{B}) \end{array}$$

and use the fact that thick inclusions are stable under filtered colimits in the category Tri of triangulated categories. Now, for each index j, there exists a quasi-equiconic dg functor $\mathcal{B}''_j \to \mathcal{B}'_j$ with $\mathcal{B}'_j$ cofibrant and pre-triangulated, such that any thick inclusion into $\mathcal{B}'_j$ is in $\mathcal{E}$. As $\mathcal{B}'_j$ and

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$\mathcal{B}_j$ are isomorphic in Hec, and as $\mathcal{B}'_j$ is cofibrant and $\mathcal{B}_j$ is pre-triangulated, we get a quasi-equivalence $\mathcal{B}'_j \to \mathcal{B}_j$. In conclusion, we obtain pullback squares

$$\begin{array}{ccc} \mathcal{A}'_j & \longrightarrow & \mathcal{B}'_j \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{A}_j & \longrightarrow & \mathcal{B}_j \end{array}$$

in which the vertical maps are quasi-equivalences, and the map $\mathcal{A}'_j \to \mathcal{B}'_j$ belongs to $\mathcal{E}$. □

The derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$ (see Theorem 3.2) admits a (left proper and cellular) Quillen model and so we can consider its left Bousfield localization with respect to the set of maps

$$\Theta_\epsilon : \mathsf{cone}[\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon) : \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B})] \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}/\mathcal{A}), \tag{2}$$

where $\epsilon : \mathcal{A} \to \mathcal{B}$ belongs to $\mathcal{E}$ (see Notation 5.2). We obtain then a new derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ (which admits also a left proper and cellular Quillen model) and an adjunction.

$$\begin{array}{c} \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \\ \gamma_! \downarrow \quad \uparrow \gamma^* \\ \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \end{array}$$

The morphism $\gamma_!$ preserves homotopy colimits and sends the maps Θ_ϵ to isomorphisms, and it is universal with respect to these properties; see Definition A.4.

Theorem 5.4. *The composition morphism*

$$\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}} : \mathsf{Trdgcat} \xrightarrow{\underline{\mathcal{U}}_\alpha^{\mathsf{add}}} \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} \xrightarrow{\gamma_!} \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

has the following properties.

(α-flt) *It commutes with α-filtered homotopy colimits.*

(p) *It preserves the terminal object.*

(wloc) *It sends strict exact sequences (5.1) to distinguished triangles in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$.*

Moreover, $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}$ is universal with respect to these properties, i.e. for every strong triangulated derivator $\mathbb{D}$, we have an equivalence of categories

$$(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}})^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_\alpha^{\mathsf{wloc}}(\mathsf{Trdgcat}, \mathbb{D}),$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{Trdgcat}, \mathbb{D})$ of morphisms of derivators which verify the above three conditions.

Proof. Since $\gamma_!$ preserves homotopy colimits, the morphism $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}$ satisfies conditions (α-flt), (p) and (add) stated in Theorem 3.2. Condition (wloc) follows from Proposition 5.3 and the fact that $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}$ commutes with α-filtered homotopy colimits. Finally, the universality of $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}$ is a consequence of Theorem 3.2 and Definition A.4 (notice that, as any split exact sequence is a strict exact sequence, under condition (p), condition (add) is implied by condition (wloc)). □

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Let us now prove a general result, which will be used in the proofs of Theorems 5.6 and 7.16.

Proposition 5.5. *Let $\mathbb{D}$ be a stable derivator (A.1), S a set of morphisms in the base category $\mathbb{D}(e)$ and X a compact object in $\mathbb{D}(e)$. Let us consider the left Bousfield localization of $\mathbb{D}$ with respect to S (see Definition A.4).*

$$\begin{array}{c} \mathbb{D} \\ \gamma_! \downarrow \uparrow \gamma^* \\ \mathsf{L}_S\mathbb{D} \end{array}$$

If the functor $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,-):\mathbb{D}(e)\to\mathsf{Ho}(\mathsf{Spt})$ sends the elements of S to isomorphisms, then $\gamma_!(X)$ is compact in $\mathsf{L}_S\mathbb{D}(e)$ and for every object $M\in\mathbb{D}(e)$, we have a natural isomorphism in the homotopy category of spectra:

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,M)\simeq\mathbb{R}\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}}(\gamma_!(X),\gamma_!(M)).$$

Proof. Let us start by showing the above natural isomorphism in the homotopy category of spectra. Thanks to [Tab08, Proposition 4.1], $\mathsf{L}_S\mathbb{D}(e)$ is the localization of $\mathbb{D}(e)$ with respect to the smallest class W of maps in $\mathbb{D}(e)$, which contains the class S, has the two-out-of-three property, and which is closed under homotopy colimits. Since X is compact and the functor $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,-)$ sends the elements of S to isomorphisms, we conclude that it also sends all the elements of W to isomorphisms. Finally, since for every object $M\in\mathbb{D}(e)$, the co-unit adjunction morphism $M\to\gamma^*\gamma_!(M)$ belongs to W, we obtain the following natural isomorphism in the homotopy category of spectra:

$$\mathbb{R}\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}}(\gamma_!(X),\gamma_!(M))\simeq\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,\gamma^*\gamma_!(M))\simeq\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X,M).$$

We now show that the object $\gamma_!(X)$ is compact (Definition 1.2) in the triangulated category $\mathsf{L}_S\mathbb{D}(e)$. Notice that for every object $N\in\mathsf{L}_S\mathbb{D}(e)$, the unit adjunction morphism $\gamma_!\gamma^*(N)\xrightarrow{\sim}N$ is an isomorphism. Let $(N_i)_{i\geqslant 0}$ be a family of objects in $\mathsf{L}_S\mathbb{D}(e)$. The proof follows from the following equivalences:

$$\begin{aligned}\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X),\oplus_i N_i) &\simeq \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X),\oplus_i\gamma_!\gamma^*(N_i))\\ &\simeq \mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X),\gamma_!(\oplus_i\gamma^*(N_i)))\\ &\simeq \mathsf{Hom}_{\mathbb{D}(e)}(X,\oplus_i\gamma^*(N_i))\\ &\simeq \oplus_i\,\mathsf{Hom}_{\mathbb{D}(e)}(X,\gamma^*(N_i))\\ &\simeq \oplus_i\,\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X),\gamma_!\gamma^*(N_i))\\ &\simeq \oplus_i\,\mathsf{Hom}_{\mathsf{L}_S\mathbb{D}(e)}(\gamma_!(X),N_i).\end{aligned}$$

Theorem 5.6 (Waldhausen localization theorem). *The object $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k})$ is compact (Definition 1.2) in the triangulated category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$ and for every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra (see Notation 2.27)*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}),\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))\simeq\underline{K}(\mathcal{A}).$$

Proof. The proof consists on verifying the conditions of Proposition 5.5, with $\mathbb{D}=\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}$, $S=\{\Theta_\epsilon\mid\epsilon\in\mathcal{E}\}$ (see (2)), $X=\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k})$ and $M=\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})$. Since by Proposition 3.7, $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k})$ is compact in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}(e)$, it is enough to show that the functor

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}),-):\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}(e)\longrightarrow\mathsf{Ho}(\mathsf{Spt})$$

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sends the elements of S to isomorphisms. For this, consider the following diagram.

$$\begin{array}{ccccc}
\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) & \xrightarrow{\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)} & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}) & \longrightarrow & \mathsf{cone}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)) \\
\| & & \| & & \downarrow{\scriptstyle \Theta_\epsilon} \\
\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A}) & \xrightarrow[\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\epsilon)]{} & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}) & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{B}/\mathcal{A})
\end{array}$$

By virtue of Theorem 3.9 and Proposition 3.7, the functor $\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), -)$, applied to the above diagram, gives the following one, where the upper line is a homotopy cofiber sequence of spectra.

$$\begin{array}{ccccc}
\underline{K}(\mathcal{A}) & \xrightarrow{\underline{K}(\epsilon)} & \underline{K}(\mathcal{B}) & \longrightarrow & \mathsf{cone}(\underline{K}(\epsilon)) \\
\| & & \| & & \downarrow \\
\underline{K}(\mathcal{A}) & \xrightarrow[\underline{K}(\epsilon)]{} & \underline{K}(\mathcal{B}) & \longrightarrow & \underline{K}(\mathcal{B}/\mathcal{A})
\end{array}$$

Now, consider the following sequence of Waldhausen categories (see Notation 2.27):

$$\mathsf{tri}^{\mathcal{W}}(\mathcal{A}) \longrightarrow \mathsf{tri}^{\mathcal{W}}(\mathcal{B}) \longrightarrow \mathsf{tri}^{\mathcal{W}}(\mathcal{B}/\mathcal{A}).$$

Let us denote by $v\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$ the Waldhausen category with the same cofibrations as $\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$, but whose weak equivalences are the maps with cone in $\mathsf{tri}^{\mathcal{W}}(\mathcal{A})$. Since $\mathsf{tri}(\mathcal{A})$ is thick in $\mathsf{tri}(\mathcal{B})$, we conclude by Waldhausen's fibration theorem [Wal85, Theorem 1.6.4] (applied to the inclusion $\mathsf{tri}^{\mathcal{W}}(\mathcal{B}) \subset v\mathsf{tri}^{\mathcal{W}}(\mathcal{B})$) that the lower line of the above diagram is a homotopy fiber sequence of spectra. Since the (homotopy) category of spectra is stable, we conclude that the right vertical map is a weak equivalence of spectra. Finally, since by Theorem 3.9, we have a natural isomorphism

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{A})) \simeq \underline{K}(\mathcal{A}),$$

in the stable homotopy category of spectra, the proof is finished. □

5.2 Morita invariance

Let $\mathcal{S}$ be the set of morphisms in Hec of the form $\mathcal{B} \to \mathcal{B}^\wedge$ (see Notation 4.1), with $\mathcal{B}$ a α-small dg cell (2.7).

Proposition 5.7. *For any dg category $\mathcal{A}$, there exists an α-filtered direct system $\{\mathcal{B}_j \to \mathcal{B}_j^\wedge\}_{j\in J}$ of elements of $\mathcal{S}$, such that*

$$\mathrm{hocolim}_j \{\mathcal{B}_j \longrightarrow \mathcal{B}_j^\wedge\} \xrightarrow{\sim} (\mathcal{A} \longrightarrow \mathcal{A}^\wedge).$$

Proof. Thanks to Proposition 3.6(iii), there exits an α-filtered direct system $\{\mathcal{B}_j\}_{j\in J}$ of α-small dg cells such that

$$\mathrm{hocolim}_j\, \mathcal{B}_j \xrightarrow{\sim} \mathcal{A}.$$

By Proposition 4.2(i), the functor $(-)^\wedge$ preserves α-filtered homotopy colimits, and so the proof is finished. □

Since the derivator $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ (see Theorem 5.4) admits a (left proper and cellular) Quillen model, we can localize it with respect to the image of the set $\mathcal{S}$ under $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}$. We obtain then a

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new derivator $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$ and an adjunction as in the following diagram.

$$\begin{array}{c} \underline{\mathrm{Mot}}^{\mathsf{wloc}}_\alpha \\ {\scriptstyle l_!}\big\downarrow \big\uparrow{\scriptstyle l^*} \\ \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha \end{array}$$

PROPOSITION 5.8. *The composition*

$$\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha : \mathsf{Trdgcat} \xrightarrow{\underline{\mathcal{U}}^{\mathsf{wloc}}_\alpha} \underline{\mathrm{Mot}}^{\mathsf{wloc}}_\alpha \xrightarrow{l_!} \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$$

sends the derived Morita equivalences (Definition 2.21) to isomorphisms.

