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CONVEXLY ORDERABLE GROUPS AND VALUED FIELDS

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Abstract. We consider the model theoretic notion of convex orderability, which fits strictly between the notions of VC-minimality and dp-minimality. In some classes of algebraic theories, however, we show that convex orderability and VC-minimality are equivalent, and use this to give a complete classification of VC-minimal theories of ordered groups and abelian groups. Consequences for fields are also considered, including a necessary condition for a theory of valued fields to be quasi-VC-minimal. For example, the p-adics are not quasi-VC-minimal.

§1. Introduction. After many of the advancements in modern stability theory, some model theorists have been seeking to adapt techniques from stable model theory to other families of unstable, yet still well-behaved theories. These include o-minimal theories as well as theories without the independence property.

As these notions of model-theoretic tameness proliferate, in each case, two natural questions arise: what are the useful consequences of the property, and which interesting theories have the property? As an example of the latter line of inquiry, an ordered group is weakly o-minimal if and only if it is abelian and divisible, and an ordered field is weakly o-minimal if and only if it is real closed [10]. Similar characterizations of dp-minimality for abelian groups can be found in [3], and results on dp-minimal ordered groups can be found in [13].

Resting comfortably among these conditions is VC-minimality, introduced by Adler in [2]. Most of the classical variations on minimality, such as (weak) o-minimality, strong minimality, and C-minimality, imply VC-minimality. On the other hand, VC-minimality is strong enough to imply many properties of recent interest, such as dependence and dp-minimality.

The question of consequences of VC-minimality has been addressed elsewhere (see e.g., [4, 7, 8]). In this paper, we seek to identify the VC-minimal theories among some basic classes of algebraic structures. Here a problem quickly arises. While it tends to be straightforward to verify that a theory is VC-minimal, the definition of VC-minimality does not lend itself easily to negative results. Except in some special cases, previously it had only been possible to show a theory is not VC-minimal by showing that it is not dp-minimal or dependent.

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To sidestep this problem, we explore the intermediate notion of convex orderability, first introduced in [8]. All VC-minimal theories are also convexly orderable, and while the converse fails in general, in many cases it is, in a sense, close enough. The strategy, thus, is twofold. Given a class of algebraic theories, we use known results (e.g., on o-minimal ordered groups) to produce a list of VC-minimal theories from the class. We then study convex orderability in relation to the class of theories to establish that the list is exhaustive.

In this way, we give a complete classification of VC-minimal theories of ordered groups (Section 3) and abelian groups (Section 5). Partial results, in the form of necessary conditions for VC-minimality, are given for ordered fields (Section 3) and valued fields (Section 4). For valued fields, the weaker condition of quasi-VC-minimality is also evaluated.

The remainder of this section gives the necessary background on VC-minimality, and Section 2 presents some useful facts about convex orderability.

1.1. VC-minimality. Let X be any set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. We say that $\mathcal{B}$ is *directed* if, for all $A, B \in \mathcal{B}$, one of the following conditions holds:

(1) $A \subseteq B$,
(2) $B \subseteq A$, or
(3) $A \cap B = \emptyset$.

Let T be a first-order $\mathcal{L}$-theory, and fix a set of formulas

$$\Psi = \{\psi_i(x; \bar{y}_i) \mid i \in I\}$$

(note that the singleton x is a free variable in every formula of Ψ, but the parameter variables $\bar{y}_i$ may vary). Then Ψ is *directed* if, for all $\mathfrak{M} \models T$,

$$\left\{ \psi_i(\mathfrak{M}; \bar{a}) \;\middle|\; i \in I, \bar{a} \in M^{|\bar{y}_i|} \right\}$$

is directed, where $\psi_i(\mathfrak{M}; \bar{a}) = \{b \in M \mid \mathfrak{M} \models \psi_i(b; \bar{a})\} \subseteq M$.

We say that T is *VC-minimal* if there exists a directed Ψ such that all (parameter-definable) formulas $\varphi(x)$ are T-equivalent to a boolean combination of instances of formulas from Ψ (i.e., formulas of the form $\psi(x; \bar{a})$ for $\psi \in \Psi$). In this case, Ψ is called a *generating family* for T.

For example, it is easy to see that strongly minimal theories are VC-minimal; take $\Psi = \{x = y\}$. Similarly, o-minimal theories are VC-minimal; take $\Psi = \{x \leq y, x = y\}$. A prototypical example of a VC-minimal theory which is neither stable nor o-minimal is the theory of algebraically closed valued fields; take $\Psi = \{v(z) < v(x - y), v(z) \leq v(x - y)\}$, recalling the swiss cheese decomposition of Holly [9]. By a simple type-counting argument, one can see that formulas $\varphi(x; \bar{y})$ in VC-minimal theories have VC-density ≤ 1 (see [3]). From this, one can conclude that VC-minimal theories are dp-minimal (see, for instance, [6]).

Finally, T is *quasi-VC-minimal* if there exists a directed Ψ such that all formulas $\varphi(x)$ are T-equivalent to a boolean combination of instances of formulas from Ψ and parameter-free formulas. Clearly, all VC-minimal theories are quasi-VC-minimal. Moreover, the theory of Presburger arithmetic, $\mathrm{Th}(\mathbb{Z}; +, \leq)$, is quasi-VC-minimal; take $\Psi = \{x \leq y, x = y\}$. Again, by the same type-counting argument, one can check that quasi-VC-minimal theories are dp-minimal.

§**2. Convex orderability.** VC-minimality is a powerful condition having many consequences (see, e.g., [2, 4, 7, 8]). However, it can be difficult to verify that a theory is not VC-minimal. In attempting to classify VC-minimal theories of certain kinds, therefore, we instead look at a related notion called convex orderability.

DEFINITION 2.1. *An $\mathcal{L}$-structure $\mathfrak{M}$ is* convexly orderable *if there exists a linear order $\trianglelefteq$ on M (not necessarily definable) such that, for all $\varphi(x;\bar{y})$, there exists $k < \omega$ such that, for all $\overline{b} \in M^{|\bar{y}|}$, $\varphi(\mathfrak{M};\overline{b})$ is a union of at most k $\trianglelefteq$-convex subsets of M.*

Note in the above that k may depend on φ, but $\trianglelefteq$ does not. In [8], it is shown that if $\mathfrak{M}$ is convexly orderable and $\mathfrak{M} \equiv \mathfrak{N}$, then $\mathfrak{N}$ is convexly orderable as well. Therefore, convex orderability is a property of a theory. Moreover, the next proposition follows immediately from the definition.

PROPOSITION 2.2. *The property of convex orderability is closed under reducts. That is, if T is a convexly orderable $\mathcal{L}$-theory and $\mathcal{L}' \subseteq \mathcal{L}$, then the reduct $T \restriction \mathcal{L}'$ is also convexly orderable.*

For later reference, we cite the following from [8].

PROPOSITION 2.3 (Corollary 2.9 of [8]). *If T is convexly orderable, then T is dp-minimal.*

Furthermore, the following proposition is a simple modification of Proposition 2.5 of [8].

PROPOSITION 2.4. *Suppose X is a set and $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is directed. Then, there exists a linear ordering $\trianglelefteq$ on X so that every $B \in \mathcal{B}$ is a $\trianglelefteq$-convex subset of X.*

From this, a simple compactness argument gives the corollary.

