Macroeconomic Dynamics, **6**, 2002, 665–686. Printed in the United States of America.
DOI: 10.1017.S1365100501010124

OPTIMAL LONG-RUN BUDGETARY POLICIES SUBJECT TO THE MAASTRICHT CRITERIA OR A STABILITY PACT

Maria Dworak
Austrian Academy of Sciences

Franz Wirl
University of Vienna

Alexia Prskawetz
Max Planck Institute for Demographic Research

Gustav Feichtinger
Vienna University of Technology

We investigate the optimal path of the primary surplus that a government will choose to minimize costs that derive from exceeding the Maastricht criteria and generally of a "stability pact," where we assume three components of costs that are related to (1) the debt-to-GDP ratio, (2) the overall deficit-to-GDP ratio, and (3) the acceptance level of savings in the economy. We show that various political-economic settings can result in completely different equilibrium strategies of the debt-to-GDP ratio and the primary deficit. The spectrum of possible optimal strategies ranges from no stationary solutions to multiple equilibrium and cyclical solutions and from positive to negative levels of the optimal debt-to-GDP ratio. Our results emphasize the importance of macroeconomic and behavioral (acceptance rate of a policy) variables in order to explain complex economic time series.

Keywords: Dynamic Programming/Optimal Control, Multiple Equilibria, Local Bifurcation, Limit Cycles, Budgetary Policies, Maastricht Criteria

1. INTRODUCTION

This paper is motivated by the fact that most governments of EU countries had to implement a retrenchment program to meet the Maastricht criteria in order to participate in the European Monetary Union (EMU). A number of papers

This research was partly financed by the Austrian Science Foundation under Contract No. P12745-OEK. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation or the Max Planck Institute for Demographic Research. Address correspondence to Gustav Feichtinger, Institute for Econometrics, Operations Research and Systems Theory, Vienna University of Technology, Argentinierstr. 8/119, A-1040 Vienna, Austria; e-mail: or@eos.tuwien.ac.at.

 1365-1005/02 $9.50

[recently, Prskawetz (1998) and Prskawetz et al. (1998)] consider the dynamics of this or similar adjustment processes. However, problems of deficits and excessive spending will not disappear in the near future. Moreover, the constraints are soft despite all the rhetoric. This motivates us to investigate the long run, where parts of the Maastricht criteria are compatible with the (endogenous) steady states of a dynamic system. However, as we demonstrate, the evolution may never reach, or at least may not remain in, this equilibrium, or it may attain this equilibrium in a nontrivial pattern.

There are various reasons for missing the targets, predominately positive explanations from Public Choice and normative, if one believes in Keynesian control of economies. In both cases the reason is that overachieving the target produces little benefit, if any. On the other hand, failing to meet a criterion marginally because of increased spending has no costs, but substantial benefits, either because a constituency, which is crucial for reelection, receives a subsidy, or because deficit spending reduces unemployment. These last remarks highlight the connection of this paper with the political business cycle (PBC) literature, which dates back at least to Nordhaus (1975) and Hibbs' (1977) partisan model; for more recent surveys, see Schneider and Frey (1988), Mueller (1989), and Nordhaus (1989). The approaches of both Nordhaus and Hibbs rely on voters who can be fooled; Alesina (1987), however, is a first attempt to derive PBC's when voters have rational expectations.

Although definitely related to this literature, this paper differs in important aspects. First, the results in the PBC literature depend on the exogenous election date and the nonmonotonicity of the optimal strategies of governments seeking reelection. Thus, if the election date were very far away, the cycle would disappear in all of these models and be replaced by monotonic strategies. Second, the PBC literature focuses on macroeconomic variables, which is problematic because important policy variables, such as monetary growth, cannot be chosen by the government, either because of independent central banks (e.g., in Germany) or because of pegging the currency to another currency (such as Austria and the Netherlands did with respect to the DM). Moreover, transfers seem more important to support-seeking politicians than do elusive (from the voter's point of view) "macroeconomic" variables [see Buchanan and Wagner (1977)]. A recent empirical investigation of PBC's with respect to spending patterns is that of van Dalen and Swank (1996). They find—as expected—increasing expenditures prior to elections for both left- and right-wing governments, albeit geared to a different clientele. Although the paper of van Dalen and Swank (1996) and, in fact, the entire PBC literature are concerned about a short planing horizon (with a maximum of 4 to 5 years), the accumulation of debt and the expansion of deficits is a highly sluggish process and retrenchment will be even more sluggish. Figure 1 shows the debt-to-GDP ratios and overall deficit-to-GDP ratios, subsequently denoted as debt ratios and overall deficit ratios, of some countries, which highlights the large time constant of this process compared with expenditures.

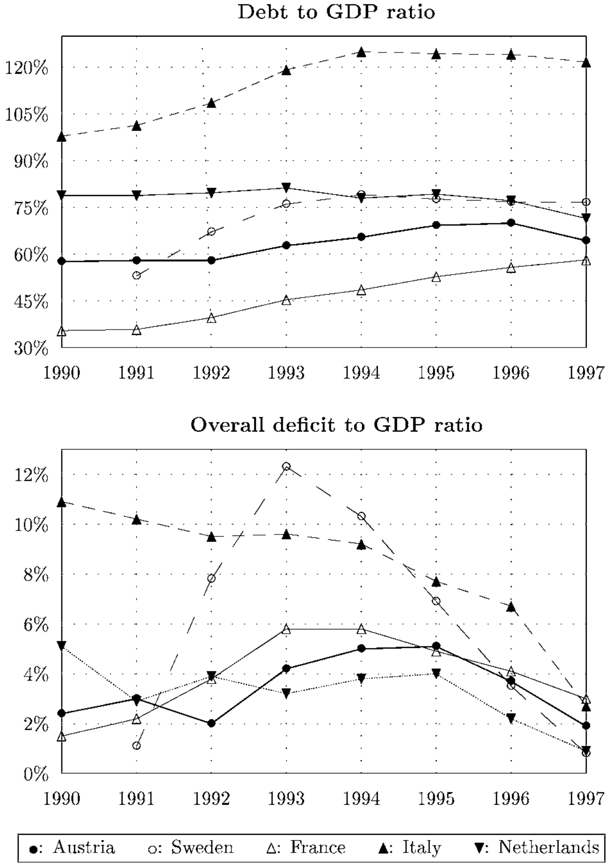

Figure 1. Debts and overall deficits from 1990 through 1997 for Austria, Sweden, France, Italy, and the Netherlands [Source: Statistik Austria (1999)].

2. MODEL

We consider a long-lasting government that lacks any reelection constraint. However, to meet the Maastricht criteria (indicated by the superscript M in the following) the government needs to adjust its debt to GDP ratio d and the overall deficit ratio $rd - u$, where r denotes the nominal interest rate and u is the primary deficit ($u < 0$) or primary surplus ($u > 0$), respectively.