Proof. Observe first that $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ sends the maps of shape $\mathcal{A} \to \mathcal{A}^\wedge$ to isomorphisms. This follows from Proposition 5.7 and from the fact that $\underline{\mathcal{U}}^{\mathsf{wloc}}_\alpha$ commutes with α-filtered homotopy colimits. Thanks to Propositions 2.18 and 2.24, a dg functor $F : \mathcal{A} \to \mathcal{B}$ is a derived Morita equivalence if and only if $F^\wedge : \mathcal{A}^\wedge \to \mathcal{B}^\wedge$ is a quasi-equivalence (Definition 2.12). The proof is then finished. □

Thanks to Proposition 5.8, the morphism $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ descends to the derivator $\mathsf{HO}(\mathsf{dgcat})$ associated with the Morita model structure on dgcat; see Theorem 2.22.

THEOREM 5.9. *The morphism*

$$\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$$

has the following properties.

(α-flt) *It commutes with α-filtered homotopy colimits.*

($\underline{\mathrm{p}}$) *It preserves the terminal object.*

(loc) *It sends the exact sequences in* Hmo *(Definition 3.1) to distinguished triangles in* $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$.

Moreover, $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ *is universal with respect to these properties, i.e. for every strong triangulated derivator* $\mathbb{D}$*, we have an equivalence of categories*

$$(\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha)^* : \underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}^{\mathsf{loc}}_\alpha(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}),$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ *of morphisms of derivators which verify the above three conditions.*

Notation 5.10. The objects of the category $\underline{\mathsf{Hom}}^{\mathsf{loc}}_\alpha(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ will be called *α-localizing invariants* and $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ the *universal α-localizing invariant.*

Proof. Since $l_!$ preserves homotopy colimits, the morphism $\underline{\mathcal{U}}^{\mathsf{loc}}_\alpha$ satisfies conditions (α-flt) and ($\underline{\mathrm{p}}$). In what concerns condition (loc), we can suppose by [Kel06, Theorem 4.11] that we have a exact sequence in Hmo of the form

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{B}/\mathcal{A}.$$

Consider the diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{I} & \mathcal{B} & \longrightarrow & \mathcal{B}/\mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}^\wedge & \xrightarrow[I^\wedge]{} & \mathcal{B}^\wedge & \longrightarrow & \mathcal{B}^\wedge/\mathcal{A}^\wedge \end{array}$$

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where the right vertical map is the induced one. Since the first two vertical maps are derived Morita equivalences, the right vertical one is also a derived Morita equivalence. Furthermore, since the lower line is a strict exact sequence (Definition 5.1), Proposition 5.8 and Theorem 5.4 imply that $\underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}$ satisfies condition (loc). The universality of $\underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}$ is now a clear consequence of Theorem 5.4 and Proposition 5.8. □

6. Schlichting's set-up

In this section we will build a set-up (see [Sch06, § 2.2]) in Hec, that will be used in § 7 to construct the non-connective K-theory spectrum. In Proposition 6.6, we will relate it with Schlichting's set-up on Frobenius pairs.

6.1 Set-up

We start by constructing a infinite sums completion functor. Recall from [Tab09, § 6], the construction of the following Quillen adjunctions (where we have taken $\alpha = \aleph_1$)

$$\begin{array}{c} \mathsf{dgcat}_{\mathrm{ex},\aleph_1} \\ {\scriptstyle F_1}\uparrow \;\; \downarrow{\scriptstyle U_1} \\ T_{\aleph_1}\text{-}\mathsf{alg} \\ {\scriptstyle F}\uparrow \;\; \downarrow{\scriptstyle U} \\ \mathsf{dgcat} \end{array}$$

where dgcat is endowed with the model structure of Theorem 2.13. Consider the composed functor

$$\mathsf{dgcat} \xrightarrow{(-)^{\mathsf{pre\text{-}tr}}} \mathsf{dgcat} \xrightarrow{F_1 \circ F} \mathsf{dgcat}_{\mathrm{ex},\aleph_1} \xrightarrow{U \circ U_1} \mathsf{dgcat}, \tag{3}$$

where $(-)^{\mathsf{pre\text{-}tr}}$ denotes Bondal–Kapranov's pre-triangulated envelope construction; see [BK90] for details.

Lemma 6.1. *The above composed functor (3) sends quasi-equiconic dg functors (Definition 2.15) between cofibrant dg categories to quasi-equivalences (Definition 2.12).*

Proof. Note first that by [Tab07, Lemma 2.13] $(-)^{\mathsf{pre\text{-}tr}}$ is a fibrant resolution functor with respect to the quasi-equiconic model structure; see Proposition 2.19. Therefore, by [Tab07, Proposition 2.14], $(-)^{\mathsf{pre\text{-}tr}}$ sends quasi-equiconic dg functors (between cofibrant dg categories) to quasi-equivalences (between cofibrant dg categories). Since $F \circ F_1$ is a left Quillen functor, quasi-equivalences between cofibrant dg categories are sent to weak equivalences in $\mathsf{dgcat}_{\mathrm{ex},\aleph_1}$. Finally, the composed functor $U \circ U_1$ maps weak equivalences to quasi-equivalences, and so the proof is finished. □

Thanks to Lemma 6.1 and [Hir03, Proposition 8.4.8], the above composed functor (3) admits a total left derived functor (with respect to the quasi-equiconic model structure), denoted by

$$\mathcal{F}(-) : \mathsf{Hec} \longrightarrow \mathsf{Hec}.$$

Notice that by construction, we have also a natural transformation $\mathrm{Id} \Rightarrow \mathcal{F}(-)$.

Lemma 6.2. *The functor $\mathcal{F}(-)$ gives rise to a morphism of derivators*

$$\mathcal{F}(-) : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat},$$

which preserves $\aleph_1$-filtered homotopy colimits.

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Proof. This follows from the fact that the functors $(-)^{\mathsf{pre\text{-}tr}}$ and $U \circ U_1$ preserve $\aleph_1$-filtered homotopy colimits. The case of $(-)^{\mathsf{pre\text{-}tr}}$ is clear. In what concerns $U \circ U_1$, please see [Tab09, Proposition 6.7]. □

PROPOSITION 6.3. *Let $\mathcal{A}$ be a small dg category.*

(i) *The small dg category $\mathcal{F}(\mathcal{A})$ has countable sums.*

(ii) *The small dg category $\mathcal{F}(\mathcal{A})$ is Morita fibrant (see Proposition 2.25).*

(iii) *The dg functor $\mathcal{A} \to \mathcal{F}(\mathcal{A})$ induces a fully-faithful triangulated functor (see Notation 2.10)*

$$\mathsf{tri}(\mathcal{A}) \to \mathsf{tri}(\mathcal{F}(\mathcal{A})) \simeq \mathsf{H}^0(\mathcal{F}(\mathcal{A})).$$

Moreover, the image of $\mathsf{H}^0(\mathcal{A})$ in $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$ forms a set of compact generators.

Proof. Condition (i) follows from [Tab09, Proposition 3.8]. Thanks to [Tab09, 6.4], $\mathcal{F}(\mathcal{A})$ is pre-triangulated (see Proposition 2.19). Since in any triangulated category with countable sums (for instance $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$) every idempotent splits (see [Nee01, Proposition 1.6.8]), we conclude that $\mathcal{F}(\mathcal{A})$ is Morita fibrant (see Proposition 2.25). Finally, condition (iii) follows from the construction of $\mathcal{F}(-)$. □

Remark 6.4. By Proposition 6.3(iii) and [Sch06, § 3.1, Lemma 2] we have an equivalence $\widetilde{\mathsf{tri}(\mathcal{A})} \simeq \mathsf{H}^0(\mathcal{F}(\mathcal{A}))_c$ between the idempotent completion of $\mathsf{tri}(\mathcal{A})$ (see Notation 1.4) and the triangulated subcategory of compact objects in $\mathsf{H}^0(\mathcal{F}(\mathcal{A}))$.

We can then associate to every small dg category $\mathcal{A}$, an exact sequence

$$\mathcal{A} \longrightarrow \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{A}) := \mathcal{F}(\mathcal{A})/\mathcal{A},$$

where $\mathcal{F}(\mathcal{A})/\mathcal{A}$ denotes the Drinfeld's dg quotient [Dri04]. In this way, we obtain also a functor

$$\mathcal{S}(-) : \mathsf{Hec} \longrightarrow \mathsf{Hec}.$$

PROPOSITION 6.5. *The category Hec endowed with $\mathcal{F}(-)$, $\mathcal{S}(-)$ and with the functor (see Notation 2.10)*

$$\mathsf{tri} : \mathsf{Hec} \longrightarrow \mathsf{Tri} \quad \mathcal{A} \mapsto \mathsf{tri}(\mathcal{A}),$$

satisfies the hypothesis of Schlichting's set-up [Sch06, § 2.2].

