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ON THE PERIODICITY OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract

According to a conjecture by Yang, if $f(z)f^{(k)}(z)$ is a periodic function, where $f(z)$ is a transcendental entire function and k is a positive integer, then $f(z)$ is also a periodic function. We propose related questions, which can be viewed as difference or differential-difference versions of Yang's conjecture. We consider the periodicity of a transcendental entire function $f(z)$ when differential, difference or differential-difference polynomials in $f(z)$ are periodic. For instance, we show that if $f(z)^n f(z+\eta)$ is a periodic function with period c, then $f(z)$ is also a periodic function with period $(n+1)c$, where $f(z)$ is a transcendental entire function of hyper-order $\rho_2(f) < 1$ and $n \geq 2$ is an integer.

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1. Introduction and main results

Periodicity is an important and easy to recognise property for meromorphic functions. Rényi and Rényi [15] proved that if $f(z)$ is an arbitrary nonconstant entire function and $P(z)$ is an arbitrary polynomial with $\deg(P(z)) \geq 3$, then the entire function $f(P(z))$ cannot be a periodic function. If $\deg(P(z)) = 2$, then there exists a transcendental entire function $f(z)$ such that $f(P(z))$ is periodic. For example, if $P(z) = Az^2 + Bz + C$, where $A \neq 0$, B, C are constants and

$$f(z) = \cos\sqrt{4A(z-C)+B^2} = \sum_{k=0}^{\infty}(-1)^k \frac{(4A(z-C)+B^2)^k}{(2k)!},$$

then

$$f(P(z)) = \cos(2Az+B)$$

is a periodic function with period π/A. Rényi and Rényi [15] also proved that if $Q(z)$ is a nonconstant polynomial and $g(z)$ is entire and nonperiodic, then $Q(g(z))$ cannot be periodic either. Thus, if $Q(g(z))$ is a periodic function, then also $g(z)$ must be a periodic

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function. Further investigations on the periodicity of entire functions can be found in [1, 5, 6, 18].

Ozawa [14, Theorem 1] has shown that for any $\rho \in [1, +\infty)$ there exists a prime periodic entire function h of order $\rho(h) = \rho$. We assume that the reader is familiar with the basic symbols and fundamental results of Nevanlinna theory [8, 19]. Recall that the order of $f(z)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and the hyper-order of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Given a nonconstant meromorphic function f, the family of all meromorphic functions w such that $T(r, w) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, is denoted by $S(f)$. Let $\widehat{S}(f) = S(f) \cup \{\infty\}$. Suppose that f, g are meromorphic and $a \in \widehat{S}(f)$. Denoting by $E(a, f)$ the set of those points $z \in \mathbb{C}$ where $f(z) = a$, we say that f, g share a IM (ignoring multiplicities) if $E(a, f) = E(a, g)$. Provided that $E(a, f) = E(a, g)$ and the multiplicities of the zeros of $f(z) - a$ and $g(z) - a$ are the same at each $z \in \mathbb{C}$, then f, g share a CM (counting multiplicities).

Heittokangas *et al.* [9, Theorem 2] obtained the periodicity of $f(z)$ under the condition that $f(z)$ and $f(z + c)$ share three small periodic functions.

Theorem A. *Let $f(z)$ be a finite-order transcendental meromorphic function and let $a_1, a_2, a_3 \in \widehat{S}(f)$ be three distinct periodic functions with period c. If $f(z)$ shares a_1, a_2 CM and a_3 IM with $f(z + c)$, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

We consider the periodicity of an entire function $f(z)$ when a differential, difference or differential-difference polynomial in $f(z)$ is periodic. We assume that n, k are integers in the following. Note that $f^{(k)}(z)$ $(k \geq 1)$ can be a periodic function even if $f(z)$ is not periodic. For instance, $f(z) = e^z + z$ is an example of such a function. However, replacing $f(z)$ by $f(z)^n$ $(n \geq 2)$ or $f(z)^n f(z + c)$ $(n \geq 2)$, the periodicity can be determined partially. The questions posed in the present paper are inspired by Yang's conjecture, which appeared firstly in [16, Conjecture 1.1].