Associated with the necessary retrenchment policies, the government faces costs for debt, $k(d, d^M)$, and for the overall deficit, $g(rd - u, (rd - u)^M)$, and domestic costs associated with the necessity to save, $c(u, x)$, where x denotes the equilibrium acceptance rate of savings by the public as defined later.[1]

Reasons for the international costs associated with large debts and deficits can be of a normative and a positive nature. For example, high debts reduce the ratings

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

of an economy, which will increase the interest costs. Positively, politicians of a country notoriously failing to achieve the criteria might be treated at international meetings if not like the pariah, then like poor beggars. Domestic costs associated with deficits may be due to fiscal conservatism prevailing in some parts of the constituency, despite decades of Keynesian policies.[2]

Since a retrenchment policy cuts the population's real income either directly through increased taxes or indirectly through a reduction in public services and transfers, we introduce the last cost item to purely reflect political costs. These costs depend on the population's understanding of the necessity and thus of the willingness to save, x. As a consequence, politicians, who would like to reduce debt and deficits, which requires dramatic changes in the primary surplus u, face adjustment costs [for a recent survey of adjustment, albeit in a different and largely empirical context, see Hamermesh and Pfann (1996)].[3]

Summarizing, a government tries to devise a strategy of primary surpluses $\{u(t), t \in [0, \infty)\}$, deficits if $u(t)$ is negative, such that the present value of the overall costs, using the subjective discount rate $\rho > 0$, becomes minimal:

$$\min_{u(t)} \int_0^{\infty} e^{-\rho t}[k(d, d^M) + g(rd - u, (rd - u)^M) + c(u, x)]dt. \qquad \textbf{(1)}$$

Formulation (1) incorporates the Maastricht criteria, $d^M = 0.6$ and $(rd - u)^M = 0.03$, as parameters to refer to the motivation of this investigation.[4]

Figure 2 is a graph of the three cost items in (1). The costs arising from debt, k, and from deficit, g, are asymmetric with respect to the Maastricht target. In Figure 2, a nonlinear example of the function k and a linear example of the function

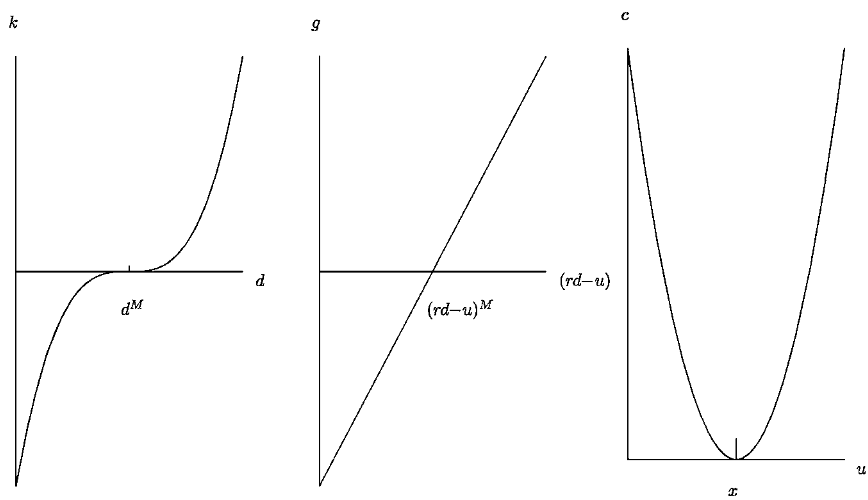

FIGURE 2. Graphical representation of the costs for debt (k), deficits (g), and retrenchment (c).

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

g are drawn. Finally, a symmetric example is drawn for the political costs of retrenchment, c, which might be asymmetric as well since excessive spending hurts less than excessive saving.

The maximization of the objective (1) is subject to two dynamic constraints. The first constraint is a pure accounting relation for d, the debt-to-GDP ratio (in short, the debt) as given by[5]

$$\dot{d} = (r - \theta)d - u = (rd - u) - \theta d. \tag{2}$$

That is, the debt-to-GDP ratio grows by the difference between the nominal interest cost r and the nominal economic growth θ minus the primary "surplus" u.[6] We assume that $\delta := r - \theta$ is positive and less than the subjective discount rate, $0 < \delta < \rho$.[7]

The second dynamic constraint concerns the costs to the politicians if they "surprise" the electorate with the necessity to increase saving for Maastricht or any other purpose. In the objective (1), we assume that voters penalize the politicians for savings, that is, primary surpluses, exceeding the people's willingness to support retrenchment. These ideas are captured by the differential equation

$$\dot{x} = \tau(\bar{x} - x), \qquad x(0) = x_0. \tag{3}$$

That is, the voters change their acceptance level of saving—this is what $x(t)$ describes in contrast to the actual saving as given by the primary surplus, $u(t)$—proportional to the deviation from the equilibrium level, $\bar{x}$, which depends on some fundamentals. This relation (3) can be explained as an adaptive expectation mechanism, where the voters, reading new information about some fundamentals, update their acceptance for primary surpluses depending on the time constant τ. Another, equivalent interpretation is that the present acceptance level of retrenchment x depends on the history of readings and other information about debt and savings (the primary surplus ratio or, respectively deficit if $u < 0$). Assuming a linear relation, we obtain

$$x(t) = \alpha \int_{-\infty}^{t} \exp(-\tau(t - \nu))d(\nu)d\nu + \beta \int_{-\infty}^{t} \exp(-\tau(t - \nu))u(\nu)d\nu. \tag{4}$$

Differentiating (4) with respect to time implies (3) with a linear relation for the equilibrium acceptance level

$$\bar{x} = \tilde{\alpha}d + \tilde{\beta}u, \tag{5}$$

where $0 < \tilde{\alpha} := \alpha/\tau$, $0 < \tilde{\beta} := \beta/\tau$.

Summing up, the objective of the government is to choose an optimal level of the primary surplus (respectively, primary deficit) to minimize aggregate present value of losses, (1), subject to the evolution of the government debt, (2), and the equilibrium acceptance rate of savings, (3), and given an initial level of the debt-to-GDP ratio, $d(0)$, and the equilibrium acceptance rate of savings, $x(0)$.

3. OPTIMALITY CONDITIONS AND STABILITY ANALYSIS

In this section we apply Pontryagin's maximum principle [see Feichtinger and Hartl (1986) or Leonard and Long (1992)] to solve for the optimal time path of the debt, the primary deficit, and the equilibrium acceptance rate of savings. By locally linearizing the resulting dynamical system around its steady state, we determine the local stability properties.

To simplify, we assume that the costs of debt, $k(d, d^M)$, and the domestic costs associated with the necessity to save, $c(u, x)$, depend on the difference of their respective arguments:

$$k(d, d^M) = \kappa \tilde{k}(d - d^M), \qquad \tilde{k}' > 0, \tag{6}$$

$$c(u, x) = \gamma \tilde{c}(u - x), \qquad \tilde{c}' > 0, \tag{7}$$

where κ and γ are constant parameters and $\tilde{k}$ and $\tilde{c}$ are scalar functions, which vanish for the argument zero.

The costs associated with a deficit are assumed to be linear:

$$g(rd - u, (rd - u)^M) = \varphi[(rd - u) - (rd - u)^M], \tag{8}$$

with the proportionality factor φ.

To solve the optimization problem, we define the current-value Hamiltonian $H(d, x, u, \lambda, \mu)$ letting λ and μ denote the current-value costate variables corresponding to d and x, respectively. The Hamiltonian maximizing condition[8]

$$\frac{\partial H}{\partial u} = 0$$

yields

$$-\gamma \tilde{c}'(u - x) + \varphi - \lambda + \mu\beta = 0, \tag{9}$$

from which we derive for interior points

$$\begin{aligned} u &= x + (\tilde{c}')^{-1}((\varphi - \lambda + \mu\beta)/\gamma) \\ &= x + h(\mu\beta - \lambda), \end{aligned} \tag{10}$$

where we define

$$h(\cdot) := (\tilde{c}')^{-1}((\varphi + \cdot)/\gamma). \tag{11}$$

The optimal intertemporal evolution of the system—taking into account condition (10)—is given by the canonical system

$$\begin{aligned} \dot{d} &= (r - \theta)d - x - h(\mu\beta - \lambda) \\ \dot{x} &= \alpha d + (\beta - \tau)x + \beta h(\mu\beta - \lambda) \\ \dot{\lambda} &= \rho\lambda - \frac{\partial H}{\partial d} = (\rho - r + \theta)\lambda - \mu\alpha + \kappa \tilde{k}'(d - d^M) + \varphi r \\ \dot{\mu} &= \rho\mu - \frac{\partial H}{\partial x} = (\rho + \tau - \beta)\mu + \lambda - \varphi. \end{aligned} \tag{12}$$