Proof. By construction, we have for every $\mathcal{A} \in \mathsf{Hec}$ an exact sequence

$$\mathcal{A} \longrightarrow \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{S}(\mathcal{A}).$$

Since $\mathcal{F}(\mathcal{A})$ has countable sums, the Grothendieck group $\mathbb{K}_0(\mathcal{F}(\mathcal{A}))$ is trivial (see Notation 2.29). It remains to show that $\mathcal{F}(-)$ and $\mathcal{S}(-)$ preserve exact sequences. For this, consider the following diagram in Hec

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{I} & \mathcal{B} & \xrightarrow{P} & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\mathcal{A}) & \xrightarrow{\mathcal{F}(I)} & \mathcal{F}(\mathcal{B}) & \xrightarrow{\mathcal{F}(P)} & \mathcal{F}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}(\mathcal{A}) & \xrightarrow[\mathcal{S}(I)]{} & \mathcal{S}(\mathcal{B}) & \xrightarrow[\mathcal{S}(P)]{} & \mathcal{S}(\mathcal{C})
\end{array}$$

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where the upper horizontal line is a exact sequence. Notice that by Remark 6.4, the triangulated functors $\mathsf{tri}(\mathcal{F}(I))$ and $\mathsf{tri}(\mathcal{F}(P))$ preserve compact objects. Moreover, since these functors preserve also countable sums (see [Tab09, 6.4]), [Sch06, § 3.1, Corollary 3] implies that the middle line is also an exact sequence. Finally, by [Sch06, § 3.1, Lemma 3] and Remark 6.4, we conclude that the lower line is also an exact sequence. □

6.2 Relation with Frobenius pairs

Let us denote by $(\mathrm{Frob}; \mathcal{F}_{\mathcal{M}}, \mathcal{S}_{\mathcal{M}})$ Schlichting's set-up on the category of Frobenius pairs; see [Sch06, § 5] for details. Recall from [Sch06, § 4.3, Definition 6] that we have a *derived category* functor

$$D : \mathrm{Frob} \longrightarrow \mathsf{Tri}, \quad \mathbf{A} = (\mathcal{A}, \mathcal{A}_0) \mapsto D\mathbf{A} = \underline{\mathcal{A}}/\underline{\mathcal{A}_0},$$

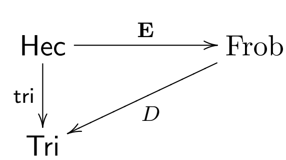
where $\underline{\mathcal{A}}/\underline{\mathcal{A}_0}$ denotes the Verdier's quotient of the stable categories. Notice that we have a natural functor

$$\mathbf{E} : \mathsf{Hec} \longrightarrow \mathrm{Frob}, \quad \mathcal{A} \mapsto (E(\mathcal{A}), E(\mathcal{A})\text{-prinj}),$$

where $E(\mathcal{A}) = \mathsf{perf}^{\mathcal{W}}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$ (see Notation 2.27) is the Frobenius category associated to the cofibrant $\mathcal{A}$-modules which become isomorphic in $\mathcal{D}(\mathcal{A})$ to elements of $\mathsf{tri}(\mathcal{A})$, and $E(\mathcal{A})$-prinj is the full subcategory of projective–injective objects of $E(\mathcal{A})$. Since

$$D\mathbf{E}(\mathcal{A}) = \underline{E(\mathcal{A})}/\underline{E(\mathcal{A})\text{-prinj}} \simeq \mathsf{tri}(\mathcal{A}),$$

we have an essentially commutative diagram.



Proposition 6.6. *For any dg category $\mathcal{A}$, we have canonical equivalences of triangulated categories*

$$D\mathcal{F}_{\mathcal{M}}\mathbf{E}(\mathcal{A}) \simeq \mathsf{tri}(\mathcal{F}\mathcal{A}), \quad D\mathcal{S}_{\mathcal{M}}\mathbf{E}(\mathcal{A}) \simeq \mathsf{tri}(\mathcal{S}\mathcal{A}).$$

Proof. We start by showing that $\mathbf{E} \circ \mathcal{F}$ and $\mathcal{F}_{\mathcal{M}} \circ \mathbf{E}$ are weakly isomorphic. Let $\mathcal{A}$ be a pre-triangulated dg category (see Proposition 2.19) and $\mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$ the left Bousfield localization of $\mathcal{C}(\mathcal{F}\mathcal{A})$ (see Notation 2.8), with respect to the set

$$S := \left\{ \bigoplus_{i \in I} \underline{h}(x_i) \longrightarrow \underline{h}\left(\bigoplus_{i \in I} x_i\right) \,\middle|\, x_i \in \mathcal{F}\mathcal{A}, |I| < \aleph_1 \right\}.$$

Let us now introduce an 'intermediate' Frobenius pair $\mathbf{E}'(\mathcal{F}\mathcal{A}) := (E'(\mathcal{F}\mathcal{A}), E'(\mathcal{F}\mathcal{A})_0)$. The Frobenius category $E'(\mathcal{F}\mathcal{A}) \subset \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$ consists of the cofibrant $\mathcal{F}\mathcal{A}$-modules which become isomorphic in $\mathsf{Ho}(\mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A}))$ to representable ones, and $E'(\mathcal{F}\mathcal{A})_0$ consists of the $\mathcal{F}\mathcal{A}$-modules M whose morphism $M \to 0$ to the terminal object is a S-equivalence. Notice that since every representable $\mathcal{F}\mathcal{A}$-module is S-local in $\mathcal{C}(\mathcal{F}\mathcal{A})$, we have a natural weak

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equivalence $\mathbf{E}(\mathcal{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{E}'(\mathcal{F}\mathcal{A})$ of Frobenius pairs. Consider the following (solid) commutative diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad\text{colim}\quad} & & \\
\mathcal{F}_{\mathcal{M}}(E(\mathcal{A})) & \longleftarrow & E(\mathcal{A}) & \hookrightarrow & \mathcal{C}(\mathcal{A}) \\
 & \searrow^{\Phi} & \downarrow & & \downarrow \\
 & & E(\mathcal{F}\mathcal{A}) & \hookrightarrow & \mathcal{C}(\mathcal{F}\mathcal{A}) \\
 & & \downarrow\wr & & \downarrow \\
 & & E'(\mathcal{F}\mathcal{A}) & \hookrightarrow & \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})
\end{array}$$

where Φ is the composition $\mathcal{F}_{\mathcal{M}}(E(\mathcal{A})) \xrightarrow{\text{colim}} \mathcal{C}(\mathcal{A}) \to \mathsf{L}_S\mathcal{C}(\mathcal{F}\mathcal{A})$. We obtain then the following commutative diagram of Frobenius pairs.

$$\begin{array}{ccc}
\mathbf{E}(\mathcal{A}) & \longrightarrow & \mathbf{E}(\mathcal{F}\mathcal{A}) \\
\downarrow & & \downarrow\wr \\
\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A})) & \longrightarrow & \mathbf{E}'(\mathcal{F}\mathcal{A})
\end{array}$$

Thanks to [Sch06, § 5.2, Proposition 1] and Proposition 6.3(iii) the triangulated categories $\mathcal{D}\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathsf{tri}(\mathcal{F}(\mathcal{A})) \simeq \mathcal{D}\mathbf{E}(\mathcal{F}\mathcal{A})$ (and so $\mathcal{D}\mathbf{E}'(\mathcal{F}(\mathcal{A}))$ are compactly generated (see [Nee01, § 8.1]) by $\mathsf{tri}(\mathcal{A})$. Therefore, the lower map in the above square is also a weak equivalence of Frobenius pairs. In conclusion, we obtain a (two steps) zig-zag of weak equivalences relating $\mathcal{F}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathbf{E}(\mathcal{F}\mathcal{A})$.

We now show that $\mathbf{E} \circ \mathcal{S}$ and $\mathcal{S}_{\mathcal{M}} \circ \mathbf{E}$ are weakly isomorphic. For this, notice that we have the following zig-zag of weak equivalences of Frobenius pairs relating $\mathcal{S}_{\mathcal{M}}(\mathbf{E}(\mathcal{A}))$ and $\mathbf{E}(\mathcal{S}(\mathcal{A}))$

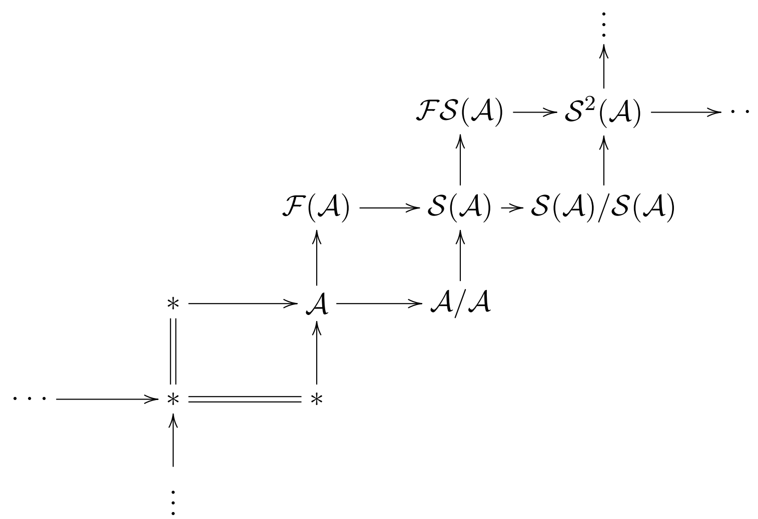
$$\begin{array}{ccccc}
 & & (E(\mathcal{F}\mathcal{A}), S_0\mathbf{E}(\mathcal{F}\mathcal{A})) & \xrightarrow{\sim} & \mathbf{E}(\mathcal{S}(\mathcal{A})) \\
 & & \downarrow\wr & & \\
\mathcal{S}_{\mathcal{M}}(\mathbf{E}(\mathcal{A})) & \xrightarrow{\sim} & (E'(\mathcal{F}\mathcal{A}), S_0\mathbf{E}'(\mathcal{F}\mathcal{A})) & &
\end{array}$$

where $S_0\mathbf{E}(\mathcal{F}\mathcal{A})$ (respectively $S_0\mathbf{E}'(\mathcal{F}\mathcal{A})$) is the full subcategory of $E(\mathcal{F}\mathcal{A})$ (respectively of $E'(\mathcal{F}\mathcal{A})$) consisting of those objects sent to zero in the quotient $\mathcal{D}\mathbf{E}(\mathcal{F}\mathcal{A})/\mathcal{D}\mathbf{E}(\mathcal{A})$ (respectively in $\mathcal{D}\mathbf{E}'(\mathcal{F}\mathcal{A})/\mathcal{D}\mathbf{E}(\mathcal{A})$). □

7. Non-connective K-theory

7.1 Construction

Let $\mathbf{V} = W^{\mathsf{op}}$, where W is the poset $\{(i, j) \mid |i - j| \leqslant 1\} \subset \mathbb{Z} \times \mathbb{Z}$ considered as a small category (using the product partial order of $\mathbb{Z} \times \mathbb{Z}$). We can construct for every small dg category $\mathcal{A}$, a diagram $\mathrm{Dia}(\mathcal{A}) \in \mathsf{Trdgcat}(\mathbf{V})$ as follows.