COROLLARY 2.5 (Theorem 2.4 of [8]). *If T is VC-minimal and $\mathfrak{M} \models T$, then $\mathfrak{M}$ is convexly orderable.*

By contrast, the above corollary does not hold for quasi-VC-minimal theories, as the $\emptyset$-definable sets may be quite complicated. However, restricting our attention to a single formula, we obtain a localized result for quasi-VC-minimal theories. In the following, notice that $\trianglelefteq$ *does* depend on the formula φ.

COROLLARY 2.6. *If T is a quasi-VC-minimal theory, $\mathfrak{M} \models T$, and $\varphi(x;\overline{y})$ is a formula, then there exists a linear ordering $\trianglelefteq$ on M and $k < \omega$ such that, for all $\overline{b} \in M^{|\overline{y}|}$, $\varphi(\mathfrak{M};\overline{b})$ is a union of at most k $\trianglelefteq$-convex subsets of M. That is, T is 'locally convexly orderable'.*

PROOF. By compactness, there exists $k_0 < \omega$, $\delta(x;\overline{z})$ a directed formula, and a $\emptyset$-definable partition of M via the finite set of formulas $\Theta(x)$ so that, for each $\overline{b} \in M^{|\overline{y}|}$, $\varphi(\mathfrak{M};\overline{b})$ is a boolean combination of at most k_0 instances of δ and formulas from Θ. (More precisely, compactness yields k_0 and a finite set of formulas, while coding tricks allow one to compress a finite set of directed formulas into the single formula δ.)

Let $k = k_0|\Theta| + 1$ and, for each $\theta \in \Theta$, let $\delta_\theta(x;\overline{z})$ be the formula $\delta(x;\overline{z}) \wedge \theta(x)$. Note that each δ_θ is directed, as δ is. Hence, by Proposition 2.4, for each $\theta \in \Theta$, there exists $\trianglelefteq_\theta$ a linear ordering on $\theta(\mathfrak{M})$ so that every instance of δ_θ is $\trianglelefteq_\theta$-convex. We then concatenate the orderings $\trianglelefteq_\theta$ in an arbitrary (but fixed) sequence to form a single linear ordering $\trianglelefteq$ on M.

Now, for any $\overline{b} \in M^{|\overline{y}|}$ and $\theta \in \Theta$, $\varphi(x;\overline{b}) \wedge \theta(x)$ is a boolean combination of at most k_0 instances of δ_θ, each of which is $\trianglelefteq$-convex. Therefore, $\varphi(\mathfrak{M};\overline{b})$ is a union of at most $k = k_0|\Theta| + 1$ $\trianglelefteq$-convex subsets of M. ⊣

One of the original motives for defining convex orderability was to give an analog to VC-minimality which is closed under reducts. However, the converse to Corollary 2.5 does not hold. The dense circle order is convexly orderable but not VC-minimal (for more information, see [2]). It is, in fact, a reduct of (a definitional expansion of) dense linear orders without endpoints, which is o-minimal and hence VC-minimal. On the other hand, the dense circle order becomes VC-minimal if one allows a single parameter in the generating family.

Let us call a theory *VC-minimal with parameters* if there exists a directed generating family as in the original definition, but allowing parameters from some distinguished model in the formulas. One could then ask whether VC-minimality with parameters is closed under reducts. An example in [1] shows that this is still not the case. Recalling Proposition 2.2, therefore, there are convexly orderable theories which are not VC-minimal even with parameters.

Nevertheless, in the following sections we will see several instances where convex orderability serves as a useful proxy for VC-minimality. In particular, we use Corollaries 2.5 and 2.6 to answer questions about which algebraic structures of various kinds are convexly orderable, VC-minimal, and quasi-VC-minimal.

§3. Ordered groups.

Let $\mathfrak{G} = (G;\cdot,\leq)$ be an infinite ordered group and let $T = \mathrm{Th}(\mathfrak{G})$. We prove the following theorem.

THEOREM 3.1. *The following are equivalent*:

(1) *$\mathfrak{G}$ is abelian and divisible.*
(2) *T is o-minimal,*
(3) *T is VC-minimal,*
(4) *T is convexly orderable.*

This is a generalization of Theorem 5.1 of [10], which is itself a generalization of Theorem 2.1 of [11]. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are well-known (or clear from the previous section), so it will suffice to show that (4) ⇒ (1).

Thus, suppose that T is convexly orderable. By Proposition 3.3 of [13], all dp-minimal ordered groups are abelian. Using Proposition 2.3, therefore, we already have that $\mathfrak{G}$ is abelian and it remains only to show that it is divisible. We begin with a general lemma about convexly orderable ordered structures.

LEMMA 3.2. *If $\mathfrak{M} = (M;\leq,\dots)$ is a linearly ordered structure that is convexly orderable, then there do not exist definable sets $X_0, X_1, \dots \subseteq M$ that are pairwise disjoint and coterminal (that is, cofinal or coinitial) in M.*

PROOF. Suppose that $\mathfrak{M}$ is convexly ordered by $\trianglelefteq$. Suppose that there exists definable sets $X_0, X_1, \dots \subseteq M$ that are pairwise disjoint and $\leq$-coterminal in M. By the pigeonhole principle, we may assume that all X_i are either $\leq$-cofinal or $\leq$-coinitial in M. Without loss of generality, suppose all are $\leq$-cofinal in M. By convex orderability, for each i, X_i is a union of finitely many $\trianglelefteq$-convex subsets of M. Therefore, there exists some $\trianglelefteq$-convex subset $C_i \subseteq X_i$ such that C_i is $\leq$-cofinal in M.

Because the rays $[a,\infty)_\leq$ are uniformly definable, there is a natural number k such that every $[a,\infty)_\leq$ is the union of at most k $\trianglelefteq$-convex sets. Now consider the sets $C_1,\dots,C_{2k+1}$. Since these are $\trianglelefteq$-convex and pairwise disjoint, we may arrange the indices so that

$$C_{i_1} \lhd C_{i_2} \lhd \dots \lhd C_{i_{2k+1}}.$$

For each $j \leq 2k+1$, choose $b_j \in C_{i_j}$, and fix $a > \max\{b_j \mid 1 \leq j \leq 2k+1\}$. By $\leq$-cofinality of C_{i_j}, for each j we may also choose $c_j \in C_{i_j} \cap [a,\infty)_\leq$. Thus we have

$$c_1 \lhd b_2 \lhd c_3 \lhd \dots \lhd b_{2k} \lhd c_{2k+1}$$

with each $c_j \in [a,\infty)_\leq$ and each $b_j \notin [a,\infty)_\leq$. It follows that for $j = 0,\dots,k$, each c_{2j+1} lies in a separate $\trianglelefteq$-convex component of $[a,\infty)_\leq$. This contradiction implies that $\mathfrak{M}$ is not convexly orderable, as required. ⊣

We return to the case of $T = \mathrm{Th}(\mathfrak{G})$, where $\mathfrak{G} = (G;+,\leq)$ is a convexly orderable ordered group. For $k < \omega$, let $k \mid x$ be the formula $\exists y\,(k \cdot y = x)$. For each natural number $n \geq 1$ and prime p, define the set

$$D_{p,n} = \left\{x \in G \;\middle|\; x > 0,\, p^n \mid x \text{ and } p^{n+1} \nmid x\right\}.$$

Lemma 3.3. *Suppose for some prime p that $pG \neq G$. Then for each n, $D_{p,n}$ is cofinal in G.*

Proof. Since $pG \neq G$, there is some $c > 0$ with $p \nmid c$. Consider $0 < a \in G$. We show that there is $x \geq a$ such that $x \in D_{p,n}$. First, if $p \nmid a$, let $b = a$; if $p \mid a$, set $b = a + c$. So, $b \geq a$ and $p \nmid b$. Now $x = p^n \cdot b \geq a$ and $x \in D_{p,n}$. ⊣

Combining this with Lemma 3.2, we can now easily establish Theorem 3.1.