The stationary solution $(d^*, x^*, \lambda^*, \mu^*)$ of system (12), if it exists, is given by the solution of the following set of equations:

$$\begin{aligned} d^* &= \frac{\tau}{(r-\theta)(\tau-\beta)-\alpha} h\left((\rho+\tau)\frac{\kappa\tilde{k}'(d^*-d^M)+\varphi(\rho+\theta)}{\alpha+(\rho-r+\theta)(\rho+\tau-\beta)}-\varphi\right) \\ x^* &= \frac{\beta(r-\theta)+\alpha}{\tau} d^* \\ \lambda^* &= \varphi-(\rho+\tau-\beta)\mu^* \\ \mu^* &= \frac{\kappa\tilde{k}'(d^*-d^M)+\varphi(\rho+\theta)}{\alpha+(\rho-r+\theta)(\rho+\tau-\beta)}. \end{aligned} \tag{13}$$

To determine the stability properties of the steady state, we proceed according to Dockner (1985), where a classification of the equilibria is made depending on the values of the determinant of the Jacobian of the dynamical system (12), det J and K, the latter being defined as follows:

$$K := \left\| \begin{matrix} \frac{\partial \dot{d}}{\partial d} & \frac{\partial \dot{d}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial d} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{matrix} \right\| + \left\| \begin{matrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \mu} \\ \frac{\partial \dot{\mu}}{\partial x} & \frac{\partial \dot{\mu}}{\partial \mu} \end{matrix} \right\| + 2 \left\| \begin{matrix} \frac{\partial \dot{d}}{\partial x} & \frac{\partial \dot{d}}{\partial \mu} \\ \frac{\partial \dot{\lambda}}{\partial x} & \frac{\partial \dot{\lambda}}{\partial \mu} \end{matrix} \right\|. \tag{14}$$

In our model, the Jacobian is given as

$$J = \begin{pmatrix} r-\theta & -1 & h'(\beta\mu^*-\lambda^*) & -\beta h'(\beta\mu^*-\lambda^*) \\ \alpha & \beta-\tau & -\beta h'(\beta\mu^*-\lambda^*) & \beta^2 h'(\beta\mu^*-\lambda^*) \\ \kappa\tilde{k}''(d^*-d^M) & 0 & \rho-r+\theta & -\alpha \\ 0 & 0 & 1 & \rho+\tau-\beta \end{pmatrix}. \tag{15}$$

The determinant of the Jacobian, det J, and K are then given, respectively, by

$$\begin{aligned} \det J &= [\alpha+(\rho-r+\theta)(\rho+\tau-\beta)][\alpha+(r-\theta)(\beta-\tau)] \\ &\quad +\tau(\rho+\tau)\kappa\tilde{k}''(d^*-d^M)h'(\beta\mu^*-\lambda^*) \end{aligned} \tag{16}$$

$$\begin{aligned} K &= (r-\theta)(\rho-r+\theta)+(\beta-\tau)(\rho+\tau-\beta)+2\alpha \\ &\quad -\kappa\tilde{k}''(d^*-d^M)h'(\beta\mu^*-\lambda^*). \end{aligned} \tag{17}$$

To obtain specific results on the stability property of the steady state, we specify functions for the cost of debt, $\tilde{k}(d-d^M)$, and for the domestic cost associated with the necessity to save, $\tilde{c}(u-x)$, in the next section.

4. NUMERICAL EXAMPLES

For simplicity, we assume polynomials for the costs of debt and the domestic costs associated with the necessity to save; that is,

$$\tilde{k}(d - d^M) = \frac{1}{b}(d - d^M)^b, \qquad b \geq 1 \tag{18}$$

and

$$\tilde{c}(u - x) = \frac{1}{a}(u - x)^a, \qquad a > 1. \tag{19}$$

Furthermore, we assume b to be an odd number since we postulated asymmetric costs due to debt. The power a can be even or odd, since the domestic costs due to deficit may be symmetric or asymmetric. The polynomial for $\tilde{c}(u - x)$ satisfies all properties assumed in the preceding sections.[9]

In what follows, we distinguish between the cases of linear costs of debt and nonlinear costs of debt, that is, $b = 1$ versus $b > 1$.[10]

4.1. Model with Linear Costs of Debt

Upon substituting equations (18) with $b = 1$ and (19) into equations (13), (16), and (17), the expressions for $\det J$, K, and the equilibrium $(d^*, x^*, \lambda^*, \mu^*)$ simplify to

$$\det J = [\alpha + (\rho - r + \theta)(\rho + \tau - \beta)][\alpha + (r - \theta)(\beta - \tau)], \tag{20}$$

$$K = (r - \theta)(\rho - r + \theta) + (\beta - \tau)(\rho + \tau - \beta) + 2\alpha \tag{21}$$

and

$$\begin{aligned} d^* &= \frac{\tau}{(r - \theta)(\tau - \beta) - \alpha}\left\{(\rho + \tau)\frac{\kappa + \varphi(\rho + \theta)}{\gamma[\alpha + (\rho - r + \theta)(\rho + \tau - \beta)]}\right\}^{\frac{1}{a-1}} \\ x^* &= \frac{\beta(r - \theta) + \alpha}{\tau} d^* \\ \lambda^* &= \varphi - (\rho + \tau - \beta)\mu^* \\ \mu^* &= \frac{\kappa + \varphi(\rho + \theta)}{\alpha + (\rho - r + \theta)(\rho + \tau - \beta)}. \end{aligned} \tag{22}$$

This leads to Proposition 1.

PROPOSITION 1. *In the case of linear costs of debt, the steady state is unique. The stability of the equilibrium is independent of the functional form (symmetry/asymmetry) of the costs associated with the necessity to save. There is an even stronger result: The stability of the equilibrium is independent of the degree of nonlinearity of the domestic costs associated with the necessity to save. Furthermore, the stability of the unique steady state does not depend on the parameters of the costs of deficit.*

To ease the stability analysis, we fix the parameters r, θ, α, β, ρ, and take τ as the bifurcation parameter. By choosing τ as the bifurcation parameter, we aim to investigate the influence of the speed of adjustment of the public's acceptance of savings on the optimal levels of government debt and its primary deficit.

In Figure 3, a plot of the stationary values d^*, x^*, and u^* versus τ is shown, where the other parameters are fixed at $a = 4$, $\gamma = 2{,}000$, $\kappa = 1$, $\varphi = 0.1$, $\alpha = 0.9$, $\beta = 0.1$, $r = 0.05$, $\theta = 0.031$, and $\rho = 0.055$.[11] An enlargement of this plot for small values of τ is given in Figure 4.

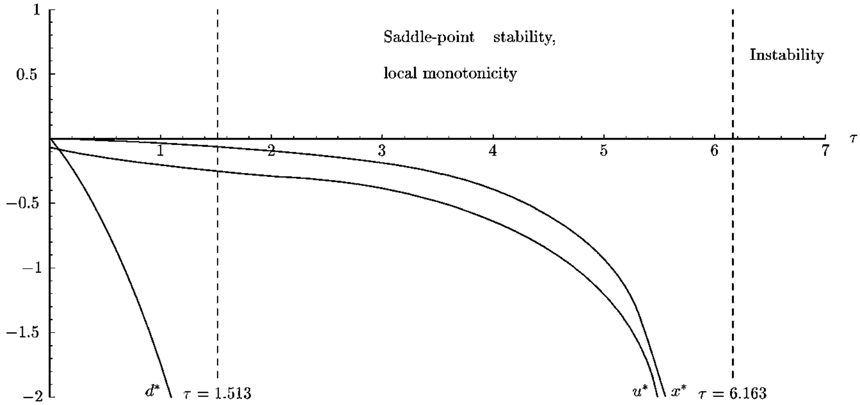

FIGURE 3. Plot of the stationary values of debt-to-GDP ratio, d^*, and of the deficit-to-GDP ratio, x^*, and the optimal value of the primary deficit in the steady state, u^* versus τ. For $\tau < 1.513$, the stability of the steady state is shown in Figure 4.