Notice that the assignment $\mathcal{A} \mapsto \mathrm{Dia}(\mathcal{A})$ is functorial. We obtain then a morphism of derivators

$$\mathrm{Dia}(-) : \mathsf{Trdgcat} \longrightarrow \mathsf{Trdgcat}_{\mathbf{V}}.$$

Lemma 7.1. *Let $\mathcal{A}$ be a pre-triangulated dg category (see Proposition 2.19) with countable sums. Then for any dg category $\mathcal{B}$, $\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, \mathcal{A})$ (see Proposition 2.20) has countable sums.*

Proof. Recall that $\mathcal{A}$ has countable sums if and only if the diagonal functor $\mathcal{A} \to \prod_{\mathbb{N}} \mathcal{A}$ has a left adjoint. Since for any dg category $\mathcal{B}$, $\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, -)$ is a 2-functor which preserves (derived) products, the proof is finished. □

Proposition 7.2. *Let $\mathcal{A}$ be a dg category and $n \geqslant 0$. Then $\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A})$ and $\mathcal{F}\mathcal{S}^n(\mathcal{A})$ become trivial after application of $\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}$ (see Theorem 3.2).*

Proof. Since $\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A})$ is clearly isomorphic to the terminal object in Hec and $\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}$ preserves the terminal object, $\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{S}^n(\mathcal{A})/\mathcal{S}^n(\mathcal{A}))$ is trivial. In what concerns $\mathcal{F}\mathcal{S}^n(\mathcal{A})$, it is enough by Proposition 3.7, to show that for every α-small dg cell $\mathcal{B}$ (Definition 2.7), the spectrum (see Notation 2.27)

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{B}), \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{F}\mathcal{S}^n(\mathcal{A}))) \simeq \underline{K}\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, \mathcal{F}\mathcal{S}^n(\mathcal{A}))$$

is (homotopically) trivial. By Proposition 6.3(i), the dg category $\mathcal{F}\mathcal{S}^n(\mathcal{A})$ has countable sums, and so by Lemma 7.1 $\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, \mathcal{F}\mathcal{S}^n(\mathcal{A}))$ has also countable sums. This implies, by additivity, that the identity of $\underline{K}\mathsf{rep}_{\mathrm{tr}}(\mathcal{B}, \mathcal{F}\mathcal{S}^n(\mathcal{A}))$ is homotopic to zero. □

Notation 7.3. Let $V(-)$ be the composed morphism

$$\mathsf{Trdgcat} \xrightarrow{\mathrm{Dia}(-)^{\wedge}} \mathsf{Trdgcat}_{\mathbf{V}} \xrightarrow{\underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}} (\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha})_{\mathbf{V}},$$

where $\mathrm{Dia}(-)^{\wedge}$ is obtained from $\mathrm{Dia}(-)$ by applying $(-)^{\wedge}$ objectwise (see Notation 4.1). Given any dg category $\mathcal{A}$, Proposition 7.2 implies that $V(\mathcal{A})$ is a spectrum in $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$; see § A.3. We define $V_a(\mathcal{A}) = \Omega^{\infty} V(\mathcal{A})$ as the infinite loop object associated to $V(\mathcal{A})$. This construction defines

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a morphism of derivators

$$V_a : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}.$$

We also define V_l as the composition $\gamma_! \circ V_a : \mathsf{Trdgcat} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$.

Remark 7.4. For any dg category $\mathcal{A}$ we have

$$V_a(\mathcal{A}) = \Omega^\infty V(\mathcal{A}) \simeq \mathrm{hocolim}_n\, \underline{\mathcal{U}}_\alpha^{\mathsf{add}}(\mathcal{S}^n(\mathcal{A})^\wedge)[-n].$$

Furthermore, since we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Spec}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{add}}) & \xrightarrow{\mathsf{Spec}(\gamma_!)} & \mathsf{Spec}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}) \\ \downarrow{\scriptstyle \Omega^\infty} & & \downarrow{\scriptstyle \Omega^\infty} \\ \underline{\mathrm{Mot}}_\alpha^{\mathsf{add}} & \xrightarrow[\gamma_!]{} & \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}} \end{array}$$

$V_l(\mathcal{A})$ can be described as

$$V_l(\mathcal{A}) = \mathrm{hocolim}_n\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge)[-n].$$

Notice also that we have a natural 2-morphism of derivators $\underline{\mathcal{U}}_\alpha^{\mathsf{add}}(-) \Rightarrow V_a(-)$.

PROPOSITION 7.5. *For any dg category $\mathcal{A}$, there is a canonical isomorphism*

$$\mathbb{K}(\mathcal{A}) \simeq \mathrm{hocolim}_n\, K(\mathcal{S}^n(\mathcal{A}))[-n]$$

in the stable homotopy category of spectra (see Notation 2.29).

Proof. This follows immediately from the very definition of the non-connective K-theory spectrum and from Proposition 6.6. □

PROPOSITION 7.6. *For any dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy category of spectra (see Notation 2.29)*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A}).$$

Proof. We already know from Proposition 7.5 that we have a weak equivalence of spectra

$$\mathbb{K}(\mathcal{A}) \simeq \mathrm{hocolim}_n\, K(\mathcal{S}^n(\mathcal{A}))[-n]$$

we have also a weak equivalence of spectra

$$\mathrm{hocolim}_n\, K(\mathcal{S}^n(\mathcal{A}))[-n] \simeq \mathrm{hocolim}_n\, K(\mathcal{S}^n(\mathcal{A})^\wedge)[-n]$$

(see the proof of [Sch06, § 12.1, Thmeorem 8]). We deduce from these facts the following computations:

$$\begin{aligned} \mathbb{K}(\mathcal{A}) &\simeq \mathrm{hocolim}_n\, K(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ &\simeq \mathrm{hocolim}_n\, \underline{K}\mathsf{rep}_{\mathrm{tr}}(\underline{k}, \mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ &\simeq \mathrm{hocolim}_n\, \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge))[-n] \qquad (4) \\ &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})). \qquad (5) \end{aligned}$$

Equivalence (4) comes from Theorem 5.6, and equivalence (5) from the compactness of $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(k)$ in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$; see Theorem 5.6 and Remark 3.8. □

7.2 Co-representability in $\underline{\mathsf{Mot}}_\alpha^{\mathsf{loc}}$

PROPOSITION 7.7. *The morphism $V_l(-): \mathsf{Trdgcat} \to \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$ sends derived Morita equivalences (Definition 2.21) to isomorphisms.*

Proof. As in the proof of Theorem 5.9, it is enough to show that $V_l(-)$ sends the maps of shape $\mathcal{A} \to \mathcal{A}^\wedge$ to isomorphisms. For this consider the following diagram.

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{F}(\mathcal{A}) & \longrightarrow & \mathcal{S}(\mathcal{A}) := \mathcal{F}(\mathcal{A})/\mathcal{A} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(P)} & & \downarrow{\scriptstyle \mathcal{S}(P)} \\ \mathcal{A}^\wedge & \longrightarrow & \mathcal{F}(\mathcal{A}^\wedge) & \longrightarrow & \mathcal{S}(\mathcal{A}^\wedge) := \mathcal{F}(\mathcal{A}^\wedge)/\mathcal{A}^\wedge \end{array}$$

Thanks to Proposition 6.3, $\mathcal{F}(P)$ is an isomorphism. Observe that the induced morphism $\mathcal{S}(P)$, between the Drinfeld dg quotients, is also an isomorphism. Finally, using the description of $V_l(-)$ of Remark 7.4, the proof is finished. □

Notice that by Proposition 7.7, the morphism $V_l(-)$ descends to the derivator $\mathsf{HO}(\mathsf{dgcat})$ associated to the Morita model structure.

PROPOSITION 7.8. *Assume that $\alpha \geqslant \aleph_1$. The morphism of derivators*

$$V_l(-): \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

is an α-localizing invariant (Notation 5.10).

Proof. We start by showing condition (α-flt). Since, by Lemma 6.2, the functor $\mathcal{F}(-)$ preserves α-filtered colimits, the functors $\mathcal{S}^n(-), n \geqslant 0$, also preserves α-filtered colimits. Now, Proposition 4.2(i), Theorem 5.4, and the classical Fubini rule on homotopy colimits imply the claim. Condition ($\underline{\mathrm{p}}$) is clear. In what concerns condition (loc), let

$$0 \longrightarrow \mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C} \longrightarrow 0$$

be an exact sequence (Definition 3.1) in Hmo. Consider the following diagram in $\mathsf{Trdgcat}(\mathbf{V})$,

$$\begin{array}{ccccc} \mathrm{Dia}(\mathcal{A})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{B})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge := \mathrm{Dia}(\mathcal{A})^\wedge/\mathrm{Dia}(\mathcal{B})^\wedge \\ \| & & \| & & \downarrow \\ \mathrm{Dia}(\mathcal{A})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{B})^\wedge & \longrightarrow & \mathrm{Dia}(\mathcal{C})^\wedge \end{array}$$

where $\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge$ is obtained by applying Drinfeld's dg quotient [Dri04] objectwise. Since the upper horizontal line is objectwise a strict exact sequence (Definition 5.1), we obtain a distinguished triangle

$$V_l(\mathcal{A}) \longrightarrow V_l(\mathcal{B}) \longrightarrow \Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \longrightarrow V_l(\mathcal{A})[1]$$

in $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$. We have the following explicit description of $\Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)$:

$$\Omega^\infty \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) = \mathrm{hocolim}_n\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge)[-n].$$

We now show that the morphism

$$D: \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)$$

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becomes an isomorphism after application of the infinite loop functor Ω^∞. For this, consider the following solid diagram,

$$\begin{array}{ccccc}
\mathcal{S}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{A})^\wedge \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}^n(\mathcal{B})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{B})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{B})^\wedge \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{B})^\wedge/\mathcal{FS}^n(\mathcal{A})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{B})^\wedge/\mathcal{S}^{n+1}(\mathcal{A})^\wedge \\
\downarrow & \overset{\psi_n}{\nearrow} & \downarrow \phi & & \downarrow \\
\mathcal{S}^n(\mathcal{C})^\wedge & \longrightarrow & \mathcal{FS}^n(\mathcal{C})^\wedge & \longrightarrow & \mathcal{S}^{n+1}(\mathcal{C})^\wedge
\end{array}$$

where the composition in each line is trivial. Since the dg functor $\mathcal{FS}^n(\mathcal{A})^\wedge \to \mathcal{FS}^n(\mathcal{B})^\wedge$ preserves countable sums, [Sch06, § 3.1, Theorem 2] implies that the dg category $\mathcal{FS}^n(\mathcal{B})^\wedge/\mathcal{FS}^n(\mathcal{A})^\wedge$ is Morita fibrant (see Proposition 2.25) and so ϕ is an isomorphism. We obtain then a well-defined morphism ψ_n, which induces an interpolation map

$$\Psi_n : \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{C})^\wedge)[-n] \longrightarrow \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^{n+1}(\mathcal{B})^\wedge/\mathcal{S}^{n+1}(\mathcal{A})^\wedge)[-(n+1)].$$

Notice that we have the following commutative diagrams.

$$\begin{array}{ccc}
\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)_n[-n] & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge)_{n+1}[-(n+1)] \\
{\scriptstyle D_n[-n]}\downarrow & \overset{\Psi_n}{\nearrow} & \downarrow{\scriptstyle D_{n+1}[-(n+1)]} \\
\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)_n[-n] & \longrightarrow & \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge)_{n+1}[-(n+1)]
\end{array}$$

The existence of such a set of interpolation maps $\{\Psi_n\}_{n\in\mathbb{N}}$ implies that the map

$$\mathrm{hocolim}_n\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{B})^\wedge/\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \xrightarrow{\sim} \mathrm{hocolim}_n\, \underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{C})^\wedge)[-n]$$

is an isomorphism; see Corollary A.14. In other words, the induced morphism

$$\Omega^\infty(D) : \Omega^\infty\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{A},\mathcal{B})^\wedge) \xrightarrow{\sim} \Omega^\infty\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathrm{Dia}(\mathcal{C})^\wedge) = V_l(\mathcal{C})$$

is an isomorphism. □

COROLLARY 7.9 (Of Proposition 7.8). *If $\alpha \geqslant \aleph_1$, since $V_l(-)$ is an α-localizing invariant, Theorem 5.9 implies the existence of a morphism of derivators (which commutes with homotopy colimits)*

$$\mathsf{Loc} : \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

such that $\mathsf{Loc}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) \simeq V_l(\mathcal{A})$, for every small dg category $\mathcal{A}$.