Corollary 3.4. *If $\mathfrak{G}$ is convexly orderable, then $\mathfrak{G}$ is divisible.*

Proof. Suppose $\mathfrak{G}$ is convexly orderable but not divisible, say $pG \neq G$. For each n, $D_{p,n}$ is cofinal and pairwise disjoint in $\mathfrak{G}$. Apply Lemma 3.2 to conclude. ⊣

Although there were previously known examples of dp-minimal theories that are not VC-minimal (e.g., see [6]), this gives us a natural example of such a theory (discovered independently in [1]).

Example 3.5. *The theory of Presburger arithmetic, $T = \mathrm{Th}(\mathbb{Z};+,\leq)$, is not VC-minimal and not convexly orderable. On the other hand, it is quasi-VC-minimal, and hence also dp-minimal.*

This has interesting consequences for ordered fields.

Proposition 3.6. *Suppose $\mathfrak{F} = (F;+,\cdot,\leq)$ is an ordered field. If $\mathfrak{F}$ is convexly orderable, then every positive element has an n^{th} root for all $n \geq 1$.*

Proof. Suppose $\mathfrak{F}$ is convexly ordered by $\trianglelefteq$. Then, $\trianglelefteq$ induces a convex ordering on the ordered group $(F_+;\cdot,\leq)$ where $F_+ = \{a \in F \mid a > 0\}$. Thus, by Theorem 3.1, F_+ is divisible. In other words, for any $a \in F_+$ and $n \geq 1$, there exists $b \in F_+$ such that $b^n = a$. ⊣

Theorem 5.3 of [10] states that any weakly o-minimal ordered field is real closed. This suggests the following open question.

OPEN QUESTION 3.7. *Is it the case that an ordered field $(F; +, \cdot, \leq)$ is convexly orderable if and only if $(F; +, \cdot, \leq)$ is real closed?*

Before we get carried away, however, not all ordered structures that are convexly orderable are weakly o-minimal. For example, consider $\mathbb{Q}$ and take $D \subseteq \mathbb{Q}$ dense and codense. One can verify that the structure $\mathfrak{M} = (\mathbb{Q}; \leq, D)$ has quantifier elimination, from which it easily follows that it is VC-minimal. For instance, take as a generating family

$$\Psi = \{(D(x) \wedge x < y), (\neg D(x) \wedge x < y), D(x), x = y\}.$$

So $\mathfrak{M}$ is convexly orderable, but on the other hand, $\mathfrak{M}$ is clearly not weakly o-minimal. The issue is that Lemma 3.2 necessitates *infinitely many* coterminal disjoint sets to contradict convex orderability. This leads to another open question.

OPEN QUESTION 3.8. *If $\mathfrak{M} = (M; \leq, \ldots)$ is a linearly ordered structure that is convexly orderable, then is $\mathfrak{M}$ quasi-weakly o-minimal?*

§4. Valued fields.

4.1. Simple interpretability. In this subsection we exhibit a means of passing convex orderability from a structure to a simple interpretation in the structure. If $\mathfrak{M}$ and $\mathfrak{N}$ are models (not necessarily in the same language) and $A \subseteq M$, then $\mathfrak{M}$ *interprets* $\mathfrak{N}$ *over* A if there are $n \geq 1$, an A-definable subset $S \subseteq M^n$, and an A-definable equivalence relation ε on S such that

- the elements of $\mathfrak{N}$ are in bijection with the ε-equivalence classes of S, and
- the relations on S induced by the relations and functions of $\mathfrak{N}$ via this bijection are A-definable in $\mathfrak{M}$.

Moreover, if $n = 1$ in the above definition, we say that $\mathfrak{M}$ *simply interprets* $\mathfrak{N}$.

It is generally most convenient to identify the elements of $\mathfrak{N}$ with the equivalence classes of S, so that for instance we will write $\bar{a} \in x$ if $\bar{a} \in S$ and $x \in N$ corresponds to the ε-equivalence class containing $\bar{a}$.

REMARK 4.1. *Using the same notation as above, suppose $\varphi(\bar{x}; \bar{y})$ is a formula in the language of $\mathfrak{N}$ with $k = |\bar{x}|$. Then there is $\tilde{\varphi}(\bar{z}; \bar{w})$ in the language of $\mathfrak{M}$ (with parameters from A) with the property that, for any set $X \subseteq N^k$ defined by an instance $\varphi(\bar{x}; \bar{a})$ of φ, the set*

$$\tilde{X} = \bigcup X \subseteq S^k.$$

is defined by an instance $\tilde{\varphi}(\bar{z}; \bar{b})$ of $\tilde{\varphi}$. To see this, induct on the complexity of φ, replacing function and relation symbols from $\mathfrak{N}$ with their corresponding definitions in $\mathfrak{M}$ and $=$ with ε, and relativizing all quantifiers to S.

LEMMA 4.2. *If $\mathfrak{M}$ simply interprets $\mathfrak{N}$ and $\mathfrak{M}$ is convexly orderable, then $\mathfrak{N}$ is also convexly orderable.*

PROOF. Let $\varepsilon(x, y)$ define an equivalence relation on $S \subseteq M$ as in the definition of interpretation (possibly over parameters), and suppose that $\mathfrak{M}$ is convexly ordered by $\trianglelefteq_M$. Define on $\mathfrak{N}$ the relation $\trianglelefteq_N$ by

$$x \trianglelefteq_N y \Longleftrightarrow (\forall s \in y)(\exists r \in x)[r \trianglelefteq_M s].$$

We claim that $\mathfrak{N}$ is convexly ordered by $\trianglelefteq_N$.

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First note that $\trianglelefteq_N$ linearly orders N. Transitivity and linearity are clear. For antisymmetry, suppose that $x \trianglelefteq_N y$ and $y \trianglelefteq_N x$. Then, beginning with an arbitrary $s_0 \in y$, find $r_i \in x$, $s_i \in y$ such that for every $i < \omega$, $r_i \trianglelefteq_M s_i$ and $s_{i+1} \trianglelefteq_M r_i$. But since x is a definable subset of $\mathfrak{M}$, x must be a finite union of $\trianglelefteq_M$-convex sets. So we must have $s_i \in x$ for some i, whence $x = y$. A similar argument shows that $x \triangleleft_N y$ iff there is an $r \in x$ such that $r \triangleleft_M s$ for all $s \in y$.

Now consider a formula $\varphi(x; \bar{y})$ in the language of $\mathfrak{N}$, $\bar{a}$ a tuple from N, and $X \subseteq N$ the set defined by $\varphi(x; \bar{a})$. For $\tilde{\varphi}(x; \bar{b})$ defining $\tilde{X}$ as in Remark 4.1, since $\trianglelefteq_M$ convexly orders $\mathfrak{M}$, there is a uniform bound k on the number of $\trianglelefteq_M$-convex sets comprising an instance of $\tilde{\varphi}$ in $\mathfrak{M}$. It will suffice to show that X is also a union of at most k $\trianglelefteq_N$-convex sets in N.