Yang's conjecture. *Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Wang and Hu [16, Theorem 1.1] showed that Yang's conjecture is true for $k = 1$, while Liu and Yu [13, Theorem 1.1] proved that Yang's conjecture is also true for an arbitrary k if $f(z)$ has a nonzero Picard exceptional value, namely, if $f(z) = e^{h(z)} + d$, where $h(z)$ is a nonconstant entire function and d is a nonzero constant. Note that if $h(z)$ is a nonconstant polynomial and $d = 0$, Yang's conjecture is also true and this can be seen as follows. We assume that

$$f(z)f^{(k)}(z) = f(z + c)f^{(k)}(z + c),$$

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where c is a nonzero constant. Substituting $f(z) = e^{h(z)}$ into the equation above gives $e^{2h(z+c)-2h(z)} = H(z)/H(z+c)$, where $H(z)$ is a polynomial in $h(z)$ and its derivatives and so also a polynomial in z. Since the rational function $H(z)/H(z+c)$ has no zeros and poles, then $H(z)/H(z+c) \equiv 1$. Thus, $e^{2h(z+c)-2h(z)} \equiv 1$, that is, $f(z)$ is a periodic function with period c or $2c$. Yang's conjecture for entire functions with a Picard exceptional value remains open in the case when $h(z)$ is transcendental and $d = 0$. We obtain the following result related to this question.

Theorem 1.1. *Let $f(z) = p(z)e^{h(z)} + q(z)$, where $p(z), q(z)$ are nonzero polynomials and $h(z)$ is a nonconstant entire function. If $f(z)f^{(k)}(z)$ is a periodic function, then $p(z)$ and $q(z)$ are constants.*

Even though Yang's conjecture has not been completely solved, it inspires us to propose related questions which will be considered in the paper.

Question 1.2. Let $f(z)$ be a transcendental entire function and n, k be integers. If $f(z)^n f^{(k)}(z+\eta)$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

We begin to consider Question 1.2 in the case $\eta = 0$ when k is a positive integer (the case $\eta = 0$ and $k = 0$ is trivial). This is the differential version of Question 1.2 and a generalisation of Yang's conjecture. As we have seen, the case $n = 1$ and $k = 1$ has been solved by Wang and Hu [16, Theorem 1.1]. If $n \geq 2$ and $k = 1$, the answer to Question 1.2 is also positive. Namely, assuming that $f^n(z)f'(z)$ is a periodic function with period $c(\neq 0)$, then

$$f(z+c)^n f'(z+c) = f(z)^n f'(z),$$

which implies that

$$f(z+c)^{n+1} - f(z)^{n+1} = A, \tag{1.1}$$

where $A \in \mathbb{C}$. Equation (1.1) has no nonconstant entire solutions provided that $A \not\equiv 0$, which is a direct consequence of Yang's result [17, Theorem 1], that is, there are no nonconstant entire solutions $f(z)$ and $g(z)$ that satisfy $a(z)f(z)^n + b(z)g(z)^m = 1$ provided that $m^{-1} + n^{-1} < 1$, where $a(z), b(z) \in S(f)$. Hence, $A \equiv 0$ in (1.1) and $f(z+c) = tf(z)$, where $t^{n+1} = 1$. Thus, $f(z)$ is a periodic function with period $(n+1)c$. It remains open whether Question 1.2 is true for $k \geq 2$, $n \geq 2$.

We next consider the case $k = 0$ and $\eta \neq 0$ in Question 1.2, which is the difference version of Question 1.

Theorem 1.3. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 2$ be a positive integer. If $f(z)^n f(z+\eta)$ is a periodic function with period c, then $f(z)$ is a periodic function with period $(n+1)c$.*

Theorem 1.3 is not valid for transcendental entire functions with $\rho_2(f) \geq 1$. This can be seen by taking a nonperiodic entire function $f(z) = e^{ze^z}$ such that $e^\eta = -n$, where n is a positive integer. Then $f(z)^n f(z+\eta) = e^{-n\eta e^z}$ is a periodic function. We claim that

$f(z) = e^{ze^z}$ is not a periodic function. Otherwise, there exists a nonzero constant c such that $e^{ze^z} = e^{(z+c)e^{z+c}}$ and thus $(z + c)e^{z+c} - ze^z = 2k\pi i$, which is impossible for a nonzero constant c.

In the case $n = 1$, it is easy to see that if $f(z)f(z + \eta)$ is a periodic function with period $c_1 = \eta$, then $f(z)$ is also a periodic function with period 2η. However, the case $c_1 \neq \eta$ is still open.

We pose another question and obtain two results below.

Question 1.4. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $[f(z)^n f(z + \eta)]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

Theorem 1.5. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 4$ be a positive integer. If $[f(z)^n f(z + \eta)]^{(k)}$ is a periodic function with period c, then $f(z)$ is a periodic function with period $(n + 1)c$.*

Theorem 1.5 is not true if $n = 1$ and $k \geq 2$. This can be seen by the example $f(z) = e^z + z$ and $e^c = -1$, where $[f(z)f(z + c)]'' = -2^2e^{2z} + ce^z + 2$ and $[f(z)f(z + c)]^{(k)} = -2^k e^{2z} + ce^z$ $(k \geq 3)$ are both periodic functions with period $2c$, but $f(z)$ is not a periodic function. However, we have the following result.

