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The Frobenius semiradical, generic stabilizers, and Poisson center for nilradicals

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Abstract. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{n}$ the nilradical of a parabolic subalgebra of $\mathfrak{g}$. We consider some properties of the coadjoint representation of $\mathfrak{n}$ and related algebras of invariants. This includes (i) the problem of existence of generic stabilizers, (ii) a description of the Frobenius semiradical of $\mathfrak{n}$ and the Poisson center $\mathcal{Z}(\mathfrak{n})$ of the symmetric algebra $\mathcal{S}(\mathfrak{n})$, (iii) the structure of $\mathcal{S}(\mathfrak{n})$ as $\mathcal{Z}(\mathfrak{n})$-module, and (iv) the description of square integrable (= quasi-reductive) nilradicals. Our main technical tools are the Kostant cascade in the set of positive roots of $\mathfrak{g}$ and the notion of optimization of $\mathfrak{n}$.

1 Introduction

1.1

Let G be a simple algebraic group with $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$ a fixed triangular decomposition, and $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ the fixed Borel subalgebra. Then Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$, Δ^+ is the set of positive roots corresponding to $\mathfrak{u}$, and θ is the highest root in Δ^+. Write U, T, B for the connected subgroups of G corresponding to $\mathfrak{u}, \mathfrak{t}, \mathfrak{b}$.

Let $P = L{\cdot}N$ be a parabolic subgroup of G, with the unipotent radical N and a Levi subgroup L. Then $\mathfrak{n} = \operatorname{Lie} N$ is the nilradical of $\mathfrak{p} = \operatorname{Lie} P$. The unipotent radicals of the parabolic subgroups provide an important class of non-reductive groups. For instance, N is a *Grosshans subgroup* of G [11, Theorem 16.4]. Various results on the coadjoint representation of $\mathfrak{n}$ can be found in [9, 10, 19, 22]. Our main goal is to elaborate on invariant-theoretic properties of the coadjoint representation $(N : \mathfrak{n}^*)$, but we also consider actions of some larger unipotent groups on $\mathfrak{n}^*$.

Without loss of generality, we may assume that $\mathfrak{p}$ is *standard*, i.e., $\mathfrak{p} \supset \mathfrak{b}$. Then $\mathfrak{n} \subset \mathfrak{u}$ is a sum of root spaces and $\Delta(\mathfrak{n})$ denotes the corresponding set of positive roots. Unless otherwise stated, "a nilradical" (in $\mathfrak{g}$) means "the nilradical of a standard parabolic subalgebra" of $\mathfrak{g}$. Let $\mathcal{K} = \{\beta_1, \dots, \beta_m\}$ be the *Kostant cascade* in Δ^+. It is a poset, and $\beta_1 = \theta$ is the unique maximal element of $\mathcal{K}$ (see Section 2.2 for details). To each $\mathfrak{n}$, we attach the subposet $\mathcal{K}(\mathfrak{n}) = \mathcal{K} \cap \Delta(\mathfrak{n})$. Another ingredient is the *optimization* of $\mathfrak{n}$. By definition, it is the maximal nilradical, $\tilde{\mathfrak{n}}$, such that $\mathcal{K}(\mathfrak{n}) = \mathcal{K}(\tilde{\mathfrak{n}})$. If $\mathfrak{n} = \tilde{\mathfrak{n}}$, then $\mathfrak{n}$ is said to be *optimal*. An explicit description of $\tilde{\mathfrak{n}}$ via $\mathcal{K}(\mathfrak{n})$ is given in Section 2.3.

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Properties of the optimal nilradicals are better, and in order to approach arbitrary nilradicals, it is convenient to consider first the optimal ones. Roughly speaking, the output of this article is that to a great extent invariant-theoretic properties of $\mathfrak{n}$ are determined by $\mathcal{K}(\mathfrak{n})$ and $\tilde{\mathfrak{n}}$.

1.2

Let $\mathfrak{q} = \operatorname{Lie} Q$ be a Lie algebra. As usual, $\xi \in \mathfrak{q}^*$ is said to be *regular*, if the stabilizer $\mathfrak{q}^\xi$ has minimal dimension. Then $\mathfrak{q}^*_{\text{reg}}$ denotes the set of all regular points and $\operatorname{ind}\mathfrak{q} := \dim \mathfrak{q}^\xi$ for any $\xi \in \mathfrak{q}^*_{\text{reg}}$. If $\operatorname{ind}\mathfrak{q} = 0$, then $\mathfrak{q}$ is called *Frobenius*. Set $\boldsymbol{b}(\mathfrak{q}) := (\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$. By definition, the *Frobenius semiradical* of $\mathfrak{q}$ is $\mathcal{F}(\mathfrak{q}) = \sum_{\xi\in\mathfrak{q}^*_{\text{reg}}} \mathfrak{q}^\xi$. Hence, $\mathcal{F}(\mathfrak{q}) = 0$ if and only if $\mathfrak{q}$ is Frobenius. Clearly, $\mathcal{F}(\mathfrak{q})$ is a characteristic ideal of $\mathfrak{q}$. This notion and basic results on it are due to Ooms [18, 19].

The symmetric algebra of $\mathfrak{q}$, $\mathcal{S}(\mathfrak{q})$, is a Poisson algebra equipped with the Lie–Poisson bracket $\{\ ,\ \}$. The algebra of symmetric invariants $\mathcal{S}(\mathfrak{q})^\mathfrak{q}$ is the center of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \})$, i.e.,

$$\mathcal{S}(\mathfrak{q})^\mathfrak{q} = \mathcal{Z}(\mathfrak{q}) = \{F \in \mathcal{S}(\mathfrak{q}) \mid \{F, x\} = 0 \ \ \forall x \in \mathfrak{q}\} = \{F \in \mathcal{S}(\mathfrak{q}) \mid \{F, P\} = 0 \ \ \forall P \in \mathcal{S}(\mathfrak{q})\}.$$

If the group Q is connected, then $\mathcal{S}(\mathfrak{q})^\mathfrak{q} = \mathbb{C}[\mathfrak{q}^*]^Q$, i.e., the Poisson center $\mathcal{Z}(\mathfrak{q})$ is also the algebra of Q-invariant polynomial functions on $\mathfrak{q}^*$. If $\mathcal{P} \subset \mathcal{S}(\mathfrak{q})$ is a Poisson-commutative subalgebra, then $\operatorname{trdeg}\mathcal{P} \leqslant \boldsymbol{b}(\mathfrak{q})$ [28, 0.2] and this upper bound is always attained [25]. Therefore, if $\mathfrak{r} \subset \mathfrak{q}$ is a Lie subalgebra, then $\boldsymbol{b}(\mathfrak{r}) \leqslant \boldsymbol{b}(\mathfrak{q})$. The passage $\mathfrak{n} \rightsquigarrow \tilde{\mathfrak{n}}$ has the property that $\boldsymbol{b}(\mathfrak{n}) = \boldsymbol{b}(\tilde{\mathfrak{n}})$. This implies that $\mathcal{Z}(\tilde{\mathfrak{n}}) = \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{\mathfrak{n}}} \subset \mathcal{S}(\mathfrak{n})$ (see [22, Proposition 5.5]).

An abelian subalgebra $\mathfrak{a} \subset \mathfrak{q}$ is called a *commutative polarization* (= CP), if $\dim\mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$. Then $\boldsymbol{b}(\mathfrak{a}) = \boldsymbol{b}(\mathfrak{q})$. A complete classification of the nilradicals with CP is obtained in [22]. In Section 3.4, we use Rosenlicht's theorem to provide simple proofs of some basic properties of CP's.

1.3

For $\mathfrak{n} = \mathfrak{p}^{\text{nil}}$, let $\mathfrak{n}^- \subset \mathfrak{u}^-$ be the opposite nilradical, i.e., $\Delta(\mathfrak{n}^-) = -\Delta(\mathfrak{n})$. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}^-$. Let $\mathfrak{g}_\gamma$ denote the roots space of $\gamma \in \Delta$ and $e_\gamma \in \mathfrak{g}_\gamma$ a nonzero vector. Consider the space $\mathfrak{k} = \bigoplus_{\beta\in\mathcal{K}(\mathfrak{n})} \mathfrak{g}_{-\beta} \subset \mathfrak{n}^-$ and $\boldsymbol{\zeta} = \sum_{\beta\in\mathcal{K}(\mathfrak{n})} e_{-\beta} \in \mathfrak{k}$. Using the vector space isomorphism $\mathfrak{n}^* = \mathfrak{g}/\mathfrak{p} \simeq \mathfrak{n}^-$, one can regard $\mathfrak{k}$ as a subspace of $\mathfrak{n}^*$ and $\boldsymbol{\zeta}$ as an element of $\mathfrak{n}^*$. We say that $\mathfrak{k} \subset \mathfrak{n}^*$ is a *cascade subspace* and $\boldsymbol{\zeta} \in \mathfrak{k}$ is a *cascade point*. As usual, $\xi \in \mathfrak{n}^*$ is called *N-generic*, if there is an open subset $\Omega \in \mathfrak{n}^*$ such that $\xi \in \Omega$ and the stabilizer $\mathfrak{n}^\xi$ is N-conjugate to $\mathfrak{n}^{\xi'}$ for any $\xi' \in \Omega$. Any stabilizer $\mathfrak{n}^\nu$ with $\nu \in \Omega$ is said to be *N-generic*, too.

In Section 4, we prove that the action $(N : \mathfrak{n}^*)$ has N-generic stabilizers if and only if the stabilizer $\mathfrak{n}^{\boldsymbol{\zeta}}$ is generic (and the latter is not always the case!). Moreover, N-generic stabilizers always exist if $\mathfrak{n}$ is optimal. If $\mathfrak{n}$ is not optimal, then one can consider the linear action of the larger group $\tilde{N} = \exp(\tilde{\mathfrak{n}}) \supset N$ on $\mathfrak{n}^*$. We prove that the action $(\tilde{N} : \mathfrak{n}^*)$ always has an $\tilde{N}$-generic stabilizer, and $\tilde{\mathfrak{n}}^{\boldsymbol{\zeta}}$ is such a stabilizer. Actually, the equality $\tilde{\mathfrak{n}}^{\boldsymbol{\zeta}} = \mathfrak{n}^{\boldsymbol{\zeta}}$ holds here. For any $\mathfrak{n}$, we give an explicit formula for $\mathfrak{n}^{\boldsymbol{\zeta}}$ via $\mathcal{K}(\mathfrak{n})$ and $\tilde{\mathfrak{n}}$, which shows that $\mathfrak{n}^{\boldsymbol{\zeta}}$ is T-stable. Using that formula and the criterion for the coadjoint representations [27, Corollary 1.8(i)], one easily verifies whether

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the stabilizer $\mathfrak{n}^\xi$ is generic in each concrete example. For $\mathbf{A}_n$, the nilradicals having a generic stabilizer for $(N : \mathfrak{n}^*)$ are explicitly described, while for $\mathbf{C}_n$, all nilradicals have a generic stabilizer (see Section 5). A general construction of nilradicals without generic stabilizers is also provided.

We prove that $\mathcal{F}(\mathfrak{n})$ is the $\mathfrak{b}$-stable ideal of $\mathfrak{n}$ generated by $\mathfrak{n}^\xi$ (regardless of the presence of generic stabilizers). Then our formula for $\mathfrak{n}^\xi$ allows us to explicitly describe $\mathcal{F}(\mathfrak{n})$ for $\mathbf{A}_n$ and $\mathbf{C}_n$. For any $\mathfrak{g}$, we provide a criterion for the equality $\mathcal{F}(\mathfrak{n}) = \mathfrak{n}$ and give the complete list of nilradicals with this property. Another observation is that if $\mathfrak{n} \subset \mathfrak{n}' \subset \tilde{\mathfrak{n}}$, then $\mathcal{F}(\mathfrak{n}') \subset \mathcal{F}(\mathfrak{n})$.

1.4

Since $\mathfrak{n}$ is B-stable, one can consider the algebra of Q-invariants $\mathcal{S}(\mathfrak{n})^Q = \mathbb{C}[\mathfrak{n}^*]^Q$ for any subgroup $Q \subset B$, and we are primarily interested in the unipotent subgroups U and $\tilde{N} = \exp(\tilde{\mathfrak{n}}) \subset U$. In Section 6, we prove that

$$\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{n})^{\tilde{N}} = \mathcal{S}(\tilde{\mathfrak{n}})^U = \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}} \tag{1.1}$$

and this common algebra is polynomial, of Krull dimension $\#\mathcal{K}(\mathfrak{n})$. In particular, for any optimal nilradical $\tilde{\mathfrak{n}}$, the Poisson center of $\mathcal{S}(\tilde{\mathfrak{n}})$ is a polynomial algebra. If $\mathfrak{n} \neq \tilde{\mathfrak{n}}$, then $\mathcal{S}(\mathfrak{n})^N$ does not occur in (1.1). This algebra is not always polynomial, and its Krull dimension equals $\#\mathcal{K}(\mathfrak{n}) + \dim(\tilde{\mathfrak{n}}/\mathfrak{n})$. Nevertheless, $\mathcal{S}(\mathfrak{n})^N$ shares many properties with algebras of invariants of reductive groups. For instance, $\mathcal{S}(\mathfrak{n})^N$ is finitely generated [12, Lemma 4.6], and we prove that the affine variety $\mathfrak{n}^* /\!\!/ N := \operatorname{Spec} \mathcal{S}(\mathfrak{n})^N$ has rational singularities.

Using results on $\mathcal{K}(\mathfrak{n})$ and $\mathfrak{n}^\xi$, we describe the nilradicals having the property that $\operatorname{ind}\mathfrak{n} = \dim \mathfrak{z}(\mathfrak{n})$, where $\mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$. By [15], this property is equivalent to that the Lie group N has a "square integrable representation." Therefore, such nilpotent Lie algebras are sometimes called "square integrable" (see [9, 10]). From a modern point of view, the square integrable nilradicals are precisely the "quasi-reductive" ones (see [1, 7, 16]). Our description shows that all square integrable nilradicals are metabelian.

As $\mathcal{S}(\mathfrak{n})^U$ is a polynomial algebra, we are interested in question whether $\mathcal{S}(\mathfrak{n})$ is a free $\mathcal{S}(\mathfrak{n})^U$-module. Equivalently, when is the quotient map $\pi : \mathfrak{n}^* \to \mathfrak{n}^* /\!\!/ U = \operatorname{Spec} \mathcal{S}(\mathfrak{n})^U$ equidimensional? We prove several assertions for U or (what is the same) $\tilde{N}$.

• If $\mathfrak{n}$ has a CP, then $\mathcal{S}(\mathfrak{n})$ is a free module over $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{n})^{\tilde{N}}$. This includes **all** nilradicals for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$.

• In particular, if $\mathfrak{n}$ is the optimization of a nilradical with CP (any $\mathfrak{g}$) or any optimal nilradical in $\mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$, then $\mathcal{S}(\mathfrak{n})$ is a free module over its Poisson center $\mathcal{Z}(\mathfrak{n})$. This also implies that in these cases, the enveloping algebra $\mathcal{U}(\mathfrak{n})$ is a free module over its center.

1.5 Structure of the article

In Section 2, we recall basic facts on $\mathcal{K}$ and (optimal) nilradicals. In Section 3, the necessary information is gathered on generic stabilizers, the Frobenius semiradical, and commutative polarizations. We also include invariant-theoretic proofs for some properties of commutative polarizations. Our results on generic stabilizers and $\mathcal{F}(\mathfrak{n})$ for a nilradical $\mathfrak{n} \subset \mathfrak{g}$ are gathered in Section 4, whereas Section 5 contains explicit results for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$. In Section 6, we study various algebras of invariants related to the coadjoint representation of $\mathfrak{n}$, and in Section 7, we classify the square integrable nilradicals. Topics related to the equidimensionality of the quotient map $\pi : \mathfrak{n}^* \to \mathfrak{n}^* /\!\!/ U$ are treated in Section 8. The lists of cascade roots for all $\mathfrak{g}$ and the Hasse diagrams of some posets $\mathcal{K}$ are presented in Appendix A.

Main notation. Throughout, $\mathfrak{g} = \mathrm{Lie}\,(G)$ is a simple Lie algebra. Then:

- $\mathfrak{b}$ is a fixed Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$.
- $\mathfrak{t}$ is a fixed Cartan subalgebra in $\mathfrak{b}$ and Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$.
- Δ^+ is the set positive roots corresponding to $\mathfrak{u}$, and $\theta \in \Delta^+$ is the highest root.
- $\Pi = \{\alpha_1, \dots, \alpha_{\mathrm{rk}\,\mathfrak{g}}\}$ is the set of simple roots in Δ^+.
- $\mathfrak{t}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector subspace of $\mathfrak{t}^*$ spanned by Δ, and $(\ ,\ )$ is the positive-definite form on $\mathfrak{t}^*_{\mathbb{Q}}$ induced by the Killing form on $\mathfrak{g}$.
- If $\gamma \in \Delta$, then $\mathfrak{g}_\gamma$ is the root space in $\mathfrak{g}$ and $e_\gamma \in \mathfrak{g}_\gamma$ is a nonzero vector.
- If $\mathfrak{c} \subset \mathfrak{u}^\pm$ is a $\mathfrak{t}$-stable subspace, then $\Delta(\mathfrak{c}) \subset \Delta^\pm$ is the set of roots of $\mathfrak{c}$.
- $\boldsymbol{b}(\mathfrak{q}) = (\dim \mathfrak{q} + \mathrm{ind}\,\mathfrak{q})/2$ for a Lie algebra $\mathfrak{q}$.
- In the explicit examples, the Vinberg–Onishchik numbering of simple roots of $\mathfrak{g}$ is used (see [17, Table 1]).

2 Generalities on the cascade and nilradicals

2.1 The root order in Δ^+ and the Heisenberg subset

We identify Π with the vertices of the Dynkin diagram of $\mathfrak{g}$. For any $\gamma \in \Delta^+$, let $[\gamma : \alpha]$ be the coefficient of $\alpha \in \Pi$ in the expression of γ via Π. The *support* of γ is $\mathrm{supp}(\gamma) = \{\alpha \in \Pi \mid [\gamma : \alpha] \neq 0\}$. As is well known, $\mathrm{supp}(\gamma)$ is a connected subset of the Dynkin diagram. For instance, $\mathrm{supp}(\theta) = \Pi$ and $\mathrm{supp}(\alpha) = \{\alpha\}$. Let "$\preccurlyeq$" denote the *root order* in Δ^+, i.e., we set $\gamma \preccurlyeq \gamma'$ if $[\gamma : \alpha] \leqslant [\gamma' : \alpha]$ for all $\alpha \in \Pi$. Then $(\Delta^+, \preccurlyeq)$ is a graded poset, and we write $\gamma \prec \gamma'$ if $\gamma \preccurlyeq \gamma'$ and $\gamma \neq \gamma'$.

An *upper ideal* of $(\Delta^+, \preccurlyeq)$ is a subset I such that if $\gamma \in I$ and $\gamma \prec \gamma'$, then $\gamma' \in I$. Therefore, I is an upper ideal of Δ^+ if and only if $\mathfrak{r} = \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma$ is a $\mathfrak{b}$-stable ideal of $\mathfrak{u}$, i.e., $[\mathfrak{b}, \mathfrak{r}] \subset \mathfrak{r}$.

For a dominant weight $\lambda \in \mathfrak{t}^*_{\mathbb{Q}}$, set $\Delta_\lambda = \{\gamma \in \Delta \mid (\lambda, \gamma) = 0\}$ and $\Delta^\pm_\lambda = \Delta_\lambda \cap \Delta^\pm$. Then Δ_λ is the root system of a semisimple subalgebra $\mathfrak{g}^{\perp\lambda} \subset \mathfrak{g}$ and $\Pi_\lambda = \Pi \cap \Delta^+_\lambda$ is the set of simple roots in Δ^+_λ. Then:

- $\mathfrak{p}_\lambda = \mathfrak{g}^{\perp\lambda} + \mathfrak{b}$ is a standard parabolic subalgebra of $\mathfrak{g}$.
- The set of roots of the nilradical $\mathfrak{n}_\lambda = \mathfrak{p}^{\mathrm{nil}}_\lambda$ is $\Delta^+ \setminus \Delta^+_\lambda$. It is also denoted by $\Delta(\mathfrak{n}_\lambda)$.

If $\lambda = \theta$, then $\mathfrak{n}_\theta$ is a *Heisenberg Lie algebra* (Heisenberg nilradical) and $\mathcal{H}_\theta := \Delta(\mathfrak{n}_\theta)$ is called the *Heisenberg subset* (of Δ^+).

2.2 The cascade poset

The recursive construction of the Kostant cascade in Δ^+ begins with $\beta_1 = \theta$. On the next step, we take the highest roots in the irreducible subsystems of Δ_θ. These roots

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are called the *descendants* of β_1. The same construction is then applied to every descendant of β_1, and so on. This procedure eventually terminates and yields a set $\mathcal{K} = \{\beta_1, \beta_2, \dots, \beta_m\} \subset \Delta^+$, which is called *the Kostant cascade*. The roots in $\mathcal{K}$ are *strongly orthogonal*, which means that $\beta_i \pm \beta_j \notin \Delta$ for all i, j. We make $\mathcal{K}$ a poset by letting that β_i covers β_j if and only if β_j is a descendant of β_i. Then β_1 is the unique maximal element of $\mathcal{K}$. We refer to [12, Section 2], [13], and [22, 2.2] for more details. Let us summarize the main features of $\mathcal{K}$.