Suppose not, so that there are

$$c_0 \trianglelefteq_N c_1 \trianglelefteq_N \dots \trianglelefteq_N c_{2k}$$

such that $c_i \in X$ iff i is even. For each $i < 2k$, since $c_i \neq c_{i+1}$ there is $\tilde{c}_i \in c_i$ such that $\tilde{c}_i \triangleleft_M d$ for all $d \in c_{i+1}$. Take also any element $\tilde{c}_{2k} \in c_{2k}$. Now

$$\tilde{c}_0 \trianglelefteq_M \tilde{c}_1 \trianglelefteq_M \dots \trianglelefteq_M \tilde{c}_{2k}$$

and $\tilde{c}_i \in \tilde{X}$ iff i is even. This contradicts the fact that $\tilde{X}$ is a union of k (or fewer) $\trianglelefteq_M$-convex sets.

We conclude that in $\mathfrak{N}$, every instance of φ defines a union of k or fewer $\trianglelefteq_N$-convex sets. Since any formula in the language of $\mathfrak{N}$ admits such a uniform bound, $\trianglelefteq_N$ convexly orders $\mathfrak{N}$. ⊣

Lemma 4.2 allows us to show that a theory is not convexly orderable (hence not VC-minimal) by simply interpreting a structure that is not convexly orderable. We can apply this to theories of valued fields. Let K be a valued field with value group Γ, residue field k, and valuation $v : K \to \Gamma \cup \{\infty\}$, and let $T = \mathrm{Th}(K; +, \cdot, |)$. Here $x|y$ means $v(x) \leq v(y)$. Though we work in the one-sorted language $\mathcal{L} = \{+, \cdot, |\}$, the statements could be adapted to other languages of valued fields.

Corollary 4.3. *If T is convexly orderable, then both the value group Γ and the residue field k are convexly orderable.*

Proof. Both Γ and k are simply interpretable (over $\emptyset$) in K. For example, Γ is interpreted on $S = K \setminus \{0\}$ via $\varepsilon(x, y) \equiv x \mid y \wedge y \mid x$ (i.e., $v(x) = v(y)$). Since $v(xy) = v(x) + v(y)$, the addition in Γ is interpreted by multiplication in K, and the ordering is explicitly given by $|$. We use Lemma 4.2 to conclude. ⊣

We know that the theory of algebraically closed valued fields is convexly orderable. Also, the theory of real closed valued fields is weakly o-minimal [5], hence also convexly orderable. This leads to an interesting open question: Under which circumstances does the converse of Corollary 4.3 hold?

Open Question 4.4. *Is it true that, for any Henselian valued field K with value group Γ and residue field k, K is convexly orderable if and only if Γ and k are convexly orderable?*

We understand when Γ is convexly orderable by Theorem 3.1, but we do not currently have a characterization for when k is convexly orderable. Answering Open Question 4.4 would probably require first understanding when a field is convexly orderable in general.

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We can apply Corollary 4.3 to the case of the p-adics.

COROLLARY 4.5. *If Γ is not divisible, then T is not convexly orderable, hence not VC-minimal. In particular, the theory of the p-adics is not VC-minimal.*

PROOF. By Theorem 3.1, Γ is convexly orderable if and only if Γ is divisible. Hence, if Γ is not divisible, then Corollary 4.3 implies that T is not convexly orderable. In particular, the theory of the p-adics, $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$, has value group $(\mathbb{Z}; +, \leq)$, which is not divisible. Hence, the theory of the p-adics is not VC-minimal. ⊣

By Section 6 of [6], the theory of the p-adics is dp-minimal. So this corollary gives us another natural example of a theory that is dp-minimal but not VC-minimal. In the next subsection, we exhibit a means of producing examples of theories that are dp-minimal but not quasi-VC-minimal.

4.2. Quasi-VC-minimality. For this subsection, fix K a valued field with value group Γ and let $T = \mathrm{Th}(K; +, \cdot, |)$ as in the previous subsection. First, recall that if K is algebraically closed, then T is VC-minimal. Notice that if K is algebraically closed, then Γ is divisible. The main goal of this section is to prove the following stronger result.

THEOREM 4.6. *If T is quasi-VC-minimal, then Γ is divisible.*

Suppose then that Γ is not divisible, say $p\Gamma \neq \Gamma$. Fix some positive $\gamma_1 \in \Gamma \setminus p\Gamma$. Define $\gamma_n \in \Gamma$ by

$$\gamma_n = \begin{cases} k \cdot p \cdot \gamma_1 & \text{if } n = 2k, \\ \gamma_1 + k \cdot p \cdot \gamma_1 & \text{if } n = 2k+1. \end{cases}$$

Notice that $0 = \gamma_0 < \gamma_1 < ... < \gamma_n < ...$ and $p \mid \gamma_n$ if and only if n is even.

We now construct, for each $n < \omega$, $\mathcal{A}_n \subseteq K$ as follows. Set $\mathcal{A}_0 = \{0\}$. For each $a \in \mathcal{A}_n$, choose $a' \in K$ such that $v(a - a') = \gamma_n$. Let

$$\mathcal{A}_{n+1} = \mathcal{A}_n \cup \{a' \mid a \in \mathcal{A}_n\}.$$

Note that $a' \notin \mathcal{A}_n$ (to see this, show inductively that for distinct $b_1, b_2 \in \mathcal{A}_n$, $v(b_1 - b_2) \leq \gamma_{n-1}$). Therefore $|\mathcal{A}_n| = 2^n$. Moreover, for all $a \in \mathcal{A}_n$ and all $i < n$, there exists $b \in \mathcal{A}_n$ such that $v(a - b) = \gamma_i$.

Suppose that $\trianglelefteq$ is a linear ordering on K. In this case, each $\mathcal{A}_n$ is also linearly ordered by $\trianglelefteq$. For each $b \in K$, define

$$X_b = \{a \in K \mid p \mid v(a - b)\}.$$

LEMMA 4.7. *For each $n < \omega$, there exists $b \in K$ such that X_b is the union of no fewer than $n + 1$ $\trianglelefteq$-convex subsets of K.*

PROOF. Fix $n < \omega$ and let $\mathcal{A} = \mathcal{A}_{2n+1}$, which is a finite linear order (under $\trianglelefteq$).

Let $a_0 \in \mathcal{A}$ be the $\trianglelefteq$-minimal element. In general, we inductively construct a sequence $a_0, ..., a_{2n+1} \in \mathcal{A}$ such that

(1) $v(a_j - a_i) = \gamma_j$ for all $j < i$,
(2) $a_0 \triangleleft a_1 \triangleleft ... \triangleleft a_{2n+1}$, and
(3) for all $a \in \mathcal{A}$ with $v(a - a_i) \geq \gamma_i$, $a_i \trianglelefteq a$.

Suppose that $a_0, \ldots, a_i$ with the above properties have been found, and choose $a_{i+1} \in \mathcal{A}$ $\trianglelefteq$-minimal such that $v(a_{i+1} - a_i) = \gamma_i$. This exists by definition of

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$\mathcal{A} = \mathcal{A}_{2n+1}$. By condition (3), $a_i \lhd a_{i+1}$, so condition (2) holds up to a_{i+1}. Condition (1) and $v(a_{i+1} - a_i) = \gamma_i > \gamma_j$ implies that $v(a_j - a_{i+1}) = \gamma_j$ for all $j < i$. Therefore, condition (1) holds for a_{i+1}. Finally, fix $a \in \mathcal{A}$ and suppose $v(a - a_{i+1}) \geq \gamma_{i+1}$. Since $v(a_{i+1} - a_i) = \gamma_i$, we have $v(a - a_i) = \gamma_i$ as well. However, since a_{i+1} was chosen $\unlhd$-minimal in the set $\{x \in \mathcal{A} \mid v(x - a_i) = \gamma_i\}$ and a belongs to this set, we must have that $a_{i+1} \unlhd a$. Thus, condition (3) holds for a_{i+1}.

