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ON VARIABILITY OF SERIES AND PARALLEL SYSTEMS WITH HETEROGENEOUS COMPONENTS

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This paper studies the variability of both series and parallel systems comprised of heterogeneous (and dependent) components. Sufficient conditions are established for the star and dispersive orderings between the lifetimes of parallel [series] systems consisting of dependent components having multiple-outlier proportional hazard rates and Archimedean [Archimedean survival] copulas. We also prove that, without any restriction on the scale parameters, the lifetime of a parallel or series system with independent heterogeneous scaled components is larger than that with independent homogeneous scaled components in the sense of the convex transform order. These results generalize some corresponding ones in the literature to the case of dependent scenarios or general settings of components lifetime distributions.

Keywords: archimedean copula; convex transform order; parallel system; series system; star order

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1. INTRODUCTION

Order statistics play a crucial role in statistical inference, reliability theory, life testing, operations research, and many other areas. In reliability context, the k-th order statistic $X_{k:n}$ from random variables $X_1, \ldots, X_n$ corresponds to the lifetime of a $(n-k+1)$-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems. In particular, $X_{n:n}$ and $X_{1:n}$ represent the lifetimes of parallel and series systems, respectively, and $X_{2:n}$ characterizes the lifetime of a fail-safe system. For comprehensive discussions on various properties of order statistics and their applications, one may refer to Balakrishnan and Rao; David and Nagaraj [3,4,10].

As a powerful tool, stochastic ordering has been generally employed to compare the magnitude and variability of order statistics from heterogeneous and/or dependent random variables. Pledger and Proschan [32] pioneered comparisons on the order statistics arising from heterogeneous independent exponential variables. After that, many researchers have paid attention to this research direction and its generalization, to name a few, including Balakrishnan and Torrado; Cali, Longobardi, and Navarro; Di Crescenzo; Mesfioui, Kayid, and Izadkhah; Navarro and Spizzichino; Navarro, Torrado, and del Aguila; Proschan and Sethuraman; Zhang and Zhao; Zhang, Amini-Seresht, and Zhao; Zhang et al. [5,8,11,26,28, 29,33,37–40], and a comprehensive review article by Balakrishnan and Zhao [6].

Recall that a random variable X is said to belong to the proportional hazard rates (PHR) and scale families if it has the survival function $\overline{F}^{\lambda}(x)$ and $\overline{F}(\lambda x)$ with $\lambda > 0$, respectively, where the baseline distribution F is an absolutely continuous distribution function. The PHR and scale models play an important role in various fields of probability and statistics. For example, the PHR model, including exponential, Weibull, Pareto, and Lomax distributions as special cases, describes that the hazard rate functions of concerned components are proportional, while the scale model, for which the scale parameter acts to control the rate at which time passes, is termed "accelerated life" model in the context of life testing. For this reason, many researchers have attempted to find out sufficient conditions for comparing k-out-of-n systems with heterogeneous PHR or scaled components by various stochastic orders; see, for example, Cai, Zhang, and Zhao; Kochar and Xu [7,17].

Kochar and Xu [19] studied the effects of heterogeneity among hazard rate parameters on the skewness of the k-th order statistics arising from two sets of independent multiple-outlier exponential samples by means of the star order, which was partially strengthened by Amini-Seresht et al. [2] to the framework of independent multiple-outlier PHR models. For the case of dependent heterogeneous and homogeneous PHR random variables, Li and Fang [22] established sufficient conditions for the dispersive ordering between the largest order statistics. Under this setup, Fang, Li, and Li [14] also discussed the dispersiveness and skewness of the smallest order statistics. Very recently, Fang, Li, and Li [15] studied the dispersive ordering between minima from heterogeneous and homogeneous samples with scale proportional hazards and common Archimedean survival copulas. To the best of our knowledge, there is no related study on the variability of extreme order statistics arising from two sets of heterogeneous and dependent random variables. Motivated by this, we shall establish sufficient conditions for comparing the lifetimes of series and parallel systems consisting of dependent multiple-outlier PHR distributed components in the sense of the star and dispersive orderings, which partially extends some corresponding results established in Amini-Seresht et al. [2] to the dependent setting, and serves as a nice complement to Fang et al.; Li and Fang; Mesfioui et al. [14,15,22,26].

In the context of reliability theory, the convex transform order is called the more increasing failure rate (IFR) order, and it can be said that one random variable ages faster than the other one in some sense (see Section 2). Kochar and Xu [18] proved that the largest order statistics from independent heterogeneous exponential samples is larger than that from independent homogeneous exponential samples according to the convex transform order. Along this line, Kochar and Xu [20] established the star order between the largest order statistics from independent heterogeneous and homogeneous PHR samples. Da, Xu, and Balakrishnan [9] proved that the k-th order statistics from heterogeneous independent exponential random variables is always larger than that from homogeneous independent exponential random variables in the sense of the Lorenz order. Recently, Ding, Yang, and Ling [12] showed that the skewness of extreme order statistics from independent heterogeneous scale samples is larger than that from independent homogeneous scale samples in terms of the star ordering, which solved the open problem proposed by Kochar and Xu [20]. In this paper, we shall investigate the convex transform ordering between the lifetimes of series and parallel systems comprised of independent heterogeneous and homogeneous scaled components, which not only extends the results of Kochar and Xu [18] to the scale model but also strengthens the star ordering results of Ding et al. [12].

The rest of the paper is rolled out as follows. Section 2 collects some pertinent definitions and notions used in the sequel. Section 3 examines the skewness and dispersiveness of the lifetimes of series and parallel systems consisting of dependent multiple-outlier PHR distributed components. Section 4 establishes the convex transform ordering between two series or parallel systems having independent heterogeneous and homogeneous scaled components. Section 5 concludes the paper with some remarks.

2. PRELIMINARIES

Assume that nonnegative random variables X and Y have distribution functions F and G, survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, and density functions f and g, respectively. In the sequel, we use $\tilde{r}_F := f/F$ and $\tilde{r}_G := g/G$ to denote the reversed hazard rate functions of F and G, and F^{-1} and G^{-1} to denote the right continuous inverses of the distribution functions F and G, respectively. All random variables are assumed to be continuous and defined on $\mathbb{R}_+ = [0, +\infty)$. We use '$\overset{\text{sgn}}{=}$' to denote that both sides of the equality have the same sign and $\boldsymbol{I}_n$ to denote a n-dimensional real vector with all of its components equal to 1. The terms *increasing* and *decreasing* mean *non-decreasing* and *non-increasing*, respectively.

Definition 2.1: *X is said to be smaller than Y in the*

- (i) *usual stochastic order (denoted by $X \le_{\text{st}} Y$), if $\overline{F}(x) \le \overline{G}(x)$ for all $x \in \mathbb{R}_+$;*
- (ii) *convex transform order (denoted by $X \le_{\text{c}} Y$) if and only if $G^{-1}F(x)$ is convex in $x \in \mathbb{R}_+$, or equivalently, $X \le_{\text{c}} Y$ if and only if $F^{-1}G(x)$ is concave in $x \in \mathbb{R}_+$;*
- (iii) *dispersive order (denoted by $X \le_{\text{disp}} Y$) if $F^{-1}(v) - F^{-1}(u) \le G^{-1}(v) - G^{-1}(u)$ for all $0 \le u \le v \le 1$;*
- (iv) *star order (denoted by $X \le_{\star} Y$) if $G^{-1}F(x)/x$ is increasing in $x \in \mathbb{R}_+$;*
- (v) *Lorenz order (denoted by $X \le_{\text{Lorenz}} Y$) if $L_X(p) \ge L_Y(p)$ for all $p \in [0, 1]$, where the Lorenz curve L_X is defined as $L_X(p) = \int_0^p F^{-1}(u)\mathrm{d}u/\mu_X$, and $\mu_X = \mathbb{E}[X]$.*

If $X \le_{\text{c}} Y$ then Y is more skewed than X, as explained in Marshall and Olkin; Van Zwet [23,36]. The convex transform order is also called the more *IFR* order in reliability theory, since when f and g exist, the convexity of $G^{-1}F(x)$ means that

$$\frac{f(F^{-1}(u))}{g(G^{-1}(u))} = \frac{\tilde{r}_F(F^{-1}(u))}{\tilde{r}_G(G^{-1}(u))}$$

is increasing in $u \in [0, 1]$. Thus, $X \le_{\text{c}} Y$ can be interpreted that X ages faster than Y in some sense. It should be also mentioned that this partial order is scale invariant.