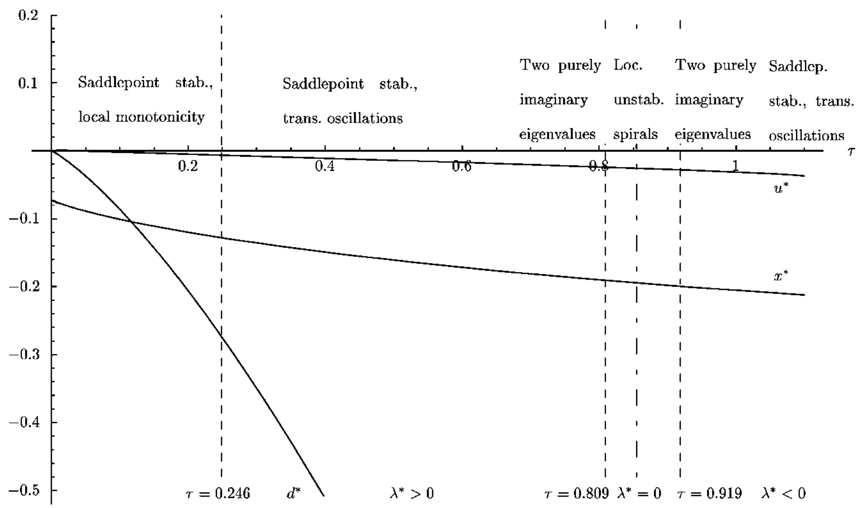

FIGURE 4. Enlargement of Figure 3 for small values of τ.

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

For $\tau < \beta - (r - \theta) - 2\sqrt{\alpha}$ (which equals 0.246 for the assumed parameter values in Figure 4) the steady state exhibits saddlepoint stability with local monotonicity. This means that, for given initial states, choosing the corresponding initial conditions of the costates from the stable, two-dimensional manifold ensures convergence to the steady state, where the time path into the steady state is locally monotone. All other initial conditions of the costates lead to divergence. For $\beta - (r - \theta) - 2\sqrt{\alpha} < \tau < \beta + r - \theta - 2\rho$ ($0.246 < \tau < 0.809$), the steady state still exhibits saddlepoint stability but with transient oscillations. This means that the time path into the steady state is now oscillating.

For $\tau = \beta + r - \theta - 2\rho$ and for $\tau = \beta + r - \theta$ ($\tau = 0.809$ and $\tau = 0.919$, respectively, in Figure 4), the eigenvalues are purely imaginary and thus these parameter values are candidates for a Hopf bifurcation. Although we could not prove the existence of limit cyles for $\tau = \beta + r - \theta$, despite the existence of purely imaginary eigenvalues, we can prove that limit cycles cannot emerge at the bifurcation point $\tau = \beta + r - \theta - 2\rho$. This analysis is provided in the Appendix. Consequently, for values of τ between these bifurcation points, the system exhibits instability. Note that the length of this interval of instability only depends on the politician's discount rate ρ. Saddlepoint stability with transient oscillations is again obtained for $\beta + r - \theta < \tau < \beta - r + \theta + 2\sqrt{\alpha}$ ($0.919 < \tau < 1.513$ in Figures 3 and 4, respectively). For $\beta - r + \theta + 2\sqrt{\alpha} \leq \tau < \beta + \alpha/(r - \theta)$ ($1.513 < \tau < 6.163$ in Figure 3), the system still exhibits saddlepoint stability, but with a locally monotone path into the steady state.

At $\tau = \beta + \alpha/(r - \theta)$ ($\tau = 6.163$ in Figure 3), the determinant of the Jacobian is equal to zero. In equation (22), it can be seen that d^* and x^* exhibit a pole at $\tau = \beta + \alpha/(r - \theta)$ ($\tau = 6.163$ in Figure 3). For values of τ greater than this parameter value, $\det J$ is negative, and the dynamical system is unstable except for the nongeneric case in which the initial conditions stem from a particular, one-dimensional manifold of the state space. Consequently, the values of d^* and x^* greater than this threshold are economically uninteresting. The fact that $\det J$ and d^* have opposite signs, leads to Proposition 2.

PROPOSITION 2. *Stability requires the government to be a lender.*

Proof. See Appendix.

4.2. Model with Nonlinear Costs of Debt

Upon substituting (18) and (19) into equations (13), (16), and (17), the expressions for $\det J$, K, and the system of implicit equations to determine the equilibria $(d^*, x^*, \lambda^*, \mu^*)$ become

$$\det J = [\alpha + (\rho - r + \theta)(\rho + \tau - \beta)][\alpha + (r - \theta)(\beta - \tau)] + (b-1)\frac{\tau}{\gamma}\frac{\rho + \tau}{a-1}\kappa(d^* - d^M)^{b-2}\left(\frac{\varphi + \beta\mu^* - \lambda^*}{\gamma}\right)^{\frac{2-a}{a-1}}, \quad (23)$$

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

$$K = (r-\theta)(\rho - r + \theta) + (\beta - \tau)(\rho + \tau - \beta) + 2\alpha$$
$$- \frac{b-1}{a-1}\frac{\kappa}{\gamma}(d^* - d^M)^{b-2}\left(\frac{\varphi + \beta\mu^* - \lambda^*}{\gamma}\right)^{\frac{2-a}{a-1}}, \quad (24)$$

and

$$d^* = \frac{\tau}{(r-\theta)(\tau-\beta)-\alpha}\left\{(\rho+\tau)\frac{\kappa(d^*-d^M)^{b-1} + \varphi(\rho+\theta)}{\gamma[\alpha + (\rho - r + \theta)(\rho + \tau - \beta)]}\right\}^{\frac{1}{a-1}}$$
$$x^* = \frac{\beta(r-\theta)+\alpha}{\tau}d^*$$
$$\lambda^* = \varphi - (\rho + \tau - \beta)\mu^* \quad (25)$$
$$\mu^* = \frac{\kappa(d^*-d^M)^2 + \varphi(\rho+\theta)}{\alpha + (\rho - r + \theta)(\rho + \tau - \beta)}.$$

This leads to Proposition 3.

PROPOSITION 3. *In the case of nonlinear costs of debt, one has to distinguish whether the degree of nonlinearity of the costs of debt, b, is higher than the degree of nonlinearity of the costs associated with the necessity to save, a. If $b \geq a$, there exist either none, one, or exactly two equilibria, whereas for $b < a$, there exist at least one equilibrium and at most three equilibria.*

The proposition can be verified by using the rule of Descartes, which allows one to estimate the number of zeros of a polynomial, and neglecting the solutions that would lead to a minimum of the Hamiltonian. Since the proof is extremely technical and provides no further insight into the dynamics of the system, we refrain from presenting the proof, but it can be obtained on request from the authors.

Similar to the model with linear costs to debt, we can state some stability properties of the equilibria from equations (23) and (24).

PROPOSITION 4. *Stability requires the government either to be a lender or to hold debt exceeding the Maastricht threshold.*

The proof is given in the Appendix. For further stability analysis, we consider first the case of a higher order of costs of debt than for the costs of deviating from the public's acceptance level of savings and set $a = 2$ and $b = 3$, where, at most, two equilibria exist. Furthermore, we again fix the other parameter values at $r = 0.05$, $\theta = 0.031$, $\alpha = 0.1$, $\beta = 0.9$, $\varphi = 0.1$, and $\kappa = 1$, and alternatively take τ, γ, and ρ as bifurcation parameters. Hence, we investigate the effects of changes in the speed of adjustment of the public's acceptance of savings, τ, the weight of the domestic cost associated with the necessity to save, γ, and the social time discount rate, ρ, on the equilibrium value of national debt.

In Figures 5–7, the stationary values of debt, d^*, are plotted versus τ, γ, and ρ respectively, with baseline levels set at $\tau = 7$, $\gamma = 2{,}000$, and $\rho = 0.055$. These

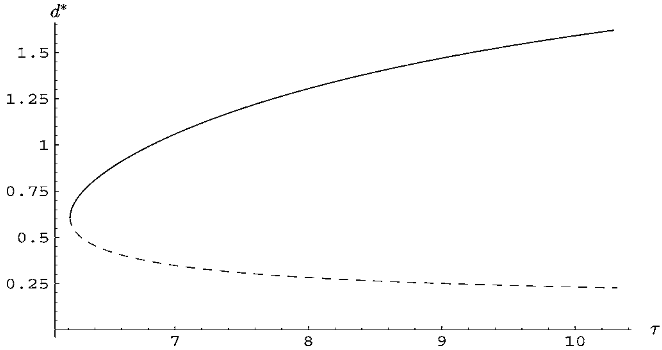

FIGURE 5. Bifurcation diagram of the stationary value of d^* versus τ. The unstable equilibrium is indicated by a dashed line; the stable equilibrium is represented by a solid line.