Finally, set $b = a_{2n+1}$. Then, for $i \leq 2n$, $a_i \in X_b$ if and only if $p \mid v(a_i - b)$ if and only if $p \mid \gamma_i$. Recall, moreover, that $p \mid \gamma_i$ if and only if i is even. Therefore, $a_i \in X_b$ if and only if i is even. By condition (2), X_b is the union of no fewer than $n + 1$ $\unlhd$-convex subsets of K. ⊣

Proof of Theorem 4.6. Suppose $\Gamma \neq p\Gamma$. Fix the formula

$$\varphi(x; y) = \exists z(z^p \mid (x - y)).$$

Towards a contradiction, suppose T were quasi-VC-minimal. By Corollary 2.6, there exists a linear order $\unlhd$ on K and $n < \omega$ such that each instance of φ is a union of at most n $\unlhd$-convex subsets of K. By Lemma 4.7, there exists $b \in K$ such that $X_b = \varphi(K; b)$ is a union of no fewer than $n + 1$ $\unlhd$-convex subsets of K, a contradiction. ⊣

Corollary 4.8. *The following theories are not quasi-VC-minimal*: $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$ *for any prime* p, *and* $\mathrm{Th}(k((t)); +, \cdot, |)$ *for any field* k.

Since the p-adics are dp-minimal, this gives us a natural example of a theory that is dp-minimal and not quasi-VC-minimal. Combining this observation with Corollary 3.5, we get strict implications

$$\text{VC-minimal} \Rightarrow \text{quasi-VC-minimal} \Rightarrow \text{dp-minimal}$$

where strictness is witnessed by Presburger arithmetic and the p-adics respectively.

§5. Abelian Groups.

Let $\mathfrak{A} = (A; +)$ be an abelian group and $T = \mathrm{Th}(\mathfrak{A})$. Throughout this section we work exclusively in the pure group language $\mathcal{L} = \{+\}$. For each $k, m < \omega$, consider the formula

$$\varphi_{k,m}(x) = \exists y \, (k \cdot y = m \cdot x) \, .$$

Notice that $\varphi_{k,m}(\mathfrak{A})$ is a subgroup of A. For $k = 0$, $\varphi_{0,m}(\mathfrak{A})$ is the subgroup of m-torsion elements of A, which we will also denote by $A[m]$. For $m = 1$, $\varphi_{k,1}(\mathfrak{A})$ is the subgroup of k-multiples of A, which we will also denote by kA.

Proposition 5.1 (Corollary 2.13 of [12]). *All definable subsets of A are boolean combinations of cosets of* $\varphi_{k,m}(\mathfrak{A})$ *for various* $k, m < \omega$.

Let $\mathrm{PP}(A)$ be the set of all the p.p.-definable subgroups of A, which are namely the finite intersections of subgroups of the form $\varphi_{k,m}(\mathfrak{A})$ for various $k, m < \omega$. Define a quasi-order $\precsim$ on all subgroups of A by setting, for each subgroup B_0 and B_1 of A:

$$B_0 \precsim B_1 \text{ if and only if } [B_0 : B_0 \cap B_1] < \aleph_0.$$

Think of this as B_0 being almost a subgroup of B_1 (missing only by a finite index). This quasi-order generates an equivalence relation $\sim$, which is called *commensurability*. For any $B_0 \sim B_1$, notice that $B_0 \cap B_1 \sim B_0$, so $\sim$-classes are closed under intersection. We denote by $\widetilde{\mathrm{PP}}(A)$ the set $\mathrm{PP}(A)/\sim$ of equivalence classes. Thus, $\precsim$ induces a partial order on $\widetilde{\mathrm{PP}}(A)$. In [3], this partial order is used to characterize dp-minimality of T as follows.

PROPOSITION 5.2 (Corollary 4.12 of [3]). *The theory T is dp-minimal if and only if $\left(\widetilde{\mathrm{PP}}(A); \precsim\right)$ is linear.*

This is then used as the main tool for proving a classification of dp-minimal theories of abelian groups. In the following, a *nonsingular* group B is one for which $B[p]$ and B/pB are finite for all primes p.

PROPOSITION 5.3 (Proposition 5.27 of [3]). *The theory T is dp-minimal if and only if $\mathfrak{A}$ is elementarily equivalent to one of the following abelian groups*:

(1) $\bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_i)} \oplus \mathbb{Z}\left(p^\infty\right)^{(\beta)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma)} \oplus B$ *for some prime p, a nonsingular abelian group B, and α_i, β, and γ cardinals with $\alpha_i < \aleph_0$ for all i.*

(2) $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\alpha)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\beta)} \oplus B$ *for some prime p, $k \geq 1$, finite abelian group B, and cardinals α and β, at least one of which is infinite.*

In this section, we will prove a characterization for when T is VC-minimal (and convexly orderable) analogous to Proposition 5.2, and likewise use it to obtain a complete list of VC-minimal theories of abelian groups.

LEMMA 5.4. *Suppose that there exists $\mathcal{H} \subseteq \mathrm{PP}(A)$ such that*

(1) *$(\mathcal{H}; \subseteq)$ is a linear order; and*

(2) *For all k and m, $\varphi_{k,m}(\mathfrak{A})$ is a boolean combination of cosets of elements $H \in \mathcal{H}$.*

Then, T is VC-minimal.

PROOF. For each $H \in \mathcal{H}$, let $\psi_H(x; y)$ be the formula $x - y \in H$, and let $\Psi = \{\psi_H \mid H \in \mathcal{H}\}$. The instances of Ψ define precisely the cosets of members of $\mathcal{H}$. We claim that Ψ is a generating family for T.

First, to see that Ψ is directed, fix $H_1, H_2 \in \mathcal{H}$ and $a_1, a_2 \in A$. By (1), we may assume without loss of generality that $H_1 \subseteq H_2$. Then each coset of H_1 is a subset of a coset of H_2, so that either $a_1 + H_1 \subseteq a_2 + H_2$ or $(a_1 + H_1) \cap (a_2 + H_2) = \emptyset$ as required.

By Proposition 5.1, all definable subsets of A are boolean combinations of cosets of $\varphi_{k,m}(\mathfrak{A})$ for various $k, m < \omega$. So (2) implies that all parameter-definable subsets of A are in fact boolean combination of cosets of elements $H \in \mathcal{H}$. ⊣

COROLLARY 5.5. *The theory $T = \mathrm{Th}(\mathbb{Z}; +)$ is VC-minimal.*

PROOF. Let $\mathcal{H} = \{(n!) \cdot \mathbb{Z} \mid 1 \leq n < \omega\} \cup \{0\}$. This satisfies the conditions in Lemma 5.4. ⊣

For a prime p, let $\mathbb{Z}_{(p)}$ be the additive group of the ring $\mathbb{Z}$ localized at the prime ideal $(p) = p\mathbb{Z}$. Let $\mathbb{Z}(p^\infty)$ be the Prüfer p-group, which is the direct limit of $(\mathbb{Z}/p^k\mathbb{Z})$ for all $k \geq 1$. For an abelian group A and cardinal κ, let $A^{(\kappa)}$ be the direct sum of κ copies of A.