The star order is also called the more *IFRA* (increasing failure rate in average) order in reliability theory and is one of the partial orders which are scale invariant. It is known that

$$X \le_{\text{c}} Y \Longrightarrow X \le_{\star} Y \Longrightarrow X \le_{\text{Lorenz}} Y \Longrightarrow \gamma_X \le \gamma_Y,$$

where $\gamma_X = \sqrt{\text{Var}[X]}/\mathbb{E}[X]$ denotes the *coefficient of variation* of X. For more details on these stochastic orders, we refer interested reader to Shaked and Shanthikumar [35].

Let $x_{1:n} \le \ldots \le x_{n:n}$ be the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$. For two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$, $\boldsymbol{x}$ is said to *majorize* $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^n x_{i:n} = \sum_{i=1}^n y_{i:n}$ and $\sum_{i=1}^j x_{i:n} \le \sum_{i=1}^j y_{i:n}$ for $j = 1, \ldots, n-1$; $\boldsymbol{x}$ is said to *weakly majorize* $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\text{w}}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^j x_{i:n} \le \sum_{i=1}^j y_{i:n}$ for $j = 1, \ldots, n$.

It is known that the majorization order implies the weak majorization order, but the reverse is not true. The notion of majorization is quite useful in establishing various inequalities. For more details on their properties and applications, one may refer to Marshall and Olkin [24].

Archimedean copulas have been widely used in reliability theory, actuarial science, and many other areas due to its mathematical tractability and the capability of capturing wide ranges of dependence. By definition, for a decreasing and continuous function $\phi : [0, +\infty) \mapsto [0, 1]$ such that $\phi(0) = 1$ and $\phi(+\infty) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse,

$$C_\phi(u_1, \cdots, u_n) = \phi\left(\psi(u_1) + \cdots + \psi(u_n)\right), \quad \text{for all } u_i \in [0,1],\ i = 1, 2, \ldots, n,$$

is called an *Archimedean copula* with the generator ϕ if $(-1)^k \phi^{(k)}(x) \geq 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex. It is common knowledge that the Archimedean family contains a great many useful copulas, including the independence (product) copula, the Clayton copula, and the Ali–Mikhail–Haq (AMH) copula. According to Corollary 8.23(b) of Joe [16], copula C_ϕ is *positive lower orthant dependent* (PLOD) if $-\log \phi(t)$ is concave, and *negative lower orthant dependent* (NLOD) if $-\log\phi(t)$ is convex. For more discussions on copulas and their properties, one may refer to McNeil and Noslehova; Nelsen [25,30].

3. STAR AND DISPERSIVE ORDERS

In this section, we investigate the skewness and dispersiveness of parallel [series] systems consisting of dependent multiple-outlier PHR components whose lifetimes are assembled with Archimedean [survival] copula.

3.1. Parallel System

The following useful lemma, which is originally due to Saunders and Moran [34], is introduced to prove the main results.

Lemma 3.1: *Let $\{F_\lambda | \lambda \in \mathbb{R}_+\}$ be a class of distribution functions, such that F_λ is supported on some interval $(a, b) \subseteq (0, \infty)$ and has density f_λ which does not vanish on any subinterval of (a, b). Then, $F_\lambda \leq_\star F_{\lambda^*}$ for $\lambda \leq \lambda^*$, if and only if $((\partial F_\lambda(x))/(\partial\lambda))/(x f_\lambda(x))$ is decreasing in x, where $(\partial F_\lambda(x))/(\partial\lambda)$ is the partial derivative of F_λ with respect to λ.*

Now, we present a star ordering result for comparing the lifetimes of parallel systems with dependent multiple-outlier exponential distributed components.

Theorem 3.2: *Let $X_1, \ldots, X_n$ $[Y_1, \ldots, Y_n]$ be a set of exponential random variables having the Archimedean copula with generator ϕ, where X_i $[Y_i]$ has hazard rate λ_1 $[\mu_1]$ for $i = 1, \ldots, p$, and X_j $[Y_j]$ has hazard rate λ_2 $[\mu_2]$ for $j = p+1, \ldots, n$. Suppose that*

$$\left[1 - \frac{t\psi''(1-t)}{\psi'(1-t)}\right] \ln t \text{ is increasing in } t \in [0, 1].$$

If $(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) \geq 0$ and $(\lambda_{2:2})/(\lambda_{1:2}) \geq (\mu_{2:2})/(\mu_{1:2})$, we have $X_{n:n} \geq_\star Y_{n:n}$.

Proof: Without loss of generality, it is assumed that $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$. The distribution function of $X_{n:n}$ can be written as

$$F_{X_{n:n}}(x) = \phi[p\psi(1 - e^{-\lambda_1 x}) + q\psi(1 - e^{-\lambda_2 x})], \quad x \in \mathbb{R}_+,$$

where $q = n - p$. In this case, the proof can be completed by the following two parts.

Case 1: $\lambda_1+\lambda_2=\mu_1+\mu_2$. For ease of convenience, we denote $F_\lambda(x)=F_{X_{n:n}}(x)$ and assume that $\lambda_1+\lambda_2=\mu_1+\mu_2=1$. Let $\lambda_2=\lambda$ and $\mu_2=\mu$, where $\lambda,\mu\in[1/2,1)$. According to Lemma 3.1, we need to prove that $(\partial F_\lambda(x))/(\partial\lambda)/(xf_\lambda(x))$ is decreasing in $x\in\mathbb{R}_+$ for $\lambda\in[1/2,1)$, where

$$\begin{aligned}\frac{\partial F_\lambda(x)}{\partial\lambda}&=\phi'[p\psi(1-e^{-(1-\lambda)x})+q\psi(1-e^{-\lambda x})]\\&\quad\times[-pxe^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})+qxe^{-\lambda x}\psi'(1-e^{-\lambda x})]\end{aligned}$$

and

$$\begin{aligned}f_\lambda(x)&=\phi'[p\psi(1-e^{-(1-\lambda)x})+q\psi(1-e^{-\lambda x})]\\&\quad\times[p(1-\lambda)e^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})+q\lambda e^{-\lambda x}\psi'(1-e^{-\lambda x})].\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial F_\lambda(x)/\partial\lambda}{xf_\lambda(x)}&=\frac{-pe^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})+qe^{-\lambda x}\psi'(1-e^{-\lambda x})}{p(1-\lambda)e^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})+q\lambda e^{-\lambda x}\psi'(1-e^{-\lambda x})}\\&=\left(\lambda+\frac{pe^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})}{qe^{-\lambda x}\psi'(1-e^{-\lambda x})-pe^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})}\right)^{-1}\\&=\left[\lambda+\left(\frac{q}{p}\times\frac{e^{-\lambda x}\psi'(1-e^{-\lambda x})}{e^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})}-1\right)^{-1}\right]^{-1}.\end{aligned}$$

Thus, it suffices to show that, for $\lambda\in[1/2,1)$,

$$\Delta(x)=\frac{e^{-\lambda x}\psi'(1-e^{-\lambda x})}{e^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})}$$

is decreasing in $x\in\mathbb{R}_+$. Taking the derivative of $\Delta(x)$ with respective to x gives rise to

$$\begin{aligned}\Delta'(x)&\overset{\text{sgn}}{=}\left[\lambda xe^{-2\lambda x}\psi''(1-e^{-\lambda x})-\lambda xe^{-\lambda x}\psi'(1-e^{-\lambda x})\right]\times e^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})\\&\quad-\left[(1-\lambda)xe^{-2(1-\lambda)x}\psi''(1-e^{-(1-\lambda)x})-(1-\lambda)xe^{-(1-\lambda)x}\psi'(1-e^{-(1-\lambda)x})\right]\\&\quad\times e^{-\lambda x}\psi'(1-e^{-\lambda x})\\&\overset{\text{sgn}}{=}(1-\lambda)x\left[1-e^{-(1-\lambda)x}\frac{\psi''(1-e^{-(1-\lambda)x})}{\psi'(1-e^{-(1-\lambda)x})}\right]-\lambda x\left[1-e^{-\lambda x}\frac{\psi''(1-e^{-\lambda x})}{\psi'(1-e^{-\lambda x})}\right]\\&=\left[1-\frac{t_2\psi''(1-t_2)}{\psi'(1-t_2)}\right]\ln t_2-\left[1-\frac{t_1\psi''(1-t_1)}{\psi'(1-t_1)}\right]\ln t_1,\end{aligned}\qquad\textbf{(1)}$$

where $t_1=e^{-(1-\lambda)x}$ and $t_2=e^{-\lambda x}$. Since $\lambda\in[1/2,1)$, we have $t_1\geq t_2$ and this implies the right hand of (1) is non-positive based on the condition that

$$\left[1-\frac{t\psi''(1-t)}{\psi'(1-t)}\right]\ln t\ \text{ is increasing in } t\in[0,1].$$

Hence, the proof of this part is completed.