Theorem 1.6. *Suppose that $[f(z)^n f(z + \eta)]^{(k)}$ is a periodic function with period η. If $f(z)$ is a transcendental entire function of finite order and $n \geq 1$, then $f(z)$ is a periodic function with period $(n + 1)\eta$. If $f(z)$ is a transcendental entire function of infinite order and $n = 1$, $k = 1$, then $f(z)$ is a periodic function with period 2η.*

Yang's conjecture and Question 1.2 are related to differential (difference or differential-difference) monomials and Question 1.4 is related to differential-difference polynomials. We will next consider the following Question 1.7 related to the derivatives of difference polynomials.

Question 1.7. Let $f(z)$ be a transcendental entire function and $\Delta_\eta f := f(z + \eta) - f(z)$. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

Theorem 1.8. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and $n \geq 5$ be a positive integer. If $[f(z)^n \Delta_\eta f]^{(k)}$ is a periodic function with period η, then $f(z)$ is a periodic function with period $(n + 1)\eta$.*

Finally, observe that $f^{(k)}(z) + f^{(l)}(z)$ $(k > l)$ may be a periodic function, even if $f(z)$ is not a periodic function. This can be seen, for instance, by taking $f(z) = e^z + z$. On the relation between the periodicity of $f^{(k)}(z) + f^{(l)}(z)$ and $f(z)$, we give the following result.

Theorem 1.9. *Let $f(z)$ be a transcendental entire function and let $(f(z)^2)^{(k)} + (f(z)^2)^{(l)}$ be a periodic function with period c. If $k = 1$ and $l = 0$, then $f(z)$ is a periodic function with period c, $2c$, $4c$ or $4i\pi$. If $\rho(f) \geq 2$ and $k > l$, then $f(z)$ is a periodic function with period $2c$ or $4c$.*

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We see that Theorem 1.9 is not true for $\rho(f) = 1$, $k > l \geq 2$. Take $f(z) = e^{-z} + z + 1$. By an elementary computation, we see that $(f(z)^2)''' + (f(z)^2)'' = -4e^{-2z} + 2e^{-z} + 2$ is a periodic function, but $f(z)$ is not periodic. The case of $k = l$ is [16, Theorem 1.1].

2. Lemmas

The relations between the characteristic functions of a meromorphic function f and its difference polynomials will play important roles in our proofs. We firstly recall that if $f(z)$ is a transcendental entire function such that $\rho_2(f) < 1$, then

$$T(r, f(z+c)) = T(r, f) + S(r, f) \tag{2.1}$$

and

$$T(r, f(z+c) - f(z)) \leq T(r, f) + S(r, f). \tag{2.2}$$

These can be obtained by the difference analogue of the logarithmic derivative lemma [7, Lemma 8.3]. In the proofs of Theorem 1.5 and Theorem 1.8 below, the following three lemmas are needed.

Lemma 2.1 [12, Lemma 2.4]. *Let $f(z)$ be a transcendental entire function such that $\rho_2(f) < 1$. If $n \geq 1$, then*

$$T(r, f(z)^n f(z+c)) = (n+1)T(r, f) + S(r, f).$$

Lemma 2.2 [12, Lemma 2.6]. *Let $f(z)$ be a transcendental entire function such that $\rho_2(f) < 1$. If $n \geq 1$, then*

$$T(r, f(z)^n \Delta_\eta f) \geq nT(r, f) + S(r, f).$$

Lemma 2.3 [19, Theorem 1.62]. *Suppose that $n \geq 3$ and f_j $(j = 1, 2, \ldots, n)$ are meromorphic functions which are not constants except possibly for f_n. Let $\sum_{j=1}^{n} f_j = 1$. If $f_n \not\equiv 0$ and*

$$\sum_{j=1}^{n} N\Big(r, \frac{1}{f_j}\Big) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k), \quad \text{where } r \in I,$$

I is a set whose linear measure is infinite, $k \in \{1, 2, \ldots, n-1\}$ and $\lambda < 1$, then $f_n \equiv 1$.

Gross [4] proved that the Fermat functional equation $f(z)^2 + g(z)^2 = 1$ has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, and no other solutions exist. The following lemma concerns equations with small modifications to the Fermat-type difference equations

$$f(z+c)^2 + f(z)^2 = h(z).$$

Some results on the above equation can be found in [2, 11], where $h(z)$ is an entire function with finitely many zeros or a nonzero constant.

Lemma 2.4. *Let c be a nonzero constant. All entire solutions of*

$$f(z+c)^2 - f(z)^2 = e^{-z} \tag{2.3}$$

are periodic functions with period $4i\pi$, $2c$ or $4c$.