- $\mathcal{K}$ is a maximal set of strongly orthogonal roots in Δ^+.
- Each β_i is the highest root of the irreducible root system $\Delta\langle i\rangle \subset \Delta$ with simple roots $\mathrm{supp}(\beta_i)$.
- $\mathcal{K}$ is also a subposet of $(\Delta^+, \preccurlyeq)$, which provides the same poset structure as above.
- One has $\beta_j \prec \beta_i$ if and only if $\mathrm{supp}(\beta_j) \subsetneq \mathrm{supp}(\beta_i)$.
- β_j and β_i are incomparable in $\mathcal{K}$ if and only if $\mathrm{supp}(\beta_j) \cap \mathrm{supp}(\beta_i) = \varnothing$.
- The numbering of $\mathcal{K}$ is not canonical. It is only required to be a linear extension of $(\mathcal{K}, \preccurlyeq)$, i.e., if $\beta_j \prec \beta_i$, then $j > i$. In specific examples considered below, we use the numbering of cascade roots given in Appendix A.

Using the decomposition $\Delta^+ = \Delta^+_\theta \sqcup \mathcal{H}_\theta$ and induction on $\mathrm{rk}\,\mathfrak{g}$, one obtains the disjoint union parametrized by $\mathcal{K}$:

$$\Delta^+ = \bigsqcup_{i=1}^{m} \mathcal{H}_{\beta_i}, \tag{2.1}$$

where $\mathcal{H}_{\beta_i}$ is the Heisenberg subset of $\Delta\langle i\rangle^+$ and $\mathcal{H}_{\beta_1} = \mathcal{H}_\theta$. For $1 \leqslant i \leqslant m$, let $\mathfrak{g}\langle i\rangle \subset \mathfrak{g}$ be the simple Lie algebra with root system $\Delta\langle i\rangle$. The geometric counterpart of (2.1) is the vector space sum $\mathfrak{u} = \bigoplus_{i=1}^m \mathfrak{h}_i$, where $\mathfrak{h}_i$ is the Heisenberg Lie algebra in $\mathfrak{g}\langle i\rangle$ and $\Delta(\mathfrak{h}_i) = \mathcal{H}_{\beta_i}$. In particular, $\mathfrak{h}_1 = \mathfrak{n}_\theta$. For each $\beta_i \in \mathcal{K}$, we set $\Phi(\beta_i) = \Pi \cap \mathcal{H}_{\beta_i}$. It then follows from (2.1) that $\Pi = \bigsqcup_{\beta_i \in \mathcal{K}} \Phi(\beta_i)$. Note that $\#\Phi(\beta_i) \leqslant 2$ and $\#\Phi(\beta_i) = 2$ if and only if the algebra $\mathfrak{g}\langle i\rangle$ is of type $\mathbf{A}_n$ with $n \geqslant 2$. Our definition of the subsets $\Phi(\beta_i)$ yields the well-defined map $\Phi^{-1} : \Pi \to \mathcal{K}$, where $\Phi^{-1}(\alpha) = \beta_i$ if $\alpha \in \Phi(\beta_i)$. Note that $\alpha \in \mathrm{supp}(\Phi^{-1}(\alpha))$ and $\alpha \in \Phi(\Phi^{-1}(\alpha))$. We think of the cascade poset as a triple $(\mathcal{K}, \preccurlyeq, \Phi)$. The corresponding Hasse diagrams, with subsets $\Phi(\beta_i)$ attached to every node, are depicted in [22, Section 6]. Some of them are included in Appendix A.

Obviously, $\#\mathcal{K} \leqslant \mathrm{rk}\,\mathfrak{g}$, and $\#\mathcal{K} = \mathrm{rk}\,\mathfrak{g}$ if and only if each β_i is a multiple of a fundamental weight for $\mathfrak{g}\langle i\rangle$. Recall that θ is a multiple of a fundamental weight of $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not of type $\mathbf{A}_n$, $n \geqslant 2$. It is well known that $\#\mathcal{K} = \mathrm{rk}\,\mathfrak{g}$ if and only if $\mathrm{ind}\,\mathfrak{b} = 0$. This happens exactly if $\mathfrak{g} \notin \{\mathbf{A}_n, \mathbf{D}_{2n+1}, \mathbf{E}_6\}$ and then Φ^{-1} yields a bijection between $\mathcal{K}$ and Π.

2.3 Nilradicals and optimal nilradicals

Let $\mathfrak{p} \supset \mathfrak{b}$ be a standard parabolic subalgebra of $\mathfrak{g}$, with nilradical $\mathfrak{n} = \mathfrak{p}^{\mathrm{nil}}$. If $\Pi \cap \Delta(\mathfrak{n}) = \mathcal{T}$, then we write $\mathfrak{n} = \mathfrak{n}_\mathcal{T}$ and $\mathfrak{p} = \mathfrak{p}_\mathcal{T}$. Here, $\mathcal{T}$ is the set of minimal elements of the poset $(\Delta(\mathfrak{n}), \preccurlyeq)$ and $\Pi \backslash \mathcal{T}$ is the set of simple roots for the standard Levi subalgebra $\mathfrak{l}_\mathcal{T} \subset \mathfrak{p}_\mathcal{T}$. Clearly, $\mathcal{T} \neq \varnothing$ if and only if $\mathfrak{n}_\mathcal{T} \neq \{0\}$.

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The integer $d_{\mathcal{T}} = \sum_{\alpha\in\mathcal{T}}[\theta:\alpha]$ is the *depth* of $\mathfrak{n}_{\mathcal{T}}$. Letting

$$\Delta_{\mathcal{T}}(i) = \left\{\gamma\in\Delta^+ \mid \sum_{\alpha\in\mathcal{T}}[\gamma:\alpha]=i\right\} \ \text{and} \ \mathfrak{n}_{\mathcal{T}}(i) = \bigoplus_{\gamma\in\Delta_{\mathcal{T}}(i)}\mathfrak{g}_\gamma,$$

one obtains the partition $\Delta(\mathfrak{n}_{\mathcal{T}}) = \bigsqcup_{i=1}^{d_{\mathcal{T}}}\Delta_{\mathcal{T}}(i)$ and the canonical $\mathbb{Z}$-grading

$$\mathfrak{n}_{\mathcal{T}} = \bigoplus_{i=1}^{d_{\mathcal{T}}}\mathfrak{n}_{\mathcal{T}}(i).$$

The following is well known and easy.

Lemma 2.1 *If* $\{\mathfrak{n}_{\mathcal{T}}^{(i)}\}_{i\geqslant1}$ *denotes the lower central series of* $\mathfrak{n}_{\mathcal{T}}$, *then* $\mathfrak{n}_{\mathcal{T}}^{(i)} = \bigoplus_{j\geqslant i}\mathfrak{n}_{\mathcal{T}}(j)$. *The center of* $\mathfrak{n}_{\mathcal{T}}$ *is* $\mathfrak{z}(\mathfrak{n}) = \mathfrak{n}_{\mathcal{T}}(d_{\mathcal{T}})$. *Hence,* $\mathfrak{n}_{\mathcal{T}}$ *is abelian if and only if* $d_{\mathcal{T}}=1$, *i.e.,* $\mathcal{T}=\{\alpha\}$ *and* $[\theta:\alpha]=1$.

Set $\mathcal{K}_{\mathcal{T}} = \mathcal{K}\cap\Delta(\mathfrak{n}_{\mathcal{T}})$. Then $\theta=\beta_1\in\mathcal{K}_{\mathcal{T}}$ for any nonzero nilradical $\mathfrak{n}_{\mathcal{T}}$.

Lemma 2.2 [22, Section 2] *For any* $\mathcal{T}\subset\Pi$, *one has*

(1) $\mathcal{K}_{\mathcal{T}}$ *is an upper ideal of* $(\mathcal{K},\preccurlyeq)$.
(2) $\mathcal{T}\subset\bigcup_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\Phi(\beta_j)$ *and* $\mathfrak{n}_{\mathcal{T}}\subset\bigoplus_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\mathfrak{h}_j$.

A standard parabolic subalgebra $\mathfrak{p}_{\mathcal{T}}$ is said to be *optimal* if

$$\mathcal{T} = \bigcup_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\Phi(\beta_j).$$

This goes back to [12, 4.10], and we also apply this term to $\mathfrak{n}_{\mathcal{T}}$. Then $\mathfrak{n}_{\mathcal{T}}$ is optimal if and only if $\mathfrak{n}_{\mathcal{T}} = \bigoplus_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\mathfrak{h}_j$. For a nonempty $\mathcal{T}\subset\Pi$, set $\tilde{\mathcal{T}} = \bigcup_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\Phi(\beta_j)$ and consider the nilradical $\mathfrak{n}_{\tilde{\mathcal{T}}}$. Then $\mathcal{K}_{\mathcal{T}} = \mathcal{K}_{\tilde{\mathcal{T}}}$ and $\mathfrak{n}_{\mathcal{T}}\subset\mathfrak{n}_{\tilde{\mathcal{T}}} = \bigoplus_{\beta_j\in\mathcal{K}_{\mathcal{T}}}\mathfrak{h}_i$. Hence, $\mathfrak{n}_{\tilde{\mathcal{T}}}$ is optimal, it is the minimal optimal nilradical containing $\mathfrak{n}_{\mathcal{T}}$, and it is the maximal element of the set of nilradicals $\{\mathfrak{n}' \mid \Delta(\mathfrak{n}')\cap\mathcal{K} = \mathcal{K}_{\mathcal{T}}\}$.

Definition 1 The nilradical $\mathfrak{n}_{\tilde{\mathcal{T}}}$ is called the *optimization* of $\mathfrak{n}_{\mathcal{T}}$.

If $\mathcal{T}\subset\Pi$ is not specified for a given nilradical $\mathfrak{n}$, then $\mathcal{K}(\mathfrak{n}) := \mathcal{K}\cap\Delta(\mathfrak{n})$ and we write $\tilde{\mathfrak{n}}$ for the optimization of $\mathfrak{n}$.

Proposition 2.3 (cf. [12, 2.4] and [22, 2.3]) *Let* $\tilde{\mathfrak{n}}$ *be the optimization of a nilradical* $\mathfrak{n}$. *Then:*

- $\operatorname{ind}\mathfrak{n} = \dim(\tilde{\mathfrak{n}}/\mathfrak{n}) + \#\mathcal{K}(\mathfrak{n})$.
- $\operatorname{ind}\tilde{\mathfrak{n}} = \#\mathcal{K}(\tilde{\mathfrak{n}}) = \#\mathcal{K}(\mathfrak{n})$ *and* $\boldsymbol{b}(\mathfrak{n}) = \boldsymbol{b}(\tilde{\mathfrak{n}})$.

Remark 2.4 The merit of optimization is that the passage from $\mathfrak{n}$ to $\tilde{\mathfrak{n}}$ does not change $\mathcal{K}(\mathfrak{n})$ and $\boldsymbol{b}(\mathfrak{n})$. More generally, if two nilradicals $\mathfrak{n}'$ and $\mathfrak{n}$ have the same optimization, then $\mathcal{K}(\mathfrak{n}') = \mathcal{K}(\mathfrak{n})$ and $\boldsymbol{b}(\mathfrak{n}') = \boldsymbol{b}(\mathfrak{n})$. For instance, this happens if $\mathfrak{n}\subset\mathfrak{n}'\subset\tilde{\mathfrak{n}}$.

Example 2.5 (1) If $\mathfrak{g} = \mathfrak{sl}_{n+1}$, then $\mathfrak{u}$ is the set of strictly upper-triangular matrices, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1\leqslant i\leqslant n)$, and $\mathcal{K} = \{\beta_1,\dots,\beta_t\}$, where $t = [(n+1)/2]$ and $\beta_i = \alpha_i + \dots + \alpha_{n+1-i} = \varepsilon_i - \varepsilon_{n+2-i}$. Here, $(\mathcal{K},\preccurlyeq)$ is a chain and $\Phi(\beta_i) = \{\alpha_i,\alpha_{n+1-i}\}$.

(2) Take $\mathcal{T} = \{\alpha_2,\alpha_6\}$ and the nilradical $\mathfrak{n}_{\mathcal{T}}\subset\mathfrak{sl}_7$. Then $\mathcal{K}_{\mathcal{T}} = \{\beta_1,\beta_2\}$ and therefore $\tilde{\mathcal{T}} = \{\alpha_1,\alpha_2,\alpha_5,\alpha_6\}$ (cf. the matrices below).

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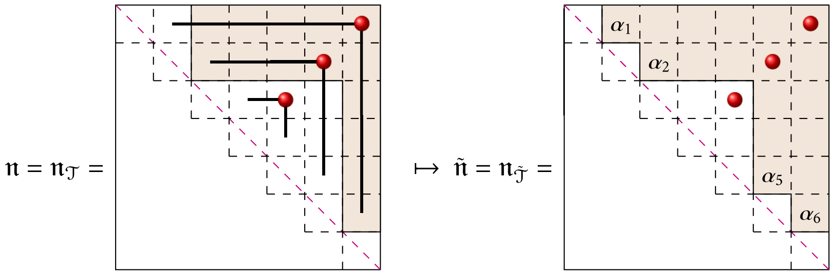

The cells with ball represent the cascade, and the thick lines depict the Heisenberg subset attached to an element of the cascade. By Proposition 2.3, we have $\operatorname{ind}\tilde{\mathfrak{n}} = 2$, $\operatorname{ind}\mathfrak{n} = 6$, and $\boldsymbol{b}(\mathfrak{n}) = \boldsymbol{b}(\tilde{\mathfrak{n}}) = 10$.

Example 2.6 For a square matrix A, let $\hat{A}$ denote its transpose with respect to the antidiagonal. Choose the skew-symmetric form defining $\mathfrak{g} = \mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$ such that

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} A & M \\ M' & -\hat{A} \end{pmatrix} \mid M = \hat{M} \ \& \ M' = \hat{M}' \right\},$$

where A, M, M' are $n \times n$ matrices. Then $\mathfrak{u}$ (resp. $\mathfrak{t}$) is the set of symplectic strictly upper triangular (resp. diagonal) matrices. Hence, $\mathfrak{t} = \{\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n, -\varepsilon_n, \dots, -\varepsilon_1) \mid \varepsilon_i \in \mathbb{C}\}$. Recall that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < n)$ and $\alpha_n = 2\varepsilon_n$. Then $\mathcal{K} = \{\beta_1, \dots, \beta_n\}$ is a chain, where $\beta_i = 2\varepsilon_i$ and $\Phi(\beta_i) = \{\alpha_i\}$ for all i. Here, $\mathfrak{n}_{\{\alpha_n\}} = \left\{ \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \mid M = \hat{M} \right\}$ is the nilradical of the maximal parabolic subalgebra with $\mathcal{T} = \{\alpha_n\}$. It is the only (standard) **abelian** nilradical in $\mathfrak{sp}_{2n}$ and $\mathcal{K} \subset \Delta(\mathfrak{n}_{\{\alpha_n\}})$ corresponds to the antidiagonal entries of M.

3 Generic stabilizers, the Frobenius semiradical, and commutative polarizations

Let Q be a connected algebraic group with $\mathfrak{q} = \operatorname{Lie} Q$. If $\rho : Q \to GL(\mathbb{V})$ is a representation of Q, then the corresponding Q-action on $\mathbb{V}$ is denoted by $(Q : \mathbb{V})$. For $q \in Q$ and $v \in \mathbb{V}$, we write $q{\cdot}v$ in place of $\rho(q)v$. Likewise, $(\mathfrak{q} : \mathbb{V})$ corresponds to $d\rho : \mathfrak{q} \to \mathfrak{gl}(\mathbb{V})$.

3.1 Generic stabilizers

Let $(Q : \mathbb{V})$ be a linear action. We say that $v \in \mathbb{V}$ is *Q-generic*, if there is a dense open subset $\Omega \subset \mathbb{V}$ such that $v \in \Omega$ and the stabilizer $\mathfrak{q}^x$ is Q-conjugate to $\mathfrak{q}^v$ for any $x \in \Omega$. Then any $\mathfrak{q}^x$ $(x \in \Omega)$ is called a *Q-generic stabilizer* for the representation $(\mathfrak{q} : \mathbb{V})$, and we say that $(\mathfrak{q} : \mathbb{V})$ has a Q-generic stabilizer. (One can consider similar notions for non-connected groups, for arbitrary actions of Q, and for stationary subgroups $Q^x \subset Q$, but we do need it now.) By semi-continuity of orbit dimensions, the set Q-generic points is contained in the set of Q-regular points

$$\mathbb{V}_{\text{reg}} = \{v \in \mathbb{V} \mid \dim Q{\cdot}v \text{ is maximal}\}, \tag{3.1}$$

but usually, this inclusion is proper. By a result of Richardson [24], if Q is reductive, then Q-generic stabilizers exist for any action of Q on a smooth affine variety. But this is no longer true for non-reductive groups, and one of our goals is to study (the presence of) generic stabilizers for the coadjoint representation of a nilradical in $\mathfrak{g}$.

A practical method for proving the existence of Q-generic points and finding Q-generic stabilizers is given by Elashvili [8, Lemma 1]. Let $\mathrm{T}_v(Q{\cdot}v) = \mathfrak{q}{\cdot}v$ be the tangent space of the orbit $Q{\cdot}v$ at v and $\mathbb{V}^{\mathfrak{q}_v}$ the fixed point subspace of $\mathfrak{q}^v$ in $\mathbb{V}$. Then

$$v \in \mathbb{V} \text{ is } Q\text{-generic if and only if } \mathbb{V} = \mathfrak{q}{\cdot}v + \mathbb{V}^{\mathfrak{q}_v}. \tag{3.2}$$

The main case of interest for us is the coadjoint representation of Q, when $\mathbb{V} = \mathfrak{q}^*$. For the coadjoint representation, we usually skip "Q" from notation and refer to "generic" and "regular" points (in $\mathfrak{q}^*$) and "generic" stabilizers (in $\mathfrak{q}$). Translating Elashvili's criterion (3.2) into the setting of coadjoint representations and taking annihilators, one obtains the following nice formula (see [27, Corollary 1.8(i)]). Given $\xi \in \mathfrak{q}^*$, the stabilizer $\mathfrak{q}^\xi$ is generic (i.e., ξ is a Q-generic point) if and only if

$$[\mathfrak{q}, \mathfrak{q}^\xi] \cap \mathfrak{q}^\xi = \{0\}. \tag{3.3}$$

The reason is that $(\mathfrak{q}{\cdot}\xi)^\perp = \mathfrak{q}^\xi$ and $((\mathfrak{q}^*)^{\mathfrak{q}_\xi})^\perp = [\mathfrak{q}, \mathfrak{q}^\xi]$, where $(\cdot)^\perp$ stands for the annihilator in the dual space.

3.2 The Frobenius semiradical

For the Q-module $\mathbb{V} = \mathfrak{q}^*$, the set of Q-regular (or just "regular") points $\mathfrak{q}^*_{\mathrm{reg}}$ consists of all $\xi \in \mathfrak{q}^*$ such that the stabilizer $\mathfrak{q}^\xi$ has the minimal possible dimension. If $\xi \in \mathfrak{q}^*_{\mathrm{reg}}$, then $\mathrm{ind}\,\mathfrak{q} := \dim \mathfrak{q}^\xi$ is the *index* of (a Lie algebra) $\mathfrak{q}$. The *Frobenius semiradical* $\mathcal{F}(\mathfrak{q})$ of a Lie algebra $\mathfrak{q}$ is introduced by Ooms (see [18, 19]). By definition,

$$\mathcal{F}(\mathfrak{q}) = \sum_{\xi \in \mathfrak{q}^*_{\mathrm{reg}}} \mathfrak{q}^\xi.$$

Obviously, $\mathcal{F}(\mathfrak{q})$ is a characteristic ideal of $\mathfrak{q}$, and $\mathcal{F}(\mathfrak{q}) = 0$ if and only $\mathrm{ind}\,\mathfrak{q} = 0$ (i.e., $\mathfrak{q}$ is a *Frobenius* Lie algebra). Note that if $(\mathfrak{q} : \mathfrak{q}^*)$ has a generic stabilizer, then any Q-generic point in the sense of Section 3.1 is regular, but not vice versa.