Case 2: $\lambda_1+\lambda_2 \neq \mu_1+\mu_2$. In this case, one can note that $\lambda_1+\lambda_2 = c(\mu_1+\mu_2)$, where c is a scalar. It then holds that $(\lambda_1,\lambda_2) \overset{\mathrm{m}}{\succeq} (c\mu_1, c\mu_2)$. Let $Z_{n:n}$ be the lifetime of a parallel system having n dependent exponential distributed components whose lifetimes are connected with an Archimedean copula with generator ϕ, where $Z_1,\ldots,Z_p$ have hazard rate $c\mu_1$, and $Z_{p+1},\ldots,Z_n$ have hazard rate $c\mu_2$. From the result of Case 1, we have $X_{n:n} \geq_\star Z_{n:n}$. On the other hand, since the star order is scale invariant, it follows that $X_{n:n} \geq_\star Y_{n:n}$. ■

For the case of dependent multiple-outlier PHR distributed components, the following star ordering result can be derived.

THEOREM 3.3: *Let $X_1,\ldots,X_n$ [$Y_1,\ldots,Y_n$] be a set of PHR variables having the Archimedean copula with generator ϕ, where X_i [Y_i] has survival function $\overline{F}^{\lambda_1}(x)$ [$\overline{F}^{\mu_1}(x)$] for $i=1,\ldots,p$, and X_j [Y_j] has survival function $\overline{F}^{\lambda_2}(x)$ [$\overline{F}^{\mu_2}(x)$] for $j=p+1,\ldots,n$. Let $R(x)=\int_0^x h(t)\mathrm{d}t$ be the cumulative hazard rate function of the baseline distribution $F(x)$. Suppose that $R(x)/xh(x)$ is increasing in $x\in\mathbb{R}_+$, and*

$$-t\psi'(1-t) \quad \text{and} \quad \left[1-\frac{t\psi''(1-t)}{\psi'(1-t)}\right]\ln t \quad \text{are both increasing in } t\in[0,1].$$

If $\lambda_1\leq\mu_1\leq\mu_2\leq\lambda_2$ and $(\lambda_1\boldsymbol{I}_p,\lambda_2\boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1\boldsymbol{I}_p,\mu_2\boldsymbol{I}_q)$, we have $X_{n:n}\geq_\star Y_{n:n}$.

PROOF: Under the conditions that $\lambda_1\leq\mu_1\leq\mu_2\leq\lambda_2$ and $(\lambda_1\boldsymbol{I}_p,\lambda_2\boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1\boldsymbol{I}_p,\mu_2\boldsymbol{I}_q)$, one can observe that $p\lambda_1+q\lambda_2\leq p\mu_1+q\mu_2$.

(i) Suppose $p\lambda_1+q\lambda_2 = p\mu_1+q\mu_2$. Without loss of generality, it is assumed that $p\lambda_1+q\lambda_2=1$. Let $\lambda_2=\lambda$ and $\lambda_1=(1-q\lambda)/p$. Upon using a similar proof method of Case 1 in Theorem 3.2, we know that, for $\lambda\in[1/(p+q),1/q)$,

$$\begin{aligned}\frac{\partial F_\lambda(x)/\partial\lambda}{xf_\lambda(x)} &= \frac{R(x)}{xh(x)}\times\frac{-qe^{-(1-q\lambda/p)R(x)}\psi'(1-e^{-(1-q\lambda/p)R(x)})+qe^{-\lambda R(x)}\psi'(1-e^{-\lambda R(x)})}{(1-q\lambda)e^{-(1-q\lambda/p)R(x)}\psi'(1-e^{-(1-q\lambda/p)R(x)})+q\lambda e^{-\lambda R(x)}\psi'(1-e^{-\lambda R(x)})}\\ &= \frac{R(x)}{xh(x)}\times\Omega(x).\end{aligned}$$

Since $\psi'(t)\leq 0$, $t\psi'(1-t)$ is decreasing in $t\in[0,1]$ and $(1-q\lambda)/p\leq\lambda$, we have $\Omega(x)\leq 0$. Based on the increasing property of $R(x)/(xh(x))$, we just need to prove that $\Omega(x)$ is decreasing in $x\in\mathbb{R}_+$. The proof can be completed in a similar way with that of Theorem 3.2.

(ii) Suppose $p\lambda_1+q\lambda_2 < p\mu_1+q\mu_2$. For this case, there must exist some λ_1' such that $\lambda_1\leq\lambda_1'<\mu_1$ and $p\lambda_1'+q\lambda_2=p\mu_1+q\mu_2$. Let $Z_1,\ldots,Z_n$ be a set of PHR random variables having the Archimedean copula with generator ϕ, where Z_i has survival function $\overline{F}^{\lambda_1'}(x)$ for $i=1,\ldots,p$, and Z_j have survival function $\overline{F}^{\lambda_2}(x)$ for $j=p+1,\ldots,n$. Then, from (i) we know that $Z_{n:n}\geq_\star Y_{n:n}$. On the other hand, we want to show that $X_{n:n}\geq_\star Z_{n:n}$. Let $d=\lambda_2-\lambda_1$ and $d'=\lambda_2-\lambda_1'$.

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Clearly, it follows that $d \geq d' > 0$. According to Lemma 3.1, it is enough to show that

$$\frac{\partial F_d(x)/\partial d}{x f_d(x)} = \frac{R(x)}{xh(x)} \times \frac{-pe^{-(\lambda_2-d)R(x)}\psi'(1-e^{-(\lambda_2-d)R(x)})}{p(\lambda_2-d)e^{-(\lambda_2-d)R(x)}\psi'(1-e^{-(\lambda_2-d)R(x)}) + q\lambda_2 e^{-\lambda_2 R(x)}\psi'(1-e^{-\lambda_2 R(x)})} = \frac{R(x)}{xh(x)} \times \Gamma(x)$$

is decreasing in $x \in \mathbb{R}_+$. From the analysis in (i), we need to prove that $\Gamma(x)$ is decreasing in $x \in \mathbb{R}_+$, which can be verified easily by following the same method of Theorem 3.2. To sum up, the desired result can be reached. ■

It should be mentioned here that the requirement imposed on the cumulative hazard rate function, that $R(x)/(xh(x))$ is increasing in $x \in \mathbb{R}_+$, is very general and satisfies many distributions; Amimi-Seresht et al.; Kochar and Xu see [2,20].

The next example provides three specified generators fulfilling the conditions in Theorems 3.2 and 3.3.

Example 3.4:

(i) For the independent case, the generator becomes $\phi(t) = e^{-t}$, $u \geq 0$. Then, we have $\psi(t) = -\ln t$, $t \in (0, 1]$. It can be calculated that

$$t\psi'(1-t) = \frac{t}{t-1}$$

is decreasing in $t \in [0, 1]$, and

$$\Phi_1(t) = \left[1 - \frac{t\psi''(1-t)}{\psi'(1-t)}\right] \ln t = \frac{\ln t}{1-t}, \quad t \in (0, 1).$$

Note that

$$\Phi_1'(t) \overset{\text{sgn}}{=} t\ln t - t + 1 = \Phi_2(t),$$

and $\Phi_2'(t) = \ln t \leq 0$ for $t \in (0, 1]$. Thus, we know that $\Phi_1'(t) \geq \Phi_1'(1) = 0$, which means that $\Phi_1(t)$ is increasing in $t \in (0, 1)$ as stated both in Theorems 3.2 and 3.3.