COROLLARY 5.6. *The theories of the following abelian groups are VC-minimal*:

(1) $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$ *for some $k < \omega$ and prime p,*

(2) $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}$ *for some $k < \omega$ and prime p, and*

(3) $\mathbb{Z}(p^\infty)^{(\beta)} \oplus \mathbb{Z}_{(p)}^{(\gamma)}$ *for cardinals β and γ and prime p.*

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Proof. (1) Since $p^i A = \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$ and $A[p^i] = \left(p^{k-i}\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$, we see that

$$\mathrm{PP}(A) = \left\{ \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \;\middle|\; 0 \leq i \leq k \right\},$$

which is itself a chain. We conclude that T is VC-minimal by Lemma 5.4.

(2) Notice that, for each i,

$$p^i A = \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(p^i\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}$$
$$A[p^i] = \left(p^{k-i}\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(p^{k+1-i}\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}.$$

So, let $\mathcal{H}$ be the chain $0 \subset p^k A \subset A[p] \subset p^{k-1}A \subset A[p^2] \subset \ldots$ and use Lemma 5.4 to conclude.

(3) In this case, we have

$$A[p^i] = (\mathbb{Z}(p^\infty)[p^i])^{(\beta)} \oplus 0$$
$$p^i A = \mathbb{Z}(p^\infty)^{(\beta)} \oplus \left(p^i\mathbb{Z}_{(p)}\right)^{(\gamma)}.$$

Use the chain $0 \subseteq A[p] \subseteq A[p^2] \subseteq \ldots \subseteq p^2 A \subseteq pA \subseteq A$ along with Lemma 5.4 to conclude. ⊣

However, not every dp-minimal abelian group is VC-minimal or even convexly orderable.

Lemma 5.7. *Suppose that there exists a chain of $\emptyset$-definable subgroups $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ and a $\emptyset$-definable subgroup $B \subseteq A$ such that*

(1) *for each $i < \omega$, $A_i \cap B \neq A_{i+1} \cap B$, and*
(2) *for each $i < \omega$, $[A_i : A_i \cap B] \geq \aleph_0$.*

Then, $T = \mathrm{Th}(A; +)$ is not convexly orderable. Hence, T is not VC-minimal.

Proof. By way of contradiction, suppose that $\mathfrak{A}$ is convexly ordered by $\trianglelefteq$. In particular, suppose that each instance of the formula $x - y \in B$ is a union of at most k $\trianglelefteq$-convex subsets of A for some fixed $k < \omega$.

Since $[A_k : A_k \cap B] \geq \aleph_0$, the set

$$\mathcal{C} = \{a + B \mid a \in A_k\}$$

of cosets of B is infinite. On the other hand, for each $1 \leq i \leq k$, $A_i \cap B \subsetneq A_{i-1} \cap B$. So, for any choice of $b \in (A_{i-1} \setminus A_i) \cap B$ and $a \in A_k$, $a + b \in (A_{i-1} \setminus A_i)$. Therefore, for all $a \in A_k$ and $1 \leq i \leq k$, $(a + B) \cap (A_{i-1} \setminus A_i)$ is nonempty. That is, $(A_{i-1} \setminus A_i)$ intersects nontrivially each element of $\mathcal{C}$.

By convex orderability, for each $i \leq k$, A_i is a finite union of $\trianglelefteq$-convex subsets of A. Let $\mathcal{C}_i$ denote the elements $a + B \in \mathcal{C}$ such that, for some $\trianglelefteq$-convex component C of $a + B$, $C \nsubseteq A_i$ and $C \cap A_i \neq \emptyset$. By convexity, there can be only finitely many such $a + B$, namely the ones covering the finitely many "endpoints" of A_i. Hence, $\mathcal{C}_i$ is finite for each $i \leq k$. Finally, set

$$\mathcal{C}^* = \mathcal{C} \setminus \left(\bigcup_{i \leq k} \mathcal{C}_i \right).$$

Since $\mathcal{C}$ is infinite, $\mathcal{C}^*$ is also infinite and, in particular, nonempty.

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We claim that each A_i contains at most $k - i$ $\trianglelefteq$-convex components of each element of $\mathcal{C}^*$. By choice of k, this clearly holds for $i = 0$. So suppose that $i > 0$ and that the claim holds for A_{i-1}. Consider $a + B \in \mathcal{C}^*$. By construction, for each $\trianglelefteq$-convex component C of $a+B$, either $C \subseteq A_i$ or $C \cap A_i = \emptyset$. However, as observed above $(a + B) \cap (A_{i-1} \setminus A_i) \neq \emptyset$, so at least one of the $\trianglelefteq$-convex components of $a + B$ contained in A_{i-1} must be disjoint from A_i. By assumption, A_{i-1} contains at most $k - (i - 1)$ $\trianglelefteq$-convex components of $a + B$. Thus A_i contains at most $k - i$. The conclusion follows by induction.

Therefore, for all $a + B \in \mathcal{C}^*$, $(a + B) \cap A_k = \emptyset$. On the other hand, A_k intersects every coset $a + B \in \mathcal{C}$ by definition of $\mathcal{C}$. This gives the desired contradiction. ⊣

We use this to produce an example of an abelian group whose theory is dp-minimal but not VC-minimal.

Corollary 5.8. *Fix some $\alpha_i < \aleph_0$ for each $i \geq 1$ such that the set $\{i \mid \alpha_i > 0\}$ is infinite. Then the theory of the abelian group*

$$A = \bigoplus_{i \geq 1} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i}$$

is not convexly orderable.

Proof. Let $I = \{i \mid \alpha_i > 0\}$, let $i_0 = 0$, and let $i_1 < i_2 < \ldots$ enumerate I. It is straightforward to check that the $\emptyset$-definable subgroups

$$A_\ell = p^{i_\ell} A \text{ for all } \ell < \omega, \text{ and } B = A[p]$$

satisfy the hypotheses of Lemma 5.7. ⊣

By Proposition 5.3 (1), we see that this A is, in fact, dp-minimal.

Definition 5.9. *For $X \in \widetilde{\mathrm{PP}}(A)$ (i.e., X is a $\sim$-class of* PP(A)), *we say that X is* upwardly coherent *if there exists $H \in X$ such that, for all $H_1 \in \mathrm{PP}(A)$ with $H \precsim H_1$, we have that $H \subseteq H_1$.*

By extension, we say that the group A is upwardly coherent *if every $X \in \widetilde{\mathrm{PP}}(A)$ is.*

Intuitively, upward coherence means the class contains a particular subgroup for which being *almost* a proper subgroup is sufficient to be, in fact, a subgroup. In the presence of dp-minimality, this condition implies VC-minimality as shown in the next lemma.

Lemma 5.10. *Suppose $T = \mathrm{Th}(A; +)$ is dp-minimal. If A is upwardly coherent, then T is VC-minimal.*

Proof. For each $X \in \widetilde{\mathrm{PP}}(A)$, let $H_X \in X$ witness that X is upwardly coherent. Since PP(A) is countable, so is X, so let $X = \{H_i \mid i < \omega\}$ enumerate X. Define $H_X^i \in X$ inductively as follows:

- $H_X^0 = H_X$.
- For $i \geq 0$, $H_X^{i+1} = H_X^i \cap H_i$.

Since X is closed under intersection, each H_X^i is still an element of X.