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Remark 2.5. Consider the following equation:

$$\left(\frac{f(z+c)}{e^{-z/n}}\right)^n + \left(\sqrt[n]{-1}\,\frac{f(z)}{e^{-z/n}}\right)^n = 1. \tag{2.4}$$

If $n \geq 3$, Yang's result [17, Theorem 1] shows that (2.4) has no entire solutions. If $n = 1$ in (2.4), then $f(z) = H(z) + f_1(z)$, where $H(z)$ is a periodic function with period c and $f_1(z)$ is a special solution of $f(z+c) - f(z) = e^{-z}$. This equation has entire solutions, but not all of them are periodic functions with period c. For example, $f_1(z) = (\alpha e/(1-\alpha e))e^{-z}$ with $c = 2k\pi i + \ln\alpha + 1$, $\alpha \neq 0, 1/e$, $k \in \mathbb{Z}$ is not c-periodic. Further details on finite-order transcendental entire solutions of (2.3) can be found in [2].

Remark 2.6. We recall the definition of a quasi-periodic entire function F with module g, that is, F satisfies $F(z+\tau) - F(z) = g(z)$. Chuang and Yang [3, Theorem 3.3] showed that, if $F(z+\tau) - F(z) = h(z)$, where $F(z) = f \circ g$ and $h(z)$ is a polynomial or $\rho(h) \leq 1$, then $g(z) = H_1(z) + q(z)e^{H_2(z)+Cz}$, where $H_1(z), H_2(z)$ are periodic functions with period τ, C is a constant and $q(z)$ is a polynomial. The above result is also related to the entire solution of (2.3) by taking $F(z) = f(z)^2$, which takes (2.3) into the form $F(z+\tau) - F(z) = e^{-z}$. However, this result does not give information on the periodicity of $f(z)$.

Proof of Lemma 2.4. Using Gross' result stated above,

$$\frac{f(z+c)}{e^{-z/2}} = \sin(h(z)), \qquad \frac{if(z)}{e^{-z/2}} = \cos(h(z)), \tag{2.5}$$

where $h(z)$ is any entire function such that $\sin h(z)\cos h(z) \not\equiv 0$. A basic computation from (2.5) shows that

$$e^{-c/2}\sin\left(h(z+c) + \frac{\pi}{2} + 2k\pi\right) = i\sin h(z), \quad k \in \mathbb{Z},$$

and hence

$$\frac{e^{-c/2}}{i}e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} - \frac{e^{-c/2}}{i}e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} + e^{-2ih(z)} = 1. \tag{2.6}$$

Case 1. If $h(z)$ is a constant h, then h satisfies $e^{-2ih} = (e^{c/2}-1)/(e^{c/2}+1)$ $(\neq 0, 1, -1)$ by (2.6). Thus, $f(z) = -ie^{-z/2}\cos h$, a periodic function with period $4i\pi$.

Case 2. If $h(z)$ is not a constant, then $e^{-2ih(z)}$ is not a constant, and both $h(z+c) + h(z)$ and $h(z+c) - h(z)$ are not constants at the same time. Using Lemma 2.3, we discuss the following two subcases.

Subcase 2.1. If

$$\frac{e^{-c/2}}{i}e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} \equiv 1, \quad -\frac{e^{-c/2}}{i}e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} + e^{-2ih(z)} \equiv 0,$$

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then $e^{-c} = -1$. By shifting the equation (2.3) forward, $f(z+2c)^2 - f(z+c)^2 = -e^{-z}$ and so $f(z+2c)^2 = f(z)^2$, which implies that $f(z)$ is a periodic function with period $2c$ or $4c$.

Subcase 2.2. If

$$-\frac{e^{-c/2}}{i}e^{-i(h(z+c)+\frac{1}{2}\pi+2k\pi+h(z))} \equiv 1, \quad \frac{e^{-c/2}}{i}e^{i(h(z+c)+\frac{1}{2}\pi+2k\pi-h(z))} + e^{-2ih(z)} \equiv 0,$$

then $e^{-c} = -1$. By the same discussion as in Subcase 2.1, it follows that $f(z)$ is a periodic function with period $2c$ or $4c$. □

3. Proofs of the theorems

PROOF OF THEOREM 1.1. Assume that $f(z)f^{(k)}(z)$ is a periodic function with period c ($\not\equiv 0$). Then