(ii) Consider the Clayton copula with generator $\phi(t) = (\theta t + 1)^{-1/\theta}$, where $\theta \in (0, 1]$. Through some simplifications, one can see that

$$\Upsilon_1(t) = t\psi'(1-t) = -t(1-t)^{-\theta-1}$$

and

$$\Upsilon_2(t) = \left[1 - \frac{t\psi''(1-t)}{\psi'(1-t)}\right] \ln t = \left(\frac{1+\theta t}{1-t}\right) \ln t.$$

Obviously, $-t\psi'(1-t)$ is increasing in $t \in [0, 1)$. For $\Upsilon_2(t)$, observe that

$$\Upsilon_2'(t) \overset{\text{sgn}}{=} (\theta+1)\ln t + (1+\theta t)\left(\frac{1}{t} - 1\right) = \kappa(t).$$

Taking the derivative of $\kappa(t)$ with respective to t gives rise to

$$\kappa'(t) \overset{\text{sgn}}{=} \frac{\theta+1}{t} + \theta\left(\frac{1}{t}-1\right) - \frac{\theta t+1}{t^2} \overset{\text{sgn}}{=} (t-1)(1-\theta t) \le 0,$$

which means that $\kappa(t)$ is decreasing in $t \in (0,1]$ for $\theta \in (0,1]$. Thus, $\kappa(t) \ge \kappa(1) = 0$, i.e., $\Upsilon_2'(t) \ge 0$ for all $t \in (0,1)$ and $\theta \in (0,1]$. Hence, the assumptions given in Theorems 3.2 and 3.3 are satisfied. Besides, it can be checked that $-\log\phi(t)$ is concave, which means that the lifetimes of the components are PLOD and exhibit positive dependence.

(iii) Consider the Gumbel–Hougaard copula with $\psi(t) = (-\ln t)^\theta$, $\theta > 1$. One can compute that

$$t\psi'(1-t) = -\frac{\theta t}{1-t}[-\ln(1-t)]^{\theta-1} =: -\theta\Lambda_1(t).$$

Note that

$$\begin{aligned}\Lambda_1'(t) &\overset{\text{sgn}}{=} (1-t)\left[(-\ln(1-t))^{\theta-1} + \frac{(\theta-1)t}{1-t}(-\ln(1-t))^{\theta-2}\right] + t(-\ln(1-t))^{\theta-1}\\ &\overset{\text{sgn}}{=} (1-t)\left[-\ln(1-t) + (\theta-1)\frac{t}{1-t}\right] - t\ln(1-t)\\ &= (\theta-1)t - \ln(1-t) \ge 0,\end{aligned}$$

which means that $t\psi'(1-t)$ is decreasing in $t \in [0,1)$. On the other hand, it can be calculated that

$$\Lambda_2(t) := \left[1 - \frac{t\psi''(1-t)}{\psi'(1-t)}\right]\ln t = \frac{[\ln(1-t) - (\theta-1)t]\ln t}{(1-t)\ln(1-t)}.$$

Figure 1 plots the function $\Lambda_2(t)$ with respect to $t \in (0,1)$ for different values of $\theta = 1.5, 2, 3, 4, 5, 6$, from which we can see that $\Lambda_2(t)$ is always increasing in $t \in (0,1)$. Moreover, one can verify that $-\log\phi(t)$ is concave, which implies that the lifetimes of the components are also PLOD.

Next, we present one numerical example to illustrate Theorem 3.3. Let X be a Weibull random variable, denoted by $X \sim W(a,b)$, with survival function $\overline{F}(x;a,b) = e^{-(bx)^a}$, where b and a are the scale and shape parameters, respectively. It is east to check that $R(x)/xh(x) = 1/a$ is increasing in x.

Example 3.5: Suppose that $X_i \sim W(a,b_i)$ and $Y_i \sim W(a,b_i^*)$, for $i = 1,2,3$. Let $a = 2$, $(b_1,b_2,b_3) = (2,2,5)$ and $(b_1^*,b_2^*,b_3^*) = (3,3,4)$. Note that $\lambda_i = b_i^a$, $\lambda_i^* = (b_i^*)^a$ and $(2^2,2^2,5^2) \overset{\text{w}}{\succeq} (3^2,3^2,4^2)$. According to Example 3.4, we take three kinds of Archimedean copulas with $\psi_1(t) = -\ln t$, $\psi_2(t) = 2(t^{-0.5}-1)$, and $\psi_3(t) = (-\ln t)^5$. Figure 2 displays the density functions of $X_{3:3}$ and $Y_{3:3}$ for these three specified situations, from which one can see that $X_{3:3}$ is always more skewed than $Y_{3:3}$, which means that, for a parallel system comprised of three Weibull distributed dependent components, more heterogeneity among these two types leads to larger skewness of the system lifetime distribution.

Since the Lorenz order plays an important role in many research areas, the following result is of independent interest and can be derived from Theorem 3.3.

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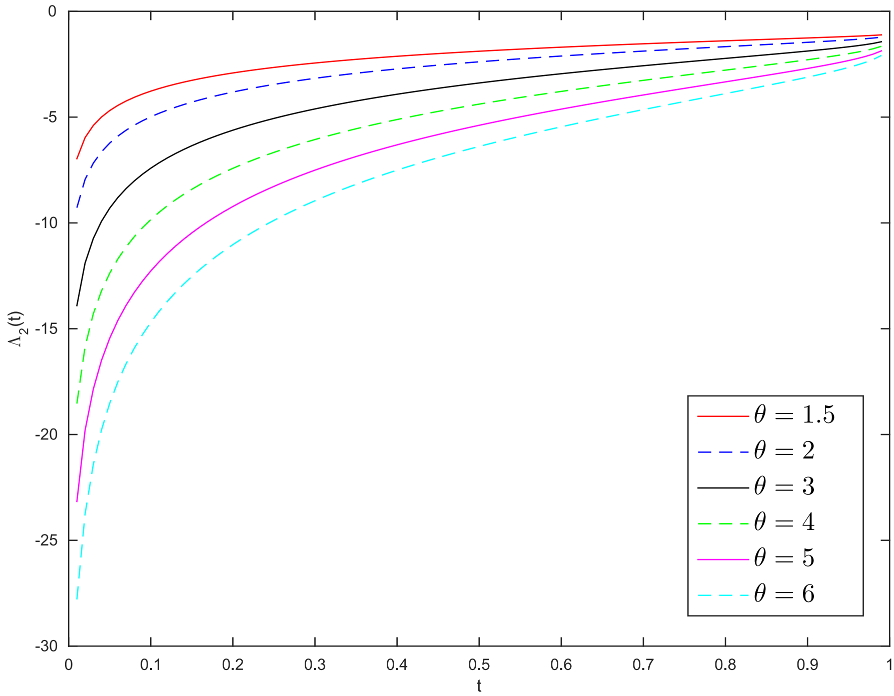

FIGURE 1. Plot of $\Lambda_2(t)$ on $t \in [0,1]$ for different values of θ.

COROLLARY 3.6: *Under the same setup of Theorem 3.3, if* $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$ *and* $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, *we have* $X_{n:n} \ge_{\mathrm{Lorenz}} Y_{n:n}$.

The following theorem studies the dispersiveness of the lifetime of a parallel system with dependent multiple-outlier PHR components.

THEOREM 3.7: *Under the same setup of Theorem 3.3, if* $\lambda_1 \le \mu_1 \le \mu_2 \le \lambda_2$ *and* $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, *we have* $X_{n:n} \ge_{\mathrm{disp}} Y_{n:n}$.

PROOF: Ahmed et al. [1] proved that, for two continuous random X and Y, if $X \le_\star Y$, then $X \le_{\mathrm{st}} Y$ implies that $X \le_{\mathrm{disp}} Y$. According to Theorem 4.1 of Li and Fang [22], it follows that $X_{n:n} \ge_{\mathrm{st}} Y_{n:n}$. Then, the desired result can be obtained by applying Theorem 3.3. ■

3.2. Series System

In this subsection, we study the skewness and dispersiveness of the lifetime of a series system with dependent multiple-outlier PHR components. First, we discuss the case of exponential lifetime distribution.

THEOREM 3.8: *Under the setup of Theorem 3.2, it is assumed that*

$$\left[1 + \frac{t\psi''(t)}{\psi'(t)}\right] \ln t \quad \textit{is decreasing in } t \in [0,1].$$

If $(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) \ge 0$ *and* $\lambda_{2:2}/\lambda_{1:2} \ge \mu_{2:2}/\mu_{1:2}$, *we have* $X_{1:n} \le_\star Y_{1:n}$.