Let $\mathcal{H}_X = \left\{H_X^i \;\middle|\; i < \omega\right\}$. By construction, $\mathcal{H}_X$ is a chain under $\subseteq$ with maximal element H_X. Moreover, by definition of $\sim$, every $H \in X$ is a *finite* union of cosets of a member of $\mathcal{H}_X$. Finally, set

$$\mathcal{H} = \bigcup \left\{\mathcal{H}_X \;\middle|\; X \in \widetilde{\mathrm{PP}}(A)\right\}.$$

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For any distinct $X, Y \in \widetilde{\mathrm{PP}}(A)$, by Proposition 5.2 either $X \precsim\!\!\!\!\!\not\sim Y$ or $Y \precsim\!\!\!\!\!\not\sim X$. Without loss, suppose $X \precsim\!\!\!\!\!\not\sim Y$. Therefore, by upward coherence, $H \supseteq H_X$ for all $H \in Y$. Hence, $\mathcal{H}_X \cup \mathcal{H}_Y$ is a chain under $\subseteq$. It follows that $\mathcal{H}$ is itself a chain under $\subseteq$. We thus conclude that $\mathcal{H}$ satisfies the hypotheses of Lemma 5.4, showing that T is VC-minimal. ⊣

Putting this all together, we arrive at the desired characterization of convexly orderable (and VC-minimal) abelian groups.

THEOREM 5.11. *The following are equivalent*:

(1) *T is VC-minimal*;
(2) *T is convexly orderable*;
(3) *T is dp-minimal and A is upwardly coherent*.

PROOF. We have (1) ⇒ (2) by Corollary 2.5. Lemma 5.10 gives (3) ⇒ (1). Thus, it remains only to show (2) ⇒ (3).

If T is convexly orderable, then T is dp-minimal by Proposition 2.3. So, suppose that there exists some $X \in \widetilde{\mathrm{PP}}(A)$ that is not upwardly coherent. Fixing any $B \in X$, we construct $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ from $\mathrm{PP}(A)$ such that, for all $i < \omega$:

(1) $[B : A_i \cap B] < \aleph_0$, so that $A_i \cap B \in X$;
(2) If $i > 0$, then $A_{i-1} \cap B \neq A_i \cap B$; and
(3) $[A_i : A_i \cap B] \geq \aleph_0$.

By Lemma 5.7, this implies that T is not convexly orderable, as required.

First, set $A_0 = A$. If $B \sim A$, then $A \in X$ trivially witnesses upward coherence, contrary to assumption. Therefore, $[A : B] \geq \aleph_0$, giving condition (3) for $i = 0$. Clearly condition (1) and (2) also hold for $i = 0$.

Now fix $i \geq 0$ and suppose that A_i has been constructed satisfying (1), (2), and (3). Consider $A_i \cap B$. Since $A_i \cap B \in X$ and X is not upwardly coherent, there exists $H \in \mathrm{PP}(A)$ such that $A_i \cap B \precsim\!\!\!\!\!\not\sim H$ and $A_i \cap B \not\subseteq H$. Set $A_{i+1} = H \cap A_i$. We show that A_{i+1} satisfies (1), (2), and (3).

Since $A_i \cap B \precsim H$,

$$[A_i \cap B : A_{i+1} \cap B] = [A_i \cap B : H \cap A_i \cap B] < \aleph_0,$$

giving condition (1). Suppose $A_i \cap B = A_{i+1} \cap B$. Then $H \cap (A_i \cap B) = A_i \cap B$ implies $(A_i \cap B) \subseteq H$, contrary to assumption. Therefore, condition (2) holds.

Finally, consider the inclusions

$$(A_{i+1} \cap B) \subseteq A_{i+1} \subseteq A_i \text{ and } (A_{i+1} \cap B) \subseteq A_{i+1} \subseteq H.$$

Since $[A_i : A_i \cap B] \geq \aleph_0$, $[A_i : A_{i+1} \cap B] \geq \aleph_0$. Moreover, since $A_i \cap B \nsim H$, $[H : A_{i+1} \cap B] \geq \aleph_0$. However, by Proposition 5.2, at least one of $[H : A_{i+1}]$ and $[A_i : A_{i+1}]$ is finite, as either $H \precsim A_i$ or $A_i \precsim H$. Therefore, from

$$[A_i : A_{i+1} \cap B] = [A_i : A_{i+1}][A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$$
$$[H : A_{i+1} \cap B] = [H : A_{i+1}][A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$$

we obtain $[A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$. Hence, condition (3) holds. This completes the construction, showing that T is not convexly orderable. ⊣

Before turning to the classification of VC-minimal abelian groups, we will need two lemmas. Both address the question of transferring VC-minimality between an abelian group and its direct summands. For groups $\mathfrak{A} = (A; +)$ and $\mathfrak{B} = (B; +)$, let $\mathfrak{A} \oplus \mathfrak{B} = (A \oplus B; +)$.

Lemma 5.12. *If* $\mathfrak{B}$ *is any abelian group and* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *is VC-minimal, then* $\mathrm{Th}(\mathfrak{A})$ *is VC-minimal.*

Proof. Assume $T^* = \mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ is VC-minimal. By Theorem 5.11 (3), T^* is dp-minimal and $A \oplus B$ is upwardly coherent. By the proof of Lemma 5.10, there exists $\mathcal{H} \subseteq \mathrm{PP}(A \oplus B)$ such that $(\mathcal{H}; \subseteq)$ is a linear order and, for all k and m, $\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})$ is a finite union of cosets of some $H_{k,m} \in \mathcal{H}$. Thus, we may write

$$\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}) = \bigcup_{i \leq n} (a_i \oplus b_i) + H_{k,m}$$

for some choice of $a_i \in A$, $b_i \in B$.

If π_A denotes the projection of $\mathfrak{A} \oplus \mathfrak{B}$ onto $\mathfrak{A}$, note that $\pi_A(\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})) = \varphi_{k,m}(\mathfrak{A})$. So, clearly, $\mathcal{H}_A = \pi_A(\mathcal{H})$ is also linearly ordered by $\subseteq$. Moreover, we have

$$\varphi_{k,m}(\mathfrak{A}) = \bigcup_{i \leq n} a_i + \pi_A(H_{k,m}).$$

Therefore, using $\mathcal{H}_A$ in Lemma 5.4, we see that $T = \mathrm{Th}(\mathfrak{A})$ is VC-minimal. ⊣

Lemma 5.13. *If* $\mathfrak{B}$ *is a finite abelian group, then* $\mathrm{Th}(\mathfrak{A})$ *is VC-minimal if and only if* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *is VC-minimal.*

Proof. Suppose $T = \mathrm{Th}(\mathfrak{A})$ is VC-minimal. Again recalling the proof of Lemma 5.10, there exists $\mathcal{H} \subseteq \mathrm{PP}(A)$ so that $(\mathcal{H}; \subseteq)$ is a chain and, for all k and m, $\varphi_{k,m}(\mathfrak{A})$ is a finite union of cosets of some $H_{k,m} \in \mathcal{H}$. For each $H \in \mathcal{H}$, choose a subgroup $B(H) \subseteq B$ minimal (with respect to $\subseteq$) such that $H \oplus B(H) \in \mathrm{PP}(A \oplus B)$. Finally, let

$$\mathcal{H}^* = \{H \oplus B(H) \mid H \in \mathcal{H}\}.$$

We verify that $\mathcal{H}^*$ satisfies the hypotheses of Lemma 5.4 for $\mathfrak{A} \oplus \mathfrak{B}$.