PROPOSITION 7.10. *If $\alpha \geqslant \aleph_1$, the two morphisms of derivators*

$$\mathsf{Loc}, l^* : \underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}} \longrightarrow \underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}$$

are canonically isomorphic.

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Proof. Consider the composed morphism $L := \mathsf{Loc} \circ l_!$. Thanks to Theorem 5.9 and to Proposition 7.7, the 2-morphism of derivators $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}} \Rightarrow V_l(-)$ can be naturally extended to a 2-morphism $\eta : \mathrm{Id} \Rightarrow L$. We show first that the couple (L, η) defines a left Bousfield localization of the category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$, i.e. we prove that the natural transformations $L\eta$ and η_L are equal isomorphisms. By construction, the functor L commutes with homotopy colimits, and so by Theorem 5.9 it is enough to prove it for the objects of shape $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})$, with $\mathcal{A}$ a (pre-triangulated) dg category. Thanks to Proposition 7.7 we have $V_l(\mathcal{A}) \simeq V_l(\mathcal{A}^\wedge)$, and Proposition 7.8, applied recursively to the exact sequences

$$\mathcal{S}^n(\mathcal{A}) \longrightarrow \mathcal{F}\mathcal{S}^n(\mathcal{A}) \longrightarrow \mathcal{S}^{n+1}(\mathcal{A}), \quad n \geqslant 0,$$

shows us that we have $V_l(\mathcal{S}^n(\mathcal{A})) \simeq V_l(\mathcal{A})[n]$. This implies the following isomorphisms:

$$\begin{aligned} L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) &= L(V_l(\mathcal{A})) \\ &\simeq \mathrm{hocolim}_n\, V_l(\mathcal{S}^n(\mathcal{A})^\wedge)[-n] \\ &\simeq \mathrm{hocolim}_n\, V_l(\mathcal{S}^n(\mathcal{A}))[-n] \\ &\simeq \mathrm{hocolim}_n (V_l(\mathcal{A})[n])[-n] \\ &\simeq V_l(\mathcal{A}). \end{aligned}$$

More precisely, a careful description of the above isomorphisms shows that the morphisms

$$\eta_{L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))},\, L(\eta_{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})}) : L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \longrightarrow L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))$$

become equal isomorphisms after composing them with the canonical isomorphism

$$L^2(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})) \xrightarrow{\sim} \mathrm{hocolim}_{n,m}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})[m+n])[-m-n].$$

Hence $\eta_{L(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A}))} = L(\eta_{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})})$ is an isomorphism, which proves that (L, η) defines a left Bousfield localization of the category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$.

The general formalism of left Bousfield localizations makes that we are now reduced to prove the following property: a morphism of $\underline{\mathrm{Mot}}_\alpha^{\mathsf{wloc}}(e)$ becomes an isomorphism after applying the functor L if and only if it becomes an isomorphism after applying the functor $l_!$. For this purpose, it is even sufficient to prove that the induced morphism $l_!\eta : l_! \Rightarrow l_!(L)$ is an isomorphism. Once again, it is enough to show it for the objects of the form $\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{A})$. By virtue of Theorem 5.9, the spectrum $\{\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\mathcal{S}^n(\mathcal{A})^\wedge)\}_{n \geqslant 0}$ becomes an Ω-spectrum in $\mathsf{St}(\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}})$ after application of $l_!$. Therefore, we obtain an isomorphism $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}) \simeq l_! V_l(\mathcal{A})$. □

Remark 7.11. Since the morphism of derivators Loc preserves homotopy colimits, Proposition 7.10 and Theorem 5.6 imply that the object $\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(k)$ is compact (Definition 1.2) in the triangulated category $\underline{\mathrm{Mot}}_\alpha^{\mathsf{loc}}(e)$.

Theorem 7.12. *Let $\mathcal{A}$ be a small dg category. If $\alpha \geqslant \aleph_1$, we have a natural isomorphism in the stable homotopy category of spectra (see Notation 2.29)*

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A}).$$

Proof. We have the following isomorphisms:

$$\begin{aligned} \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A})) &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), l^*(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}))) \\ &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), \mathsf{Loc}(\underline{\mathcal{U}}_\alpha^{\mathsf{loc}}(\mathcal{A}))) \qquad (6) \\ &\simeq \mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_\alpha^{\mathsf{wloc}}(\underline{k}), V_l(\mathcal{A})) \qquad (7) \\ &\simeq \mathbb{K}(\mathcal{A}). \qquad (8) \end{aligned}$$

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Equivalence (6) comes from Proposition 7.10, equivalence (7) follows from Corollary 7.9, and equivalence (8) is Proposition 7.6. □

7.3 Filtered homotopy colimits

Recall from [Tab08, § 10] the construction of the *universal localizing invariant*

$$\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}.$$

THEOREM 7.13 [Tab08, Theorem 10.5]. *The morphism $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}$ has the following properties.*

- (flt) *It commutes with filtered homotopy colimits.*
- (p) *It preserves the terminal object.*
- (loc) *It sends the exact sequences in Hmo (Definition 3.1) to distinguished triangles in $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$.*

Moreover, $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}$ is universal with respect to these properties, i.e. for every strong triangulated derivator $\mathbb{D}$, we have an equivalence of categories

$$(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}})^* : \underline{\mathsf{Hom}}_!(\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}^{\mathsf{loc}}_{\mathsf{dg}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D}),$$

where the right-hand side consists of the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ of morphisms of derivators which verify the above three conditions.

Notation 7.14. The objects of $\underline{\mathsf{Hom}}^{\mathsf{loc}}_{\mathsf{dg}}(\mathsf{HO}(\mathsf{dgcat}), \mathbb{D})$ will be called *localizing invariants*.

Note that $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} = \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\aleph_0}$, so that Theorem 7.13 is a particular case of Theorem 5.9 with $\alpha = \aleph_0$. We will consider now a fixed infinite regular cardinal $\alpha \geqslant \aleph_1$. Thanks to Theorem 5.9, we have a unique morphism of derivators (which preserves all homotopy colimits),

$$t_! : \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \longrightarrow \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}},$$

such that $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} = t_! \circ \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}$.

PROPOSITION 7.15. *The morphism $t_! : \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha} \to \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ describes the derivator $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ as the left Bousfield localization of the derivator $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$ by all maps of shape*

$$\mathrm{hocolim}_j \, \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{loc}}_{\alpha}(\mathcal{A}), \tag{9}$$

where $\mathcal{A}$ is a dg category, J is a filtered category, and $D : J \to \mathsf{dgcat}$ is a functor such that $\mathrm{hocolim}\, D_j \simeq \mathcal{A}$ *in Hmo. Moreover, the morphism $t_!$ has a fully-faithful right adjoint.*

Proof. The first assertion follows directly from Theorem 7.13 and from the universal property of left Bousfield localizations of derivators; see Definition A.4.

In order to prove the second assertion, we start by replacing $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ (respectively $\underline{\mathrm{Mot}}^{\mathsf{loc}}_{\alpha}$) by $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ (respectively by $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$). We now describe the derivator $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ as the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ by an explicit small set of maps, namely, the set T of maps of shape

$$\mathrm{hocolim}_j \, \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{add}}_{\alpha}(\mathcal{A}), \tag{10}$$

where $\mathcal{A}$ is an α-small dg cell, J is a α-small filtered category, and $D : J \to \mathsf{dgcat}$ is a functor with values in homotopically finitely presented dg categories, such that $\mathrm{hocolim}\, D_j \simeq \mathcal{A}$ in Hec.

Notice that, by construction of $\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}$ (see Theorem 3.2), for any triangulated strong derivator $\mathbb{D}$ there is a canonical equivalence of categories between $\underline{\mathsf{Hom}}_!(\underline{\mathrm{Mot}}^{\mathsf{add}}_{\alpha}, \mathbb{D})$ and the full subcategory of $\underline{\mathsf{Hom}}(\mathsf{dgcat}_{\alpha}, \mathbb{D}) = \mathbb{D}(\mathsf{dgcat}_{\alpha}^{\mathsf{op}})$ which consists of morphisms $\mathsf{dgcat}_{\alpha} \to \mathbb{D}$ which

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preserve the terminal object, send derived Morita equivalences to isomorphisms, and send split exact sequences of α-small dg cells to split distinguished triangles in $\mathbb{D}$.

The map $\mathsf{dgcat}_\alpha \to \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ corresponding to the identity of $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ can be restricted to finite dg cells (i.e. $\aleph_0$-small dg cells), hence defines a canonical map $\mathsf{dgcat}_f \to \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ which preserves the terminal object, sends derived Morita equivalences to isomorphisms, and sends split exact sequences of finite dg cells to direct sums. In other words, it defines a canonical homotopy colimit preserving map $\varphi : \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0} \to \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$. Let $\mathrm{loc}_T : \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha \to \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$ be the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha$ by T. Note that any α-small dg cell is a filtered union of its finite sub dg cells. In other words, for any α-small dg cell $\mathcal{A}$, there exists an α-small filtered category J, a functor $D : J \to \mathsf{dgcat}$ with values in homotopically finitely presented dg categories, such that hocolim $D_j \simeq \mathcal{A}$ in Hec. Therefore, by definition of loc_T, we get the following essentially commutative diagram of derivators,

$$\begin{array}{ccccccc}
\mathsf{dgcat}_\alpha & \xrightarrow{\underline{\mathcal{U}}^{\mathsf{add}}_\alpha} & \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha & & & \xrightarrow{\mathrm{loc}_T} & \\
\downarrow & & & & & & \searrow \\
\mathsf{dgcat} & \xrightarrow[\underline{\mathcal{U}}^{\mathsf{add}}_{\aleph_0}]{} & \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0} & \xrightarrow[\varphi]{} & \underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha & \xrightarrow[\mathrm{loc}_T]{} & L_T\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha
\end{array}$$

from which we deduce that the morphism $\mathrm{loc}_T \circ \underline{\mathcal{U}}^{\mathsf{add}}_\alpha$ preserves filtered homotopy colimits. This implies that the canonical map $L_T\underline{\mathrm{Mot}}^{\mathsf{add}}_\alpha \to \underline{\mathrm{Mot}}^{\mathsf{add}}_{\aleph_0}$, obtained from Theorem 3.2 and from the universal property of the left Bousfield localization by T, is an equivalence of derivators. This in turns implies that $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}} = \underline{\mathrm{Mot}}^{\mathsf{loc}}_{\aleph_0}$ is the left Bousfield localization of $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$ by the set of maps of shape

$$\mathrm{hocolim}_j\, \underline{\mathcal{U}}^{\mathsf{loc}}_\alpha(D_j) \longrightarrow \underline{\mathcal{U}}^{\mathsf{loc}}_\alpha(\mathcal{A}), \tag{11}$$

where $\mathcal{A}$ is an α-small dg cell, J is a α-small filtered category, and $D : J \to \mathsf{dgcat}$ is a functor with values in homotopically finitely presented dg categories, such that hocolim $D_j \simeq \mathcal{A}$ in Hmo. In particular, $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ is obtained from $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha$ as a left Bousfield localization by a small set of maps. Therefore it comes from a left Bousfield localization at the level of the underlying model categories (see § A.4). Hence the canonical map $\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha \to \mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$ has a fully-faithful right adjoint.[1] □