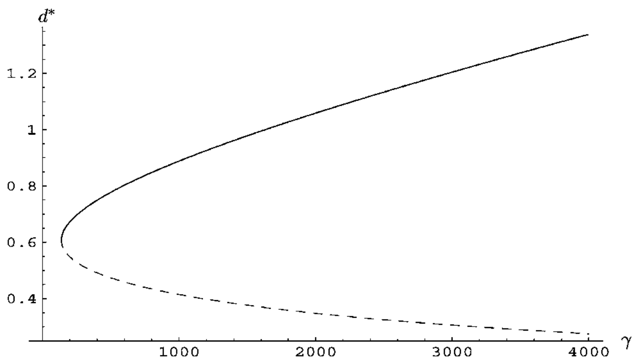

FIGURE 6. Bifurcation diagram of the stationary value of d^* versus γ. The unstable equilibrium is indicated by a dashed line; the stable equilibrium is represented by a solid line.

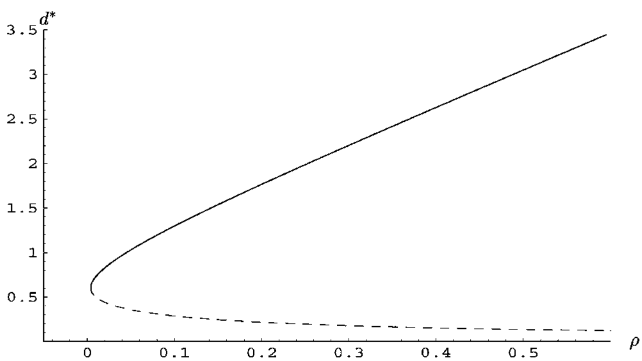

FIGURE 7. Bifurcation diagram of the stationary value of d^* versus ρ. The unstable equilibrium is indicated by a dashed line; the stable equilibrium is represented by a solid line.

bifurcation diagrams show that the steady state undergoes a saddle-node bifurcation, which means that two equilibria collide and annihilate. For parameter values to the left of the bifurcation point, there does not exist any equilibrium, whereas two equilibria exist to the right of the bifurcation point.

In these examples, the lower equilibrium is unstable, so that there exists a threshold (around the lower equilibrium) above which the system converges toward the upper equilibrium, and below which the system can converge toward a boundary solution, say, of balanced budgets.

Next, we look for a numerical example in the case of $b < a$. In this case, there exist at least one equilibrium and at most three equilibria. For instance, setting $a = 4$ and $b = 3$, fixing the other parameters as in the examples above, that is, $\gamma = 2{,}000$, $\kappa = 1$, $\varphi = 0.1$, $\alpha = 0.9$, $\beta = 0.1$, $r = 0.05$, $\theta = 0.031$, and $\rho = 0.055$, and taking τ as the bifurcation parameter yields a unique equilibrium. Figure 8 provides a plot of the stationary values of d^*, x^*, and u^* versus τ.

For $\tau < 0.245$, the equilibrium exhibits saddlepoint stability with locally monotonic paths, whereas for $0.245 < \tau < 0.604$ the equilibrium is still saddlepoint stable but the path in the steady state is now oscillating.

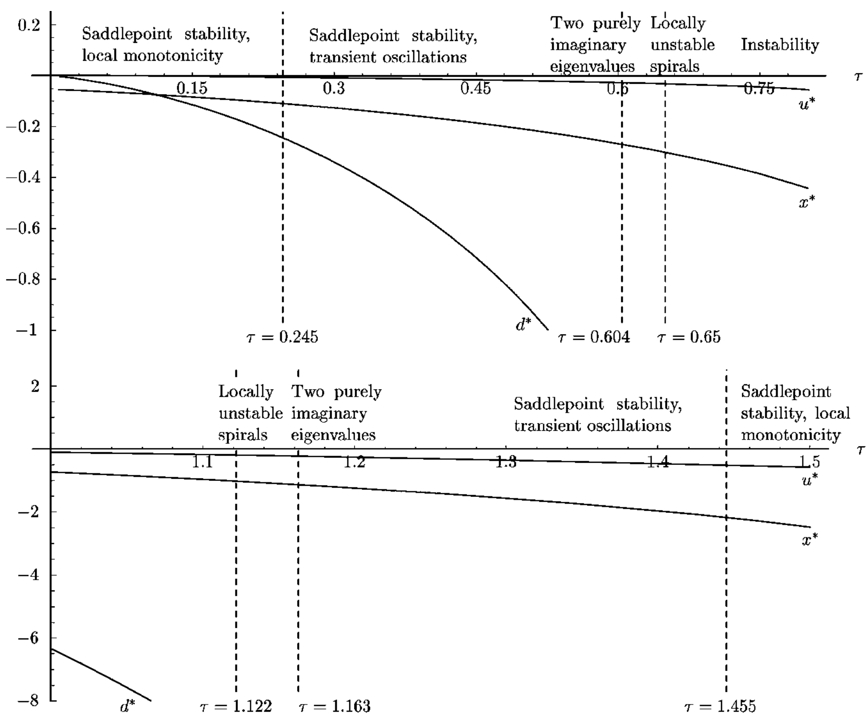

FIGURE 8. Plot of the stationary values of d^*, x^*, and u^* versus τ. For $\tau \geq 6.163$ the equilibrium is unstable.

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

At $\tau = 0.604$, a simple pair of eigenvalues becomes purely imaginary, which defines a candidate for a Hopf bifurcation. In fact, a subcritical Hopf bifurcation occurs, which means that the stable equilibrium and a coexisting unstable limit cycle collapse into an unstable equilibrium. For $0.604 < \tau < 1.163$, the steady state is unstable, where either all eigenvalues exhibit positive real parts or one eigenvalue is negative and the other ones are positive or have positive real parts. For $\tau = 1.163$, the eigenvalues are again purely imaginary, but the corresponding first Lyapunov coefficient of the Hopf bifurcation, as derived by the program package CONTENT, is of size $10^{(-7)}$. Therefore, no reliable statements about the behavior at $\tau = 1.163$ can be given.

For $\tau > 1.163$, saddlepoint stability is again obtained, where for $1.163 < \tau < 1.445$ the path into the steady state is oscillating, and for $\tau > 1.445$ the path is locally monotone.

According to numerical computations, for $\tau \geq 6.163$, the equilibrium loses stability as $\det J$ becomes negative. Since d^* exhibits a pole at $\tau = 6.163$, the exact value of the zero of $\det J$ cannot be computed.

Analyzing the unstable limit cycle, which collides with the equilibrium at $\tau = 0.604$, for values below that point, one encounters a subcritical flip bifurcation at $\tau = 0.589089$. At a subcritical flip-bifurcation point, an unstable limit cycle bifurcates to an unstable limit cycle with approximately twice the period of the unstable limit cycle before the bifurcation. Furthermore, the latter changes stability; that is, the limit cycle exhibits saddlepoint stability with a two-dimensional stable manifold. Figure 9 provides a plot of the stable limit cycle and the unstable limit cycle, which, shortly after the bifurcation has approximately twice the period than before the bifurcation.[12] It can be seen that the limit cycles lie entirely in the domain $d < 0$.