Lemma 3.1 (cf. [19, Proposition 1.7]) *If $(\mathfrak{q} : \mathfrak{q}^*)$ has a generic stabilizer and $\xi \in \mathfrak{q}^*$ is any generic point, then $\mathcal{F}(\mathfrak{q})$ is the $\mathfrak{q}$-ideal generated by the sole stabilizer $\mathfrak{q}^\xi$.*

Proof By [19, Lemma 1.2], if Ψ is open and dense in $\mathfrak{q}^*_{\mathrm{reg}}$, then $\mathcal{F}(\mathfrak{q}) = \sum_{\eta \in \Psi} \mathfrak{q}^\eta$. Applying this to the set of generic points $\Omega \subset \mathfrak{q}^*_{\mathrm{reg}}$, we obtain $\mathcal{F}(\mathfrak{q}) = \sum_{\eta \in \Omega} \mathfrak{q}^\eta = \sum_{g \in Q} \mathfrak{q}^{g \cdot \xi}$. Clearly, the last sum yields the ideal of $\mathfrak{q}$ generated by $\mathfrak{q}^\xi$. ■

If $\mathfrak{q}$ is *quadratic*, i.e., $\mathfrak{q} \simeq \mathfrak{q}^*$ as Q-module, then $\mathcal{F}(\mathfrak{q}) = \sum_{x \in \mathfrak{q}_{\mathrm{reg}}} \mathfrak{q}^x$. Since $x \in \mathfrak{q}^x$ for any $x \in \mathfrak{q}$, we see that here $\mathcal{F}(\mathfrak{q}) = \mathfrak{q}$ (cf. [10, Theorem 3.2]). In particular, this is the case if $\mathfrak{q}$ is reductive. Following Ooms, $\mathfrak{q}$ is said to be *quasi-quadratic* if $\mathcal{F}(\mathfrak{q}) = \mathfrak{q}$. Another interesting property of the functor $\mathcal{F}(\cdot)$ is that $\mathrm{ind}\,\mathfrak{q} \leqslant \mathrm{ind}\,\mathcal{F}(\mathfrak{q})$ and $\mathcal{F}(\mathcal{F}(\mathfrak{q})) = \mathcal{F}(\mathfrak{q})$ [19].

3.3 Commutative polarizations

If $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{q}$, then $\dim \mathfrak{a} \leqslant \boldsymbol{b}(\mathfrak{q})$ (see [28, 0.2] or [18, Theorem 14]). If $\dim \mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$, then $\mathfrak{a}$ is called a *commutative polarization* (=CP) of $\mathfrak{q}$, and we say that $\mathfrak{q}$ has a CP. If $\mathfrak{a}$ is a CP and also an ideal of $\mathfrak{q}$, then it is called a CP-*ideal*. If $\mathfrak{q}$ is solvable and has a CP, then it also has a CP-ideal (see [10, Theorem 4.1]). More generally, a similar argument shows that if $\mathfrak{q}$ is an ideal of a solvable Lie algebra $\mathfrak{r}$ and $\mathfrak{q}$ has a CP, then $\mathfrak{q}$ has a CP-ideal that is $\mathfrak{r}$-stable. A standard nilradical $\mathfrak{n}$ is an ideal of $\mathfrak{b}$. Therefore, if $\mathfrak{n}$ has a CP, then it also has a CP-ideal that is $\mathfrak{b}$-stable. Henceforth, "a CP-ideal of $\mathfrak{n}$" means "a $\mathfrak{b}$-stable CP-ideal of $\mathfrak{n}$."

Basic results on commutative polarizations are presented in [10]. It is also shown therein that if $\mathfrak{g}$ is of type $\mathbf{A}_n$ or $\mathbf{C}_n$, then every nilradical in $\mathfrak{g}$ has a CP. A complete classification of the nilradicals having a CP is obtained in [22]. By Lemma 2.1, $\mathfrak{n}_{\mathcal{T}}$ is abelian if and only if $\mathcal{T} = \{\alpha\}$ and $[\theta : \alpha] = 1$. The abelian nilradical $\mathfrak{n}_{\{\alpha\}}$ play a key role in our theory. By Theorems 3.10 and 4.1 in [22], a nilradical $\mathfrak{n}$ has a CP if and only if at least one of the following two conditions is satisfied:

(1) $\mathfrak{n} = \mathfrak{n}_\theta = \mathfrak{h}_1$ is the Heisenberg nilradical. In this case, if $\mathfrak{a}$ is any maximal abelian ideal of $\mathfrak{b}$, then $\mathfrak{a} \cap \mathfrak{n}$ is a CP-ideal of $\mathfrak{n}$, and vice versa.
(2) There is an abelian nilradical $\mathfrak{n}_{\{\alpha\}}$ such that $\mathfrak{n}$ is contained in $\widetilde{\mathfrak{n}_{\{\alpha\}}}$, the optimization of $\mathfrak{n}_{\{\alpha\}}$. (There can be several abelian nilradicals with this property, and, for a "right" choice of such $\check{\alpha} \in \Pi$, $\mathfrak{n} \cap \mathfrak{n}_{\{\check{\alpha}\}}$ is a CP-ideal of $\mathfrak{n}$ (cf. also Section 8)).

If $\mathfrak{g}$ has no parabolic subalgebras with abelian nilradicals, then the Heisenberg nilradical $\mathfrak{n}_\theta = \mathfrak{h}_1$ is the only nilradical with CP. This happens precisely if $\mathfrak{g}$ is of type $\mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_8$. Another result of [22] is that $\mathfrak{n}$ has a CP if and only if $\tilde{\mathfrak{n}}$ has.

3.4 The role of commutative polarizations

If $\mathfrak{a}$ is a CP of a Lie algebra $\mathfrak{q}$, then:

(1) $\mathcal{F}(\mathfrak{q}) \subset \mathfrak{a}$ [18, Proposition 20] and thereby $\mathcal{F}(\mathfrak{q})$ is an abelian ideal. (However, it can happen that $\mathcal{F}(\mathfrak{q})$ is abelian, whereas $\mathfrak{q}$ has no CP (see Example 7.3).)
(2) Since $\mathfrak{a}$ is abelian, $\boldsymbol{b}(\mathfrak{a}) = \dim \mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$. Therefore, the Poisson center $\mathcal{Z}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is contained in $\mathcal{S}(\mathfrak{a})$ [22, Proposition 5.5]. Hence, if $\mathfrak{a}$ is a CP-ideal of $\mathfrak{q}$, then $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{a})^{\mathfrak{q}}$.
(3) Thus, if $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ are different CP in $\mathfrak{q}$, then $\mathcal{F}(\mathfrak{q}) \subset \bigcap_{i=1}^s \mathfrak{a}_i$ and $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \subset \mathcal{S}(\bigcap_{i=1}^s \mathfrak{a}_i)$.

In many cases, the presence of a CP in $\mathfrak{n}$ allows us to quickly describe the Frobenius semiradical for $\mathfrak{n}$ (see Sections 5.1 and 5.2). For the reader's convenience, we provide an invariant-theoretic proof for two basic results on abelian subalgebras mentioned above.

Proposition 3.2 *Let* $\mathfrak{a}$ *be an abelian subalgebra of* $\mathfrak{q}$. *Then:*

(1) $\dim \mathfrak{a} \leqslant \boldsymbol{b}(\mathfrak{q})$.
(2) *If* $\dim \mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$, *then* $\mathfrak{q}^\xi \subset \mathfrak{a}$ *for any* $\xi \in \mathfrak{q}^*_{reg}$, *i.e.*, $\mathcal{F}(\mathfrak{q}) \subset \mathfrak{a}$.

Proof We show that both assertions are immediate consequences of Rosenlicht's theorem [4, Chapter 1.6]. Let $A \subset Q$ be the connected (abelian) subgroup with $\operatorname{Lie} A = \mathfrak{a}$.

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For the A-action on $\mathfrak{q}^*$, let $\mathbb{C}(\mathfrak{q}^*)^A$ denote the field of A-invariant rational functions on $\mathfrak{q}^*$.

(1) Since $\mathfrak{a}$ is abelian, we have $\mathbb{C}[\mathfrak{q}^*]^A = \mathcal{S}(\mathfrak{q})^A \supset \mathcal{S}(\mathfrak{a})$; hence, $\operatorname{trdeg}\mathbb{C}(\mathfrak{q}^*)^A \geqslant \dim\mathfrak{a}$. By Rosenlicht's theorem,

$$\operatorname{trdeg}\mathbb{C}(\mathfrak{q}^*)^A = \dim\mathfrak{q} - \max_{\xi\in\mathfrak{q}^*} A{\cdot}\xi = \dim\mathfrak{q} - \dim\mathfrak{a} + \min_{\xi\in\mathfrak{q}^*}\dim\mathfrak{a}^\xi.$$

Therefore,

$$2\dim\mathfrak{a} \leqslant \dim\mathfrak{q} + \min_{\xi\in\mathfrak{q}^*}\dim\mathfrak{a}^\xi \leqslant \dim\mathfrak{q} + \operatorname{ind}\mathfrak{q} = 2\boldsymbol{b}(\mathfrak{q}).$$

(2) If $\dim\mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$, then $\min_{\xi\in\mathfrak{q}^*}\dim\mathfrak{a}^\xi = \operatorname{ind}\mathfrak{q}$. For $\xi\in\mathfrak{q}^*_{\rm reg}$, one has $\dim\mathfrak{q}^\xi = \operatorname{ind}\mathfrak{q}$. That is, $\mathfrak{a}^\xi = \mathfrak{q}^\xi \subset \mathfrak{a}$ for any $\xi\in\mathfrak{q}^*_{\rm reg}$. ∎

The proof above also implies that if $\dim\mathfrak{a} = \boldsymbol{b}(\mathfrak{q})$, then $\operatorname{trdeg}\mathbb{C}(\mathfrak{q}^*)^A = \dim\mathfrak{a}$. Therefore, $\mathbb{C}[\mathfrak{q}^*]^A = \mathcal{S}(\mathfrak{a})$ and $\mathbb{C}(\mathfrak{q}^*)^A$ is the fraction field of $\mathcal{S}(\mathfrak{a})$.

4 Generic stabilizers and the Frobenius semiradical for the nilradicals

In this section, we study the Frobenius semiradical, $\mathcal{F}(\mathfrak{n})$, of a nilradical $\mathfrak{n}\subset\mathfrak{g}$ and the existence of generic stabilizers for the coadjoint representation of $N = \exp(\mathfrak{n})$. We also say that "$\mathfrak{n}$ has a generic stabilizer," if the coadjoint representation $(N:\mathfrak{n}^*)$ has.

Let $\mathfrak{n} = \mathfrak{p}^{\rm nil}$ be a standard nilradical and $\mathfrak{n}^- = (\mathfrak{p}^-)^{\rm nil}$ the opposite nilradical, i.e., $\Delta(\mathfrak{n}^-) = -\Delta(\mathfrak{n})$. Using the vector space sum $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{n}^-$ and the P-module isomorphism $\mathfrak{n}^*\simeq\mathfrak{g}/\mathfrak{p}$, we identify $\mathfrak{n}^*$ with $\mathfrak{n}^-$ and thereby regard $\mathfrak{n}^-$ as P-module. In terms of $\mathfrak{n}^-$, the $\mathfrak{p}$-action on $\mathfrak{n}^*$ is given by the Lie bracket in $\mathfrak{g}$, with the subsequent projection to $\mathfrak{n}^-$. In particular, the coadjoint representation $\operatorname{ad}^*_{\mathfrak{n}}$ of $\mathfrak{n}$ has the following interpretation. If $x\in\mathfrak{n}$ and $\xi\in\mathfrak{n}^-$, then $(\operatorname{ad}^*_{\mathfrak{n}}x){\cdot}\xi = \operatorname{pr}_{\mathfrak{n}^-}([x,\xi])$, where $\operatorname{pr}_{\mathfrak{n}^-}:\mathfrak{g}\to\mathfrak{n}^-$ is the projection with kernel $\mathfrak{p}$.

Recall that $\mathcal{K}(\mathfrak{n}) = \mathcal{K}\cap\Delta(\mathfrak{n})$. Set $\mathfrak{k} = \bigoplus_{\beta\in\mathcal{K}(\mathfrak{n})}\mathfrak{g}_{-\beta}\subset\mathfrak{n}^-\simeq\mathfrak{n}^*$ and $\boldsymbol{\zeta} = \sum_{\beta\in\mathcal{K}(\mathfrak{n})} e_{-\beta}\in\mathfrak{k}$. We say that $\mathfrak{k}$ is a *cascade subspace* of $\mathfrak{n}^*$ and $\boldsymbol{\zeta}$ is a *cascade point* of $\mathfrak{n}^*$. Clearly, $\boldsymbol{\zeta}$ depends on the choice of nonzero root vectors $e_{-\beta}\in\mathfrak{g}_{-\beta}$, but all such points $\boldsymbol{\zeta}$ form a sole dense T-orbit in $\mathfrak{k}$, which is denoted by $\mathfrak{k}_0$. That is, $\mathfrak{k}_0 = \{\sum_{\beta\in\mathcal{K}(\mathfrak{n})} a_\beta e_{-\beta} \mid a_\beta\in\mathbb{C}\setminus\{0\}\}$. It was proved by Joseph [12, 2.4] that upon the identification of $\mathfrak{n}^-$ and $\mathfrak{n}^*$ as P-modules and hence B-modules, $B{\cdot}\boldsymbol{\zeta}$ is dense in $\mathfrak{n}^*$ and $B{\cdot}\boldsymbol{\zeta}\subset\mathfrak{n}^*_{\rm reg}$. In particular, $\mathfrak{k}_0\subset\mathfrak{n}^*_{\rm reg}$.

Proposition 4.1 *Let $\mathfrak{n}$ be an arbitrary nilradical and $\boldsymbol{\zeta}\in\mathfrak{n}^*$ a cascade point. Then:*

(i) *$\mathcal{F}(\mathfrak{n})$ is the $\mathfrak{b}$-stable ideal of $\mathfrak{n}$ generated by $\mathfrak{n}^{\boldsymbol{\zeta}}$.*
(ii) *$(\mathfrak{n}:\mathfrak{n}^*)$ has a generic stabilizer if and only if the stabilizer $\mathfrak{n}^{\boldsymbol{\zeta}}$ is generic.*

Proof (i) Since $B{\cdot}\boldsymbol{\zeta}$ is dense in $\mathfrak{n}^*_{\rm reg}$, a result of Ooms [19, Lemma 1.2] implies that $\mathcal{F}(\mathfrak{n}) = \sum_{b\in B}\mathfrak{n}^{b\cdot\boldsymbol{\zeta}}$. Clearly, the last sum is the smallest $\mathfrak{b}$-stable ideal containing $\mathfrak{n}^{\boldsymbol{\zeta}}$.

(ii) If $\boldsymbol{\zeta}$ is not generic, then $[\mathfrak{n},\mathfrak{n}^{\boldsymbol{\zeta}}]\cap\mathfrak{n}^{\boldsymbol{\zeta}}\neq\{0\}$ and hence $[\mathfrak{n},\mathfrak{n}^{b\cdot\boldsymbol{\zeta}}]\cap\mathfrak{n}^{b\cdot\boldsymbol{\zeta}}\neq\{0\}$ for any $b\in B$. Therefore, neither of the points $b{\cdot}\boldsymbol{\zeta}$ $(b\in B)$ can be generic. Since $B{\cdot}\boldsymbol{\zeta}$ is dense in $\mathfrak{n}^*$, this means that generic points cannot exist in such a case. ∎

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The point of Proposition 4.1(i) is that the existence of a generic stabilizer for $\mathfrak{n}$ is not required (cf. Lemma 3.1). We use instead the action of B on $\mathfrak{n}^*$.

Proposition 4.2 *If $\mathfrak{n}$ is an optimal nilradical, then:*

(i) $\mathfrak{n}^{\zeta}$ *is a generic stabilizer for* $(\mathfrak{n} : \mathfrak{n}^*)$.
(ii) *The N-saturation of $\mathfrak{k}$ is dense in $\mathfrak{n}^*$, i.e., $\overline{N{\cdot}\mathfrak{k}} = \mathfrak{n}^*$.*

Proof (i) For an optimal nilradical $\mathfrak{n}$, one has $\mathfrak{n}^{\zeta} = \bigoplus_{\beta\in\mathcal{K}(\mathfrak{n})} \mathfrak{g}_\beta$ [12, 2.4]. Since the roots in $\mathcal{K}(\mathfrak{n})$ are strongly orthogonal, condition (3.3) is satisfied for $\xi = \zeta$.

(ii) Recall that $\operatorname{ind}\mathfrak{n} = \#\mathcal{K}(\mathfrak{n}) = \dim\mathfrak{k}$ and $\operatorname{codim}_{\mathfrak{n}^*}(\mathfrak{n}{\cdot}\xi) = \operatorname{ind}\mathfrak{n}$ for any $\xi \in \mathfrak{k}_0$. Furthermore, $\mathfrak{n}{\cdot}\xi \cap \mathfrak{k} = \{0\}$ for any $\xi \in \mathfrak{k}$. (Use again the strong orthogonality.) Hence, $\mathfrak{n}{\cdot}\xi \oplus \mathfrak{k} = \mathfrak{n}^*$ for any $\xi \in \mathfrak{k}_0$, which implies that $N{\cdot}\mathfrak{k}_0$ is open and dense in $\mathfrak{n}^*$ (cf. [8, Lemma 1.1]). ■

Recall that $\tilde{\mathfrak{n}}$ denotes the optimization of $\mathfrak{n}$. Then $\mathcal{K}(\mathfrak{n}) = \mathcal{K}(\tilde{\mathfrak{n}})$, $\operatorname{ind}\tilde{\mathfrak{n}} = \#\mathcal{K}(\mathfrak{n})$, and

$$\operatorname{ind}\mathfrak{n} = \dim(\tilde{\mathfrak{n}}/\mathfrak{n}) + \operatorname{ind}\tilde{\mathfrak{n}} = \dim(\tilde{\mathfrak{n}}/\mathfrak{n}) + \#\mathcal{K}(\mathfrak{n}). \tag{4.1}$$

The cascade subspaces and cascade points associated with $\mathfrak{n}$ or $\tilde{\mathfrak{n}}$ are the same, if regarded as objects in $\mathfrak{u}^-$. But one has to distinguish them as objects in $\mathfrak{n}^*$ or $\tilde{\mathfrak{n}}^*$. For this reason, working simultaneously with $\mathfrak{n}$ and $\tilde{\mathfrak{n}}$, we write $\tilde{\zeta}$ for a cascade point in the cascade subspace $\tilde{\mathfrak{k}} \subset \tilde{\mathfrak{n}}^*$. Note that, for the natural projection $\tau : \tilde{\mathfrak{n}}^* \to \mathfrak{n}^*$, one has $\tau(\tilde{\mathfrak{k}}) = \mathfrak{k}$ and $\tau(\tilde{\zeta}) = \zeta$. Since τ is B-equivariant, this allows us to transfer some good properties from the coadjoint action $(\tilde{N} : \tilde{\mathfrak{n}}^*)$ to the coadjoint action $(N : \mathfrak{n}^*)$.

If $\mathfrak{n}$ is not optimal, then it may or may not have a generic stabilizer. To see this, we provide an explicit description of $\mathfrak{n}^{\zeta}$ for an arbitrary nilradical $\mathfrak{n}$, which generalizes Joseph's description for the optimal nilradicals.

Because $\zeta \in \mathfrak{n}^*_{\text{reg}}$ and $\tilde{\zeta} \in \tilde{\mathfrak{n}}^*_{\text{reg}}$, it follows from (4.1) that $\dim\mathfrak{n}^{\zeta} = \dim(\tilde{\mathfrak{n}}/\mathfrak{n}) + \dim\tilde{\mathfrak{n}}^{\tilde{\zeta}}$. Since τ is B-equivariant, we have $\mathfrak{n}^{\zeta} \supset \tilde{\mathfrak{n}}^{\tilde{\zeta}} = \bigoplus_{\beta\in\mathcal{K}(\mathfrak{n})} \mathfrak{g}_\beta$. If $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \dots, \beta_k\}$, then $\mathfrak{n} \subset \tilde{\mathfrak{n}} = \bigoplus_{j=1}^k \mathfrak{h}_j$ and $\mathfrak{n} = \bigoplus_{j=1}^k (\mathfrak{h}_j \cap \mathfrak{n})$ (see Section 2.3). For any $\gamma \in \Delta(\mathfrak{h}_j)\setminus\{\beta_j\} =: \overline{\Delta(\mathfrak{h}_j)}$, we have $\beta_j - \gamma \in \Delta^+$.

Lemma 4.3 *Under the previous notation, if $\mathfrak{n} \ne \tilde{\mathfrak{n}}$ and $\gamma \in \Delta(\mathfrak{h}_j)\setminus\Delta(\mathfrak{n}) =: \mathcal{C}_{\mathfrak{n}}(j) \subset \overline{\Delta(\mathfrak{h}_j)}$, then the root space $\mathfrak{g}_{\beta_j-\gamma}$ belongs to $\mathfrak{n}^{\zeta}$.*

Proof Assume that $\beta_j - \gamma \notin \Delta(\mathfrak{n})$. Then $[\gamma : \alpha] = [\beta_j - \gamma : \alpha] = 0$ for all $\alpha \in \mathcal{T} = \Delta(\mathfrak{n}) \cap \Pi$. Hence, $[\beta_j : \alpha] = 0$. However, since $\beta_j \in \Delta(\mathfrak{n})$, there is an $\alpha' \in \mathcal{T}(\mathfrak{n})$ such that $\beta_j \succ \alpha'$, i.e., $[\beta_j : \alpha'] > 0$. This contradiction shows that $\gamma' := \beta_j - \gamma \in \Delta(\mathfrak{n})$.

Let us prove that $\mathfrak{g}_{\gamma'} \subset \mathfrak{n}^{\zeta}$. Since $\gamma' \in \Delta(\mathfrak{h}_j)$ and hence $\operatorname{supp}(\gamma') \subset \operatorname{supp}(\beta_j)$, properties of $\mathcal{K}$ imply that β_j is the only element β_i of $\mathcal{K}$ such that $\beta_i - \gamma'$ is a root. Indeed,

- if β_i and β_j are incomparable, then $\operatorname{supp}(\beta_i) \cap \operatorname{supp}(\beta_j) = \varnothing$;
- if $\beta_i \prec \beta_j$, then $\operatorname{supp}(\beta_i) \subset \operatorname{supp}(\beta_j)\setminus\Phi(\beta_j)$, while $\operatorname{supp}(\gamma') \cap \Phi(\beta_j) \ne \varnothing$;
- if $\beta_i \succ \beta_j$, then $(\beta_i, \gamma') = 0$ and γ' does not belong to the Heisenberg subset of $\Delta\langle i\rangle$.