We obtain then an adjunction as follows.

$$\begin{array}{c}
\underline{\mathrm{Mot}}^{\mathsf{loc}}_\alpha \\
t_! \downarrow \;\; \uparrow t^* \\
\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}
\end{array}$$

7.4 Co-representability in $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}$

Theorem 7.16. *The object $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})$ is compact (Definition 1.2) in the triangulated category $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ and for every dg category $\mathcal{A}$, we have a natural isomorphism in the stable homotopy*

[1] The existence of this fully-faithful right adjoint can be obtained directly from the general theory of left Bousfield localizations and from the fact that any homotopy colimit preserving morphism of derivators which are obtained from combinatorial model categories has a right adjoint; see [Ren09]. However, our proof is more explicit.

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category of spectra (see Notation 2.29)

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A}).$$

Proof. The proof consists on verifying the conditions of Proposition 5.5, with $\mathbb{D} = \underline{\mathrm{Mot}}_{\alpha}^{\mathsf{loc}}$, S the set of maps of shape (11), $X = \underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\underline{k})$ and $M = \underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\mathcal{A})$. Since by Remark 7.11, the object $\underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\underline{k})$ is compact in the triangulated category $\underline{\mathrm{Mot}}_{\alpha}^{\mathsf{loc}}(e)$, it is enough to show that the functor

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\underline{k}), -) : \underline{\mathrm{Mot}}_{\alpha}^{\mathsf{loc}}(e) \longrightarrow \mathsf{Ho}(\mathsf{Spt})$$

sends the elements of S to isomorphisms. This follows from the fact that non-connective K-theory preserves filtered homotopy colimits (see [Sch06, § 7, Lemma 6]) and that, by virtue of Theorem 7.12, we have a natural isomorphism in the stable homotopy category of spectra

$$\mathbb{R}\mathsf{Hom}(\underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\underline{k}), \underline{\mathcal{U}}_{\alpha}^{\mathsf{loc}}(\mathcal{A})) \simeq \mathbb{K}(\mathcal{A}).$$

Corollary 7.17. *We have natural isomorphisms of abelian groups*

$$\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})[n], \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \simeq \mathbb{K}_n(\mathcal{A}), \quad n \in \mathbb{Z}.$$

8. Higher Chern characters

In this section we will show how our co-representability Theorem 7.16 furnishes us for free higher Chern characters and higher trace maps; see Theorem 8.4. Recall that, throughout the article, we have been working over a commutative base ring k. Consider a localizing invariant (7.14)

$$E : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt}),$$

with values in the triangulated derivator of spectra. Non-connective K-theory furnishes us also a localizing invariant:

$$\mathbb{K} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt}).$$

We will use Theorem 7.16 and its Corollary 7.17 to understand the natural transformations from $\mathbb{K}$ to E. Since E and $\mathbb{K}$ are localizing invariants, they descend by Theorem 7.13 to homotopy colimit preserving morphisms of derivators

$$E_{\mathsf{loc}}, \mathbb{K}_{\mathsf{loc}} : \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{loc}} \longrightarrow \mathsf{HO}(\mathsf{Spt}),$$

such that $E_{\mathsf{loc}} \circ \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} = E$ and $\mathbb{K}_{\mathsf{loc}} \circ \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}} = \mathbb{K}$.

Theorem 8.1. *There is a canonical bijection between the following sets.*

(i) *The set of 2-morphisms of derivators $\mathbb{K}_{\mathsf{loc}} \Rightarrow E_{\mathsf{loc}}$.*
(ii) *The set of 2-morphisms of derivators $\mathbb{K} \Rightarrow E$.*
(iii) *The set $E_0(\underline{k}) := \pi_0(E(\underline{k}))$ (see Definition 2.4).*

Proof. The bijection between sets (i) and (ii) comes directly from Theorem 7.13. By virtue of Theorem 7.16, we have

$$\mathbb{K}(\mathcal{A}) \simeq \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})).$$

Therefore, as E_{loc} is a functor enriched in spectra (see A.3), we have a natural map

$$\mathbb{K}(\mathcal{A}) = \mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k}), \mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A})) \longrightarrow \mathbb{R}\mathsf{Hom}(E_{\mathsf{loc}}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})), E_{\mathsf{loc}}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A}))).$$

However, by definition of E_{loc} we also have

$$\mathbb{R}\mathsf{Hom}(E_{\mathsf{loc}}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\underline{k})), E_{\mathsf{loc}}(\mathcal{U}_{\mathsf{dg}}^{\mathsf{loc}}(\mathcal{A}))) \simeq \mathbb{R}\mathsf{Hom}(E(\underline{k}), E(\mathcal{A})) \simeq E(\mathcal{A})^{E(\underline{k})}.$$

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In other words, we have a 2-morphism of derivators

$$\mathbb{K} \Rightarrow F = \mathbb{R}\mathsf{Hom}(E(\underline{k}), E(-)).$$

Any class $x \in E_0(\underline{k})$, seen as a map $x : S^0 \to E(\underline{k})$ in the stable homotopy category of spectra, obviously defines a 2-morphism of derivators $F \Rightarrow E$ (using the formula $E = \mathbb{R}\mathsf{Hom}(S^0, E(-))$), hence gives rise to a 2-morphism of derivators $c_x : \mathbb{K} \longrightarrow E$. On the other hand, given a natural transformation $c : \mathbb{K} \Rightarrow E$, we have in particular a map

$$c(k) : \mathbb{K}_0(k) = \pi_0\mathbb{K}(k) \longrightarrow \pi_0 E(\underline{k}) = E_0(\underline{k}),$$

which furnishes us an element $c(k)(1)$ in $E_0(\underline{k})$. One checks easily that the maps $c \mapsto c(k)(1)$ and $x \mapsto c_x$ are inverse to each other. □

Remark 8.2. As $\mathbb{K}_{\mathsf{loc}}$ and E_{loc} are enriched over spectra (see § A.3), there is a spectra $\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}})$ of natural transformations. Theorem 8.1 and the enriched Yoneda lemma leads to an enriched version of the preceding theorem (as it was suggested in its proof).

Corollary 8.3. *There is a natural isomorphism in the stable homotopy category of spectra*

$$\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}}) \simeq E(\underline{k}).$$

Proof. Theorem 8.1 implies that the natural map

$$\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}}) \longrightarrow \overline{E}(\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\underline{k})) = E(\underline{k})$$

induced by the unit $1 \in \mathbb{K}(k)$ leads to a bijection $\pi_0\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}}) \simeq \pi_0 E(\underline{k})$ since the set of 2-morphisms $\mathbb{K}_{\mathsf{loc}} \Rightarrow E_{\mathsf{loc}}$ identifies with $\pi_0\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}})$. By applying this to the morphism of derivators $E(-)[-n] : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$ (i.e. E composed with the loop functor iterated n times), one deduces that the following map is bijective for any integer n

$$\pi_n\mathbb{R}\mathsf{Nat}(\mathbb{K}_{\mathsf{loc}}, E_{\mathsf{loc}}) \longrightarrow \pi_n E(\underline{k}).$$ □

As an illustration, let us consider

$$\mathbb{K}_n(-), \quad HC_j(-), \quad THH_j(-) : \mathsf{Ho}(\mathsf{dgcat}) \longrightarrow \mathsf{Mod}\text{-}\mathbb{Z}, \quad n \in \mathbb{Z} j \geqslant 0,$$

to be respectively, the nth algebraic K-theory group functor (see [Sch06, § 12]), the jth cyclic homology group functor (see [Tab08, Theorem 10.7]), and the jth topological Hochschild homology group functor (see [Tab10, § 11]).

Theorem 8.4. *We have the following functorial morphisms of abelian groups:*

(i) *higher Chern characters*

$$\mathrm{ch}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow HC_{n+2r}(\mathcal{A}), \quad n \in \mathbb{Z}, r \geqslant 0,$$

such that $\mathrm{ch}_{0,r}$ *sends* $1 \in \mathbb{K}_0(k)$ *to a generator of the* k*-module of rank one* $HC_{2r}(k)$*;*

(ii) *when* $k = \mathbb{Z}$*, higher trace maps*

$$\mathrm{tr}_n : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_n(\mathcal{A}), \quad n \in \mathbb{Z},$$

such that tr_0 *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to* $1 \in THH_0(\mathbb{Z})$*, and*

$$\mathrm{tr}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r-1}(\mathcal{A}), \quad n \in \mathbb{Z}, r \geqslant 1,$$

such that $\mathrm{tr}_{0,r}$ *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to a generator in the cyclic group* $THH_{2r-1}(\mathbb{Z}) \simeq \mathbb{Z}/r\mathbb{Z}$*;*

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(iii) *when* $k = \mathbb{Z}/p\mathbb{Z}$, *with* p *a prime number, higher trace maps*

$$\mathrm{tr}_{n,r} : \mathbb{K}_n(\mathcal{A}) \longrightarrow THH_{n+2r}(\mathcal{A}), \quad n, r \in \mathbb{Z},$$

such that $\mathrm{tr}_{0,r}$ *sends* $1 \in \mathbb{K}_0(\mathbb{Z})$ *to a generator in the cyclic group* $THH_0(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.