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c).$$

Substituting $f(z) = p(z)e^{h(z)} + q(z)$ into the equation above,

$$\begin{aligned} &p(z)H_k(z)e^{2h(z)} + [q(z)H_k(z) + p(z)q^{(k)}(z)]e^{h(z)} - p(z+c)H_k(z+c)e^{2h(z+c)} \\ &\quad - [q(z+c)H_k(z+c) + p(z+c)q^{(k)}(z+c)]e^{h(z+c)} = q(z+c)q^{(k)}(z+c) - q(z)q^{(k)}(z), \end{aligned} \tag{3.1}$$

where $H_k(z) = p(z)[h'(z)]^k + H(z)$ is a differential polynomial in $p(z)$ and $h(z)$ with the degree in $h(z)$ and its derivatives less than k. From (3.1),

$$T(r, e^{h(z)}) = T(r, e^{h(z+c)}) + S(r, e^{h(z)})$$

and so

$$T(r, h(z)) = T(r, h(z+c)) = S(r, e^{h(z)}).$$

We discuss two cases as follows.

Case 1. If $q(z)$ is a polynomial with $\deg(q(z)) \geq k$, then $q^{(k)}(z) \not\equiv 0$ and

$$q(z+c)q^{(k)}(z+c) - q(z)q^{(k)}(z) \not\equiv 0.$$

Therefore, $h(z)$ must be a constant by Lemma 2.3 and (3.1), which is a contradiction to the hypothesis that $h(z)$ is a nonconstant entire function.

Case 2. If $q(z)$ is a polynomial with $\deg(q(z)) < k$, then $q^{(k)}(z) \equiv 0$. From (3.1),

$$\frac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} + \frac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} - \frac{p(z+c)}{q(z+c)}e^{h(z+c)} = 1.$$

Now $(q(z+c)/p(z+c))e^{-h(z+c)}$ is not a constant because $h(z)$ is not a constant. From Lemma 2.3, we have two subcases.

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Subcase 2.1. Assume that

$$\begin{cases} \dfrac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} \equiv 1, \\ \dfrac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} - \dfrac{p(z+c)}{q(z+c)}e^{h(z+c)} \equiv 0. \end{cases} \tag{3.2}$$

Then $e^{h(z)+h(z+c)} \equiv q(z)q(z+c)/p(z)p(z+c)$. This implies that $h(z+c) \equiv B - h(z)$, where B is a constant. Thus, $(p(z)H_k(z)/q(z+c)H_k(z+c))e^{3h(z)-B} \equiv 1$ from the first equation of (3.2). So, $T(r, e^{h(z)}) = S(r, e^{h(z)})$, which is impossible.

Subcase 2.2. Suppose that

$$\begin{cases} \dfrac{q(z)H_k(z)}{q(z+c)H_k(z+c)}e^{h(z)-h(z+c)} \equiv 1, \\ \dfrac{p(z)H_k(z)}{q(z+c)H_k(z+c)}e^{2h(z)-h(z+c)} - \dfrac{p(z+c)}{q(z+c)}e^{h(z+c)} \equiv 0. \end{cases} \tag{3.3}$$

Now $e^{h(z+c)-h(z)} \equiv p(z)q(z+c)/q(z)p(z+c)$. Since $p(z)$ and $q(z)$ are polynomials, this implies that $h(z+c) - h(z) \equiv 2mi\pi$, where m is an integer, and it follows that

$$\frac{q(z+c)}{q(z)} \equiv \frac{p(z+c)}{p(z)}. \tag{3.4}$$

We will show that $p(z)$ and $q(z)$ are constants. If $h(z)$ is a nonconstant polynomial, then it must be a linear polynomial and so $H_k(z)$ is also a polynomial. From the first equation of (3.3), $q(z)$ and $H_k(z)$ are constants and so $p(z)$ is also a constant. If $h(z)$ is a transcendental entire function, then $q(z)H_k(z)/q(z+c)H_k(z+c) \equiv 1$ and $h'(z) \equiv h'(z+c)$. From the second equation of (3.3),

$$[p(z)^2 - p(z+c)^2][h'(z)]^k \equiv p(z+c)H(z+c) - p(z)H(z),$$

where $H(z)$ is a differential polynomial in $h'(z)$ with polynomial coefficients and degree less than k. If $p(z)^2 - p(z+c)^2 \not\equiv 0$, using the Clunie lemma [10, Lemma 2.4.2], we get $m(r, h') = S(r, h)$, which contradicts $h(z)$ being transcendental entire. Hence, $p(z)^2 \equiv p(z+c)^2$ from (3.4) and $p(z)$ and $q(z)$ are constants. □