First, to see that $\mathcal{H}^*$ is a linear order under $\subseteq$, suppose $H_1 \subseteq H_2$ from $\mathcal{H}$. As

$$(H_1 \oplus B(H_1)) \cap (H_2 \oplus B(H_2)) = H_1 \oplus (B(H_1) \cap B(H_2))$$

is again an element of $\mathrm{PP}(A \oplus B)$, the minimality of $B(H_1)$ implies $B(H_1) = B(H_1) \cap B(H_2)$. Thus $B(H_1) \subseteq B(H_2)$ and $H_1 \oplus B(H_1) \subseteq H_2 \oplus B(H_2)$.

Second, we wish to show that $\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})$ is a boolean combination of cosets of elements of $\mathcal{H}^*$. Since we already know that $\varphi_{k,m}(\mathfrak{A})$ is a finite union of cosets of $H_{k,m}$, and B is finite, it suffices to show that

$$H_{k,m} \oplus B(H_{k,m}) \subseteq \varphi_{k,m}(\mathfrak{A}) \oplus \varphi_{k,m}(\mathfrak{B}) = \varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}).$$

That is, we need to show $B(H_{k,m}) \subseteq \varphi_{k,m}(\mathfrak{B})$. If not, however,

$$\left(H_{k,m} \oplus B(H_{k,m})\right) \cap \varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}) = H_{k,m} \oplus \left(B(H_{k,m}) \cap \varphi_{k,m}(\mathfrak{B})\right)$$

would be in $\mathrm{PP}(A \oplus B)$, in which case $B(H_{k,m}) \cap \varphi_{k,m}(\mathfrak{B})$ would contradict the minimality of $B(H_{k,m})$.

Therefore, $\mathcal{H}^*$ satisfies the conditions of Lemma 5.4, proving VC-minimality of $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$. The converse follows immediately from Lemma 5.12. ⊣

We are now ready to prove an analog to Proposition 5.3 for VC-minimal (and convexly orderable) theories of abelian groups. The proposition gives a strong starting point, a complete list of dp-minimal theories of abelian groups. Theorem 5.11 and the above lemmas provide a set of tools for determining which of these are VC-minimal.

THEOREM 5.14. *T is VC-minimal (and convexly orderable) if and only if $\mathfrak{A}$ is elementarily equivalent to one of the following abelian groups*:

(1) $\bigoplus_{p \text{ prime}} \left(\mathbb{Z}(p^\infty)^{(\beta_p)}\right) \oplus \left(\mathbb{Z}_{(q)}\right)^{(\gamma)} \oplus \mathbb{Q}^{(\delta)} \oplus B$ *for a fixed prime q, finite abelian group B, and cardinals β_p, γ, and δ such that $\beta_p < \aleph_0$ for all $p \neq q$;*

(2) $\bigoplus_{p \text{ prime}} \left(B_p \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}\right) \oplus \mathbb{Q}^{(\delta)}$ *for a fixed prime q, finite p-groups B_p, and cardinals β_p, γ_p, and δ such that $\beta_p < \aleph_0$ for all $p \neq q$ and $\gamma_p < \aleph_0$ for all p (including q);*

(3) $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\alpha)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\beta)} \oplus B$ *for some prime p, $k \geq 1$, finite abelian group B, and cardinals α and β, at least one of which is infinite.*

PROOF. Suppose T is dp-minimal. By Proposition 5.3, it falls under one of two categories. T is either the theory of a group as in (3) above; or, $\mathfrak{A}$ is elementarily equivalent to

$$\bigoplus_{i \geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)} \oplus B \qquad (\star)$$

for a prime p, nonsingular abelian group B, and cardinals $\alpha_{p,i}$, β_p, and γ_p with each $\alpha_{p,i}$ finite.

For the former category, it follows from Corollary 5.6 and Lemma 5.13 that the group in (3) is also VC-minimal.

For the latter, first recall that by results of Szmielew [14], any abelian group is elementarily equivalent to one of the form

$$\bigoplus_{p \text{ prime}} \left(\bigoplus_{i \geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)} \right) \oplus \mathbb{Q}^{(\delta)}.$$

It is straightforward to verify that such a group is only nonsingular if each $\alpha_{p,i}$, β_p, γ_p, and $\{i \mid \alpha_{p,i} > 0\}$ is finite. For instance, for $B = \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}$, we have $B/pB = (\mathbb{Z}/p\mathbb{Z})^{(\gamma_p)}$, which is finite iff γ_p is. Hence, $(\star)$ becomes

$$\bigoplus_{p \text{ prime}} \left(\bigoplus_{i \geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)} \right) \oplus \mathbb{Q}^{(\delta)} \qquad (\dagger)$$

with each $\alpha_{p,i}$ finite and β_p, γ_p, and $\{i \mid \alpha_{p,i} > 0\}$ finite for $p \neq q$. In other words, writing $B_p = \bigoplus_{i \geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})}$, we have that B_p is a finite p-group for all $p \neq q$.

Suppose, then, that $(\dagger)$ is VC-minimal. We show that $(\dagger)$ is as in (1) or (2). By Corollary 5.8 and Lemma 5.12, B_q must also be finite. If $\gamma_q < \aleph_0$, then we are in case (2).

Thus, suppose that $\gamma_p \geq \aleph_0$. Notice that $qA \precnsim A$. We must show that $B = \bigoplus_p B_p$ is finite and $\gamma_p = 0$ for $p \neq q$.

If $\gamma_p > 0$ for some $p \neq q$, then $qA \precnsim p^n A$ for all n. However, there is no $H \in \mathrm{PP}(A)$ with $H \sim qA$ such that $H \subseteq p^n A$ for all n. Therefore, the $\sim$-class of qA is not upwardly coherent, contradicting Theorem 5.11.

If B_p is nonzero for infinitely many primes p, let $p_0, p_1, \dots$ enumerate all such primes, excluding q. Then we have

$$qA \precnsim \left(\prod_{i\leq n} p_i\right) A.$$

But there is no $H \in \mathrm{PP}(A)$ with $H \sim qA$ such that $H \subseteq \left(\prod_{i\leq n} p_n\right) A$ for every n, again contradicting upward coherence of the $\sim$-class of qA.

We have thus established that the theory of a VC-minimal abelian group belongs to one of the cases (1), (2), or (3). It remains only to show that the groups in (1) and (2) are indeed VC-minimal.

For both cases, $A[q^n]$ witnesses the upward coherence of its $\sim$-class for every n. In case (2), $kA \sim A$ for all k, so the chain of $\widetilde{\mathrm{PP}}(A)$ is given by

$$0 \precnsim A[q] \precnsim A[q^2] \precnsim \dots \precnsim A,$$

and each $\sim$-class is upwardly coherent. Furthermore, each group in $\mathrm{PP}(A)$ is a boolean combination of cosets of groups in this chain. The details of this computation can be found in Lemma 5.28 of [3].

In case (1), in addition to $A[q^n]$, we also have that $q^n A$ witnesses the upward coherence of its $\sim$-class. The chain of $\widetilde{\mathrm{PP}}(A)$ is given by

$$0 \precnsim A[q] \precnsim A[q^2] \precnsim \dots \precnsim q^2 A \precnsim qA \precnsim A.$$

Again, we refer to Lemma 5.28 of [3] to see that the groups in this chain generate every member of $\mathrm{PP}(A)$.

In both cases, therefore, A is upwardly coherent. By Theorem 5.11, T is VC-minimal. ⊣

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