Therefore, $\gamma' - \beta_j = -\gamma$ is the only root that might occur in $[e_{\gamma'}, \zeta]$. But $\gamma \notin \Delta(\mathfrak{n})$, i.e., $-\gamma \notin \Delta(\mathfrak{n}^*)$ and therefore $\operatorname{ad}^*_{\mathfrak{n}}(e_{\gamma'})(\zeta) = 0$. ■

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Theorem 4.4 *Let $\mathfrak{n}$ be an arbitrary nilradical and $\boldsymbol{\zeta} \in \mathfrak{k}_0 \subset \mathfrak{n}^*$ a cascade point. Then:*

(i) *$\mathfrak{n}^{\boldsymbol{\zeta}}$ is T-stable and*

$$\mathfrak{n}^{\boldsymbol{\zeta}} = \left(\bigoplus_{j=1}^{k} \bigoplus_{\gamma \in \mathcal{C}_{\mathfrak{n}}(j)} \mathfrak{g}_{\beta_j - \gamma} \right) \oplus \tilde{\mathfrak{n}}^{\tilde{\boldsymbol{\zeta}}} = \left(\bigoplus_{j=1}^{k} \bigoplus_{\gamma \in \mathcal{C}_{\mathfrak{n}}(j)} \mathfrak{g}_{\beta_j - \gamma} \right) \oplus \left(\bigoplus_{\beta \in \mathcal{K}(\mathfrak{n})} \mathfrak{g}_{\beta} \right).$$

(ii) $\Delta(\mathfrak{n}^{\boldsymbol{\zeta}}) = \{\beta_j - \gamma \mid 1 \leqslant j \leqslant k \ \& \ \gamma \in \mathcal{C}_{\mathfrak{n}}(j)\} \sqcup \mathcal{K}(\mathfrak{n})$.

(iii) *For any $\xi \in \mathfrak{k}_0$, we have $\mathfrak{n}^{\xi} = \mathfrak{n}^{\boldsymbol{\zeta}}$.*

Proof The number of roots in $\sqcup_{j=1}^{k} \mathcal{C}_{\mathfrak{n}}(j)$ equals $\dim(\tilde{\mathfrak{n}}/\mathfrak{n}) = \dim \mathfrak{n}^{\boldsymbol{\zeta}} - \dim \tilde{\mathfrak{n}}^{\tilde{\boldsymbol{\zeta}}}$, and each such root yields a root subspace in $\mathfrak{n}^{\boldsymbol{\zeta}}$ (Lemma 4.3), hence (i). Clearly, (ii) is just a reformulation of (i). The last assertion follows from the fact that $\mathfrak{n}^{\boldsymbol{\zeta}}$ is T-stable and $T{\cdot}\boldsymbol{\zeta} = \mathfrak{k}_0$. ■

The advantage of $\boldsymbol{\zeta} \in \mathfrak{k}_0$ is that $\mathfrak{n}^{\boldsymbol{\zeta}}$ is a sum of root spaces. Therefore, Eq. (3.3) is easily verified in practice. It is convenient to restate it as follows.

Condition 4.5 *The stabilizer $\mathfrak{n}^{\boldsymbol{\zeta}}$ of $\boldsymbol{\zeta} \in \mathfrak{k}_0 \subset \mathfrak{n}^*$ is generic (equivalently, $(\mathfrak{n} : \mathfrak{n}^*)$ has a generic stabilizer) if and only if the difference of any two roots in $\Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ does not belong to $\Delta(\mathfrak{n})$.*

Example 4.6 (1) If $\mathfrak{g} = \mathfrak{sl}_7$ and $\mathcal{T}_1 = \{\alpha_2, \alpha_6\}$, then $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}_1}$ is not optimal and $\tilde{\mathfrak{n}} = \mathfrak{n}_{\tilde{\mathcal{T}}}$, where $\tilde{\mathcal{T}} = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}$ (see Example 2.5). Here, $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \beta_2\} = \{\alpha_1 + \dots + \alpha_6, \alpha_2 + \dots + \alpha_5\}$ and the matrices therein show that $\dim(\tilde{\mathfrak{n}}/\mathfrak{n}) = 4$. More precisely,

$$\mathcal{C}_{\mathfrak{n}}(1) = \Delta(\mathfrak{h}_1) \backslash \Delta(\mathfrak{n} \cap \mathfrak{h}_1) = \{\alpha_1\},$$
$$\mathcal{C}_{\mathfrak{n}}(2) = \Delta(\mathfrak{h}_2) \backslash \Delta(\mathfrak{n} \cap \mathfrak{h}_2) = \{\alpha_5, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 =: [3,5]\}.$$

Therefore, $\Delta(\mathfrak{n}^{\boldsymbol{\zeta}}) = \{\alpha_2, \alpha_2 + \alpha_3, [2,4], [2,5], [2,6], [1,6]\}$. Since $[2,6] - \alpha_2 = [3,6] \in \Delta(\mathfrak{n})$, Condition 4.5 is not satisfied for $\boldsymbol{\zeta}$, and $(\mathfrak{n} : \mathfrak{n}^*)$ does not have a generic stabilizer.

(2) If $\mathfrak{g} = \mathfrak{sl}_7$ and $\mathcal{T}_2 = \{\alpha_1, \alpha_2, \alpha_6\}$, then $\mathfrak{n}_{T_2}$ has the same optimization $\tilde{\mathfrak{n}}$ as $\mathfrak{n}_{\mathcal{T}_1}$, but now $\dim(\tilde{\mathfrak{n}}/\mathfrak{n}_{\mathcal{T}_2}) = 3$ and $\Delta(\mathfrak{n}_{\mathcal{T}_2}^{\boldsymbol{\zeta}}) = \{\alpha_2, \alpha_2 + \alpha_3, [2,4], [2,5], [1,6]\}$. Here, Condition 4.5 is satisfied and $\mathfrak{n}_{\mathcal{T}_2}^{\boldsymbol{\zeta}}$ is a generic stabilizer.

Thus, the action $(N : \mathfrak{n}^*)$ does not always have N-generic stabilizers. A remedy is to consider the action of the larger group $\tilde{N}$ on $\mathfrak{n}^*$.

Theorem 4.7 *For any nilradical $\mathfrak{n}$ and the cascade subspace $\mathfrak{k} \subset \mathfrak{n}^*$, we have:*

(i) $\dim N{\cdot}\mathfrak{k} = 2\dim \mathfrak{n} - \dim \tilde{\mathfrak{n}}$, *i.e.*, $\operatorname{codim}_{\mathfrak{n}^*} N{\cdot}\mathfrak{k} = \dim(\tilde{\mathfrak{n}}/\mathfrak{n})$*;*

(ii) *the $\tilde{N}$-saturation of $\mathfrak{k}$ is dense in $\mathfrak{n}^*$, i.e., $\overline{\tilde{N}{\cdot}\mathfrak{k}} = \mathfrak{n}^*$;*

(iii) *$\tilde{\mathfrak{n}}^{\boldsymbol{\zeta}} = \mathfrak{n}^{\boldsymbol{\zeta}}$ is a generic stabilizer for the linear action $(\tilde{N} : \mathfrak{n}^*)$.*

Proof Recall that $\dim \mathfrak{k} = \operatorname{ind} \tilde{\mathfrak{n}}$ and $\max_{\xi \in \mathfrak{n}^*} \dim N{\cdot}\xi = \dim N{\cdot}\boldsymbol{\zeta} = \dim \mathfrak{n} - \operatorname{ind} \mathfrak{n}$.

(i) Since $\mathsf{T}_{\boldsymbol{\zeta}}(N{\cdot}\boldsymbol{\zeta}) = \mathfrak{n}{\cdot}\boldsymbol{\zeta} = (\mathfrak{n}^{\boldsymbol{\zeta}})^{\perp}$, it follows from Theorem 4.4(iii) that all tangent spaces $\mathfrak{n}{\cdot}\xi$, $\xi \in \mathfrak{k}_0$, are the same. Therefore, using the differential of the map

$$\kappa : N \times \mathfrak{k} \to \overline{N{\cdot}\mathfrak{k}} \subset \mathfrak{n}^*$$

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at $(1,\boldsymbol{\zeta})$, we obtain $\dim \overline{N{\cdot}\mathfrak{k}} = \dim(\operatorname{Im} d\kappa_{(1,\zeta)}) = \dim(\mathfrak{n}{\cdot}\boldsymbol{\zeta} + \mathfrak{k})$. Because the roots in $\mathcal{K}$ are strongly orthogonal, we have $\mathfrak{n}{\cdot}\boldsymbol{\zeta} \cap \mathfrak{k} = \{0\}$ and hence

$$\dim(\mathfrak{n}{\cdot}\boldsymbol{\zeta} + \mathfrak{k}) = \dim \mathfrak{n} - \operatorname{ind} \mathfrak{n} + \operatorname{ind} \tilde{\mathfrak{n}} = 2 \dim \mathfrak{n} - \dim \tilde{\mathfrak{n}}.$$

(ii) Since $\tilde{N}{\cdot}\tilde{\mathfrak{k}}$ is dense in $\tilde{\mathfrak{n}}^*$ (Proposition 4.2) and τ is B-equivariant, we conclude that $\tau(\tilde{N}{\cdot}\tilde{\mathfrak{k}}) = \tilde{N}{\cdot}\mathfrak{k}$ is dense in $\tau(\tilde{\mathfrak{n}}^*) = \mathfrak{n}^*$.

(iii) Let us first prove that

$$\max_{\xi\in\mathfrak{n}^*} \dim \tilde{N}{\cdot}\xi = \max_{\xi\in\mathfrak{n}^*} \dim N{\cdot}\xi + \dim(\tilde{\mathfrak{n}}/\mathfrak{n}). \tag{4.2}$$

Clearly, inequality "$\leqslant$" holds. By Eq. (4.1), the RHS equals $\dim \tilde{\mathfrak{n}} - \operatorname{ind} \mathfrak{n} = \dim \mathfrak{n} - \dim \mathfrak{k}$. On the other hand, part (ii) implies that $\dim \mathfrak{n} = \dim \tilde{N}{\cdot}\mathfrak{k} \leqslant \max_{\xi\in\mathfrak{k}} \dim \tilde{N}{\cdot}\xi + \dim \mathfrak{k}$. This proves Eq. (4.2) and also shows that almost all $\xi \in \mathfrak{k}$ satisfy this relation. Moreover, since $\mathfrak{k}$ is T-stable and $\mathfrak{k}_0 = T{\cdot}\boldsymbol{\zeta}$ is dense in $\mathfrak{k}$, we obtain that

$$\dim \tilde{N}{\cdot}\xi = \dim N{\cdot}\xi + \dim(\tilde{\mathfrak{n}}/\mathfrak{n})$$

for every $\xi \in \mathfrak{k}_0$. Hence, $\tilde{\mathfrak{n}}^\xi = \mathfrak{n}^\xi$ for every $\xi \in \mathfrak{k}_0$. Combining this with Theorem 4.4(iii) and part (ii), we conclude that $\tilde{\mathfrak{n}}^{\boldsymbol{\zeta}} = \mathfrak{n}^{\boldsymbol{\zeta}}$ is a generic stabilizer for the action $(\tilde{N} : \mathfrak{n}^*)$. ■

Using Theorem 4.4, we describe certain nilradicals that do not have a generic stabilizer.

Proposition 4.8 *Let β_j be a descendant of $\beta_i \in \mathcal{K}$ and $\alpha \in \Phi(\beta_i)$. Suppose that $\alpha \notin \mathcal{T}$ and $[\beta_i : \nu] > [\beta_j : \nu] > 0$ for some $\nu \in \mathcal{T}$. Then $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ does not have a generic stabilizer.*

Proof Recall that β_i is the highest root in $\Delta\langle i\rangle^+$ and β_j is a maximal root in $\Delta\langle i\rangle^+ \backslash \Delta(\mathfrak{h}_i)$. If $\Phi(\beta_i) = \{\alpha\}$, then $\alpha + \beta_j \in \Delta(\mathfrak{h}_i) \subset \Delta\langle i\rangle^+$, because β_j is not the highest root of $\Delta\langle i\rangle^+$. If $\Phi(\beta_i) = \{\alpha, \alpha'\}$, then $\Delta\langle i\rangle$ is of type $\mathbf{A}_p$ and hence both $\beta_j + \alpha$ and $\beta_j + \alpha'$ belong to $\Delta(\mathfrak{h}_i)$. In any case, if $\alpha \in \Phi(\beta_i)$, then $\beta_i - \alpha - \beta_j \in \Delta^+$.

Since $\nu \in \mathcal{T}$ and $[\beta_j : \nu] > 0$, we have $\beta_j \in \Delta(\mathfrak{n})$ and also $\beta_i \in \Delta(\mathfrak{n})$. That is, $\beta_i, \beta_j \in \mathcal{K}(\mathfrak{n})$. Because $\alpha \in \Delta(\mathfrak{h}_i)\backslash\Delta(\mathfrak{n})$, we have $\beta_i - \alpha \in \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ (see Theorem 4.4(ii)). The assumption on ν implies that $[\beta_i - \alpha - \beta_j : \nu] > 0$. Thus, we have $\beta_i - \alpha, \beta_j \in \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ and $\beta_i - \alpha - \beta_j \in \Delta(\mathfrak{n})$. By Condition 4.5, this means that $\mathfrak{n}$ has no generic stabilizers. ■

Example 4.9 For applications, it suffices to consider the case in which $\beta_i = \theta$, i.e., $i = 1$.

(1) If $\mathfrak{g}$ is exceptional, then $\#\Phi(\beta_1) = 1$ and β_1 has the unique descendant β_2. Although $\beta_1 - \beta_2$ is not a root, its support is defined as in Section 2.1. Then Proposition 4.8 applies to any $\nu \in \operatorname{supp}(\beta_1 - \beta_2)\backslash\Phi(\beta_1)$ and $\mathcal{T} = \{\nu\}$.

For instance, let $\mathfrak{g}$ be of type $\mathbf{E}_6$. Then $\Phi(\beta_1) = \{\alpha_6\}$, $\operatorname{supp}(\beta_1 - \beta_2) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$, and $\operatorname{supp}(\beta_2) \supset \{\alpha_2, \alpha_3, \alpha_4\}$ (see formulae for $\mathcal{K}$ in Appendix A). Hence, one can take $\nu = \alpha_i$ with $i \in \{2, 3, 4\}$. More generally, if $\alpha_6 \notin \mathcal{T}$ and $\mathcal{T} \cap \{\alpha_2, \alpha_3, \alpha_4\} \neq \varnothing$, then $\mathfrak{n}_{\mathcal{T}}$ has no generic stabilizers.

(2) This method also works for $\mathfrak{g} = \mathfrak{so}_n$ with $n \geqslant 7$, where $\Phi(\beta_1) = \{\alpha_2\}$.

– If $n \neq 8$, then $\beta_1 = \varepsilon_1 + \varepsilon_2$ has two descendants, $\beta_2 = \varepsilon_1 - \varepsilon_2$, and $\beta_3 = \varepsilon_3 + \varepsilon_4$ ($\varepsilon_4 = 0$, if $n = 7$) (see Appendix A). For (β_1, β_3), Proposition 4.8 applies, if one takes

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$\mathcal{T} = \{\alpha_3\}$. While for (β_1, β_2), we can take $\mathcal{T}$ such that $\alpha_2 \notin \mathcal{T}$ and $\mathcal{T} \supset \{\alpha_1, \alpha_j\}$ with $j \geqslant 3$.

– If $n = 8$, then β_1 has three descendants, and we can take $\mathcal{T} = \{\alpha_i, \alpha_j\}$ with $i, j \neq 2$.

Proposition 4.10 *If $\mathfrak{n} \subset \mathfrak{n}'$ and $\tilde{\mathfrak{n}} = \tilde{\mathfrak{n}'}$, then $\mathcal{F}(\mathfrak{n}') \subset \mathcal{F}(\mathfrak{n}) \subset \mathfrak{n}$.*

Proof Let $\mathfrak{k}$ (resp. $\mathfrak{k}'$) be the cascade subspace of $\mathfrak{n}^*$ (resp. $(\mathfrak{n}')^*$). Take $\boldsymbol{\zeta} \in \mathfrak{k}_0$ and $\boldsymbol{\zeta}' \in \mathfrak{k}'_0$. Since $\Delta(\mathfrak{n}) \subset \Delta(\mathfrak{n}')$ and $\mathcal{K}(\mathfrak{n}) = \mathcal{K}(\mathfrak{n}')$, we have $\mathcal{C}_{\mathfrak{n}}(j) \supset \mathcal{C}_{\mathfrak{n}'}(j)$ for all $\beta_j \in \mathcal{K}(\mathfrak{n})$. Then Theorem 4.4 shows that $\mathfrak{n}^{\boldsymbol{\zeta}} \supset (\mathfrak{n}')^{\boldsymbol{\zeta}'}$. By Proposition 4.1, this yields the required embedding. ∎

Our description of $\Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ yields a criterion for $\mathfrak{n}$ to be quasi-quadratic.

Theorem 4.11 *For a nilradical $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$, the following assertions are equivalent:*

- *$\mathcal{F}(\mathfrak{n}) = \mathfrak{n}$, i.e., $\mathfrak{n}$ is quasi-quadratic.*
- *For each $\alpha \in \mathcal{T}$, one of the two conditions is satisfied:*
  - *either $\alpha \in \mathcal{K}$;*
  - *or $\Phi(\Phi^{-1}(\alpha)) = \{\alpha, \alpha'\}$, and if C is the chain in the Dynkin diagram that connects α and α', then $C \cap \mathcal{T} = \{\alpha\}$ (in particular, $\alpha' \notin \mathcal{T}$).*

Proof Since $\mathcal{F}(\mathfrak{n})$ is the $\mathfrak{b}$-ideal generated by $\mathfrak{n}^{\boldsymbol{\zeta}}$ (Proposition 4.1), it is clear that $\mathcal{F}(\mathfrak{n}) = \mathfrak{n}$ if and only if $\alpha \in \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ for each $\alpha \in \mathcal{T}$.

(1) If $\alpha \in \mathcal{T} \cap \mathcal{K}$, then $\Phi(\alpha) = \alpha$ and $\alpha \in \Delta(\tilde{\mathfrak{n}}^{\boldsymbol{\zeta}}) \subset \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$.

(2) If $\alpha \in \mathcal{T} \backslash \mathcal{K}$ and $\Phi^{-1}(\alpha) = \beta_j \in \mathcal{K}_{\mathcal{T}}$, then $\alpha \neq \beta_j$ and there are two possibilities.

– $\Phi(\beta_j) = \alpha$. Then the whole Heisenberg algebra $\mathfrak{h}_j$ belongs to $\mathfrak{n}$ and hence $\mathcal{C}_{\mathfrak{n}}(j) = \varnothing$. Then it follows from Theorem 4.4(ii) that $\Delta(\mathfrak{h}_j \cap \mathfrak{n}^{\boldsymbol{\zeta}}) = \{\beta_j\}$, i.e., $\alpha \notin \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$.

– $\Phi(\beta_j) = \{\alpha, \alpha'\}$. Here, β_j is the highest root in a root system of type $\mathbf{A}_p$ $(p \geqslant 2)$. Therefore, if $C = \{\alpha = \alpha_1, \alpha_2, \dots, \alpha_p = \alpha'\}$ is the chain connecting α and α', then $\beta_j = \alpha_1 + \alpha_2 + \dots + \alpha_p$. Then $\alpha \in \Delta(\mathfrak{n}^{\boldsymbol{\zeta}})$ if and only if $\beta_j - \alpha = \alpha_2 + \dots + \alpha_p \notin \Delta(\mathfrak{n})$, and this is only possible if $\operatorname{supp}(\beta_j) \cap \mathcal{T} = C \cap \mathcal{T} = \{\alpha\}$. ∎

Using Theorem 4.11, one readily obtains the list of all quasi-quadratic nilradicals.

Proposition 4.12 *The quasi-quadratic nilradicals are as follows.*

1. *If $\mathfrak{g}$ is not of type $\boldsymbol{A}_n, \boldsymbol{D}_{2n+1}$, and $\boldsymbol{E}_6$, then $\mathcal{F}(\mathfrak{n}_{\mathcal{T}}) = \mathfrak{n}_{\mathcal{T}}$ if and only if $\mathcal{T} \subset \mathcal{K}$.*
2. *If $\mathfrak{g}$ is of type $\boldsymbol{A}_n$, then $\mathcal{F}(\mathfrak{n}_{\mathcal{T}}) = \mathfrak{n}_{\mathcal{T}}$ if and only if $\mathcal{T} = \{\alpha_i\}$, $i = 1, 2, \dots, n$.*
3. *If $\mathfrak{g}$ is of type $\boldsymbol{D}_{2n+1}$, then $\mathcal{F}(\mathfrak{n}_{\mathcal{T}}) = \mathfrak{n}_{\mathcal{T}}$ if and only if $\mathcal{T} \cap \{\alpha_2, \alpha_4, \dots, \alpha_{2n-2}\} = \varnothing$ and $\#(\mathcal{T} \cap \{\alpha_{2n-1}, \alpha_{2n}, \alpha_{2n+1}\}) \leqslant 1$.*
4. *If $\mathfrak{g}$ is of type $\boldsymbol{E}_6$, then $\mathcal{F}(\mathfrak{n}_{\mathcal{T}}) = \mathfrak{n}_{\mathcal{T}}$ if and only if $\mathcal{T} = \{\alpha_i\}$, $i \neq 6$.*

Proof (1) In this case, Φ^{-1} is a bijection; hence, $\Phi(\Phi^{-1}(\alpha)) = \{\alpha\}$ for any $\alpha \in \Pi$.