Proof. (i) Recall from [Tab08, Theorem 10.7], that we have a *mixed complex* localizing invariant

$$C : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\Lambda\text{-}\mathsf{Mod}).$$

This defines a cyclic homology functor $HC : \mathsf{HO}(\mathsf{dgcat}) \to \mathcal{D}(k)$, where $HC(\mathcal{A}) = k \otimes^{\mathbb{L}}_{\Lambda} C(\mathcal{A})$, and where $\mathcal{D}(k)$ denotes the triangulated derivator associated to the model category of unbounded complexes of k-modules (localized by quasi-isomorphisms). We thus have the following formula (see Lemma A.12):

$$\begin{aligned} HC_j(\mathcal{A}) &= \mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j]) \\ &\simeq \mathsf{Hom}_{\mathcal{D}(k)}(S^0 \otimes k, HC(\mathcal{A})[-j]) \\ &\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(S^0, \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j])) \\ &\simeq \pi_0(\mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A})[-j])). \end{aligned}$$

Since $HC_*(k) \simeq k[u]$, with u of degree two, we conclude from Theorem 8.1 applied to $E = \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(-))[-2r]$, that for each integer $r \geqslant 0$, the element $u^r \in HC_{2r}(k)$ defines a natural map

$$\mathbb{K}(\mathcal{A}) \longrightarrow \mathbb{R}\mathsf{Hom}_{\mathcal{D}(k)}(k, HC(\mathcal{A}))[-2r].$$

(ii) By Blumberg–Mandell's localization theorem [BM08, Theorem 6.1] and the connection between dg and spectral categories developed in [Tab10], we have a localizing invariant

$$THH : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt}).$$

Thanks to Bökstedt [DGM, 0.2.3], we have the following calculation:

$$THH_j(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ \mathbb{Z}/r\mathbb{Z} & \text{if } j = 2r-1, r \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the canonical generators of $\mathbb{Z}$ and $\mathbb{Z}/r\mathbb{Z}$ furnishes us the higher trace maps by applying Theorem 8.1 to $E = THH(-)[-n]$ with $n = 0$, or with $n = 2r-1$ and $r \geqslant 1$.

(iii) The proof is the same as the preceding one: over the ring $\mathbb{Z}/p\mathbb{Z}$, we have the following calculation (see [DGM, 0.2.3]):

$$THH_j(\mathbb{Z}/p\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

Hence the canonical generator $\mathbb{Z}/p\mathbb{Z}$ furnishes us the higher trace maps by applying Theorem 8.1 to $E = THH(-)[-2r]$. □

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Appendix A. Grothendieck derivators

A.1 Notations

The original reference for derivators is Grothendieck's manuscript [Gro] and Heller's monograph [Hel88] on homotopy theories. See also [Cis03, Cis08, CN08, Mal01]. A derivator $\mathbb{D}$ consists of a strict contravariant 2-functor from the 2-category of small categories to the 2-category of categories

$$\mathbb{D} : \mathsf{Cat}^{\mathsf{op}} \longrightarrow \mathsf{CAT},$$

subject to certain conditions, the main ones being that for any functor between small categories $u : X \to Y$, the *inverse image functor*

$$u^* = \mathbb{D}(u) : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$$

has a left adjoint $u_!$, called the *homological direct image functor*, as well as right adjoint u_*, called the *cohomological direct image functor*. See [Cis03] for details. The essential example to keep in mind is the derivator $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ associated to a (complete and cocomplete) Quillen model category $\mathcal{M}$ and defined for every small category X by

$$\mathsf{HO}(\mathcal{M})(X) = \mathsf{Ho}(\mathsf{Fun}(X^{\mathsf{op}}, \mathcal{M})),$$

where $\mathsf{Fun}(X^{\mathsf{op}}, \mathcal{M})$ is the category of presheaves on X with values in $\mathcal{M}$, and $\mathsf{Ho}(\mathsf{Fun}(X^{\mathsf{op}}, \mathcal{M}))$ is its homotopy category; see [Cis03, Theorem 6.11]. We denote by e the 1-point category with one object and one (identity) morphism. Heuristically, the category $\mathbb{D}(e)$ is the basic 'derived' category under consideration in the derivator $\mathbb{D}$: we might think of $\mathbb{D}$ as a structure on $\mathbb{D}(e)$ which allows to speak of homotopy limits and colimits in $\mathbb{D}(e)$. For instance, if $\mathbb{D} = \mathsf{HO}(\mathcal{M})$ then $\mathbb{D}(e)$ is the usual homotopy category $\mathsf{Ho}(\mathcal{M})$ of $\mathcal{M}$.

Definition A.1. We now recall some technical properties of derivators.

(i) A derivator $\mathbb{D}$ is called *strong* if for every finite free category X and every small category Y, the natural functor $\mathbb{D}(X \times Y) \longrightarrow \mathsf{Fun}(X^{\mathsf{op}}, \mathbb{D}(Y))$ is full and essentially surjective. See [Hel97] for details.

(ii) A derivator $\mathbb{D}$ is called *regular* if in $\mathbb{D}$, sequential homotopy colimits commute with finite products and homotopy pullbacks. See also [Hel97] for details.

(iii) A derivator $\mathbb{D}$ is *pointed* if for any closed immersion $i : Z \to X$ in Cat the cohomological direct image functor $i_* : \mathbb{D}(Z) \longrightarrow \mathbb{D}(X)$ has a right adjoint, and if, moreover and dually, for any open immersion $j : U \to X$ the homological direct image functor $j_! : \mathbb{D}(U) \longrightarrow \mathbb{D}(X)$ has a left adjoint. See [CN08, 1.13] for details.

(iv) A derivator $\mathbb{D}$ is called *triangulated* or *stable* if it is pointed and if every global commutative square in $\mathbb{D}$ is cartesian exactly when it is cocartesian. See [CN08, 1.15] for details.

A strong derivator is the same thing as a small homotopy theory in the sense of Heller [Hel97]. By [Cis10, Proposition 2.15], if $\mathcal{M}$ is a Quillen model category, its associated derivator $\mathsf{HO}(\mathcal{M})$ is strong. Moreover, if sequential homotopy colimits commute with finite products and homotopy pullbacks in $\mathcal{M}$, the associated derivator $\mathsf{HO}(\mathcal{M})$ is regular. Notice also that if $\mathcal{M}$ is pointed, then so is $\mathsf{HO}(\mathcal{M})$. Finally, a pointed model category $\mathcal{M}$ is stable if and only if its associated derivator $\mathsf{HO}(\mathcal{M})$ is triangulated.

Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators. We denote by $\underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{D}')$ the category of all morphisms of derivators. Its morphisms will be called 2-morphisms of derivators. Finally we denote by $\underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{D}')$ the category of morphisms of derivators which commute with homotopy colimits; see [Cis03, Cis08]. For instance, any left (respectively right) Quillen functor induces a morphism of derivators which preserves homotopy colimits (respectively limits); see [Cis03, Proposition 6.12].

Notation A.2. If $\mathbb{D}$ is a derivator and if A is a small category, we denote by $\mathbb{D}_A$ the derivator defined by $\mathbb{D}_A(X) = \mathbb{D}(A \times X)$. One can think of $\mathbb{D}_A$ as the derivator of presheaves on A with values in $\mathbb{D}$.

Notation A.3. If $\mathbb{D}$ is a derivator, its *opposite* $\mathbb{D}^{\mathsf{op}}$ is defined by $\mathbb{D}^{\mathsf{op}}(A) = \mathbb{D}(A^{\mathsf{op}})^{\mathsf{op}}$. In the case $\mathbb{D} = \mathsf{HO}(\mathcal{M})$, we thus have $\mathbb{D}^{\mathsf{op}} = \mathsf{HO}(\mathcal{M}^{\mathsf{op}})$.

A.2 Left Bousfield localization

Let $\mathbb{D}$ be a derivator and S a class of morphisms in the base category $\mathbb{D}(e)$.

Definition A.4. The derivator $\mathbb{D}$ admits a *left Bousfield localization* with respect to S if there exists a morphism of derivators $\gamma : \mathbb{D} \to \mathsf{L}_S\mathbb{D}$, which commutes with homotopy colimits, sends the elements of S to isomorphisms in $\mathsf{L}_S\mathbb{D}(e)$ and satisfies the following universal property: for every derivator $\mathbb{D}'$ the morphism γ induces an equivalence of categories

$$\gamma^* : \underline{\mathsf{Hom}}_!(\mathsf{L}_S\mathbb{D}, \mathbb{D}') \xrightarrow{\sim} \underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}'),$$

where $\underline{\mathsf{Hom}}_{!,S}(\mathbb{D}, \mathbb{D}')$ denotes the category of morphisms of derivators which commute with homotopy colimits and send the elements of S to isomorphisms in $\mathbb{D}'(e)$.

Theorem A.5 [Tab08, Theorem 4.4]. *Let $\mathcal{M}$ be a left proper, cellular model category and $\mathsf{L}_S\mathcal{M}$ its left Bousfield localization [Hir03, Definition 4.1.1] with respect to a set of morphisms S in $\mathsf{Ho}(\mathcal{M})$, i.e. to perform the localization we choose in $\mathcal{M}$ a representative of each element of S. Then, the induced morphism of derivators $\mathsf{HO}(\mathcal{M}) \to \mathsf{HO}(\mathsf{L}_S\mathcal{M})$ is a left Bousfield localization of derivators with respect to S. In this situation, we have moreover a natural adjunction of derivators.*

$$\begin{array}{c} \mathsf{HO}(\mathcal{M}) \\ \uparrow\downarrow \\ \mathsf{HO}(\mathsf{L}_S\mathcal{M}) \end{array}$$

Lemma A.6 [Tab08, Lemma 4.3]. *The Bousfield localization $\mathsf{L}_S\mathbb{D}$ of a* triangulated *derivator $\mathbb{D}$ remains triangulated as long as S is stable under the desuspension functor $[-1]$.*

A.3 Stabilization and spectral enrichment

Let $\mathbb{D}$ be a regular pointed strong derivator. In [Hel97], Heller constructed the universal morphism $\mathsf{stab} : \mathbb{D} \to \mathsf{St}(\mathbb{D})$ to a triangulated strong derivator in the sense of the following theorem.

Theorem A.7 (Heller [Hel97]). *Let $\mathbb{T}$ be a triangulated strong derivator. Then the morphism* stab *induces an equivalence of categories*

$$\mathsf{stab}^* : \underline{\mathsf{Hom}}_!(\mathsf{St}(\mathbb{D}), \mathbb{T}) \xrightarrow{\sim} \underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{T}).$$

The derivator $\mathsf{St}(\mathbb{D})$ is described as follows : let $\mathbf{V} = W^{\mathsf{op}}$, where W is the poset $\{(i,j) \mid |i-j| \leqslant 1\} \subset \mathbb{Z}\times\mathbb{Z}$ considered as a small category. A *spectrum* in $\mathbb{D}$ is an object X in $\mathbb{D}(\mathbf{V})$ such that $X_{i,j} \simeq 0$ for $i \neq j$. This defines a derivator $\mathsf{Spec}(\mathbb{D})$ (as a full subderivator of $\mathbb{D}_{\mathbf{V}}$). Spectra are diagrams of the following shape.

$$
\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \uparrow & & \\
 & & 0 & \longrightarrow & X_{i+1,i+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X_{i,i} & \longrightarrow & 0 & & \\
\uparrow & & \uparrow & & & & \\
\cdots \longrightarrow X_{i-1,i-1} & \longrightarrow & 0 & & & & \\
\uparrow & & & & & & \\
\vdots & & & & & &
\end{array}
$$

In particular, we get maps $X_{i,i} \to \Omega(X_{i+1,i+1})$, $i \in \mathbb{Z}$. The derivator $\mathsf{St}(\mathbb{D})$ is obtained as the full subderivator of $\mathsf{Spec}(\mathbb{D})$ which consists of Ω-spectra, i.e. spectra X such that the maps $X_{i,i} \to \Omega(X_{i+1,i+1})$ are all isomorphisms. There is a morphism $\Omega^\infty : \mathsf{Spec}(\mathbb{D}) \to \mathbb{D}$ of derivators, called the *infinite loop functor*, defined by $\Omega^\infty(X) := \operatorname{hocolim}_n \Omega^n(X_{n,n})$, where Ω^n denotes the loop space functor iterated n times. There is also a shift morphism

$$(-)\langle n\rangle : \mathsf{Spec}(\mathbb{D}) \longrightarrow \mathsf{Spec}(\mathbb{D})$$

defined by $(X\langle n\rangle)_{i,j} = X_{n+i,n+j}$ for $n \in \mathbb{Z}$. The derivator $\mathsf{St}(\mathbb{D})$ can be described as the left Bousfield localization of $\mathsf{Spec}(\mathbb{D})$ by the maps $X \to Y$ which induce isomorphisms $\Omega^\infty(X\langle n\rangle) \simeq \Omega^\infty(Y\langle n\rangle)$ for any integer n; see [Hel97]. Note that, for an Ω-spectrum X, we have a canonical isomorphism: $\Omega^\infty(X\langle n\rangle) \simeq X_{n,n}$.