Proof of Theorem 1.3. Since the period of $f(z)^n f(z+\eta)$ is c, where c is a nonzero complex number, then

$$f(z+c)^n f(z+\eta+c) = f(z)^n f(z+\eta),$$

which gives

$$\frac{f(z)^n}{f(z+c)^n} = \frac{f(z+\eta+c)}{f(z+\eta)}. \tag{3.5}$$

Let $G(z) = f(z)/f(z+c)$. From (2.1) and (3.5),

$$nT(r, G) = T\Big(r, \frac{1}{G(z+\eta)}\Big) = T(r, G(z+\eta)) + O(1) = T(r, G(z)) + S(r, G),$$

which contradicts $n \geq 2$. So, $G(z)$ must be a constant A and $A^n = A^{-1}$. Thus, $A^{n+1} = 1$, that is, $f(z)^{n+1} = f(z+c)^{n+1}$, so that $f(z)$ is a periodic function with period $(n+1)c$. □

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Proof of Theorem 1.5. Since $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period $c \neq 0$,

$$f(z+c)^n f(z+\eta+c) = f(z)^n f(z+\eta) + P(z),$$

where $P(z)$ is a polynomial with $\deg P(z) \leq k-1$. We will prove that $P(z) \equiv 0$. Since $f(z)$ is a transcendental entire function with $\rho_2(f) < 1$, Lemma 2.1 and the second main theorem for three small functions [8, Theorem 2.5] imply that

$$\begin{aligned}(n+1)T(r,f) &= T(r, f(z)^n f(z+\eta)) + S(r,f)\\ &\leq \overline{N}(r, f(z)^n f(z+\eta)) + \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta)}\Big)\\ &\quad + \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta) + P(z)}\Big) + S(r,f)\\ &\leq \overline{N}\Big(r, \frac{1}{f(z)^n f(z+\eta)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+c)^n f(z+\eta+c)}\Big) + S(r,f)\\ &\leq 4T(r,f) + S(r,f),\end{aligned}$$

which contradicts $n \geq 4$. Thus, $P(z) \equiv 0$. The same proof as for Theorem 1.3 can now be applied to show that f is a periodic function with period $(n+1)c$. □

Proof of Theorem 1.6. Since $[f(z)^n f(z+\eta)]^{(k)}$ is a periodic function with period η,

$$f(z)^n f(z+\eta) = f(z+\eta)^n f(z+2\eta) + P(z),$$

where $P(z)$ is a polynomial with $\deg P(z) \leq k-1$. Assume that $P(z) \not\equiv 0$. Then

$$f(z+\eta)[f(z)^n - f(z+\eta)^{n-1} f(z+2\eta)] = P(z).$$

Hence,

$$f(z+\eta) = P_1(z)e^{h(z)}, \quad f(z)^n - f(z+\eta)^{n-1} f(z+2\eta) = P_2(z)e^{-h(z)},$$

where $P_1(z)P_2(z) = P(z)$ and $P_1(z)$, $P_2(z)$ are nonzero polynomials. Hence,

$$P_1(z-\eta)^n e^{nh(z-\eta)} - P_1(z)^{n-1} P_1(z+\eta) e^{(n-1)h(z)+h(z+\eta)} = P_2(z)e^{-h(z)},$$

that is,

$$P_1(z-\eta)^n e^{nh(z-\eta)+h(z)} - P_1(z)^{n-1} P_1(z+\eta) e^{nh(z)+h(z+\eta)} = P_2(z). \tag{3.6}$$

Let $f_1 := P_1(z-\eta)^n e^{nh(z-\eta)+h(z)}$ and $f_2 := -P_1(z)^{n-1} P_1(z+\eta) e^{nh(z)+h(z+\eta)}$. Then (3.6) implies that $f_1(z) + f_2(z) = P_2(z)$. If f_1 and f_2 are transcendental, using the second main theorem for three small functions [8, Theorem 2.5],

$$T(r, f_1) \leq N(r, f_1) + N\Big(r, \frac{1}{f_1}\Big) + N\Big(r, \frac{1}{f_1 - P_2(z)}\Big) + S(r, f_1) \leq S(r, f_1),$$

which is impossible. Thus, f_1 and f_2 are polynomials and

$$nh(z-\eta) + h(z) = nh(z) + h(z+\eta) = B, \tag{3.7}$$

where B is a constant.

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If $f(z)$ is of finite order and $n \geq 1$, then $h(z)$ is a nonconstant polynomial and we have a contradiction from (3.7), so $P(z) \equiv 0$. As in the proof of Theorem 1.3, it follows that f is a periodic function with period $(n+1)\eta$.