(2) Here, each $\mathfrak{n}$ has a CP [10]; hence, $\mathcal{F}(\mathfrak{n})$ is abelian. That is, $\mathcal{F}(\mathfrak{n}) = \mathfrak{n}$ must be an abelian nilradical.

(3) In this case, $\Pi \cap \mathcal{K} = \{\alpha_1, \alpha_3, \dots, \alpha_{2n-1}\}$ and there is a unique $\beta \in \mathcal{K}$ such that $\#\Phi(\beta) = 2$. Namely, $\Phi(\beta_{2n-1}) = \{\alpha_{2n}, \alpha_{2n+1}\}$ (see Appendix A). The chain connecting α_{2n} and α_{2n+1} in the Dynkin diagram is $C = \{\alpha_{2n}, \alpha_{2n-1}, \alpha_{2n+1}\}$. Hence the answer.

(4) Here, $\Pi \cap \mathcal{K} = \{\alpha_3\} = \{\beta_4\}$, $\Phi(\beta_3) = \{\alpha_2, \alpha_4\}$, $\Phi(\beta_2) = \{\alpha_1, \alpha_5\}$ (see Appendix A). Since $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ form a chain in the Dynkin diagram, the result follows. ∎

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5 Generic stabilizers and the Frobenius semiradical in case of $\mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$

First, we explicitly describe the nilradicals in $\mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$ having a generic stabilizer. To state the result for $\mathfrak{sl}_{n+1}$, we need some notation. Recall that, for $\mathfrak{sl}_{n+1}$, the poset $\mathcal{K}$ is a chain $\beta_1 \succ \beta_2 \succ \cdots \succ \beta_t$, where $t = [(n+1)/2]$ and $\Phi(\beta_i) = \{\alpha_i, \alpha_{n+1-i}\}$. Let $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}} \subset \mathfrak{sl}_{n+1}$ be a nilradical. Then $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \ldots, \beta_k\}$ for some $k \leqslant t$ (cf. Lemma 2.2). Therefore, $\mathcal{T} \subset \{\alpha_1, \ldots, \alpha_k, \alpha_{n+1-k}, \ldots, \alpha_n\}$ and $\mathcal{T} \cap \{\alpha_k, \alpha_{n+1-k}\} \neq \varnothing$. Set $\mathcal{T}' = \mathcal{T} \cap \{\alpha_1, \ldots, \alpha_{k-1}\}$ and $\mathcal{T}'' = \mathcal{T} \cap \{\alpha_{n+2-k}, \ldots, \alpha_n\}$.

Theorem 5.1 *Suppose that $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \ldots, \beta_k\}$. Then*

$$\mathfrak{n}^{\zeta} \text{ is a generic stabilizer } \Leftrightarrow \sigma(\mathcal{T}') = \mathcal{T}'', \text{ where } \sigma \text{ is the symmetry of Dynkin diagram } \mathbf{A}_n.$$

(That is, $\sigma(\alpha_j) = \alpha_{n+1-j}$.) In particular, if $\sigma(\mathcal{T}) = \mathcal{T}$, then $\mathfrak{n}_{\mathcal{T}}$ has a generic stabilizer.

Proof 1^o. For $k = 1$, we have $\mathcal{T}' = \mathcal{T}'' = \varnothing$ and $\mathfrak{n} \subset \mathfrak{h}_1$. If $\mathcal{T} = \{\alpha_1, \alpha_n\}$, then $\mathfrak{n} = \mathfrak{h}_1$ is optimal. If $\mathcal{T} = \{\alpha_1\}$, then $\mathfrak{n}$ is abelian. In both cases, there is a generic stabilizer for $(\mathfrak{n} : \mathfrak{n}^*)$, as required.

2^o. Suppose that $k \geqslant 2$ and the symmetry of $\mathcal{T}'$ and $\mathcal{T}''$ fails for some $j \leqslant k-1$. W.l.o.g, we may assume that $\alpha_j \in \mathcal{T}'$, whereas $\alpha_{n+1-j} \notin \mathcal{T}''$. Then $\beta_j - \alpha_{n+1-j} = \beta_{j+1} + \alpha_j \in \Delta(\mathfrak{n}^{\zeta})$. Since $\beta_{j+1} \in \mathcal{K}(\mathfrak{n}) \subset \Delta(\mathfrak{n}^{\zeta})$ and $\alpha_j \in \Delta(\mathfrak{n})$, Condition 4.5 is not satisfied.

3^o. If $\sigma(\mathcal{T}') = \mathcal{T}''$, then one can directly describe $\mathfrak{n}^{\zeta}$ and see that Condition 4.5 is satisfied. It is necessary to distinguish two cases: **(a)** $\alpha_k, \alpha_{n+1-k} \in \mathcal{T}$ and **(b)** only $\alpha_k \in \mathcal{T}$.

(a) Here, $\sigma(\mathcal{T}) = \mathcal{T}$ and $\Phi(\beta_k) \subset \Delta(\mathfrak{n})$. Hence, $\mathfrak{h}_k \subset \mathfrak{n}$ and $\mathcal{C}_{\mathfrak{n}}(k) = \varnothing$. Theorem 4.4(ii) shows that $\mathfrak{n}^{\zeta} \subset \mathfrak{n}_{\{\alpha_k\}} \cap \mathfrak{n}_{\{\alpha_{n+1-k}\}}$, the last intersection being the north-east $k \times k$ square of $(n+1) \times (n+1)$ matrices in $\mathfrak{sl}_{n+1}$. More precisely, if $\mathcal{T}' \cup \{\alpha_k\} = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_j} = \alpha_k\}$, then $\Delta(\mathfrak{n}^{\zeta}) = \Gamma_1 \cup \cdots \cup \Gamma_j$, where

$$\Gamma_1 = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i \leqslant i_1,\ n+2-i_1 \leqslant j \leqslant n+1\},$$
$$\Gamma_2 = \{\varepsilon_i - \varepsilon_j \mid i_1 + 1 \leqslant i \leqslant i_2,\ n+2-i_2 \leqslant j \leqslant n+1-i_1\}, \text{etc.}$$

Here, $\{\Gamma_s\}_{s=1}^{j}$ is a string of square blocks located along the antidiagonal in the $k \times k$ square (see Figure 1a). The size of the sth square is $i_s - i_{s-1}$, where $i_0 = 0$. It is readily seen that if γ and γ' belong to different blocks, then $\gamma - \gamma'$ is not a root, while if γ and γ' belong to the same block, then $\gamma - \gamma'$ is either not a root or a root of the standard Levi subalgebra corresponding to $\mathfrak{n}$. Thus, $\gamma - \gamma' \notin \Delta(\mathfrak{n})$ and Condition 4.5 is satisfied.

(b) Here, $\mathfrak{n}$ is smaller than in part **(a)**, but $\mathcal{K}(\mathfrak{n})$ remains the same. Now only "half" of $\mathfrak{h}_k$ belongs to $\mathfrak{n}$. Therefore, $\mathcal{C}_{\mathfrak{n}}(k) \neq \varnothing$, and the subsets $\mathcal{C}_{\mathfrak{n}}(s)$ with $i_{j-1} < s < k$ become larger than in **(a)**. Hence, $\Delta(\mathfrak{n}^{\zeta})$ becomes larger and, along with $\Gamma_1 \cup \cdots \cup \Gamma_j$, it also contains the strip of roots

$$\tilde{\Gamma} = \{\varepsilon_i - \varepsilon_j \mid i_{j-1} + 1 \leqslant i \leqslant i_j = k,\ k+1 \leqslant j \leqslant n+1-k\}$$

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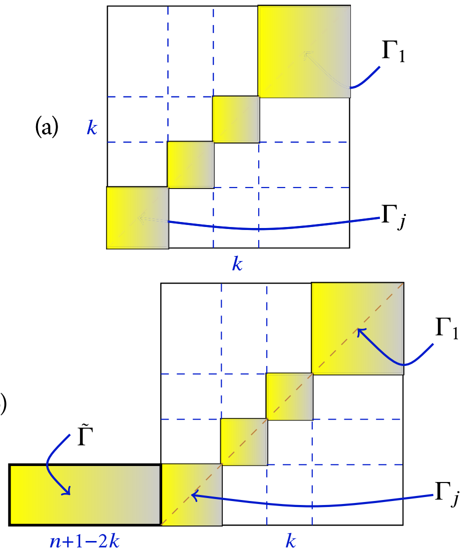

Figure 1: The generic stabilizer $\mathfrak{n}^\xi$: cases (a) and (b).

attached to Γ_j (see Figure 1b). The new feature is that the difference of roots from Γ_j and $\tilde\Gamma$ can be a root in $\bigcup_{s=i_{j-1}}^{k}\Delta(\mathfrak{h}_s)$. But such a difference belongs to the parts of sets $\Delta(\mathfrak{h}_s)$ that are missing in $\Delta(\mathfrak{n})$. Thus, Condition 4.5 is still satisfied here. ∎

The symmetry condition of Theorem 5.1 means that the matrix shape of $\mathfrak{n}\subset\mathfrak{sl}_{n+1}$ must be "almost" symmetric w.r.t. the antidiagonal. That is, the symmetry may only fail in case **(b)** for roots in $\Delta(\mathfrak{h}_s)$ with $i_{j-1}<s\leqslant i_j=k$ (if $\alpha_k\in\mathcal{T}$, but $\alpha_{n+1-k}\notin\mathcal{T}$).

Remark 5.2 Using Theorem 5.1, one easily computes the number of nontrivial (standard) nilradicals with generic stabilizers. For $\mathbf{A}_{2n-1}$, it is $2^{n+1}-3$; for $\mathbf{A}_{2n}$, it is $3(2^n-1)$. Hence, the ratio #{nilradicals with generic stabilizer}/#{all nilradicals} exponentially decreases.

Theorem 5.3 *If $\mathfrak{g}=\mathfrak{sp}_{2n}$, then a generic stabilizer exists for every nilradical $\mathfrak{n}$.*

Proof For an appropriate choice of a skew-symmetric bilinear form defining $\mathfrak{sp}_{2n}\subset\mathfrak{sl}_{2n}$, a Borel subalgebra of $\mathfrak{sp}_{2n}$, $\mathfrak{b}(\mathfrak{sp}_{2n})$, is the set of symplectic upper-triangular matrices (see Example 2.6). Then the matrix shape of **any** (standard) nilradical in $\mathfrak{sp}_{2n}$ is symmetric w.r.t. the antidiagonal, and the approach in the proof of 3^o**(a)** in Theorem 5.1 applies to any nilradical in $\mathfrak{sp}_{2n}$. If $\mathfrak{n}=\mathfrak{n}_{\mathcal{T}}$ and $\mathcal{T}=\{\alpha_{i_1},\dots,\alpha_{i_j}\}$, where $i_1<i_2<\cdots<i_j=k$, then $\mathfrak{n}^\xi$ belongs to the north-east $k\times k$ square in the abelian nilradical $\mathfrak{n}_{\{\alpha_n\}}$. Here, the blocks Γ_j inside this square represent matrices that are symmetric w.r.t. the antidiagonal (cf. Figure 1a). ∎

For these two series, we explicitly describe $\mathcal{F}(\mathfrak{n})$ for any $\mathfrak{n}$. This also demonstrates the role of commutative polarizations.

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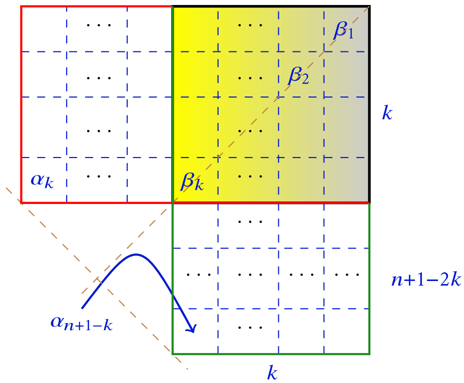

Figure 2: $\mathcal{F}(\mathfrak{n}) = \mathfrak{n}_{\{\alpha_k\}} \cap \mathfrak{n}_{\{\alpha_{n+1-k}\}}$, $\mathfrak{n} \supset \mathfrak{n}_{\{\alpha_k\}} \cup \mathfrak{n}_{\{\alpha_{n+1-k}\}}$.

5.1 The Frobenius semiradical $\mathcal{F}(\mathfrak{n})$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Let $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ be a nilradical such that $\min \mathcal{K}(\mathfrak{n}) = \{\beta_k\}$. Then $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \ldots, \beta_k\}$, $\mathcal{T} \subset \{\alpha_1, \ldots, \alpha_k, \alpha_{n+1-k}, \ldots, \alpha_n\}$, and $\Phi(\beta_k) \cap \mathcal{T} \neq \varnothing$. We may assume that $\alpha_k \in \mathcal{T}$ and then, as in the proof of Theorem 5.1, there are two possibilities.

(a) $\alpha_{n+1-k} \in \mathcal{T}$. Then $\mathfrak{n}_{\{\alpha_k\}}$ and $\mathfrak{n}_{\{\alpha_{n+1-k}\}}$ are CP-ideals of $\mathfrak{n}$ [22, Example 3.8], and hence $\mathcal{F}(\mathfrak{n}) \subset \mathfrak{n}_{\{\alpha_k\}} \cap \mathfrak{n}_{\{\alpha_{n+1-k}\}}$ (cf. Section 3.4(3)). Here, $\mathfrak{s} := \mathfrak{n}_{\{\alpha_k\}} \cap \mathfrak{n}_{\{\alpha_{n+1-k}\}}$ is the north-east square of size k in $\mathfrak{sl}_{n+1}$. On the other hand, $\mathfrak{n}^{\zeta} \subset \mathfrak{s}$ (see Theorem 5.1) and $\mathfrak{b}{\cdot}\mathfrak{n}^{\zeta} = \mathfrak{s}$. Hence, $\mathcal{F}(\mathfrak{n}) = \mathfrak{s}$ and $\dim \mathcal{F}(\mathfrak{n}) = k^2$ (see Figure 2). For the special case of $\mathfrak{n} = \mathfrak{u}$, the description of $\mathcal{F}(\mathfrak{u})$ is obtained in [19, Theorem 4.1].

Actually, here, $\mathfrak{a}_j := \mathfrak{n}_{\{\alpha_j\}} \cap \mathfrak{n}$ is a CP-ideal of $\mathfrak{n}$ for any j such that $k \leqslant j \leqslant n+1-k$ (see Example 4.8 in [22]).

(b) $\alpha_{n+1-k} \notin \mathcal{T}$. Then only $\mathfrak{n}_{\{\alpha_k\}}$ is a CP-ideal and $\mathcal{F}(\mathfrak{n}) \subset \mathfrak{n}_{\{\alpha_k\}}$. On the other hand, $\mathrm{supp}(\beta_k - \alpha_k) = \{\alpha_{k+1}, \ldots, \alpha_{n+1-k}\}$. Hence, $\beta_k - \alpha_k \notin \Delta(\mathfrak{n})$ and $\alpha_k \in \Delta(\mathfrak{n}^{\zeta})$ (see Theorem 4.4(ii)). By Proposition 4.1(i), this implies that $\mathfrak{n}_{\{\alpha_k\}} \subset \mathcal{F}(\mathfrak{n})$. Thus, here, $\mathcal{F}(\mathfrak{n}) = \mathfrak{n}_{\{\alpha_k\}}$ and $\dim \mathcal{F}(\mathfrak{n}) = k(n+1-k)$. Since $\mathcal{F}(\mathfrak{n})$ appears to be a CP, we conclude that $\mathfrak{n}_{\{\alpha_k\}}$ is the only CP for $\mathfrak{n}$.

It follows from the descriptions above that for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathcal{F}(\mathfrak{n})$ is always equal to the intersection of all CP-ideals of $\mathfrak{n}$. But this is not true for other simple Lie algebras.

5.2 The Frobenius semiradical $\mathcal{F}(\mathfrak{n})$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$

Let $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ be a nilradical such that $\min \mathcal{K}(\mathfrak{n}) = \{\beta_k\}$. Then $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \ldots, \beta_k\}$, $\mathcal{T} \subset \{\alpha_1, \ldots, \alpha_k\}$ and $\alpha_k \in \mathcal{T}$. The abelian nilradical $\mathfrak{n}_{\{\alpha_n\}}$ is identified with the space of $n \times n$ matrices that are symmetric w.r.t. the antidiagonal (see Example 2.6). Then $\mathfrak{n}^{\zeta}$ belongs to the north-east $k \times k$ square in $\mathfrak{n}_{\{\alpha_n\}}$, and the south-west corner of this square, $\mathfrak{g}_{\beta_k}$, lies in $\mathfrak{n}^{\zeta}$. Hence, $\mathcal{F}(\mathfrak{n}) = \mathfrak{b}{\cdot}\mathfrak{n}^{\zeta}$ is equal to this k-square and $\dim \mathcal{F}(\mathfrak{n}) = k(k+1)/2$. In this case, $\mathfrak{a} := \mathfrak{n}_{\{\alpha_n\}} \cap \mathfrak{n}$ is the only CP-ideal of $\mathfrak{n}$ and the inclusion $\mathcal{F}(\mathfrak{n}) \subset \mathfrak{a}$ is proper unless $k = n$ (see the following picture).

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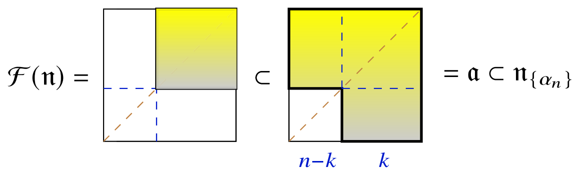

Remark 5.4 The series $\mathbf{A}_n$ and $\mathbf{C}_n$ are easily handled, because $\mathcal{K}$ is a chain for them and any nilradical has a CP. Therefore, $\mathcal{F}(\mathfrak{n})$ is an **abelian** ideal of $\mathfrak{n}$, and there is a natural upper bound on $\mathcal{F}(\mathfrak{n})$. However, it can happen that $\mathfrak{n}$ does not have a CP, but $\mathcal{F}(\mathfrak{n})$ is abelian (see Example 7.3).

6 The Poisson center and *U*-invariants

Recall that $P = L{\cdot}N$ is a standard parabolic subgroup of G and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. Since N is connected, one has $\mathcal{S}(\mathfrak{n})^N = \mathcal{S}(\mathfrak{n})^{\mathfrak{n}}$, and this algebra is the center of the Poisson algebra $(\mathcal{S}(\mathfrak{n}), \{\ ,\ \})$. Because $\mathfrak{n}$ is a P-module, one can consider algebras of invariants in $\mathcal{S}(\mathfrak{n})$ for any subgroup $Q \subset P$. Specifically, we are interested in the groups U and $\tilde{N}$, where $\operatorname{Lie} \tilde{N} = \tilde{\mathfrak{n}}$ is the optimization of $\mathfrak{n}$. We also wish to compare the algebras of Q-invariants in $\mathcal{S}(\mathfrak{n})$ and $\mathcal{S}(\tilde{\mathfrak{n}})$. The algebras of interest for us are organized in the following diagram:

$$\begin{array}{ccccc} \mathcal{S}(\tilde{\mathfrak{n}})^U & \subset & \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}} & & \\ \cup & & \cup & & \\ \mathcal{S}(\mathfrak{n})^U & \subset & \mathcal{S}(\mathfrak{n})^{\tilde{N}} & \subset & \mathcal{S}(\mathfrak{n})^N. \end{array} \tag{6.1}$$

If an algebra $\mathcal{S}(\mathfrak{n})^Q = \mathbb{C}[\mathfrak{n}^*]^Q$ is finitely generated, then we also consider the associated quotient morphism $\pi_Q : \mathfrak{n}^* \to \mathfrak{n}^* /\!\!/ Q := \operatorname{Spec}(\mathbb{C}[\mathfrak{n}^*]^Q)$.

For any nilradical $\mathfrak{n}$, one can form the solvable Lie algebra

$$\mathfrak{f}_{\mathfrak{n}} = \mathfrak{f}_{\tilde{\mathfrak{n}}} = \mathfrak{t}_{\mathfrak{n}} \oplus \tilde{\mathfrak{n}} \subset \mathfrak{b},$$

where $\mathfrak{t}_{\mathfrak{n}} = \oplus_{\beta \in \mathcal{K}(\mathfrak{n})} [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \subset \mathfrak{t}$. By [22, Proposition 5.1], the corresponding connected group $F_{\mathfrak{n}} \subset B$ has an open orbit in $\mathfrak{f}_{\mathfrak{n}}^*$, i.e., $\mathfrak{f}_n$ is a Frobenius Lie algebra. For this reason, $\mathfrak{f}_{\mathfrak{n}}$ is called the *Frobenius envelope* of $\mathfrak{n}$. Note that the unipotent radical of $F_{\mathfrak{n}}$ is $\tilde{N}$.