For initial values between the stable and unstable limit cycle, it is optimal to approach the stable limit cycle. Pursuing the optimal strategy, one passes four regions of different behavior. For instance, starting at the arrowhead in Figure 9, where the debt level reaches its maximum and the primary surplus is slightly positive, the primary surplus increases and the government gets more and more in the position of a lender. As the primary surplus reaches its maximum at approximately 0.13, one enters the second region, which can be characterized by decreasing levels of debt and primary surplus. As the primary surplus turns into deficits, the debt starts to increase. In the fourth and last region, the primary deficits decrease, but the debt still rises.

Figure 10 shows the time path on the ensuing stable limit cycle. It can be seen that the equilibrium acceptance rate of saving closely follows the time path of the primary surplus, but with an implicit time lag of approximately one period and at a substantially lower level. The first can be explained through the dynamic adjustment process of the equilibrium acceptance rate of saving in relation to prevailing and past levels of national debt and primary surpluses/deficits, respectively. Therefore, every change in the level of debt or primary surplus precedes a change in the public's acceptance level of savings.

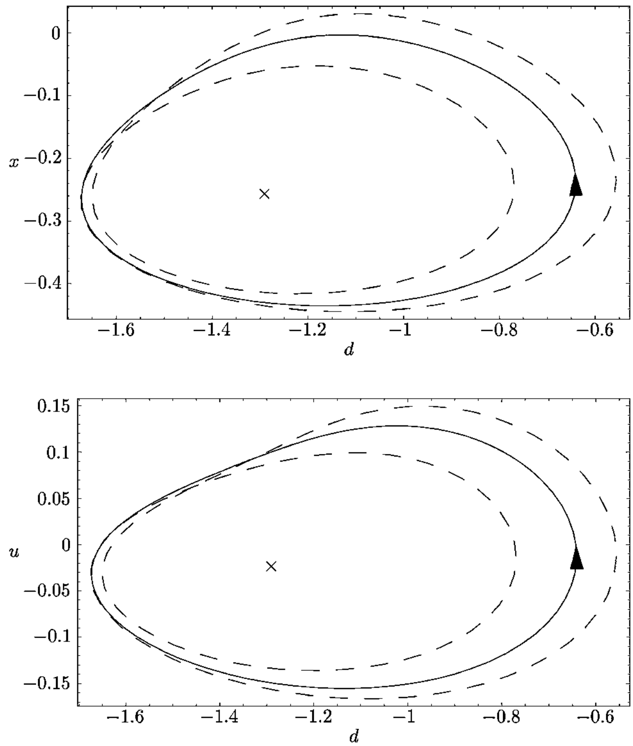

FIGURE 9. Plot of the stable (solid line) and unstable (dotted line) limit cycles for $\tau = 0.588$ in the state space (d, x) (above) and in the control state space (d, u) (below). The cross denotes the stable equilibrium.

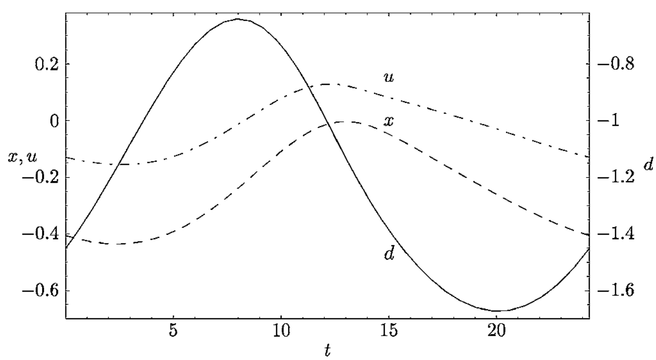

FIGURE 10. Time path of d (solid line), x (dashed line), and u (dashed-dotted line) on the stable limit cycle for $\tau = 0.588$. The left scale corresponds to x and u, and the right scale corresponds to d.

https://doi.org/10.1017/S1365100501010124 Published online by Cambridge University Press

Decreasing τ, one encounters three further flip bifurcations at $\tau = 0.587737$, $\tau = 0.587492$, and $\tau = 0.587441$. A detailed analysis of the series of flip bifurcations and further complexities would go beyond the scope of this paper and therefore it will be part of a future work.

5. DISCUSSION

The stability analysis in the preceding section has highlighted the importance of the functional form of the costs associated with the public debt, $\tilde{k}$, and the public's acceptance rate of savings, $\tilde{c}$, as related to the optimal long-run dynamics of public debt and deficit.

Linear costs of public debt (as assumed in Section 4.1) imply that the optimal strategy for the government is to accumulate an overall surplus ($rd^* - u^*$, where $u^* < 0$ and $d^* < 0$) each period for a very slow adjustment process (= very small values of τ in Figure 4). If the speed of adjustment is slightly higher, the optimal strategy is unstable, where the length of the interval of instability depends only on the discount rate. The more shortsighted that the politician is, the lower is the threshold level of the speed of adjustment from which the strategy of accumulating an overall surplus changes to an unstable equilibrium strategy. Increasing the speed of adjustment even further leads again to an optimal strategy characterized by the accumulation of a surplus. The faster the public reacts to deviations of the deficit from the equilibrium acceptance rate of savings (increasing values of τ in Figures 3 and 4), the more the government adopts the position of a lender (increasing values of d^*). This strategy enables the government to increase its social benefit as accruing through negative costs whenever $d^* < 0$. If the adjustment rate of the public surpasses a critical value (at $\tau = 6.163$), the optimal level of government surplus d^* loses stability.

Analytical calculations show that the stability of the unique equilibrium is independent of the costs due to deficit and of the domestic costs associated with the necessity to save in the case of linear costs of debt. The corresponding parameters only scale the equilibrium values.

The optimal strategy of the government is drastically altered if we consider nonlinear costs of public debt as in Section 4.2. With nonlinear, concave-convex costs, where the degree of nonlinearity of the costs of debt is higher than the degree of nonlinearity of the costs associated with the necessity to save, there exist two possible stationary solutions, of which the lower equilibrium is unstable. There exists a threshold [in the sense of Skiba (1978) and Dechert and Nishimura (1983)] such that starting with sufficiently high debts will cause convergence to the high-debt equilibrium, whereas starting with low debts can lead to a boundary solution, say of no debts. Hence, this political-economic setting provides some reasons for history dependence (e.g., debt problems persisting for decades in Latin America and in some southern European countries on the one hand and low debts or even accumulating surpluses in Switzerland and Luxembourg on the other hand) and hysteresis.

The existence of such an equilibrium strategy depends on the various parameters of the system. For instance, a slow public adjustment process and small public costs (= small values of τ and γ in Figures 5 and 6) as well as a small time discount rate for the government (= a low value of ρ in Figure 7) imply that no equilibrium strategy exists. Furthermore, the degree of nonlinearity of the costs of deviating from the public's equilibrium acceptance level of savings determines whether an equilibrium strategy exists. However, the existence of the equilibrium strategy does not depend on whether the costs due to the necessity to save are even or odd.

The equilibrium strategy drastically changes if the degree of nonlinearity of the costs associated with the necessity to save exceeds the degree of nonlinearity of the costs due to debt. For our numerical example, the optimal strategy for the government is very similar to the case of linear costs of debt, except that, now, limit cycles occur. In particular, for $0.587441 < \tau < 0.589089$, the optimal strategy might be oscillating, which means that phases of primary surpluses follow phases of primary deficits.

As our results indicate, the optimal levels of debt and deficit and their stability will be considerably influenced by the functional forms assumed for the costs of debt and for the costs associated with the necessity to save, which we postulate. With this regard, it is interesting to compare our results to the related work of Grienauer (1996) and Prskawetz (1998). In fact, our model is an extension of their work insofar as we assume an additional cost term, namely, the costs due to deficit. The functional forms of the costs and their most complex outcome in each of the papers is summarised in Table 1.

A comparison of the optimal level of debt and its stability between these three papers shows that the degree of nonlinearity of the costs due to debt mainly determines the possible modes of stability. More specifically, our first numerical example and Grienauer (1996) assume linear costs due to debt and, in both cases, the most complex behavior that can occur is saddlepoint stability. On the contrary, in our second numerical example and in Prskawetz (1998), the assumption of more pronounced nonlinearities in the costs due to debt can lead to multiple and more complex equilibria such as a limit cycle. Note that it is not the form of the cost function (i.e. even versus odd functions), but the degree of nonlinearity of the costs of debt that leads to more complex behavior such as limit cycles or multiple equilibria.