If $f(z)$ is of infinite order and $n = 1$, then $h(z)$ may be a periodic function with period 2η. The condition $k = 1$ implies that $P(z)$ and $P_1(z)$ are constants. Thus, $f(z) = P_1 e^{h(z-\eta)}$ is a periodic function and P_1 is a nonzero constant. □

Proof of Theorem 1.8. Since the period of $[f(z)^n \Delta_\eta f]^{(k)}$ is η, where η is a nonzero complex number,

$$f(z+\eta)^n[f(z+2\eta) - f(z+\eta)] = f(z)^n[f(z+\eta) - f(z)] + Q(z),$$

where $Q(z)$ is a polynomial with $\deg Q(z) \leq k-1$. If $Q(z) \not\equiv 0$, then from the first and the second main theorems for three small functions [8, Theorem 2.5] and (2.2),

$$\begin{aligned}
nT(r,f) &\leq T(r, f(z)^n[f(z+\eta)-f(z)]) + S(r,f)\\
&\leq \overline{N}(r, f(z)^n[f(z+\eta)-f(z)]) + \overline{N}\Big(r, \frac{1}{f(z)^n[f(z+\eta)-f(z)]}\Big)\\
&\quad + \overline{N}\Big(r, \frac{1}{f(z)^n[f(z+\eta)-f(z)]+Q(z)}\Big) + S(r,f)\\
&\leq \overline{N}\Big(r, \frac{1}{f(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+\eta)-f(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z+\eta)}\Big)\\
&\quad + \overline{N}\Big(r, \frac{1}{f(z+2\eta)-f(z+\eta)}\Big) + S(r,f)\\
&\leq \overline{N}\Big(r, \frac{1}{f(z)}\Big) + T(r, f(z+\eta)-f(z)) + \overline{N}\Big(r, \frac{1}{f(z+\eta)}\Big)\\
&\quad + T(r, f(z+2\eta)-f(z+\eta)) + S(r,f)\\
&\leq 4T(r,f) + S(r,f).
\end{aligned}$$

This contradicts $n \geq 5$, so $Q(z) \equiv 0$ and

$$f(z+\eta)^n[f(z+2\eta) - f(z+\eta)] = f(z)^n[f(z+\eta) - f(z)].$$

If $f(z+2\eta) - f(z+\eta) \equiv 0$, then $f(z)$ is a periodic function with period η. If $f(z+2\eta) - f(z+\eta) \not\equiv 0$, then

$$\frac{f(z+\eta)^n}{f(z)^n} = \frac{f(z+\eta)-f(z)}{f(z+2\eta)-f(z+\eta)} = \frac{\frac{f(z+\eta)}{f(z)} - 1}{\frac{f(z+2\eta)}{f(z)} - \frac{f(z+\eta)}{f(z)}} = \frac{\frac{f(z+\eta)}{f(z)} - 1}{\frac{f(z+2\eta)}{f(z+\eta)}\frac{f(z+\eta)}{f(z)} - \frac{f(z+\eta)}{f(z)}}.$$

Let $G(z) = f(z+\eta)/f(z)$. Then

$$nT(r, G(z)) \leq 2T(r, G(z)) + T(r, G(z+\eta)) \leq 3T(r, G(z)) + S(r, G).$$

Since $n \geq 5$, this is again a contradiction. So, $G(z)$ should be a constant $A(\neq 1)$ and $A^n = (A-1)/(A^2 - A) = 1/A$, so $A^{n+1} = 1$, that is, $f(z)^{n+1} = f(z+\eta)^{n+1}$, and $f(z)$ is a periodic function with period $(n+1)\eta$. □

Proof of Theorem 1.9. Since $(f(z)^2)^{(k)} + (f(z)^2)^{(l)}$ is a periodic function with period $c(\neq 0)$,

$$(f(z+c)^2)^{(k)} + (f(z+c)^2)^{(l)} = (f(z)^2)^{(k)} + (f(z)^2)^{(l)}. \tag{3.8}$$

We set

$$f(z+c)^2 - f(z)^2 := F(z). \tag{3.9}$$

Then (3.8) can be written as

$$F^{(k)}(z) = -F^{(l)}(z). \tag{3.10}$$

We discuss two cases as follows.

Case 1. If $k = 1$ and $l = 0$, by integrating (3.10), we have $F(z) = Ce^{-z}$, where C is a nonzero constant or $F(z) = 0$. If $F(z) = Ce^{-z}$, Lemma 2.4 implies that $f(z)$ is a periodic function with period $4i\pi$, $2c$ or $4c$. If $F(z) = 0$, then $f(z)$ is a periodic function with period c or $2c$.

Case 2. If $\rho(f) \geq 2$ and $k > l$, from (3.10), it follows that $F(z)$ must be an exponential polynomial satisfying $F^{(l)}(z) = \mu_1 e^{\lambda_1 z} + \cdots + \mu_{k-l} e^{\lambda_{k-l} z}$, where $\lambda_i^{k-l} = -1$ and the μ_i are constants for $i = 1, 2, \ldots, k-l$. Thus, $\rho(F(z)) \leq 1$.