Lemma 6.1 *The four algebras of invariants forming the square in (6.1) are polynomial. In particular, for any* **optimal** *nilradical $\tilde{\mathfrak{n}}$, the Poisson center $\mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}}$ is polynomial.*

Proof 1^o. For any nilradical $\mathfrak{n}$, the action $(B : \mathfrak{n}^*)$ is locally transitive and $B{\cdot}\boldsymbol{\zeta}$ is the dense orbit in $\mathfrak{n}^*$ [12, 2.4]. Therefore, $\mathcal{S}(\mathfrak{n})^U$ is a polynomial algebra and $\operatorname{trdeg} \mathcal{S}(\mathfrak{n})^U$ equals the number of prime divisors in $\mathfrak{n}^* \setminus B{\cdot}\boldsymbol{\zeta}$ (see [21, Theorem 4.4]). Hence, the algebras in the first column of (6.1) are polynomial.

2^o. More generally, the $F_{\mathfrak{n}}$-equivariant embeddings $\mathfrak{n} \hookrightarrow \tilde{\mathfrak{n}} \hookrightarrow \mathfrak{f}_{\mathfrak{n}}$ yield the $F_{\mathfrak{n}}$-equivariant projections $\mathfrak{f}_{\mathfrak{n}}^* \to \tilde{\mathfrak{n}}^* \to \mathfrak{n}^*$, which implies that $F_{\mathfrak{n}}$ has a dense orbit in $\mathfrak{n}^*$, too. Therefore, arguing as in [21, Section 4], one proves that $\mathcal{S}(\mathfrak{n})^{\tilde{N}}$ is also a polynomial algebra. Thus, the algebras in the second column of (6.1) are polynomial, too. ■

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If $\mathfrak{n} \neq \tilde{\mathfrak{n}}$, then the Poisson center $\mathcal{S}(\mathfrak{n})^N$ is not always polynomial (see examples below).

Lemma 6.2 *For any nilradical* $\mathfrak{n}$*, we have* $\mathcal{S}(\mathfrak{n})^{\tilde{N}} = \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}}$.

Proof It is clear that $\mathcal{S}(\mathfrak{n})^{\tilde{N}} \subset \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}}$. On the other hand, by [22, Proposition 5.5], the equality $\boldsymbol{b}(\mathfrak{n}) = \boldsymbol{b}(\tilde{\mathfrak{n}})$ (see Proposition 2.3) implies that $\mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}} \subset \mathcal{S}(\mathfrak{n})$. Hence, $\mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}} \subset \mathcal{S}(\mathfrak{n})^{\tilde{N}}$. ■

Lemma 6.3 *For any nilradical* $\mathfrak{n}$*, we have* $\mathcal{S}(\mathfrak{n})^{\tilde{N}} = \mathcal{S}(\mathfrak{n})^{U}$.

Proof **1**^{o}. Let us first prove that $\operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{\tilde{N}} = \operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{U}$. Since $\tilde{N}$ is unipotent, the field of invariants $\mathbb{C}(\mathfrak{n}^*)^{\tilde{N}}$ is the quotient field of the algebra of invariants $\mathbb{C}[\mathfrak{n}^*]^{\tilde{N}} = \mathcal{S}(\mathfrak{n})^{\tilde{N}}$ [4, Chapter 1.4]. Furthermore, by Rosenlicht's theorem [4, Chapter 1.6], one has

$$\operatorname{trdeg} \mathbb{C}(\mathfrak{n}^*)^{\tilde{N}} + \max_{\xi \in \mathfrak{n}^*} \dim \tilde{N}{\cdot}\xi = \dim \mathfrak{n},$$

and the same holds for U in place of $\tilde{N}$. Therefore, since

$$\max_{\xi \in \mathfrak{n}^*} \dim U{\cdot}\xi \geqslant \max_{\xi \in \mathfrak{n}^*} \dim \tilde{N}{\cdot}\xi,$$

it suffices to prove that one has equality here. By Theorem 4.7, $\tilde{N}{\cdot}\mathfrak{k}$ is dense in $\mathfrak{n}^*$. Hence, $U{\cdot}\mathfrak{k}$ is dense in $\mathfrak{n}^*$, too.

Now, $\mathfrak{u} = (\mathfrak{u} \cap \tilde{\mathfrak{l}}) \oplus \tilde{\mathfrak{n}}$, where $\tilde{\mathfrak{l}}$ is the standard Levi subalgebra of $\tilde{\mathfrak{p}} := \operatorname{norm}_{\mathfrak{g}}(\tilde{\mathfrak{n}})$. Since $\tilde{\mathfrak{n}}$ is optimal, $\mathfrak{u} \cap \tilde{\mathfrak{l}}$ stabilizes any $\xi \in \mathfrak{k}_0 \subset \mathfrak{k}$. Hence, $\mathfrak{u}^{\xi} = (\mathfrak{u} \cap \tilde{\mathfrak{l}}) \oplus \tilde{\mathfrak{n}}^{\xi}$ and $\dim U{\cdot}\xi = \dim \tilde{N}{\cdot}\xi$. Therefore, $\max_{\xi \in \mathfrak{n}^*} \dim U{\cdot}\xi = \max_{\xi \in \mathfrak{n}^*} \dim \tilde{N}{\cdot}\xi$, and we are done.

2^o. By the first part, the extension $\mathbb{C}[\mathfrak{n}^*]^{\tilde{N}} = \mathcal{S}(\mathfrak{n})^{\tilde{N}} \subset \mathcal{S}(\mathfrak{n})^{U}$ is algebraic. But the algebra of invariants of a **connected** algebraic group Q acting on an affine variety X is algebraically closed in $\mathbb{C}[X]$ (see [14, p. 100]). ■

Combining previous lemmas yields our main result on diagram (6.1).

Theorem 6.4 *The four algebras of invariants that form the square in (6.1) are equal, i.e.,*

$$\mathcal{S}(\mathfrak{n})^{U} = \mathcal{S}(\tilde{\mathfrak{n}})^{U} = \mathcal{S}(\mathfrak{n})^{\tilde{N}} = \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}}.$$

These algebras are polynomial, and their common transcendence degree is $\#\mathcal{K}(\mathfrak{n}) = \operatorname{ind} \tilde{\mathfrak{n}}$. *If* $\mathfrak{n} \neq \tilde{\mathfrak{n}}$, *then* $\operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{N} = \operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{\tilde{N}} + \dim(\tilde{\mathfrak{n}}/\mathfrak{n}) > \operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{\tilde{N}}$. *Thus, there are at most two different algebras of invariants in (6.1).*

Proof **(1)** By Lemma 6.1, these four algebras are polynomial.

(2) By Lemmas 6.2 and 6.3, these four algebras are equal. (Note that Lemma 6.3 applies also to $\tilde{\mathfrak{n}}$ in place of $\mathfrak{n}$.)

(3) Since both $\tilde{N}$ and N are unipotent, it follows from the Rosenlicht theorem that $\operatorname{trdeg} \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}} = \operatorname{ind} \tilde{\mathfrak{n}}$ and $\operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{N} = \operatorname{ind} \mathfrak{n}$. Hence, the last relation is just Eq. (4.1). ■

Remark 6.5 The algebra of U-invariants in $\mathcal{S}(\mathfrak{r})$ is polynomial for an arbitrary $\mathfrak{b}$-stable ideal $\mathfrak{r}$ of $\mathfrak{u}$. That is, if $\mathfrak{r} \subset \mathfrak{u}$ and $[\mathfrak{b}, \mathfrak{r}] \subset \mathfrak{r}$, then $\mathcal{S}(\mathfrak{r})^U$ is a polynomial algebra. The reason is that B has an open orbit in $\mathfrak{r}^*$ (see [21, Section 4] for details).

Corollary 6.6 *For any nilradical $\mathfrak{n}$ and a cascade point $\boldsymbol{\zeta} \in \mathfrak{k}_0 \subset \mathfrak{n}^*$, we have:*
(1) $\#\mathcal{K}(\mathfrak{n}) = \#\{$*the divisors in* $\mathfrak{n}^* \setminus B{\cdot}\boldsymbol{\zeta}\} = \#\{$*the divisors in* $\mathfrak{n}^* \setminus F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}\}$*;*
(2) $B{\cdot}\boldsymbol{\zeta} = F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}$.

Proof (1) It is known that:

- $\#\mathcal{K}(\mathfrak{n}) = \operatorname{ind} \tilde{\mathfrak{n}} = \operatorname{trdeg} \mathcal{S}(\tilde{\mathfrak{n}})^{\tilde{N}}$;
- $\#\{$the divisors in $\mathfrak{n}^* \setminus B{\cdot}\boldsymbol{\zeta}\} = \operatorname{trdeg} \mathcal{S}(\mathfrak{n})^U$ [21, Theorem 4.4];
- $\#\{$the divisors in $\mathfrak{n}^* \setminus F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}\} = \operatorname{trdeg} \mathcal{S}(\mathfrak{n})^{\tilde{N}}$.

The last equality relies on the facts that the orbit $F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}$ is open in $\mathfrak{n}^*$ and $\tilde{N}$ is the unipotent radical of $F_{\mathfrak{n}}$. Hence, [21, Theorem 4.4] applies also in this case.

(2) Since B and $F_{\mathfrak{n}}$ are solvable, the orbits $B{\cdot}\boldsymbol{\zeta}$ and $F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}$ are affine. Therefore, both $\mathfrak{n}^* \setminus B{\cdot}\boldsymbol{\zeta}$ and $\mathfrak{n}^* \setminus F_{\mathfrak{n}}{\cdot}\boldsymbol{\zeta}$ are the union of divisors and the assertion follows from (1). ■

If $\mathfrak{n}$ is not optimal, then $\mathcal{S}(\mathfrak{n})^N$ is not always polynomial (see Example 6.10). Actually, there are Lie algebras $\mathfrak{q}$ such that $\mathcal{S}(\mathfrak{q})^Q$ is not finitely generated! A construction of such $\mathfrak{q}$ utilizing the Nagata counterexample to Hilbert's 14th problem is given by Dixmier in [5, 4.9.20(c)]. Nevertheless, we demonstrate below that, for the nilradicals in $\mathfrak{g}$, the algebra $\mathcal{S}(\mathfrak{n})^N$ is as good as the algebras of invariants of linear actions of reductive groups. Recall that a commutative associative $\mathbb{C}$-algebra $\mathcal{A}$ equipped with action of an algebraic group Q is called a *rational* Q-algebra if, for any $a \in \mathcal{A}$, the linear span in $\mathcal{A}$ of the orbit $Q{\cdot}a$ is finite-dimensional and the representation of Q on $\langle Q{\cdot}a \rangle$ is rational (see, e.g., [11, p.1]).

Theorem 6.7 *Let $\mathfrak{n}$ be an arbitrary nilradical in $\mathfrak{g}$. Then:*
(i) *the algebra $\mathcal{S}(\mathfrak{n})^N = \mathbb{C}[\mathfrak{n}^*]^N$ is finitely generated;*
(ii) *the quotient variety $\mathfrak{n}^* /\!\!/ N$ has rational singularities.*

Proof (i) Let $\mathfrak{p}$ be the parabolic subalgebra with $\mathfrak{n} = \mathfrak{p}^{\mathsf{nil}}$. Let $\mathfrak{l}$ be the standard Levi subalgebra of $\mathfrak{p}$ and L the corresponding Levi subgroup of $P = \mathsf{Norm}_G(\mathfrak{n}) \subset G$. Since $[\mathfrak{l}, \mathfrak{n}] \subset \mathfrak{n}$, the algebra $\mathcal{S}(\mathfrak{n})^N$ is a rational L-algebra. Set $U(L) = U \cap L$. It is a maximal unipotent subgroup of L, and U is a semi-direct product of $U(L)$ and N. By Theorem 6.4, $\mathcal{S}(\mathfrak{n})^U = (\mathcal{S}(\mathfrak{n})^N)^{U(L)}$ is a polynomial algebra. Now, we can apply Corollary 4 from [23, Section 3], which asserts that if $\mathcal{A}$ is a rational L-algebra, then $\mathcal{A}$ is finitely generated if and only if $\mathcal{A}^{U(L)}$ is. In our case, $\mathcal{A} = \mathcal{S}(\mathfrak{n})^N$ and $\mathcal{A}^{U(L)}$ is a polynomial algebra, with $\#\mathcal{K}(\mathfrak{n})$ generators. Hence, $\mathcal{S}(\mathfrak{n})^N$ is finitely generated.

(ii) Let ***(P)*** be a so-called "stable property" of local rings of algebraic varieties (see [23, Section 6] for the details). By [23, Theorem 6], $\mathcal{A}$ has property ***(P)*** if and only if $\mathcal{A}^{U(L)}$ has. Since rationality of singularities is such a property, we obtain the second assertion. ■

Remark 6.8 (1) Part (i) of Theorem 6.7 appears in [12, Lemma 4.6(ii)], with another proof. Namely, in place of reference to [23], one can provide the following simple argument for the implication:

$$\mathcal{A}^{U(L)} \text{ is finitely generated} \implies \mathcal{A} \text{ is finitely generated.}$$

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Let $f_1, \dots, f_l$ be a set of generators for $\mathcal{A}^{U(L)}$. We may also assume that each f_i is a T-eigenvector. Then the $\mathbb{C}$-linear span of $L{\cdot}f_i \subset \mathcal{A}$ is a simple finite-dimensional L-module, say V_i. It is easily seen that the $\mathbb{C}$-algebra generated by $\sum_{i=1}^{l} \mathsf{V}_i$ is L-stable and contains all simple L-modules from $\mathcal{A}$. Hence, the finite-dimensional space $\sum_{i=1}^{l} \mathsf{V}_i$ generates $\mathcal{A}$.

(2) There is also an alternate approach to part (ii). By a result of Kostant, the algebra $\mathbb{C}[\mathfrak{n}^*]^N$ is a multiplicity free L-module (see [12, 4.5]). Being the algebra of invariants of a linear action, $\mathbb{C}[\mathfrak{n}^*]^N$ is also integrally closed. Therefore, $\mathfrak{n}^*/\!\!/N$ is a spherical L-variety. By [23, Theorem 10], this implies that $\mathfrak{n}^*/\!\!/N$ has rational singularities.

Remark 6.9 (1) The general treatment of "stable properties" is due to Popov [23], but the assertion that relates rationality of singularities for $\mathcal{A}$ and $\mathcal{A}^{U(L)}$ goes back to Kraft and Luna (see exposition in Brion's thesis [2, Chapter I(c)]). It is worth noting that, for a rational L-algebra $\mathcal{A}$, the equivalence

$$\mathcal{A} \textit{ is finitely generated } \Longleftrightarrow \mathcal{A}^{U(L)} \textit{ is finitely generated}$$

holds over an algebraically closed field of an arbitrary characteristic (see [11, Theorem 16.2]).

(2) A list of "stable properties" is found in [23, Section 6].

Example 6.10 (1) Take $\mathfrak{g} = \mathfrak{sl}_5$ and $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ with $\mathcal{T} = \{\alpha_3, \alpha_4\}$. Then $\dim \mathfrak{n} = 7$, $\mathcal{K}(\mathfrak{n}) = \mathcal{K} = \{\beta_1, \beta_2\}$, and $\tilde{\mathfrak{n}} = \mathfrak{u}$. Hence, $\operatorname{ind} \mathfrak{n} = 5$. Let $\{e_{ij} \mid 1 \leqslant i < j \leqslant 5, j = 4, 5\}$ be the matrix units corresponding to $\mathfrak{n}$ (i.e., a basis for $\mathfrak{n}$). We regard them as (linear) functions on $\mathfrak{n}^- \simeq \mathfrak{n}^*$. Here, $e_{15} \in \mathfrak{g}_{\beta_1}$ and $e_{24} \in \mathfrak{g}_{\beta_2}$. Obviously, e_{15} and $f_{12} = \begin{vmatrix} e_{14} & e_{15} \\ e_{24} & e_{25} \end{vmatrix}$ belong to $\mathcal{S}(\mathfrak{n})^U$. For the standard Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p} = \mathsf{norm}_{\mathfrak{g}}(\mathfrak{n})$, one has $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{sl}_3$; hence, $\mathcal{S}(\mathfrak{n})^N$ is an $\mathfrak{sl}_3$-module. Taking the $\mathfrak{sl}_3$-modules generated by e_{15} and f_{12}, one obtains six functions that generate $\mathcal{S}(\mathfrak{n})^N$:

$$e_{15}, e_{25}, e_{35}, f_{12} = \begin{vmatrix} e_{14} & e_{15} \\ e_{24} & e_{25} \end{vmatrix},\ f_{13} = \begin{vmatrix} e_{14} & e_{15} \\ e_{34} & e_{35} \end{vmatrix},\ f_{23} = \begin{vmatrix} e_{24} & e_{25} \\ e_{34} & e_{35} \end{vmatrix}.$$

Since $f_{12} \notin \mathbb{C}[e_{15}, e_{25}, e_{35}]$ and a minimal generating system can be chosen to be SL_3-invariant, the set of generators above is minimal. These generators satisfy the relation $e_{15} f_{23} - e_{25} f_{13} + e_{35} f_{12} = 0$; hence, $\mathfrak{n}^*/\!\!/N$ is a hypersurface. Under passage to $\tilde{N}$ or U, the situation improves, because $\mathcal{S}(\mathfrak{n})^{\tilde{N}} = \mathcal{S}(\mathfrak{n})^U = \mathbb{C}[e_{15}, f_{12}]$.

(2) More generally, for $\mathfrak{g} = \mathfrak{sl}_N$ and $\mathcal{T} = \{\alpha_{N-k}, \dots, \alpha_{N-1}\}$ with $k \leqslant N/2$, one has

$$\dim \mathfrak{n}_{\mathcal{T}} = \frac{k}{2}(2N - 1 - k), \quad \mathcal{K}_{\mathcal{T}} = \{\beta_1, \dots, \beta_k\}, \text{ and } \operatorname{ind} \mathfrak{n}_{\mathcal{T}} = \frac{k}{2}(2N + 1 - 3k).$$

Here, $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{sl}_{N-k}$ and $\mathcal{S}(\mathfrak{n}_{\mathcal{T}})^U = \mathbb{C}[F_1, \dots, F_k]$, where F_i is the "anti-principal" minor of degree i, i.e.,

$$F_1 = e_{1,N},\ F_2 = \begin{vmatrix} e_{1,N-1} & e_{1,N} \\ e_{2,N-1} & e_{2,N} \end{vmatrix}, \dots, F_k = \begin{vmatrix} e_{1,N-k+1} & \cdots & e_{1,N} \\ \cdots & \cdots & \cdots \\ e_{k,N-k+1} & \cdots & e_{k,N} \end{vmatrix}.$$

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The SL_{N-k}-module generated by F_i, $\langle SL_{N-k}{\cdot}F_i\rangle$, is isomorphic to $\wedge^i\mathbb{V}$, where $\mathbb{V}$ is the standard SL_{N-k}-module. It is also easily seen that F_j does not belong to the $\mathbb{C}$-algebra generated by $\sum_{i=1}^{j-1}\langle SL_{N-k}{\cdot}F_i\rangle$. Therefore, the minimal number of generators of $\mathcal{Z}(\mathfrak{n}_{\mathcal{T}})$ is

$$M_{\mathcal{T}} = \sum_{j=1}^{k}\dim\wedge^i\mathbb{V} = \sum_{j=1}^{k}\binom{N-k}{j}.$$

If $k = 1$, then $\mathfrak{n}_{\mathcal{T}}$ is abelian and $M_{\mathcal{T}} = \operatorname{ind}\mathfrak{n}_{\mathcal{T}} = N-1$. If $k \geqslant 2$, then $M_{\mathcal{T}} \geqslant \operatorname{ind}\mathfrak{n}_{\mathcal{T}}$ and the equality holds only for $k = 2, N = 4$.

(3) For $\mathbf{E}_6$ and $\mathcal{T} = \{\alpha_2\}$, one has $\dim\mathfrak{n}_{\mathcal{T}} = 25$, $\mathcal{K}_{\mathcal{T}} = \{\beta_1,\beta_2,\beta_3\}$, and $\operatorname{ind}\mathfrak{n}_{\mathcal{T}} = 13$. Here, $[\mathfrak{l},\mathfrak{l}] = \mathfrak{sl}_2\oplus\mathfrak{sl}_5$ and $\mathcal{S}(\mathfrak{n}_{\mathcal{T}})^U = \mathbb{C}[F_1,F_2,F_3]$, where $\deg F_i = 1,2,4$ for $i = 1,2,3$, respectively. Using the $\mathfrak{l}$-modules generated by F_1 and F_2, one can prove that $M_{\mathcal{T}} \geqslant 15$.

7 On the square integrable nilradicals

Recall that any nilradical $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ is equipped with the canonical $\mathbb{Z}$-grading $\mathfrak{n} = \bigoplus_{i=1}^{d_{\mathcal{T}}}\mathfrak{n}(i)$, where $d_{\mathcal{T}} = \sum_{\alpha\in\mathcal{T}}[\theta:\alpha]$ and $\mathfrak{n}(d_{\mathcal{T}}) = \mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$. Accordingly, one obtains the partition $\Delta(\mathfrak{n}_{\mathcal{T}}) = \bigsqcup_{i=1}^{d_{\mathcal{T}}}\Delta_{\mathcal{T}}(i)$ (see Section 2.3).