TABLE 1. Functional forms of the assumed cost terms and the most complex outcome in the different models

Reference	Costs due to debt	Costs due to necessity to save	Most complex outcome
This paper, Section 4.1	Linear	Nonlinear polynomial	Unique equilibrium
This paper, Section 4.2	Nonlinear polynomial, odd	Nonlinear polynomial	Multiple equilibria and limit cycles
Grienauer (1996)	Linear	Exponential	Unique equilibrium
Prskawetz (1998)	Quadratic	Exponential	Limit cycles

NOTES

1. By introducing these costs, we represent the Maastricht criteria as soft rather than as hard and binding constraints [for retrenchment under hard constraints, see Prskawetz et al. (1998)]. This seems realistic and thus plausible, given the workings of international politics, because nobody would argue for an exclusion from the existing Union if the criteria are not met literally. In fact, even prior to the Monetary Union these Maastricht criteria were not been binding, but rather were soft constraints, since movements in the right directions were accepted as well, for example, in the case of Italy.

2. For example, Mueller (1989) writes that, prior to Keynes (the legacy of the Keynesian economics lowered this esteem for balanced budgets), running a deficit was considered an immoral act by the public so that politicians at that time transgressed this norm with great peril. Switzerland and Luxembourg may be a particular modern example where budget deficits are apparently still highly penalized so that balanced budgets are the rule.

3. In fact, real-world government expenditures proceed by precedence so that the current surplus or deficit is the outcome of historically granted (or abolished) "contracts" and "rights" [compare Alt and Chrystal (1983) and Feichtinger and Wirl (1991)]. For example, Pierson (1995) recommends to reelection-seeking politicians to "make tiny, almost imperceptible cuts frequently, simple one-off change now that will result in substantial savings later on, off-load the costs to the future (generations)." Indeed, Sobel (1998) finds (for the United States and differing between Democrats and Republicans) that both tax increases and expenditure cuts carry high political costs, surprisingly of a similar magnitude.

4. The Maastricht criteria are based on the average value of $d^M = 60\%$ of the debt-to-GDP ratio for the 12 community members in 1990. Assuming a steady-state growth rate of nominal income equal to 5%, the value of the overall deficit, $rd - u$, which is consistent with this long-run stationary equilibrium is exactly $(rd - u)^M = 3\%$ [see Corsetti and Roubini (1992)].

5. Time arguments t are omitted henceforth; the dot notation in (2) refers to the derivative with respect to t.

6. The assumption that growth is independent of debts is made for reasons of simplicity, and allowing for a feedback of debt on the rate of growth will not alter the dynamics substantially. Presumably, the feedback will be small over the domain imposed by the penalties in the objective reflecting the constraints of the Maastricht treaty. Moreover, we have refrained from endogenously explaining economic growth in order to highlight the dynamics of an optimal retrenchment policy.

7. The short-sightedness of politicians is reflected by the usual assumption that the subjective discounting ρ exceeds the rate of interest r. Yet, the formal analysis can be carried out under the much less stringent demand that subjective discounting exceeds the difference between interest and growth rates.

8. To ensure a maximum, the second-order condition

$$\frac{\partial^2 H}{\partial u^2} < 0$$

must hold. This condition is equal to $\tilde{c}'' > 0$, and it will be checked as the functional forms are specified.

9. Applying equation (19) to the definition of h yields $h(\mu\beta - \lambda) = [(\varphi + \mu\beta - \lambda)/\gamma]^{[1/(a-1)]}$. For odd powers, that equation has a positive and a negative root, where only the positive root fulfills the second-order maximum condition. For even powers, the solution is unique and the second-order condition always holds.

10. All subsequent numerical computations were performed with MATHEMATICA [Wolfram (1997)], LOCBIF [Khibnik et al. (1992)] and CONTENT [Kuznetsov (1998)].

11. Since a, κ, γ, and φ do not influence the stability of the equilibrium, they are set in order to scale the equilibrium values. The $\alpha = 0.1$ and $\beta = 0.9$ values reflect the idea that people are more concerned about the primary surplus/deficit ratio than about the debt ratio in forming the acceptance level. The average of the long-run interest rates in Austria of the years 1997, 1998, and 1999 equals 0.05 and the average annual nominal GDP growth rate in Austria from 1996 through 1999 amounts to 0.031 [Austrian Institute of Economic Research (2000)]. As mentioned earlier, the time preference

rate of the politician has to fulfill the side condition $\rho > r - \theta$. On the other hand, if ρ is very high, the decisionmaker gets more and more myopic and cares less about the level of debt.

12. The numerical bifurcation analysis as well as the calculation of the limit cycles were done with the program package CONTENT [Kuznetsov (1998)].

REFERENCES

Alesina, A. (1987) Macroeconomic policy in a two-party system as repeated game. *Quarterly Journal of Economics* 102, 651–678.

Alt, J.E. & A.K. Chrystal (1983) *Political Economics*. Los Angeles: University of California Press.

Austrian Institute of Economic Research (2000) http://www.wifo.at/.

Buchanan, J.M. & R.E. Wagner (1977) *Democracy in Deficit*. New York: Academic Press.

Corsetti, G. & N. Roubini (1992) Tax Smoothing Discretion Verus Balanced Budget Rules in the Presence of Politically Motivated Fiscal Deficits: The Design of Optimal Fiscal Rules for Europe After 1992. Discussion paper 682, Center for Economic Policy Research.

Dechert, D.W. & K. Nishimura (1983) Complete characterization of optimal growth paths in an aggregate model with a non-concave production function. *Journal of Economic Theory* 31, 332–354.

Dockner, E. (1985) Local stability analysis in optimal control problems with two state veriables. In G. Feichtinger (ed.), *Optimal Control Theory and Economic Analysis 2*. Amsterdam: North-Holland.

Feichtinger, G. & R.F. Hartl (1986) *Optimale Kontrolle Ökonomischer Prozesse—Anwendungen des Maximumsprinzips in den Wirtschaftswissenschaften*. Berlin: Walter de Gruyter.

Feichtinger, G. & F. Wirl (1991) Politico-economic cycles of regulation and deregulation. *European Journal of Political Economy* 7, 469–485.

Grienauer, W.M. (1996) *Budget Stabilization in Europe: A Model of Saving and Its Acceptance*. Master's Thesis. Vienna University of Technology.

Hamermesh, D.S. & G.A. Pfann (1996) Adjustment costs in factor demand. *Journal of Economic Literature* 34, 1264–1292.

Hibbs, D. (1977) Political parties and macroeconomic policy. *American Political Science Review* 71, 1467–1487.

Leonard, D. & N.V. Long (1992) *Optimal Control Theory and Static Optimization in Economics*. Cambridge, UK: Cambridge University Press.

Khibnik, A.I., Yu. A. Kuznetsov, V.V. Levitin, & E.V. Nikolaev (1992) *Locbif*. Amsterdam: Computer Algebra Netherlands Expertise Center.

Kuznetsov, Y.A. (1998) CONTENT—Integrated Environment for Analysis of Dynamical Systems, Pushchino, Moscow Region: Institute of Mathematical Problems of Biology, Russian Academy of Sciences.

Mueller, D.C. (1989) *Public Choice II*. Cambridge, UK: Cambridge University Press.

Nordhaus, W.D. (1975) The political business cycle. *Review of Economic Studies* 42, 169–190.

Nordhaus, W.D. (1989) The political business cycle. *Brookings Papers on Economic Activity* 42, 169–190.

Pierson, P. (1995) *Dismantling the Welfare State? Reagan, Thatcher, and the Politics of Retrenchment*. Cambridge, UK: Cambridge University Press.

Prskawetz, A. (1998) Government debt, budget surplus and the popularity of politicians. *Journal of Optimization Theory and Applications* 98, 131–149.