We claim that $\rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z)) = \rho(f) \geq 2$. On the one hand, $\rho(f(z+c) - f(z)) < 2$ and $\rho(f(z+c) + f(z)) < 2$ cannot both happen simultaneously, otherwise $\rho(f) < 2$, a contradiction. On the other hand, only one of $\rho(f(z+c) - f(z))$ and $\rho(f(z+c) + f(z))$ less than 2 cannot happen, otherwise $\rho(F(z)) \geq 2$, which is a contradiction. Thus, $\rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z)) \geq 2$, by (3.9). From

$$2f(z) = f(z+c) + f(z) - (f(z+c) - f(z)),$$

we then have $\rho(f) \leq \rho(f(z+c) - f(z)) = \rho(f(z+c) + f(z))$. Combining the above with $\rho(f(z+c) - f(z)) \leq \rho(f)$ proves the claim.

If $F(z) \not\equiv 0$, from (3.9) and the Hadamard factorisation theorem,

$$f(z+c) - f(z) = h_1(z)e^{H(z)}, \quad f(z+c) + f(z) = h_2(z)e^{-H(z)}, \tag{3.11}$$

where $\max\{\rho(h_1), \rho(h_2)\} \leq 1$ and $\rho(e^H) \geq 2$. Thus, $T(r, h_i) = S(r, e^H)$, $i = 1, 2$. Then

$$f(z) = \tfrac{1}{2}(h_2(z)e^{-H(z)} - h_1(z)e^{H(z)}), \quad f(z+c) = \tfrac{1}{2}(h_2(z)e^{-H(z)} + h_1(z)e^{H(z)}).$$

Hence,

$$h_2(z)e^{-H(z)} + h_1(z)e^{H(z)} = h_2(z+c)e^{-H(z+c)} - h_1(z+c)e^{H(z+c)}. \tag{3.12}$$

Dividing (3.12) by $h_1(z)e^{H(z)}$,

$$f_1 + f_2 + f_3 = 1,$$

where we define $f_1 = -(h_2(z)/h_1(z))e^{-2H(z)}$, $f_2 = (h_2(z+c)/h_1(z))e^{-H(z+c)-H(z)}$ and $f_3 = -(h_1(z+c)/h_1(z))e^{H(z+c)-H(z)}$. Obviously, $-H(z+c) - H(z)$ and $H(z+c) - H(z)$ are not constants at the same time.

If $-H(z+c) - H(z)$ is not a constant, from Lemma 2.3, $f_3 \equiv 1$ and immediately

$$h_1(z)e^{H(z)} \equiv -h_1(z+c)e^{H(z+c)}. \tag{3.13}$$

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From the first equation of (3.11) and (3.13),

$$f(z+c) - f(z) \equiv -(f(z+2c) - f(z+c)).$$

Thus, $f(z) \equiv f(z+2c)$ and f is a periodic function with period $2c$.

If $H(z+c) - H(z)$ is not a constant, from Lemma 2.3, $f_2 \equiv 1$ and

$$h_1(z)e^{H(z)} \equiv h_2(z+c)e^{-H(z+c)}. \tag{3.14}$$

From (3.11) and (3.14),

$$f(z+c) - f(z) \equiv f(z+2c) + f(z+c).$$

Thus, $f(z) \equiv f(z+4c)$ and f is a periodic function with period $4c$. □

4. Discussion

We have considered the periodicity of transcendental entire functions mainly under the condition $\rho_2(f) < 1$. By a careful examination of the proofs of our main results, it follows that Theorem 1.3 is also valid for transcendental meromorphic functions with $\rho_2(f) < 1$. In addition, Theorem 1.5 is true for transcendental meromorphic functions with $\rho_2(f) < 1$ and $n \geq 8$, as can be seen by appropriate application of the inequality

$$T(r, f(z)^n f(z+\eta)) \geq (n-1)T(r, f) + S(r, f), \quad \eta \in \mathbb{C} \setminus \{0\}$$

(see [12, Lemma 2.5]) in the proof of Theorem 1.5. Theorem 1.8 is valid for transcendental meromorphic functions with $\rho_2(f) < 1$ and $n \geq 10$, by using

$$T(r, f(z)^n [f(z+\eta) - f(z)]) \geq (n-1)T(r, f) + S(r, f), \quad \eta \in \mathbb{C} \setminus \{0\}$$

(see [12, Lemma 2.7]) in the proof of Theorem 1.8. The other theorems cannot be directly extended to transcendental meromorphic functions in the same way.

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