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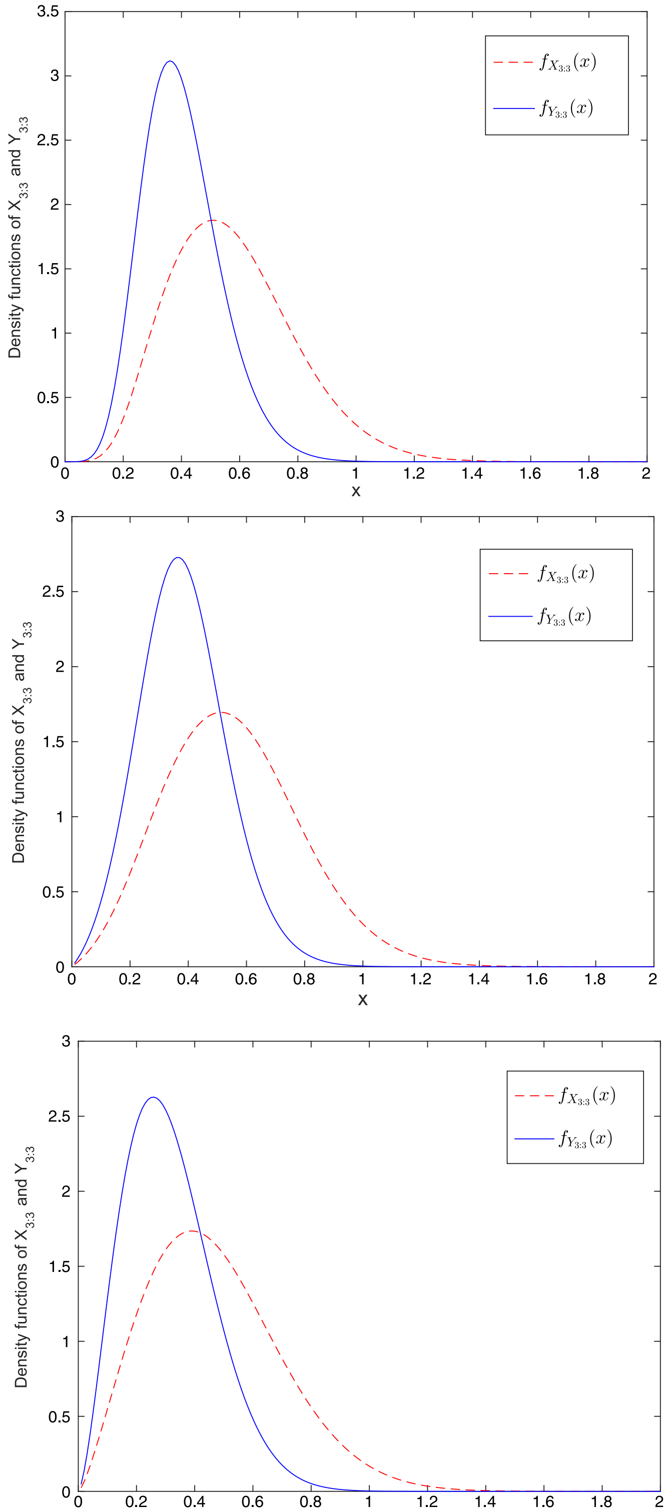

FIGURE **2.** Plot of the density functions of $X_{3:3}$ and $Y_{3:3}$ under different Archimedean copulas

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PROOF: Without loss of generality, we assume that $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$. The distribution function of $X_{n:n}$ is given by

$$F_{X_{1:n}}(x) = 1 - \phi[p\psi(e^{-\lambda_1 x}) + q\psi(e^{-\lambda_2 x})], \quad x \in \mathbb{R}_+,$$

where $q = n - p$. Similar to the proof of Theorem 3.2, we can proceed by considering the following two cases.

Case 1: $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. Without loss of generality, it is assumed that $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = 1$. Let $F_\lambda(x) = F_{X_{1:n}}(x)$, $\lambda_1 = \lambda$ and $\mu_1 = \mu$, where $\lambda, \mu \in (0, 1/2]$. From Lemma 3.1, it suffices to show that $(\partial F_\lambda(x)/\partial\lambda)/x f_\lambda(x)$ is decreasing in $x \in \mathbb{R}_+$ for $\lambda \in (0, 1/2]$, where

$$\begin{aligned}\frac{\partial F_\lambda(x)}{\partial\lambda} &= -\phi'[p\psi(e^{-\lambda x}) + q\psi(e^{-(1-\lambda)x})] \\ &\quad \times [-pxe^{-\lambda x}\psi'(e^{-\lambda x}) + qxe^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})]\end{aligned}$$

and

$$\begin{aligned}f_\lambda(x) &= -\phi'[p\psi(e^{-\lambda x}) + q\psi(e^{-(1-\lambda)x})] \\ &\quad \times [-p\lambda e^{-\lambda x}\psi'(e^{-\lambda x}) - q(1-\lambda)e^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})].\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\frac{\partial F_\lambda(x)}{\partial\lambda}}{x f_\lambda(x)} &= \frac{-pe^{-\lambda x}\psi'(e^{-\lambda x}) + qe^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})}{-p\lambda e^{-\lambda x}\psi'(e^{-\lambda x}) - q(1-\lambda)e^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})} \\ &= \left(\lambda - \frac{qe^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})}{qe^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x}) - pe^{-\lambda x}\psi'(e^{-\lambda x})}\right)^{-1} \\ &= \left[\lambda - \left(1 - \frac{p}{q} \times \frac{e^{-\lambda x}\psi'(e^{-\lambda x})}{e^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})}\right)^{-1}\right]^{-1}.\end{aligned}$$

Thus, it suffices to show that, for $\lambda \in (0, 1/2]$,

$$\Omega(x) = \frac{e^{-\lambda x}\psi'(e^{-\lambda x})}{e^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})}$$

is decreasing in $x \in \mathbb{R}_+$. By taking the derivative of $\Omega(x)$ with respective to x, we have

$$\begin{aligned}\Omega'(x) &\overset{\text{sgn}}{=} \left[(1-\lambda)xe^{-2(1-\lambda)x}\psi''(e^{-(1-\lambda)x}) + (1-\lambda)xe^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x})\right] \\ &\quad \times e^{-\lambda x}\psi'(e^{-\lambda x}) - \left[\lambda xe^{-2\lambda x}\psi''(e^{-\lambda x}) + \lambda xe^{-\lambda x}\psi'(e^{-\lambda x})\right] \times e^{-(1-\lambda)x}\psi'(e^{-(1-\lambda)x}) \\ &\overset{\text{sgn}}{=} (1-\lambda)x\left[1 + e^{-(1-\lambda)x}\frac{\psi''(e^{-(1-\lambda)x})}{\psi'(e^{-(1-\lambda)x})}\right] - \lambda x\left[1 + e^{-\lambda x}\frac{\psi''(e^{-\lambda x})}{\psi'(e^{-\lambda x})}\right] \\ &= \left[1 + \frac{t_1\psi''(t_1)}{\psi'(t_1)}\right]\ln t_1 - \left[1 + \frac{t_2\psi''(t_2)}{\psi'(t_2)}\right]\ln t_2, \end{aligned} \tag{2}$$

where $t_1 = e^{-\lambda x}$ and $t_2 = e^{-(1-\lambda)x}$. Since $\lambda \in (0, 1/2]$, we have $t_1 \geq t_2$. According to the decreasing property of $[1 + (t\psi''(t))/(\psi'(t))]\ln t$ in $t \in (0, 1]$, we know that (2) is non-negative.

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Case 2: $\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2$. The proof can be completed by adopting the proof method of Case 2 in Theorem 3.2. To sum up, the proof is finished. ∎

For the case of dependent multiple-outlier PHR components, we have the following comparison result for series systems according to the star order.

THEOREM 3.9: *Under the same setup of Theorem 3.3, it is assumed that $R(x)/xh(x)$ is increasing in $x \in \mathbb{R}_+$, and*

$$-t\psi'(t) \quad \textit{and} \quad \left[1 + \frac{t\psi''(t)}{\psi'(t)}\right] \ln t \quad \textit{are both decreasing in } t \in [0,1].$$

If $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ and $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, we have $X_{1:n} \leq_\star Y_{1:n}$.

PROOF: Based on the conditions that $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ and $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, we know that $p\lambda_1 + q\lambda_2 \leq p\mu_1 + q\mu_2$.