Prskawetz, A., G. Feichtinger, & M. Luptacik (1998) The accomplishment of the Maastricht criteria with respect to initial debt. *Journal of Economics* 68, 93–110.

Schneider, F. & B.S. Frey (1988) Politico-economic models of macroeconomic policy. In T.D. Willett (ed.), *Political Business Cycles*. Durham, NC: Duke University Press.

Skiba, A.K. (1978) Optimal growth with a convex-concave production function. *Econometrica* 46, 527–539.

Sobel, R.S. (1998) The political costs of tax increases and expenditure reductions: Evidence from the state legislative turnover. *Public Choice* 96, 61–79.

Statistik Austria (1999) Allgemeiner Auskunftsdienst und Indexauskunft.
van Dalen, H. & O.H. Swank (1996) Government spending cycles: Ideological or opportunistic? *Public Choice* 89, 183–200.
Wiggins, S. (1990) *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. New York: Springer.
Wolfram, S. (1997) *Das Mathematica Buch*, 3rd Edition. Bonn: Addison Wesley.

APPENDIX

A.1. PROOFS OF PROPOSITIONS

Proof of Proposition 2. As mentioned earlier, the case $\det J < 0$ leads to instability. Thus $\det J > 0$ is necessary (but not sufficient) for stability.

For our specific parameter set, we have already seen that $d^* < 0$ is equal to $\det J > 0$. Now we will establish this fact analytically. Equation (20) yields

$$\text{sign } \det J = \text{sign}[\alpha + (\rho + \tau - \beta)(\rho - r + \theta)]\,\text{sign}[\alpha + (\beta - \tau)(r - \theta)]. \qquad \textbf{(A.1)}$$

Neglecting the positive constants in (22), the sign of d^* is determined by

$$\text{sign } d^* = \text{sign}[(\tau - \beta)(r - \theta) - \alpha]\,\text{sign}\{[\alpha + (\rho + \tau - \beta)(\rho - r + \theta)]^{\frac{1}{a-1}}\}. \qquad \textbf{(A.2)}$$

If a is even, equation (27) simplifies to

$$\begin{aligned} \text{sign } d^* &= -\text{sign}[\alpha + (\beta - \tau)(r - \theta)]\,\text{sign}[\alpha + (\rho - r + \theta)(\rho + \tau - \beta)] \\ &= -\text{sign} \det J. \end{aligned} \qquad \textbf{(A.3)}$$

In the case of odd a, the first-order maximum condition (9) implies $\varphi - \lambda + \mu\beta \geq 0$, which transforms to

$$(\rho + \tau)\frac{\varphi(\rho + \theta)}{\alpha + (\rho - r + \theta)(\rho + \tau - \beta)} \geq 0$$

in the steady state. Therefore, a necessary condition to ensure the existence of a steady state is $\alpha + (\rho - r + \theta)(\rho + \tau - \beta) \geq 0$ for odd a. Consequently, equations (26) and (27) simplify to

$$\text{sign} \det J = \text{sign}[\alpha + (\beta - \tau)(r - \theta)] \qquad \textbf{(A.4)}$$

and

$$\text{sign } d^* = \text{sign}[(\tau - \beta)(r - \theta) - \alpha], \qquad \textbf{(A.5)}$$

which can be combined to

$$\text{sign } d^* = -\text{sign} \det J. \qquad \textbf{(A.6)}$$

■

Proof of Proposition 4. As mentioned already in the proof of Proposition 2, we want to exclude the case of $\det J < 0$, which leads to instability.

Similar to the proof of Proposition 2, one can show that the first summand of det J and d^* have opposite signs, which implies the first term of det $J < 0$ whenever $d^* > 0$. Furthermore, if d^* is below d^M, then the second term of det J is negative.

Summing up, for $0 < d^* < d^M$ both terms of det J are negative, which means that such an equilibrium will be definitively unstable. For $d^* < 0$ and $d^* > d^M$, the sign of det J is ambiguous. ■

A.2. ANALYSIS OF CANDIDATES FOR A HOPF BIFURCATION

In case of the supercritical Hopf bifurcation, the stable fixed point bifurcates into an unstable fixed point and a stable periodic orbit. Contrarily, in the case of the subcritical Hopf bifurcation, the stable fixed point and a coexisting unstable periodic orbit collapse into an unstable fixed point.

To prove the (non)-existence of limit cycles we consider the flow on the center manifold, because limit cycles can exist only on the center manifold. First, the canonical system has to be transformed into the following form:

$$\begin{aligned} \dot{y} &= Ay + \Phi(\mathbf{y}, \mathbf{z}), \\ \dot{z} &= Bz + \Psi(\mathbf{y}, \mathbf{z}), \qquad (\mathbf{y}, \mathbf{z}) \in \mathbf{R}^2 \times \mathbf{R}^2, \end{aligned} \tag{A.7}$$

where

$$\begin{aligned} \Phi(0,0) &= 0, \qquad D\Phi(0,0) = 0, \\ \Psi(0,0) &= 0, \qquad D\Psi(0,0) = 0, \end{aligned}$$

and A is a 2×2 matrix having eigenvalues with zero real parts and B is a 2×2 matrix having eigenvalues with nonzero (in our case, positive) real parts. For this purpose, we use the following transformation:

$$\begin{pmatrix} y_1 \\ y_2 \\ z_1 \\ z_2 \end{pmatrix} = T^{-1} \begin{pmatrix} d - d^* \\ x - x^* \\ \lambda - \lambda^* \\ \mu - \mu^* \end{pmatrix}, \tag{A.8}$$

where T is the matrix of eigenvectors corresponding to the real Jordan normal form. According to Wiggins (1990), the center manifold is an invariant manifold, which can be represented locally as follows:

$$W^c(0) = \{(\mathbf{y}, \mathbf{z}) \in \mathbf{R}^2 \times \mathbf{R}^2 | z = \psi(\mathbf{y}), |\mathbf{y}| < \epsilon, \psi(0) = 0, D\psi(0) = 0\},$$

for ϵ sufficiently small. The dynamics of (32) restricted to the center manifold for v sufficiently small, are given by the following two-dimensional vector field:

$$\dot{\mathbf{v}} = A\mathbf{v} + \Phi(\mathbf{v}, \psi(\mathbf{v})), \qquad \mathbf{v} \in \mathbf{R}^2.$$

Furthermore, any point on $W^c(0)$ must satisfy

$$D\psi(\mathbf{y})[A\mathbf{y} + \Phi(\mathbf{y}, \psi(\mathbf{y}))] = B\psi(\mathbf{y}) + \Psi(\mathbf{y}, \psi(\mathbf{y})). \tag{A.9}$$

Fortunately, the function ψ can be approximated by power series, which can be derived by a comparison of coefficients.

In our case, applying the transformation (33) to the canonical system yields the following form of a system of differential equations:

$$\begin{aligned}\dot{y} &= Ay\\ \dot{z} &= Bz + \Psi(y)\end{aligned}$$

for $\tau = \beta + r - \theta - 2\rho$ and

$$\begin{aligned}\dot{y} &= Ay + \Phi(z)\\ \dot{z} &= Bz\end{aligned}$$

for $\tau = \beta + r - \theta$. At the first candidate for a Hopf bifurcation, it is unnecessary to compute the center manifold, because the flow on the center manifold is already determined by

$$\dot{y} = Ay.$$

Since the flow is linear, limit cycles cannot emerge. This phenomenon is sometimes called degenerate Hopf bifurcation.

For the second candidate for a Hopf bifurcation, an invariant manifold satisfying condition (34) is given by $z \equiv 0$. The flow on this center manifold is linear and the Hopf bifurcation is degenerate. However, the center manifolds need not to be unique. However, the approximation of the center manifold with power series yields only the manifold $z \equiv 0$, because the function Φ is a cube root, which does not affect the comparison of coefficients. Summing up, there may emerge limit cycles for the $\tau = \beta + r - \theta$, but they can only be detected by luck.