Clearly, $\operatorname{ind}\mathfrak{n} \geqslant \dim\mathfrak{z}(\mathfrak{n})$ and $\mathcal{S}(\mathfrak{z}(\mathfrak{n})) \subset \mathcal{S}(\mathfrak{n})^N$. Following [9, 10], we say that $\mathfrak{n}$ is *square integrable*, if $\operatorname{ind}\mathfrak{n} = \dim\mathfrak{z}(\mathfrak{n})$. Then $\mathfrak{n}^\xi = \mathfrak{z}(\mathfrak{n})$ for any $\xi\in\mathfrak{n}^*_{\rm reg}$ and hence $\mathcal{F}(\mathfrak{n}) = \mathfrak{z}(\mathfrak{n})$ is abelian. Since $\operatorname{trdeg}\mathcal{S}(\mathfrak{n})^N = \dim\mathfrak{z}(\mathfrak{n})$, the extension $\mathcal{S}(\mathfrak{z}(\mathfrak{n}))\subset\mathcal{S}(\mathfrak{n})^N$ is algebraic. This implies that $\mathcal{S}(\mathfrak{n})^N = \mathcal{S}(\mathfrak{z}(\mathfrak{n}))$ and hence $\mathcal{S}(\mathfrak{n})$ is a free $\mathcal{S}(\mathfrak{n})^N$-module. It is easily seen that a Heisenberg algebra $\mathfrak{h}$ is square integrable, with $\operatorname{ind}\mathfrak{h} = 1$. Using $\mathcal{K}(\mathfrak{n})$ and our description of $\mathfrak{n}^\zeta$, we classify all square integrable nilradicals. Recall that $\mathcal{K}_{\mathcal{T}} = \mathcal{K}(\mathfrak{n}_{\mathcal{T}})$ is a poset.

Lemma 7.1 *For any* $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$, *we have* $\min\mathcal{K}_{\mathcal{T}} \subset \big(\Delta_{\mathcal{T}}(1)\cup\Delta_{\mathcal{T}}(2)\big)$.

Proof Take $\beta\in\min\mathcal{K}_{\mathcal{T}}$ and $\alpha\in\Phi(\beta)\cap\mathcal{T}$. Then there are the following possibilities:

- $\beta = \alpha$. Then $\beta\in\Delta_{\mathcal{T}}(1)$.
- $\beta\ne\alpha$ and $\Phi(\beta) = \{\alpha\}$. Recall that β is the highest root in the root system with basis $\operatorname{supp}(\beta)$. Since β is a minimal element of $\mathcal{K}_{\mathcal{T}}$, we have $\operatorname{supp}(\beta)\cap\mathcal{T} = \{\alpha\}$. Then $[\beta:\alpha] = 2$ and $\beta\in\Delta_{\mathcal{T}}(2)$.
- $\Phi(\beta) = \{\alpha,\alpha'\}$. Here, $\operatorname{supp}(\beta)$ is a basis for the root system $\mathbf{A}_p$ $(p\geqslant 2)$ and $\operatorname{supp}(\beta)\cap\mathcal{T} = \{\alpha,\alpha'\}$. Then either $\alpha'\in\mathcal{T}$ and $\beta\in\Delta_{\mathcal{T}}(2)$, or $\alpha'\notin\mathcal{T}$ and $\beta\in\Delta_{\mathcal{T}}(1)$. ■

Theorem 7.2 *Let* $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ *be an arbitrary nilradical. Then*

$$\mathfrak{n} \text{ is square integrable} \iff d_{\mathcal{T}}\leqslant 2 \text{ and } \mathcal{K}_{\mathcal{T}}\subset\Delta_{\mathcal{T}}(d_{\mathcal{T}}) = \Delta(\mathfrak{z}(\mathfrak{n})).$$

Proof (1) Let $\boldsymbol{\zeta}\in\mathfrak{k}_0$ be a cascade point. Then $\mathfrak{n}^\zeta\supset\bigoplus_{\beta\in\mathcal{K}_{\mathcal{T}}}\mathfrak{g}_\beta$. If $\mathfrak{n}$ is square integrable, then $\mathfrak{n}^\zeta\subset\mathfrak{n}(d_{\mathcal{T}})$. Hence, $\mathcal{K}_{\mathcal{T}}\subset\Delta_{\mathcal{T}}(d_{\mathcal{T}})$. By Lemma 7.1, this is only possible, if $d_{\mathcal{T}}\leqslant 2$. Thus, the implication "⇒" is proved.

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(2) If $d_{\mathcal{T}} = 1$, then the assumption on $\mathcal{K}_{\mathcal{T}}$ is vacuous and $\mathfrak{n}$ is abelian, hence square integrable. Therefore, to prove the implication "$\Leftarrow$," we may assume that $d_{\mathcal{T}} = 2$. Then $\mathfrak{z}(\mathfrak{n}) = \mathfrak{n}(2) = [\mathfrak{n}, \mathfrak{n}]$ and $\mathcal{K}_{\mathcal{T}} \subset \Delta_{\mathcal{T}}(2)$. Now, there are two possibilities for $\mathcal{T}$.

• Suppose that $\mathcal{T} = \{\alpha\}$ and $[\theta : \alpha] = 2$.

Set $\beta_i = \Phi^{-1}(\alpha)$. Since $\beta_i \in \mathcal{K}_{\mathcal{T}} \subset \Delta_{\mathcal{T}}(2)$, we have $[\beta_i : \alpha] = 2$. Let us prove that β_i is the unique minimal element of $\mathcal{K}_{\mathcal{T}}$. Assume that $\beta_i \notin \min \mathcal{K}_{\mathcal{T}}$, i.e., there is $\beta' \in \mathcal{K}_{\mathcal{T}}$ such that $\beta' \prec \beta$. Then $\mathrm{supp}(\beta') \subset \mathrm{supp}(\beta_i) \setminus \{\alpha\}$ and $\mathrm{supp}(\beta') \cap \mathcal{T} \neq \varnothing$. A contradiction! Hence, β_i is minimal in $\mathcal{K}_{\mathcal{T}}$. Assume that there is another minimal element of $\mathcal{K}_{\mathcal{T}}$, say $\tilde{\beta}$. Then $\mathrm{supp}(\beta_i)$ and $\mathrm{supp}(\tilde{\beta})$ are disjoint, and we must have $\mathrm{supp}(\tilde{\beta}) \cap \mathcal{T} \neq \varnothing$, which is impossible. This contradiction shows that $\min \mathcal{K}_{\mathcal{T}} = \{\beta_i\}$.

Since $B{\cdot}\boldsymbol{\zeta}$ is dense in $\mathfrak{n}^*$, $B{\cdot}\boldsymbol{\zeta} \subset \mathfrak{n}^*_{\mathrm{reg}}$, and $\mathfrak{n}(2)$ is B-stable, it suffices to prove that $\mathfrak{n}^{\zeta} \subset \mathfrak{n}(2)$ (and then, actually, $\mathfrak{n}^{\zeta} = \mathfrak{n}(2)$). Recall that $\mathfrak{n} = \bigoplus_{\beta_j \in \mathcal{K}_{\mathcal{T}}} (\mathfrak{h}_j \cap \mathfrak{n})$. For the unique minimal element $\beta_i \in \mathcal{K}_{\mathcal{T}}$, we have $\mathfrak{h}_i \subset \mathfrak{n}$ and hence $\mathfrak{h}_i \cap \mathfrak{n}^{\zeta} = \mathfrak{g}_{\beta_i}$. If $\beta_i \prec \beta_j$, then $[\beta_j : \alpha] = 2$ and the contribution from $\mathfrak{h}_j \cap \mathfrak{n}$ to $\mathfrak{n}^{\zeta}$ is

$$\mathfrak{h}_j \cap \mathfrak{n}^{\zeta} = \mathfrak{g}_{\beta_j} \oplus \Big(\bigoplus_{\gamma \in \mathcal{C}_{\mathfrak{n}}(j)} \mathfrak{g}_{\beta_j - \gamma} \Big)$$

(see Theorem 4.4). Since $\mathcal{K}_{\mathcal{T}} \subset \Delta_{\mathcal{T}}(2)$, we have $\mathfrak{g}_{\beta_j} \subset \mathfrak{n}(2)$. The very definition of $\mathcal{C}_{\mathfrak{n}}(j)$ says that $\gamma \notin \Delta_{\mathcal{T}}$. Hence, $[\gamma : \alpha] = 0$ and $[\beta_j - \gamma : \alpha] = 2$. Thus, $\beta_j - \gamma \in \Delta_{\mathcal{T}}(2)$ and $\mathfrak{h}_j \cap \mathfrak{n}^{\zeta} \subset \mathfrak{n}(2)$.

• Suppose that $\mathcal{T} = \{\alpha, \alpha'\}$ and $[\theta : \alpha] = [\theta : \alpha'] = 1$.

The argument here is similar to that in the previous part. Set $\beta = \Phi^{-1}(\alpha)$ and $\beta' = \Phi^{-1}(\alpha')$. Using the hypothesis that $\mathcal{K}_{\mathcal{T}} \subset \Delta_{\mathcal{T}}(2)$, one first proves that $\beta = \beta'$ (hence $\Phi(\beta) = \{\alpha, \alpha'\}$) and that β is the unique minimal element of $\mathcal{K}_{\mathcal{T}}$. Then the use of Theorem 4.4 allows us to check that $\mathfrak{n}^{\zeta} \subset \mathfrak{n}(2)$. ■

Using Theorem 7.2, it is not hard to get the list of square integrable nilradicals in all simple Lie algebras. For $d_{\mathcal{T}} = 1$, $\mathfrak{n}_{\mathcal{T}}$ is abelian and these cases are well known. If $d_{\mathcal{T}} = 2$, then $[\mathfrak{n}_{\mathcal{T}}, [\mathfrak{n}_{\mathcal{T}}, \mathfrak{n}_{\mathcal{T}}]] = 0$. The condition that $\mathcal{K}_{\mathcal{T}}$ belongs to the highest graded component of $\mathfrak{n}_{\mathcal{T}}$ is quite strong. Therefore, not all nilradicals with $d_{\mathcal{T}} = 2$ are square integrable.

Example 7.3 (1) The list of square integrable nilradicals with $d_{\mathcal{T}} = 2$ is given below.

- $\mathbf{A}_n$, $\mathcal{T} = \{\alpha_k, \alpha_{n+1-k}\}$ with $2k < n+1$;
- $\mathbf{C}_n$, $\mathcal{T} = \{\alpha_k\}$ with $k < n$;
- $\mathbf{B}_n$, $\mathcal{T} = \{\alpha_{2k}\}$ with $2k \leqslant n$;
- $\mathbf{D}_n$, $\mathcal{T} = \{\alpha_{2k}\}$ with $k \leqslant [n/2]-1$;
- $\mathbf{D}_{2n+1}$, $\mathcal{T} = \{\alpha_{2n}, \alpha_{2n+1}\}$;
- $\mathbf{E}_6$, $\mathcal{T} = \{\alpha_6\}$ or $\mathcal{T} = \{\alpha_1, \alpha_5\}$;
- $\mathbf{E}_7$, $\mathcal{T} = \{\alpha_6\}$ or $\mathcal{T} = \{\alpha_2\}$;
- $\mathbf{E}_8$, $\mathcal{T} = \{\alpha_1\}$ or $\mathcal{T} = \{\alpha_7\}$;
- $\mathbf{F}_4$, $\mathcal{T} = \{\alpha_4\}$ or $\mathcal{T} = \{\alpha_1\}$;
- $\mathbf{G}_2$, $\mathcal{T} = \{\alpha_2\}$.

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(2) The list contains all the cases in which $\mathfrak{n} = \mathfrak{h}_1$ is the Heisenberg nilradicals. For the exceptional algebras, this corresponds to $\mathcal{T}$ indicated first, whereas for the classical series, this corresponds to the cases with $k = 1$.

(3) Using [22], one verifies that $\mathfrak{n}_{\mathcal{T}}$ has no CP for $\mathbf{B}_n$ with $k \geqslant 2$, $(\mathbf{E}_8, \alpha_7)$, and $(\mathbf{F}_4, \alpha_1)$.

(4) For $\mathbf{D}_{2n}$ and $\mathcal{T} = \{\alpha_{2n-1}, \alpha_{2n}\}$, $\mathfrak{n}_{\mathcal{T}}$ is not square integrable. Indeed, here, $d_{\mathcal{T}} = 2$, $\mathcal{K}_{\mathcal{T}} = \{\beta_1, \beta_3, \dots, \beta_{2n-3}, \beta_{2n-1}, \beta_{2n}\}$, and $\min \mathcal{K}_{\mathcal{T}}$ contains two elements, $\alpha_{2n} = \beta_{2n-1}$ and $\alpha_{2n-1} = \beta_{2n}$. (We use the notation of Appendix A.) Hence, $\beta_{2n-1}, \beta_{2n} \in \Delta_{\mathcal{T}}(1)$, while the other elements of $\mathcal{K}_{\mathcal{T}}$ belong to $\Delta_{\mathcal{T}}(2)$. Here, $\operatorname{ind} \mathfrak{n}_{\mathcal{T}} = 2n^2 - 3n + 3$ and $\dim \mathfrak{z}(\mathfrak{n}_{\mathcal{T}}) = 2n^2 - 3n + 1$. (Of course, there are other nilradicals with $d_{\mathcal{T}} = 2$ that are not square integrable.)

Remark 7.4 A real nilpotent Lie group Q has a unitary square integrable representation if and only if $\operatorname{ind} \mathfrak{q} = \dim \mathfrak{z}(\mathfrak{q})$ [15, Theorem 1]. For this reason, Elashvili applied the term "square integrable" to the nilpotent Lie algebras $\mathfrak{q}$ satisfying that equality, over an algebraically closed field of characteristic zero [9] (cf. also [10]). Afterward, the theory of square integrable representations was extended to the setting of arbitrary Lie groups [6]. The relevant notions are those of a *quasi-reductive* Lie group and a coadjoint orbit of *reductive type* (see [7]). Therefore, Theorem 7.2 provides a classification of the quasi-reductive nilradicals in $\mathfrak{g}$. Various results on quasi-reductive *seaweed* (= biparabolic) subalgebras of $\mathfrak{g}$ are obtained in [1, 16].

8 When is the quotient morphism equidimensional?

It is well known that if a graded polynomial algebra $\mathcal{A}$ is a free module over a graded subalgebra $\mathcal{B}$, then $\mathcal{B}$ is necessarily polynomial. Furthermore, if $\mathcal{B}$ is a graded polynomial subalgebra of $\mathcal{A}$, then $\mathcal{A}$ is a free $\mathcal{B}$-module if and only if the induced morphism $\pi : \operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{B})$ is *equidimensional*, i.e., $\dim \pi^{-1}(\pi(x)) = \dim \mathcal{A} - \dim \mathcal{B}$ for any $x \in \operatorname{Spec}(\mathcal{A})$ [26, Proposition 17.29]. By a theorem of Chevalley, dominant equidimensional morphisms are open. Therefore, in the graded situation, π is also onto (see [29, 2.4]).

In this section, we point out certain nilradicals $\mathfrak{n}$ in $\mathfrak{g}$ such that $\mathcal{A} = \mathcal{S}(\mathfrak{n}) = \mathbb{C}[\mathfrak{n}^*]$ is a free module over $\mathcal{B} = \mathcal{S}(\mathfrak{n})^U = \mathbb{C}[\mathfrak{n}^*]^U$. The last algebra is always polynomial (Lemma 6.1); hence, our task is to guarantee that the quotient morphism

$$\pi : \operatorname{Spec}(\mathcal{A}) = \mathfrak{n}^* \to \mathfrak{n}^* /\!\!/ U = \operatorname{Spec}(\mathcal{B})$$

is equidimensional. Since $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{n})^{\tilde{N}}$ (Lemma 6.3), the **optimal** nilradicals of such type provide examples, where $\mathcal{S}(\mathfrak{n})$ is a free module over its Poisson center $\mathcal{Z}(\mathfrak{n}) = \mathcal{S}(\mathfrak{n})^N$. We begin with a simple observation.

Lemma 8.1 *If $\#\mathcal{K}(\mathfrak{n}) \leqslant 3$, then $\mathcal{S}(\mathfrak{n})$ is a free $\mathcal{S}(\mathfrak{n})^U$-module.*

Proof By Theorem 6.4, we have $\mathfrak{n}^* /\!\!/ U \simeq \mathbb{A}^{\#\mathcal{K}(\mathfrak{n})}$. Since $e_\theta \in \mathcal{S}(\mathfrak{n})^U$ is an element of degree 1, the hyperplane $\mathfrak{n}_0^* = \{\xi \in \mathfrak{n}^* \mid \xi(e_\theta) = 0\}$ is U-stable and $\mathfrak{n}_0^* /\!\!/ U \simeq \mathbb{A}^s$, where $s = \#\mathcal{K}(\mathfrak{n}) - 1 \leqslant 2$. By a result of Brion [3] on invariants of unipotent groups, $\pi_0 : \mathfrak{n}_0^* \to \mathfrak{n}_0^* /\!\!/ U$ is equidimensional, and this implies that π is equidimensional, too. ∎

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Theorem 8.2 *Let $\mathfrak{n}_{\{\alpha\}}$ be an abelian nilradical, i.e., $[\theta : \alpha] = 1$. Suppose that a nilradical $\mathfrak{n}$ is contained between $\mathfrak{n}_{\{\alpha\}}$ and its optimization $\widetilde{\mathfrak{n}_{\{\alpha\}}}$. Then:*

(1) *$\mathfrak{n}_{\{\alpha\}}$ is a CP-ideal of $\mathfrak{n}$;*
(2) *$\mathcal{S}(\mathfrak{n})$ is a free module over $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{n})^{\tilde{N}}$.*

Proof (1) By Remark 2.4, we have $\dim \mathfrak{n}_{\{\alpha\}} = \boldsymbol{b}(\mathfrak{n}_{\{\alpha\}}) = \boldsymbol{b}(\mathfrak{n})$.

(2) Consider the commutative diagram $\begin{array}{ccc} \mathfrak{n}^* & \xrightarrow{\varphi} & \mathfrak{n}^*_{\{\alpha\}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\{\alpha\}}} \\ \mathfrak{n}^* /\!\!/ U & \xrightarrow{\varphi /\!\!/ U} & \mathfrak{n}^*_{\{\alpha\}} /\!\!/ U \end{array}$ corresponding to the U-equivariant inclusion $\mathfrak{n}_{\{\alpha\}} \hookrightarrow \mathfrak{n}$. Since $\mathfrak{n}$ and $\mathfrak{n}_{\{\alpha\}}$ have the same optimization, Theorem 6.4 implies that $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{n}_{\{\alpha\}})^U$, i.e., $\varphi /\!\!/ U$ is an isomorphism. By a general result on U-invariants for the abelian nilradicals [20, Theorem 4.6], $\mathcal{S}(\mathfrak{n}_{\{\alpha\}})$ is a free module over $\mathcal{S}(\mathfrak{n}_{\{\alpha\}})^U$. Hence, the morphism $\pi_{\{\alpha\}}$ is equidimensional and onto. Because the projection φ is also onto and equidimensional, π must be equidimensional, too. Since $\mathfrak{n}^* /\!\!/ U$ is an affine space, the morphism π is flat and $\mathbb{C}[\mathfrak{n}^*]$ is a free $\mathbb{C}[\mathfrak{n}^*]^U$-module. ∎

Corollary 8.3 *For the optimal nilradical $\tilde{\mathfrak{n}} = \widetilde{\mathfrak{n}}_{\{\alpha\}}$, the algebra $\mathcal{S}(\tilde{\mathfrak{n}})$ is a free module over $\mathcal{Z}(\tilde{\mathfrak{n}})$.*

Theorem 8.2 implies interesting consequences for some types of simple Lie algebras.

Proposition 8.4 *For every nilradical $\mathfrak{n}$ in $\mathfrak{sl}_{n+1}$, there is an $\alpha \in \Pi$ such that $\mathfrak{n}_{\{\alpha\}} \subset \mathfrak{n} \subset \widetilde{\mathfrak{n}_{\{\alpha\}}}$. Therefore, $\mathcal{S}(\mathfrak{n})$ is a free module over $\mathcal{S}(\mathfrak{n})^U$ for any $\mathfrak{n}$.*

Proof We use the notation of Example 2.5(1). Recall that $[\theta : \alpha] = 1$ for each $\alpha \in \Pi$. If $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ and $\min \mathcal{K}(\mathfrak{n}) = \{\beta_k\}$ $(k \leqslant [(n+1)]/2)$, then $\mathcal{T} \cap \{\alpha_k, \alpha_{n+1-k}\} \neq \varnothing$. Then any α in this intersection will do. (The construction of such a simple root α is essentially contained in the proof of Theorem 6.2 in [10]. For, then $\mathfrak{n}_{\{\alpha\}}$ is a CP-ideal of $\mathfrak{n}$. But our approach that refers to $\mathcal{K}(\mathfrak{n})$ is shorter.) ∎

Of course, the first assertion of Proposition 8.4 appears to be true because **all** simple roots of $\mathfrak{sl}_{n+1}$ provide abelian nilradicals. Nevertheless, the second assertion can be proved for the nilradicals in $\mathfrak{sp}_{2n}$, although the proof becomes more involved.