(i) If $p\lambda_1 + q\lambda_2 = p\mu_1 + q\mu_2$. Without loss of generality, it is assumed that $p\lambda_1 + q\lambda_2 = 1$. Let $\lambda_1 = \lambda$ and $\lambda_2 = (1 - p\lambda)/q$. Upon using a similar proof method of Case 1 in Theorem 3.8, we know that, for $\lambda \in (0, 1/(p+q)]$,

$$\begin{aligned}\frac{\partial F_\lambda(x)/\partial\lambda}{x f_\lambda(x)} &= \frac{R(x)}{xh(x)} \\ &\quad \times \frac{pe^{-\lambda R(x)}\psi'(e^{-\lambda R(x)}) - pe^{-(1-p\lambda/q)R(x)}\psi'(e^{-(1-p\lambda/q)R(x)})}{p\lambda e^{-\lambda R(x)}\psi'(e^{-\lambda R(x)}) + (1-p\lambda)e^{-(1-p\lambda/q)R(x)}\psi'(e^{-(1-p\lambda/q)R(x)})} \\ &= \frac{R(x)}{xh(x)} \times \Lambda(x).\end{aligned}$$

Since $\psi'(t) \leq 0$, $t\psi'(t)$ is increasing in $t \in [0,1]$ and $(1-p\lambda)/q \geq \lambda$, we have $\Lambda(x) \leq 0$. Based on the increasing property of $R(x)/xh(x)$, we just need to prove that $\Lambda(x)$ is decreasing in $x \in \mathbb{R}_+$. The proof can be completed in a similar way with that of Theorem 3.8.

(ii) If $p\lambda_1 + q\lambda_2 < p\mu_1 + q\mu_2$. There must exist some λ_1' such that $\lambda_1 \leq \lambda_1' < \mu_1$ and $p\lambda_1' + q\lambda_2 = p\mu_1 + q\mu_2$. Let $Z_1, \ldots, Z_n$ be a set of PHR random variables having the Archimedean copula with generator ϕ, where Z_i have survival function $\overline{F}^{\lambda_1'}(x)$ for $i = 1, \ldots, p$, and Z_j have survival function $\overline{F}^{\lambda_2}(x)$ for $j = p+1, \ldots, n$. Then, from (i) we know that $Z_{1:n} \leq_\star Y_{1:n}$. On the other hand, we want to show that $X_{1:n} \leq_\star Z_{1:n}$. Since $\lambda_1 \leq \lambda_1'$, according to Lemma 3.1 it is enough to show that

$$\begin{aligned}\frac{\partial F_{\lambda_1}(x)/\partial\lambda_1}{x f_{\lambda_1}(x)} &= \frac{R(x)}{xh(x)} \times \frac{-pe^{-\lambda_1 R(x)}\psi'(e^{-\lambda_1 R(x)})}{p\lambda_1 e^{-\lambda_1 R(x)}\psi'(e^{-\lambda_1 R(x)}) + q\lambda_2 e^{-\lambda_2 R(x)}\psi'(e^{-\lambda_2 R(x)})} \\ &= \frac{R(x)}{xh(x)} \times \Theta(x)\end{aligned}$$

is decreasing in $x \in \mathbb{R}_+$. According to (i), we need to prove that $\Theta(x)$ is decreasing in $x \in \mathbb{R}_+$, which can be ensured by conducting the same method of Theorem 3.8. To sum up, the proof is finished.

∎

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Corollary 3.10: *Under the same setup of Theorem 3.9, if* $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ *and* $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{w}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, *we have* $X_{1:n} \leq_{\mathrm{Lorenz}} Y_{1:n}$.

The following result is a direct consequence of Theorems 3.9 and 4.1(i) of Fang et al. [14], whose proof is similar to that of Theorem 3.7 by noting that log-convex of ϕ is equivalent to the decreasing property of $-t\psi'(t)$ and thus omitted here.

Theorem 3.11: *Under the same setup of Theorem 3.9, if* $\lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ *and* $(\lambda_1 \boldsymbol{I}_p, \lambda_2 \boldsymbol{I}_q) \overset{\mathrm{m}}{\succeq} (\mu_1 \boldsymbol{I}_p, \mu_2 \boldsymbol{I}_q)$, *we have* $X_{1:n} \leq_{\mathrm{disp}} Y_{1:n}$.

In Theorem 3.4 of Fang et al. [15], the dispersive ordering is studied between minima from heterogeneous and homogeneous samples with scale proportional hazards and common Archimedean survival copulas, while Theorem 3.11 here establishes sufficient conditions for the dispersive ordering between the minima arising from two sets of dependent multiple-outlier PHR samples with common Archimedean survival copulas. It is of natural interest to extend Theorem 3.11 to the case of multiple-outlier scale proportional hazards models.

The next example illustrates the assumptions on the generator in Theorem 3.9.

Example 3.12:

(i) For the independence case, $\psi(t) = -\ln t$ for $t \in (0, 1]$. Thus, we have $t\psi'(t) = -1$ and $[1 + (t\psi''(t)/\psi'(t))] \ln t = 0$, which satisfy the conditions in Theorem 3.9.

(ii) For the Clayton copula with generator $\phi(t) = (\theta t + 1)^{-1/\theta}$, where $\theta \in (0, 1]$, it can be calculated that

$$-t\psi'(t) = t^{-\theta} \quad \text{and} \quad \left[1 + \frac{t\psi''(t)}{\psi'(t)}\right] \ln t = -\theta \ln t,$$

which are both clearly decreasing in $t \in (0, 1]$.

(iii) For the Gumbel–Hougaard copula with $\psi(t) = (-\ln t)^{\theta}$, $\theta > 1$, it can be calculated that

$$-t\psi'(t) = \theta(-\ln t)^{\theta-1} \quad \text{and} \quad \left[1 + \frac{t\psi''(t)}{\psi'(t)}\right] \ln t = \theta - 1,$$

which agrees with the conditions given in Theorem 3.9.

The next example shows the effectiveness of Theorem 3.9.

Example 3.13: Under the setup of Example 3.5, we only consider the Archimedean copula with $\psi(t) = 2(u^{-0.5} - 1)$ since the function $[1 + \frac{t\psi''(t)}{\psi'(t)}] \ln t$ is constant as demonstrated in Example 3.12 for the other two cases. The density functions of $X_{1:3}$ and $Y_{1:3}$ are displayed in Figure 3, based on which it is not easy to conclude whether $X_{1:3}$ is more skewed than $Y_{1:3}$. Instead, we can compare their coefficients of variation. It can be computed that $\gamma_{X_{1:3}} = 0.556 < \gamma_{Y_{1:3}} = 0.599$, which supports the result of Theorem 3.9.

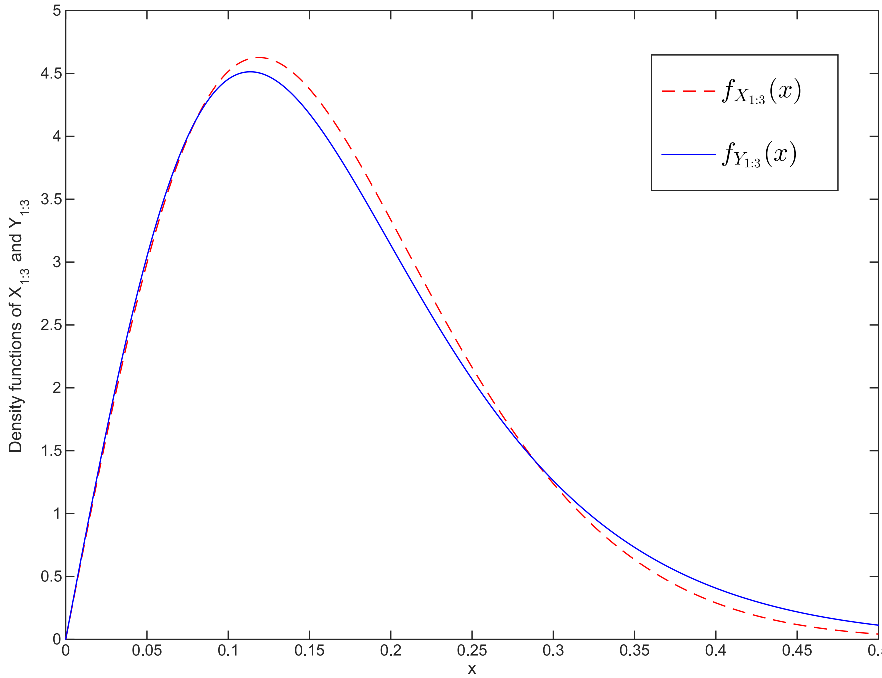

FIGURE 3. Plot of density functions of $X_{1:3}$ and $Y_{1:3}$ for $\psi(t) = 2(t^{-0.5} - 1)$.