Notation A.8. For a small category A, let Hot_A (respectively $\mathsf{Hot}_{\bullet,A}$) be the derivator associated to the projective model category structure on the category of simplicial presheaves (respectively of pointed simplicial presheaves) on A (this is compatible with the notations introduced in Notation A.2).

Remark A.9. The homotopy colimit preserving morphism

$$\mathsf{Hot}_A \longrightarrow \mathsf{Hot}_{\bullet,A}, \quad X \longmapsto X_+ = X \amalg *$$

is the universal one with target a pointed regular strong derivators; see [Cis08, Proposition 4.17].

Denote by Spt_A the stable Quillen model category of presheaves of spectra on A, endowed with the projective model structure. The infinite suspension functor defines a homotopy colimit preserving morphism $\Sigma^\infty : \mathsf{Hot}_{\bullet,A} \to \mathsf{HO}(\mathsf{Spt}_A)$ and as $\mathsf{HO}(\mathsf{Spt}_A)$ is a triangulated strong derivator, it induces a unique homotopy colimit preserving morphism $\mathsf{St}(\mathsf{Hot}_{\bullet,A}) \to \mathsf{HO}(\mathsf{Spt}_A)$ whose composition with $\mathsf{stab} : \mathsf{Hot}_{\bullet,A} \to \mathsf{St}(\mathsf{Hot}_A)$ is the infinite suspension functor. A particular case of [Tab07, Theorem 3.31] gives the following theorem.

Theorem A.10. *The canonical morphism $\mathsf{St}(\mathsf{Hot}_{\bullet,A}) \to \mathsf{HO}(\mathsf{Spt}_A)$ is an equivalence of derivators. As a consequence, the map $\Sigma^\infty : \mathsf{Hot}_{\bullet,A} \to \mathsf{HO}(\mathsf{Spt}_A)$ is the universal homotopy colimit preserving morphism from $\mathsf{Hot}_{\bullet,A}$ to a triangulated derivator.*

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The composition of the Yoneda embedding $A \to \mathsf{Hot}_A$ with the infinite suspension functor gives a canonical morphism $h: A \to \mathsf{HO}(\mathsf{Spt}_A)$. A combination of the preceding theorem, of Remark A.9, and of [Cis08, Corollary 4.19], leads to the following statement.

THEOREM A.11. *For any triangulated derivator $\mathbb{D}$, the functor*

$$h^* : \underline{\mathsf{Hom}}_!(\mathsf{HO}(\mathsf{Spt}_A), \mathbb{D}) \xrightarrow{\sim} \underline{\mathsf{Hom}}(A, \mathbb{D}) \simeq \mathbb{D}(A^{\mathsf{op}})$$

is an equivalence of categories.

Hence, given any object X in $\mathbb{D}(e)$, there is a unique homotopy colimit preserving morphism of triangulated derivators $\mathsf{HO}(\mathsf{Spt}) \longrightarrow \mathbb{D}$, $E \mapsto E \otimes X$, which sends the stable 0-sphere to X. This defines a canonical action of $\mathsf{HO}(\mathsf{Spt})$ on $\mathbb{D}$; see [Cis08, Theorem 5.22].

LEMMA A.12. *For any small category A, the functor*

$$\mathsf{Ho}(\mathsf{Spt}_A) = \mathsf{HO}(\mathsf{Spt})(A) \longrightarrow \mathbb{D}(A), \quad E \longmapsto E \otimes X$$

has a right adjoint

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -) : \mathbb{D}(A) \longrightarrow \mathsf{HO}(\mathsf{Spt})(A).$$

Proof. This follows from the Brown representability theorem applied to the compactly generated triangulated category $\mathsf{Ho}(\mathsf{Spt}_A)$; see [Nee01, Theorem 8.4.4]. □

Using Theorem A.11, one sees easily that the functors of Lemma A.12 define a morphism

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, -) : \mathbb{D} \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

which is right adjoint to the morphism $(-) \otimes X$. In particular, we have the formula

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y)) \simeq \mathsf{Hom}_{\mathbb{D}(e)}(E \otimes X, Y)$$

for any spectrum E and any objects X and Y in $\mathbb{D}(e)$. This enrichment in spectra is compatible with homotopy colimit preserving morphisms of triangulated derivators (see [Cis08, Theorem 5.22]): if $\Phi : \mathbb{D} \to \mathbb{D}'$ is a homotopy colimit preserving morphism of triangulated derivators, then for any spectrum E and any object X of $\mathbb{D}$, we have a canonical coherent isomorphism $E \otimes \Phi(X) \simeq \Phi(E \otimes X)$. As a consequence, if, moreover, Φ has a right adjoint Ψ, then we have canonical isomorphisms $\mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y) \simeq \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y))$ in the stable homotopy category of spectra. Indeed, to construct such an isomorphism, it is sufficient to construct a natural isomorphism of abelian groups, for any spectrum E:

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y)) \simeq \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y))).$$

Such an isomorphism is obtained by the following computations:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}'}(\Phi(X), Y)) &\simeq \mathsf{Hom}_{\mathbb{D}'}(E \otimes \Phi(X), Y) \\ &\simeq \mathsf{Hom}_{\mathbb{D}'}(\Phi(X \otimes E), Y) \\ &\simeq \mathsf{Hom}_{\mathbb{D}}(X \otimes E, \Psi(Y)) \\ &\simeq \mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, \Psi(Y))). \end{aligned}$$

Note finally that we can apply Theorem A.11 to $\mathbb{D}^{\mathsf{op}}$ (Notation A.3): for any object X of $\mathbb{D}$, there is a unique homotopy colimit preserving morphism

$$\mathsf{HO}(\mathsf{Spt}) \longrightarrow \mathbb{D}^{\mathsf{op}}, \quad E \longmapsto X^E$$

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such that $X^{S^0} = X$. We then have the formula

$$\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y^E) \simeq \mathbb{R}\mathsf{Hom}_{\mathsf{Ho}(\mathsf{Spt})}(E, \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(X, Y)) \simeq \mathbb{R}\mathsf{Hom}_{\mathbb{D}}(E \otimes X, Y).$$

A.4 The Milnor short exact sequence

Proposition A.13. *Let F be a strong triangulated derivator, and $F_\bullet$ an object of $\mathbb{D}(\mathbb{N}^{\mathsf{op}})$. We then have a distinguished triangle*

$$\bigoplus_n F_n \xrightarrow{\mathbf{1-s}} \bigoplus_n F_n \longrightarrow \operatorname{hocolim} F_\bullet \longrightarrow \bigoplus_n F_n[1]$$

where $\mathbf{s}$ is the morphism induced by the maps $F_n \to F_{n+1}$. As a consequence, we also have the Milnor short exact sequence

$$0 \to \lim_n^1 \mathsf{Hom}_{\mathbb{D}(e)}(F_n[1], A) \to \mathsf{Hom}_{\mathbb{D}(e)}(\operatorname{hocolim} F_\bullet, A) \to \lim_n \mathsf{Hom}_{\mathbb{D}(e)}(F_n, A) \to 0.$$

Proof. Given an object $F_\bullet$ in $\mathbb{D}(\mathbb{N}^{\mathsf{op}})$, we can consider the *weak homotopy colimit* of $F_\bullet$ (also called the sequential colimit of $F_\bullet$ in [HPS97, § 2.2])

$$F' = \mathsf{cone}\left(\mathbf{1} - \mathbf{s} : \bigoplus_n F_n \longrightarrow \bigoplus_n F_n\right).$$

For any object A of $\mathbb{D}(e)$, we then have a Milnor short exact sequence

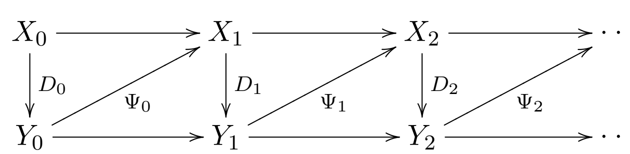
$$0 \longrightarrow \lim_n^1 \mathsf{Hom}_{\mathbb{D}(e)}(F_n[1], A) \longrightarrow \mathsf{Hom}_{\mathbb{D}(e)}(F', A) \longrightarrow \lim_n \mathsf{Hom}_{\mathbb{D}(e)}(F_n, A) \longrightarrow 0.$$

As the morphism $\mathbb{R}\mathsf{Hom}_{\mathbb{D}}(-, A)$ preserves homotopy limits, we deduce from [BK72, Theorem IX.3.1] that we also have a Milnor short exact sequence

$$0 \to \lim_n^1 \mathsf{Hom}_{\mathbb{D}(e)}(F_n[1], A) \to \mathsf{Hom}_{\mathbb{D}(e)}(\operatorname{hocolim}_n F_n, A) \to \lim_n \mathsf{Hom}_{\mathbb{D}(e)}(F_n, A) \to 0.$$

We deduce from this a (non-unique) isomorphism $\operatorname{hocolim}_n F_n \simeq F'$. □

Corollary A.14. *Let $\mathbb{D}$ a strong triangulated derivator, and let $D_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathbb{D}(\mathbb{N}^{\mathsf{op}})$. If there exists a family of interpolation maps $\{\Psi_n\}_{n\in\mathbb{N}}$, making the diagram*



commutative in $\mathbb{D}(e)$, then the induced morphism

$$\operatorname{hocolim} D_\bullet : \operatorname{hocolim}_n X_n \longrightarrow \operatorname{hocolim}_n Y_n$$

is an isomorphism.

Proof. This follows from the Proposition A.13 and from an easy cofinality argument; see [Nee01, Lemma 1.7.1]. □

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