Proposition 8.5 *For any nilradical $\mathfrak{n}$ in $\mathfrak{sp}_{2n}$, $\mathcal{S}(\mathfrak{n})$ is a free module over $\mathcal{S}(\mathfrak{n})^U$.*

Proof Here, $\mathfrak{n}_{\{\alpha_n\}}$ is the only abelian nilradical and $\mathcal{K}(\mathfrak{n}_{\{\alpha_n\}}) = \mathcal{K}$. Hence, $\widetilde{\mathfrak{n}_{\{\alpha_n\}}} = \mathfrak{u}$.

(1) If $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ and $\alpha_n \in \mathcal{T}$, then $\mathfrak{n}_{\{\alpha_n\}} \subset \mathfrak{n}$ and Theorem 8.2 applies. In particular, this shows that here $\mathcal{S}(\mathfrak{u})$ is a free module over $\mathcal{S}(\mathfrak{u})^U = \mathcal{Z}(\mathfrak{u})$.

(2) Suppose that $\alpha_n \notin \mathcal{T}$. Then $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \ldots, \beta_k\}$ for some $k < n$, $\alpha_k \in \mathcal{T}$, and $\mathfrak{a} := \mathfrak{n} \cap \mathfrak{n}_{\{\alpha_n\}}$ is a CP-ideal of $\mathfrak{n}$. If $\mathfrak{n}_{\{\alpha_n\}}$ is identified with the space $\widehat{\mathsf{Sym}}_n$ of $n \times n$ matrices that are symmetric w.r.t. the antidiagonal (see Example 2.6), then

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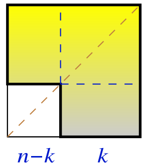
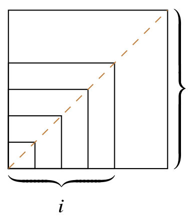
$\mathfrak{a} =$ 

$\subset \mathfrak{n}_{\{\alpha_n\}}$

Since $\mathfrak{a}$ is a $\mathfrak{b}$-stable ideal of $\mathfrak{u}$, $\mathcal{S}(\mathfrak{a})^U$ is a polynomial algebra (see Remark 6.5). Let us prove that $\mathcal{S}(\mathfrak{a})$ is a free $\mathcal{S}(\mathfrak{a})^U$-module, i.e., that $\bar{\pi} : \mathfrak{a}^* \to \mathfrak{a}^* /\!\!/ U$ is equidimensional. Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{n}^*_{\{\alpha_n\}} & \xrightarrow{\psi} & \mathfrak{a}^* \\ \downarrow \pi_{\alpha_n} & & \downarrow \bar{\pi} \\ \mathfrak{n}^*_{\{\alpha_n\}} /\!\!/ U & \xrightarrow{\psi /\!\!/ U} & \mathfrak{a}^* /\!\!/ U \end{array}$$

corresponding to the U-equivariant inclusion $\mathfrak{a} \hookrightarrow \mathfrak{n}_{\{\alpha_n\}}$. Here, $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{a})^U$ (cf. Section 3.4(2)), i.e., $\dim \mathfrak{a}^* /\!\!/ U = \dim \mathfrak{n}^* /\!\!/ U = \#\mathcal{K}(\mathfrak{n}) = k$. Upon the identification of $\mathfrak{n}^*_{\{\alpha_n\}}$ with $\mathfrak{n}^-_{\{\alpha_n\}} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C = \hat{C} \right\}$ and thereby with the dual space $\widehat{\mathsf{Sym}}_n^{\,*}$, the algebra $\mathbb{C}[\mathfrak{n}^*_{\{\alpha_n\}}]^U$ is freely generated by the principal minors of C. Let f_i be the principal minor of degree i; see the figure $C =$ 

. Then $\mathcal{S}(\mathfrak{n}_{\{\alpha_n\}})^U = \mathbb{C}[\mathfrak{n}^*_{\{\alpha_n\}}]^U = \mathbb{C}[f_1, \dots, f_n]$ and $\pi_{\alpha_n}(M) = (f_1(M), \dots, f_n(M))$. On the other hand, $f_i \subset \mathcal{S}(\mathfrak{a})$ for $i \leqslant k$ and hence $\mathcal{S}(\mathfrak{a})^U = \mathbb{C}[f_1, \dots, f_k]$. Furthermore, if $\bar{M} = \psi(M) \in \mathfrak{a}^*$, then $f_i(\bar{M}) = f_i(M)$ for $i \leqslant k$. Then $\psi /\!\!/ U$ takes $(f_1(M), \dots, f_n(M))$ to $(f_1(M), \dots, f_k(M))$ and thereby $\psi /\!\!/ U$ is surjective and equidimensional. Since we have already proved that π_{α_n} is onto and equidimensional, this yields the same conclusion for $\bar{\pi}$.

Once it is proved that $\bar{\pi}$ is onto and equidimensional, an argument similar to that in Theorem 8.2 can be applied to the embedding $\mathfrak{a} \hookrightarrow \mathfrak{n}$. Consider the diagram

$$\text{(8.1)} \qquad \begin{array}{ccc} \mathfrak{n}^* & \xrightarrow{\phi} & \mathfrak{a}^* \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \mathfrak{n}^* /\!\!/ U & \xrightarrow{\phi /\!\!/ U} & \mathfrak{a}^* /\!\!/ U \end{array} .$$

Since $\mathfrak{a}$ is a CP-ideal of $\mathfrak{n}$, we have $\mathcal{S}(\mathfrak{n})^N \subset \mathcal{S}(\mathfrak{a})$ [22, Proposition 5.5]. Hence, $\mathcal{S}(\mathfrak{n})^N = \mathcal{S}(\mathfrak{a})^N$ and thereby $\mathcal{S}(\mathfrak{n})^U = \mathcal{S}(\mathfrak{a})^U$, i.e., $\phi /\!\!/ U$ is an isomorphism. This implies that π is equidimensional, and we are done. ■

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The proofs of Propositions 8.4 and 8.5 exploit the fact that every nilradical in $\mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$ has a CP (and hence a CP-ideal). Using Theorem 6.4, we get a joint corollary to these propositions:

(8.2)
if $\mathfrak{n} = \tilde{\mathfrak{n}}$ *is an optimal nilradical in* $\mathfrak{sl}_{n+1}$ *or* $\mathfrak{sp}_{2n}$*, then* $\mathcal{S}(\tilde{\mathfrak{n}})$ *is a free* $\mathcal{Z}(\tilde{\mathfrak{n}})$*-module.*

For $\mathfrak{n} \neq \tilde{\mathfrak{n}}$, one cannot even guarantee that $\mathcal{Z}(\mathfrak{n})$ is a polynomial algebra (see Example 6.10). It might be interesting to characterize non-optimal nilradicals having a polynomial algebra $\mathcal{Z}(\mathfrak{n})$.

By [22], if $\mathfrak{n} \subset \widetilde{\mathfrak{n}}_{\{\alpha\}}$ with $[\theta : \alpha] = 1$, then $\mathfrak{n}$ has a CP. However, to point out a CP-ideal of $\mathfrak{n}$, one needs some precautions:

(1) If $\Phi(\Phi^{-1}(\alpha)) = \{\alpha\}$, then $\mathfrak{n} \cap \mathfrak{n}_{\{\alpha\}}$ is a CP-ideal of $\mathfrak{n}$.

(2) If $\Phi(\Phi^{-1}(\alpha)) = \{\alpha, \alpha'\}$ and $\mathfrak{g} \neq \mathfrak{sl}_{n+1}$, then at least one of $\mathfrak{n} \cap \mathfrak{n}_{\{\alpha\}}$ and $\mathfrak{n} \cap \mathfrak{n}_{\alpha'}$ is a CP-ideal of $\mathfrak{n}$.

(3) If $\mathfrak{g} = \mathfrak{sl}_{n+1}$, then one should choose the **minimal** $\widetilde{\mathfrak{n}}_{\{\alpha\}}$ containing $\mathfrak{n}$. More precisely, since $\mathfrak{n} \subset \widetilde{\mathfrak{n}}_{\{\alpha\}}$, we have $\mathcal{K}(\mathfrak{n}) \subset \mathcal{K}(\mathfrak{n}_{\{\alpha\}})$. Here, one has to take α such that $\mathcal{K}(\mathfrak{n}) = \mathcal{K}(\mathfrak{n}_{\{\alpha\}})$. Then item (2) applies.

In any case, $\mathfrak{n} \cap \mathfrak{n}_{\{\alpha\}}$ is a CP-ideal of $\mathfrak{n}$ for a "right" choice of $\alpha \in \Pi$, and the hypothesis that $\mathfrak{n}_{\{\alpha\}} \subset \mathfrak{n}$ is not required for the presence of CP. That is, Theorem 8.2 does not apply to all nilradicals with CP. Nevertheless, using a case-by-case argument, we can prove the following.

Theorem 8.6 *If* $\mathfrak{n} = \mathfrak{n}_{\mathcal{T}}$ *has a CP, then* $\mathcal{S}(\mathfrak{n})$ *is a free* $\mathcal{S}(\mathfrak{n})^U$*-module.*

Proof By the preceding discussion, we may assume that $\mathfrak{n} \subset \widetilde{\mathfrak{n}}_{\{\alpha\}}$ and $\mathfrak{a} := \mathfrak{n} \cap \mathfrak{n}_{\{\alpha\}}$ is a CP-ideal of $\mathfrak{n}$. As in the proof of Proposition 8.5, one has the commutative diagram (8.1), where $\phi /\!\!/ U$ is an isomorphism. Therefore, it suffices to prove that $\bar{\pi} : \mathfrak{a}^* \to \mathfrak{a}^* /\!\!/ U$ is equidimensional. Consider all simple Lie algebras having abelian nilradicals.

(1) The algebras $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$ have been considered above.

(2) For $\mathfrak{so}_{2n+1}$, the only abelian nilradical corresponds to $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\mathcal{K}(\mathfrak{n}_{\alpha_1}) = \{\beta_1, \beta_2\}$. Therefore, $\#\mathcal{K}(\mathfrak{n}) \leqslant 2$ and Lemma 8.1 applies.

(3) For $\mathfrak{so}_{2n}$, the abelian nilradicals correspond to α_1, α_{n-1}, and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$.

- For $\alpha_1 = \varepsilon_1 - \varepsilon_2$, the situation is the same as for $\mathfrak{g} = \mathfrak{so}_{2n+1}$.
- Since $\mathfrak{n}_{\{\alpha_{n-1}\}} \simeq \mathfrak{n}_{\{\alpha_n\}}$, we consider only the last possibility. The case of $(\mathfrak{so}_{2n}, \alpha_n)$ is similar to $(\mathfrak{sp}_{2n}, \alpha_n)$, which is considered in Proposition 8.5. The distinction is that now $\mathfrak{n}_{\{\alpha_n\}}$ consists of $n \times n$ matrices that are *skew-symmetric* w.r.t. the antidiagonal, and the basic U-invariants are pfaffians of the principal minors of even order. The embedding $\mathfrak{a} \subset \mathfrak{n}_{\{\alpha_n\}}$ can be described by the figure in the proof of Proposition 8.5, only now k must be even and the matrices should be skew-symmetric w.r.t. the antidiagonal.

More precisely, if $n = 2p + 1$, then $\alpha_n \notin \mathcal{K}$ and $\Phi^{-1}(\alpha_n) = \beta_{2p-1}$. If $n = 2p$, then $\alpha_n = \beta_{2p-1}$. In both cases, $\mathcal{K}(\mathfrak{n}_{\{\alpha_n\}}) = \{\beta_1, \beta_3, \ldots, \beta_{2p-1}\}$. If $\mathfrak{n} \subset \widetilde{\mathfrak{n}}_{\{\alpha_n\}}$ and $\mathfrak{n}_{\{\alpha\}} \notin \mathfrak{n}$, then $\mathcal{K}(\mathfrak{n}) = \{\beta_1, \beta_3, \ldots, \beta_{2s-1}\}$, where $s < p = [n/2]$. Then $\operatorname{trdeg} \mathcal{S}(\mathfrak{a})^U = s$ and $\mathcal{S}(\mathfrak{a})^U$ is generated by the pfaffians of order $2, 4, \ldots, 2s$. The fact that these pfaffians form a regular sequence in $\mathcal{S}(\mathfrak{a}) = \mathbb{C}[\mathfrak{a}^*]$ can be proved as in Proposition 8.5 for $\mathfrak{sp}_{2n}$.

(4) For $\mathbf{E}_6$, the abelian nilradicals correspond to α_1 and α_5. In both cases, we have $\#\mathcal{K}(\mathfrak{n}_{\{\alpha_i\}}) = 2$ and hence Lemma 8.1 applies to any $\mathfrak{n} \subset \widetilde{\mathfrak{n}_{\{\alpha_i\}}}$, $i = 1, 5$.

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(5) For $\mathbf{E}_7$, the only abelian nilradical corresponds to α_1. Here, $\mathcal{K}(\mathfrak{n}_{\{\alpha_1\}}) = \{\beta_1, \beta_2, \beta_3\}$ (see Appendix A). Therefore, Lemma 8.1 applies here. ∎

Our computations suggest that Theorem 8.6 holds in the general case.

Conjecture 8.7 *For any nilradical $\mathfrak{n}$ in a simple Lie algebra $\mathfrak{g}$, $\mathcal{S}(\mathfrak{n})$ is a free $\mathcal{S}(\mathfrak{n})^U$-module. In particular, for any optimal nilradical $\tilde{\mathfrak{n}}$, $\mathcal{S}(\tilde{\mathfrak{n}})$ is a free $\mathcal{Z}(\tilde{\mathfrak{n}})$-module.*

So far, this conjecture is proved for (i) the square integrable nilradicals (Section 7), (ii) the nilradicals with CP, (iii) the series $\mathbf{A}_n$ and $\mathbf{C}_n$, and (iv) if $\operatorname{rk}\mathfrak{g} \leqslant 3$. It is not hard to check it for $\mathbf{D}_4$. Perhaps, the first step toward a general proof is to verify the conjecture for $\mathfrak{n} = \mathfrak{u}$. Since $\mathfrak{u}$ has a CP only for $\mathbf{A}_n$ and $\mathbf{C}_n$, some fresh ideas are necessary here.

A The elements of $\mathcal{K}$

We list below the cascade roots (elements of $\mathcal{K}$) for all simple Lie algebras. The numbering of $\Pi = \{\alpha_1, \dots, \alpha_{\operatorname{rk}\mathfrak{g}}\}$ follows [17, Table 1] and, for roots of the classical Lie algebras, we use the standard ε-notation. The numbering of cascade roots yields a linear extension of the poset $(\mathcal{K}, \preccurlyeq)$, i.e., it is not unique unless $\mathcal{K}$ is a chain. In all cases, $\beta_1 = \theta$ and

$$\Phi(\beta_i) = \{\alpha \in \Pi \mid \beta_i - \alpha \in \Delta^+ \cup \{0\}\}.$$

In particular, if $\beta \in \mathcal{K} \cap \Pi$, then $\Phi(\beta) = \{\beta\}$ and β is a minimal element of $\mathcal{K}$. Conversely, if β is a minimal element of $\mathcal{K}$ and $\Phi(\beta) = \{\alpha\}$, a sole simple root, then $\alpha = \beta$.

The cascade elements for the classical Lie algebras:

$\mathbf{A}_n, n \geqslant 2$ $\beta_i = \varepsilon_i - \varepsilon_{n+2-i} = \alpha_i + \dots + \alpha_{n+1-i}$ $\left(i = 1, 2, \dots, \left[\frac{n+1}{2}\right]\right)$;
$\mathbf{C}_n, n \geqslant 1$ $\beta_i = 2\varepsilon_i = 2(\alpha_i + \dots + \alpha_{n-1}) + \alpha_n$ $(i = 1, 2, \dots, n-1)$ and $\beta_n = 2\varepsilon_n = \alpha_n$;
$\mathbf{B}_{2n}, \mathbf{D}_{2n}, \mathbf{D}_{2n+1}$ $(n \geqslant 2)$ $\beta_{2i-1} = \varepsilon_{2i-1} + \varepsilon_{2i}$, $\beta_{2i} = \varepsilon_{2i-1} - \varepsilon_{2i} = \alpha_{2i-1}$ $(i = 1, 2, \dots, n)$;
$\mathbf{B}_{2n+1}, n \geqslant 1$ here $\beta_1, \dots, \beta_{2n}$ are as above and $\beta_{2n+1} = \varepsilon_{2n+1} = \alpha_{2n+1}$.

For all orthogonal series, we have $\beta_{2i} = \alpha_{2i-1}$, $i = 1, \dots, n$, while formulae for β_{2i-1} via Π slightly differ for different series. For example, for $\mathbf{D}_{2n}$ one has $\beta_{2i-1} = \alpha_{2i-1} + 2(\alpha_{2i} + \dots + \alpha_{2n-2}) + \alpha_{2n-1} + \alpha_{2n}$ $(i = 1, 2, \dots, n-1)$ and $\beta_{2n-1} = \alpha_{2n}$.
The cascade elements for the exceptional Lie algebras:

$\mathbf{G}_2$ $\beta_1 = (32) = 3\alpha_1 + 2\alpha_2$, $\beta_2 = (10) = \alpha_1$;
$\mathbf{F}_4$ $\beta_1 = (2432) = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$, $\beta_2 = (2210)$, $\beta_3 = (0210)$, $\beta_4 = (0010) = \alpha_3$;
$\mathbf{E}_6$ $\beta_1 = \begin{smallmatrix}12321\\2\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}11111\\0\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}01110\\0\end{smallmatrix}$, $\beta_4 = \begin{smallmatrix}00100\\0\end{smallmatrix} = \alpha_3$;

$\mathbf{E}_7$ $\beta_1 = \begin{smallmatrix}123432\\2\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}122210\\1\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}100000\\0\end{smallmatrix} = \alpha_1$, $\beta_4 = \begin{smallmatrix}001210\\1\end{smallmatrix}$, $\beta_5 = \begin{smallmatrix}001000\\0\end{smallmatrix} = \alpha_3$, $\beta_6 = \begin{smallmatrix}000010\\0\end{smallmatrix} = \alpha_5$, $\beta_7 = \begin{smallmatrix}000000\\1\end{smallmatrix} = \alpha_7$;

$\mathbf{E}_8$ $\beta_1 = \begin{smallmatrix}2345642\\3\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}0123432\\2\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0122210\\1\end{smallmatrix}$, $\beta_4 = \begin{smallmatrix}0100000\\0\end{smallmatrix} = \alpha_2$, $\beta_5 = \begin{smallmatrix}0001210\\1\end{smallmatrix}$, $\beta_6 = \begin{smallmatrix}0001000\\0\end{smallmatrix} = \alpha_4$, $\beta_7 = \begin{smallmatrix}0000010\\0\end{smallmatrix} = \alpha_6$, $\beta_8 = \begin{smallmatrix}0000000\\1\end{smallmatrix} = \alpha_8$.

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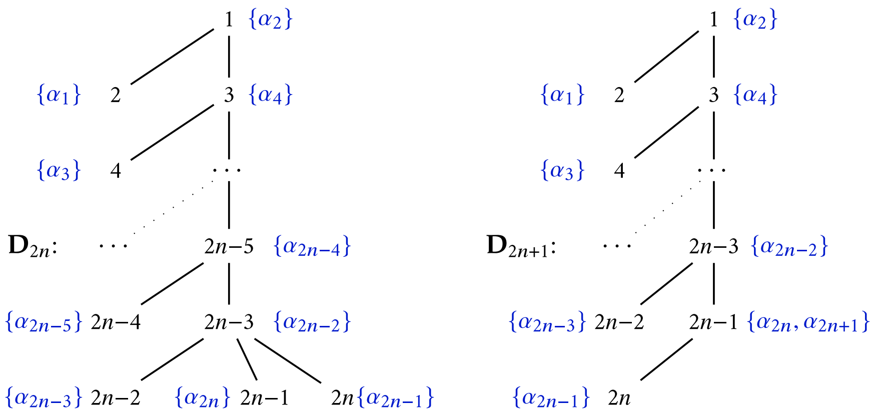
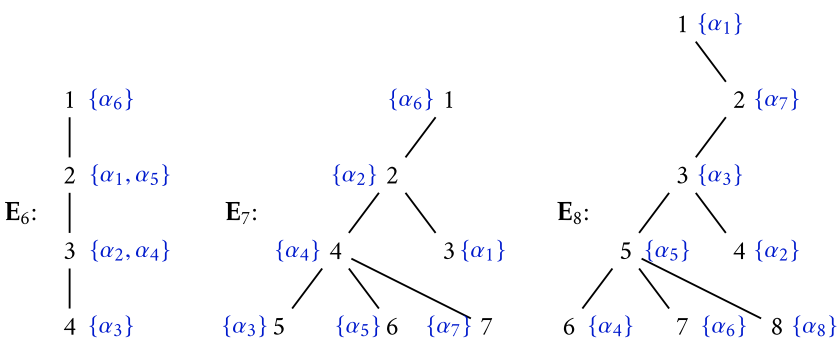
For the reader's convenience, we provide the Hasse diagram of the cascade posets for $\mathbf{D}_n$ and $\mathbf{E}_n$. The node "i" in the diagram represents $\beta_i \in \mathcal{K}$, and we attach the set $\Phi(\beta_i) \subset \Pi$ to it.





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