4. CONVEX TRANSFORM ORDER

Kochar and Xu [18] showed that a parallel system with homogeneous exponential components ages faster than a system with heterogeneous exponential components in the sense of the more IFR property (convex transform order). Due to the unique memoryless property of the exponential distribution, exponential lifetime distributions imposed on the components do not conform to reality in general. As a unity study on the scale model, we shall give some sufficient conditions under which a parallel (series) system with homogeneous components ages faster than a system with heterogeneous components in the sense of the convex transform order.

To begin with, the well-known *Cauchy–Schwarz* inequality is introduced in the following to obtain our main result.

LEMMA 4.1 ([27]): *Let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be two sequences of real numbers. Then, it must hold that*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

with equality if and only if the sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are proportional, i.e., there is a constant λ such that $a_k = \lambda b_k$ for each $k \in \{1, 2, \dots, n\}$.

Now, the main result of this section is presented as follows.

THEOREM 4.2: *Let $X_1, \dots, X_n$ be independent random variables with X_i having distribution function $F(\lambda_i x)$, $i = 1, \dots, n$, and let $Y_1, \dots, Y_n$ be a independent random sample from a*

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distribution with the common distribution function $F(\lambda x)$*. Let* $\tilde{r}(x)$ *and* $h(x)$ *be the reversed hazard rate function and hazard rate function of* $F(x)$*, respectively.*

(i) *If* $\tilde{r}(x)$ *is decreasing in* $x \in \mathbb{R}_+$ *and* $(\tilde{r}''(x)\tilde{r}(x)/(\tilde{r}'(x))^2)$ *is decreasing in* $x \in \mathbb{R}_+$*, then* $X_{n:n} \geq_{\mathrm{c}} Y_{n:n}$*.*

(ii) *If* $h(x)$ *is decreasing in* $x \in \mathbb{R}_+$ *and* $(h''(x)h(x)/(h'(x))^2)$ *is decreasing in* $x \in \mathbb{R}_+$*, then* $X_{1:n} \geq_{\mathrm{c}} Y_{1:n}$*.*

Proof: (i) The distribution functions of $X_{n:n}$ and $Y_{n:n}$ can be written as, for $x \in \mathbb{R}_+$,

$$H_{n:n}(x) = \mathbb{P}(X_{n:n} \leq x) = \prod_{i=1}^{n} F(\lambda_i x) \quad \text{and} \quad G_{n:n}(x) = \mathbb{P}(Y_{n:n} \leq x) = F^n(\lambda x).$$

From Proposition 21.A.7 of Marshall and Olkin [23], it is sufficient to show that $G_{n:n}^{-1}H_{n:n}(x)$ is concave in $x \in \mathbb{R}_+$. Note that, for $x \in \mathbb{R}_+$,

$$G_{n:n}^{-1}H_{n:n}(x) = \frac{1}{\lambda}F^{-1}\left(\prod_{i=1}^{n} F^{\frac{1}{n}}(\lambda_i x)\right).$$

Taking the derivative with respective to x, we get

$$\begin{aligned}\left[G_{n:n}^{-1}H_{n:n}(x)\right]' &\propto \frac{\prod_{i=1}^{n} F^{\frac{1}{n}}(\lambda_i x) \times \frac{1}{n}\sum_{i=1}^{n}\lambda_i \tilde{r}(\lambda_i x)}{f\left[F^{-1}\left(\prod_{i=1}^{n} F^{\frac{1}{n}}(\lambda_i x)\right)\right]} \\ &= \frac{\frac{1}{n}\sum_{i=1}^{n}\lambda_i \tilde{r}(\lambda_i x)}{\tilde{r}\left[F^{-1}\left(\prod_{i=1}^{n} F^{\frac{1}{n}}(\lambda_i x)\right)\right]} \\ &=: \Delta(x).\end{aligned}$$

It boils down to showing that $\Delta(x)$ is decreasing in $x \in \mathbb{R}_+$. Note that

$$\begin{aligned}\Delta'(x) &\overset{\text{sgn}}{=} \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 \tilde{r}'(\lambda_i x) \times \tilde{r}\left[F^{-1}\left(\prod_{i=1}^{n} F^{1/n}(\lambda_i x)\right)\right] \\ &\quad - \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i \tilde{r}(\lambda_i x)\right)^2 \times \frac{\tilde{r}'\left[F^{-1}\left(\prod_{i=1}^{n} F^{1/n}(\lambda_i x)\right)\right]}{\tilde{r}\left[F^{-1}\left(\prod_{i=1}^{n} F^{1/n}(\lambda_i x)\right)\right]}.\end{aligned}$$

Thus, $\Delta'(x) \leq 0$ equals to show that

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 \tilde{r}'(\lambda_i x) \times \tilde{r}^2\left[F^{-1}\left(\prod_{i=1}^{n} F^{1/n}(\lambda_i x)\right)\right] \\ &\quad \leq \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i \tilde{r}(\lambda_i x)\right)^2 \times \tilde{r}'\left[F^{-1}\left(\prod_{i=1}^{n} F^{1/n}(\lambda_i x)\right)\right].\end{aligned} \quad \textbf{(3)}$$

Let $F(x) = e^{-S(x)}$, for $x \geq 0$, where $S(x) = \int_x^{\infty} \tilde{r}(u)\mathrm{d}u = -\log F(x)$ denoting the cumulative reversed hazard rate function. Thus, we have $F^{-1}(u) = S^{-1}(-\log u)$ for

$u \in (0,1)$. Based on the above relationship and the decreasing property of $\tilde{r}(x)$, inequality (3) can be rewritten as

$$\frac{1}{n}\sum_{i=1}^{n}\left[\left(S^{-1}(S(\lambda_i x))\right)^2 \tilde{r}'\left(S^{-1}(S(\lambda_i x))\right)\right] \times \tilde{r}^2\left[S^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} S(\lambda_i x)\right)\right]$$
$$\leq \left\{\frac{1}{n}\sum_{i=1}^{n}\left[S^{-1}(S(\lambda_i x))\tilde{r}\left(S^{-1}(S(\lambda_i x))\right)\right]\right\}^2 \times \tilde{r}'\left[S^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} S(\lambda_i x)\right)\right],$$

i.e.,

$$\frac{\sum_{i=1}^{n}\left[\left(S^{-1}(S(\lambda_i x))\right)^2 \tilde{r}'\left(S^{-1}(S(\lambda_i x))\right)\right] \times \tilde{r}^2\left[S^{-1}\left(1/n\sum_{i=1}^{n} S(\lambda_i x)\right)\right]}{\tilde{r}'\left[S^{-1}\left(1/n\sum_{i=1}^{n} S(\lambda_i x)\right)\right]}$$
$$\geq \frac{1}{n}\left\{\sum_{i=1}^{n}\left[S^{-1}(S(\lambda_i x))\tilde{r}\left(S^{-1}(S(\lambda_i x))\right)\right]\right\}^2. \quad \textbf{(4)}$$

Based on the the classical Cauchy–Schwarz inequality in Lemma 4.1, it can be seen that

$$\left\{\sum_{i=1}^{n}\left[\lambda_i x \tilde{r}\left(\lambda_i x\right)\right]\right\}^2 \leq \sum_{i=1}^{n}\left[\left(\lambda_i x\right)^2 \tilde{r}'\left(\lambda_i x\right)\right] \times \sum_{i=1}^{n}\frac{\tilde{r}^2\left(\lambda_i x\right)}{\tilde{r}'\left(\lambda_i x\right)}. \quad \textbf{(5)}$$

By making use of (5), (4) is equivalent to showing that

$$\frac{\tilde{r}^2\left[S^{-1}\left(1/n\sum_{i=1}^{n} S(\lambda_i x)\right)\right]}{\tilde{r}'\left[S^{-1}\left(1/n\sum_{i=1}^{n} S(\lambda_i x)\right)\right]} \leq \frac{1}{n}\sum_{i=1}^{n}\frac{\tilde{r}^2\left(S^{-1}(S(\lambda_i x))\right)}{\tilde{r}'\left(S^{-1}(S(\lambda_i x))\right)}.$$

Hence, it is enough to prove that $(\tilde{r}^2(S^{-1}(u)))/(\tilde{r}'(S^{-1}(u)))$ is convex in $u \in (0,1)$, i.e.,

$$\left[\frac{\tilde{r}^2\left(S^{-1}(u)\right)}{\tilde{r}'\left(S^{-1}(u)\right)}\right]' = \frac{\tilde{r}''\left(S^{-1}(u)\right)\tilde{r}\left(S^{-1}(u)\right)}{\left[\tilde{r}'\left(S^{-1}(u)\right)\right]^2} - 2$$

is increasing in $u \in (0,1)$, which can be obtained immediately from the assumption that $\tilde{r}''(x)\tilde{r}(x)/(\tilde{r}'(x))^2$ is decreasing in $x \in \mathbb{R}_+$ and the observation that $S^{-1}(u)$ is decreasing in $u \in (0,1)$.

(ii) Observe that $\overline{F}(x) = e^{-R(x)}$, for $x \in \mathbb{R}_+$ and $F^{-1}(u) = R^{-1}(-\log(1-u))$, for $u \in (0,1)$, where $R(x) = \int_0^x h(u)\mathrm{d}u$ denotes the cumulative hazard rate function. The proof is easily completed by adopting a similar proof method of (i).

■

Under the same setup of Theorem 4.2, the results will be reversed if the decreasing properties of the given functions are all replaced by the corresponding increasingness assumptions.

Kochar and Xu [18] proved that the maximum order statistics from a set of heterogeneous exponential variables is larger than that from a set of homogeneous exponential variables according to the convex transform order, which means that the lifetime of a parallel system with heterogeneous exponential components ages slower than that of any parallel system with i.i.d. exponential components. Suppose X is an exponential random variable

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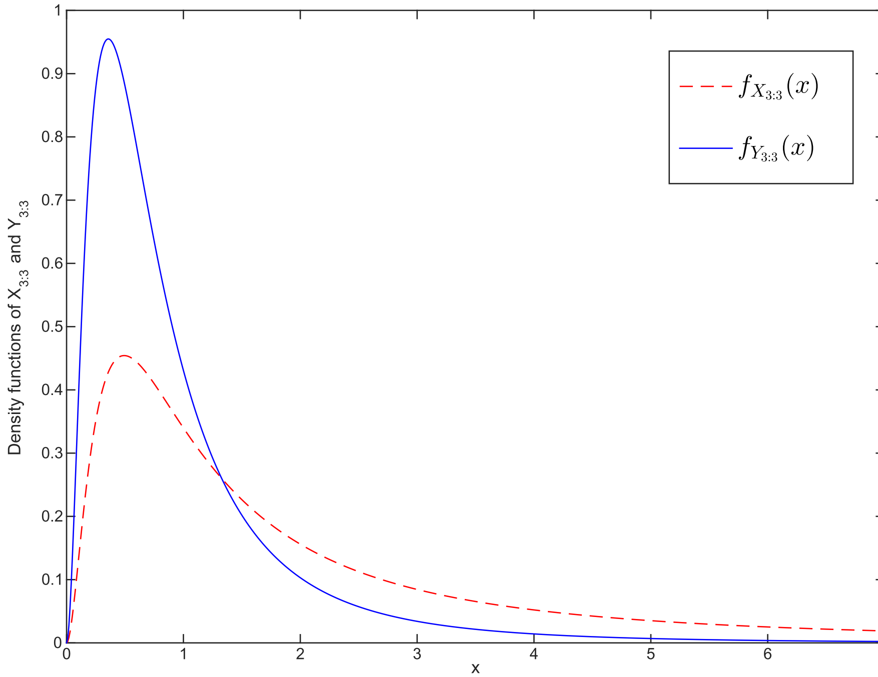

FIGURE 4. Plot of density functions of $X_{3:3}$ and $Y_{3:3}$.

with hazard rate $\lambda > 0$, then we know that $\tilde{r}(x) = (\lambda/(e^{\lambda x} - 1))$ is decreasing in $x \in \mathbb{R}_+$. It can be calculated that $(\tilde{r}''(x)\tilde{r}(x))/((\tilde{r}'(x))^2) = e^{-\lambda x} + 1$, which is also decreasing in $x \in \mathbb{R}_+$. Thus, the assumptions are satisfied in Theorem 4.2(i) and this in turn implies the result of Theorem 3.1 in Kochar and Xu [18].

The following example provides another explanation of Theorem 4.2.

Example 4.3: Take the baseline distribution as Burr distribution with $F(x) = 1 - (1+x)^{-\lambda}$ (denoted as $X \sim Burr(\lambda)$), $\lambda \geq 1$, $x \geq 0$. Through some calculation, we have

$$\tilde{r}(x) = \frac{\lambda}{(x+1)^{\lambda+1} - x - 1}, \quad \tilde{r}'(x) = -\frac{\lambda[(\lambda+1)(x+1)^{\lambda} - 1]}{[(x+1)^{\lambda+1} - x - 1]^2} \leq 0$$

and

$$\tilde{r}''(x) = \frac{\lambda(\lambda+1)(\lambda+2)(x+1)^{2\lambda} + \lambda(\lambda+1)(\lambda-4)(x+1)^{\lambda} + 2\lambda}{[(x+1)^{\lambda+1} - x - 1]^3}.$$

Then,

$$\Psi(x) := \frac{\tilde{r}''(x)\tilde{r}(x)}{(\tilde{r}'(x))^2} = \frac{(\lambda+1)(\lambda+2)(x+1)^{2\lambda} + (\lambda+1)(\lambda-4)(x+1)^{\lambda} + 2}{[(\lambda+1)(x+1)^{\lambda} - 1]^2}.$$

Observe that

$$\begin{aligned}
\Psi'(x) &\overset{\text{sgn}}{=} [2\lambda(\lambda+1)(\lambda+2)(x+1)^{2\lambda-1} + \lambda(\lambda+1)(\lambda-4)(x+1)^{\lambda-1}][(\lambda+1)(x+1)^{\lambda} - 1] \\
&\quad - 2\lambda(\lambda+1)(x+1)^{\lambda-1}[(\lambda+1)(\lambda+2)(x+1)^{2\lambda} + (\lambda+1)(\lambda-4)(x+1)^{\lambda} + 2] \\
&= -[\lambda(\lambda+1)^2(\lambda-4) + 2\lambda(\lambda+1)(\lambda+2)](x+1)^{2\lambda-1} - \lambda^2(\lambda+1)(x+1)^{\lambda-1} \\
&\overset{\text{sgn}}{=} -(\lambda-1)(x+1)^{\lambda} - 1 \leq 0,
\end{aligned}$$

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which means that $\Psi(x)$ is decreasing in $x \geq 0$ for $\lambda \geq 1$. Hence, the conditions in Theorem 4.2(i) are satisfied. On the other hand, it is easy to check that $h(x) = \lambda(1+x)^{-1}$ is decreasing in $x \in \mathbb{R}_+$, and $(h''(x)h(x))/((h'(x))^2) = 2$. Thus, the conditions in Theorem 4.2(ii) are also satisfied.

As an illustration, we suppose that $X_i \sim Burr(\lambda_i)$ and $Y_i \sim Burr(\lambda)$, for $i = 1, 2, 3$. Let $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$, and $\lambda = 3$. Figure 4 presents the figures of density functions of $X_{3:3}$ and $Y_{3:3}$. As observed, $X_{3:3}$ is more skewed than $Y_{3:3}$ and thus the result of Theorem 4.2(i) is validated.

5. CONCLUDING REMARKS

We study the effect of the heterogeneity among components on the variability of the lifetimes of series and parallel systems comprised of PHR or scaled components. Sufficient conditions are presented to compare the skewness and dispersiveness of the lifetimes of series and parallel systems with multiple-outlier PHR distributed components. It is also proved that, without any restriction on the scale parameters, the lifetime of a series or parallel system with heterogeneous scaled components is larger than that with homogeneous scaled components according to the convex transform order.

It should be mentioned that these results can be also applied in the field of auction theory to study the revenue of the first-price sealed-bid auction, which is a common type of auction in practice [c.f. 13,21,31]. Assume that all bidders submit their bids simultaneously and no bidder knows the bidding price of any other participant. The highest bidder pays the price he/she submitted, which can be characterized by the largest order statistics. To this regard, the results developed in this paper can provide insights into analyzing the effects of heterogeneity and dependence among bidders on the revenue of the first-price sealed-bid auction. For example, Theorem 4.2(i) suggests that, under appropriate conditions on the baseline distribution of the bidding prices, the revenue of the bidding group with heterogeneous bidding prices possesses more variation than that from a group of bidders having homogeneous bidding prices.

As a further study, it is of interest to generalize the results of Section 3 to the case where the components have scaled distributions or exhibit negatively dependent lifetimes (both the Clayton copula and the Gumbel–Hougaard copula merit PLOD as displayed in Example 3.4). Besides, it is worth studying the convex transform ordering for the series and parallel systems with heterogeneous and homogeneous PHR components.

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