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THE GENERALIZED ENTROPY ERGODIC THEOREM FOR NONHOMOGENEOUS MARKOV CHAINS INDEXED BY A HOMOGENEOUS TREE

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In this paper, we extend the strong laws of large numbers and entropy ergodic theorem for partial sums for tree-indexed nonhomogeneous Markov chains fields to delayed versions of nonhomogeneous Markov chains fields indexed by a homogeneous tree. At first we study a generalized strong limit theorem for nonhomogeneous Markov chains indexed by a homogeneous tree. Then we prove the generalized strong laws of large numbers and the generalized asymptotic equipartition property for delayed sums of finite nonhomogeneous Markov chains indexed by a homogeneous tree. As corollaries, we can get the similar results of some current literatures. In this paper, the problem settings may not allow to use Doob's martingale convergence theorem, and we overcome this difficulty by using Borel–Cantelli Lemma so that our proof technique also has some new elements compared with the reference Yang and Ye (2007).

Keywords: generalized entropy ergodic theorem, nonhomogeneous Markov chain, strong law of large numbers, tree

2010 Mathematics subject classification: 60C05; 05C80

1. INTRODUCTION

Let (k_n) be a sequence of positive integers and let (a_n)n be a sequence of real numbers. Zygmund [18] first gave the definition of forward delayed first arithmetic means which is defined as follows

$$r_{n,k_n} = \frac{\sum_{i=1}^{k_n} a_{n+i-1}}{k_n}.$$

Agnew [1], Safanov [11], and Chow [4] have studied the relationship between the above delayed averages and summability methods. Chow [4] has also found necessary and sufficient conditions for the Borel summability of i.i.d random variables and simplified the proofs of a number of well-known results such as Hsu–Robbins–Spitzer–Katz theorem. Lai [9] studied the law of the iterated logarithm for delayed sums of independent random variables. Recently, Gut and Stadtm*ü*ller [5,6] have studied the Laws of the single logarithm for delayed sums of random fields. Subsequently, Gut and Stadtm*ü*ller [7] also studied the strong law of large numbers for delayed sums of random fields. Now what we are interested in is the

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limit behavior of delayed averages of Markov chains fields indexed by trees. Also, Wang and Yang [12] have studied the strong limit theorems of delayed stochastic sums of the functions sequences of two variables for non-homogeneous Markov chains and a generalized entropy ergodic theorem for non-homogeneous Markov chains with convergence almost surely. In this paper, we want to find analogs of strong law of large numbers and Shannon entropy theorem for the forward delayed sums of Markov chains fields indexed by a homogeneous tree.

A tree T is a connected graph and does not contain any loop. Given any two vertices $s \neq t \in T$, let $\overline{st}$ be the unique path connecting s and t. Define the graph distance $d(s,t)$ to be the number of edges contained in the path $\overline{st}$.

Let $T_{C,N}$ be an infinite Cayley tree with root 0, in which the root 0 has only N neighbors and all other vertices have $N+1$ neighbors. For each vertex t, there is a unique path from 0 to t, and $|t| = d(0,t)$ for the number of edges on this path. We denote the first predecessor of t by ${}^1t$, the second predecessor of t by ${}^2t$, and denote by ${}^nt$ the n-th predecessor of t. We denote by $T_{(m)}^{(n)}$ the subtree of T containing the vertices from level m to level n, and L_n the set of all vertices on level n. For any two vertices s and t of tree T, write $s \leq t$ if s is on the unique path from the root 0 to t. We denote by $s \wedge t$ the vertex farthest from 0 satisfying $s \wedge t \leq s$ and $s \wedge t \leq t$. $X^A = \{X_t, t \in A\}$ and denote by $|A|$ the number of vertices of A. When the context permits, this type of tree is simply denoted by T. For good understanding of the definition of tree graph, here we draw a graph of Cayley tree $T_{C,2}$, please see the following Figure 1.

Definition 1.1 ([15]): *Let T be an infinite tree, $\mathcal{X}$ a finite state space, $\{X_t, t \in T\}$ be a collection of $\mathcal{X}$-valued random variables defined on probability space $(\Omega, \mathcal{F}, P)$. Let*

$$p = \{p(x), x \in \mathcal{X}\} \tag{1.1}$$

be a distribution on $\mathcal{X}$, and

$$P_n = (P_n(x,y)), \quad x, y \in \mathcal{X} \tag{1.2}$$

be stochastic matrices on $\mathcal{X}^2$. If for any vertex $t \in L_n$,

$$\begin{aligned} Pr(X_t = y | X_{{}^1t} = x \quad and \quad X_s \quad for \quad t \wedge s \leq^1 t) \\ = Pr(X_t = y | X_{{}^1t} = x) = P_n(x,y) \quad \forall x, y \in \mathcal{X}, \end{aligned} \tag{1.3}$$

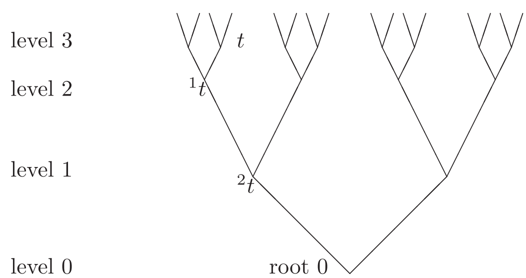

Figure 1. A Cayley tree $T_{C,2}$.

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and

$$Pr(X_0 = x) = p(x) \quad \forall x \in \mathcal{X},$$

then $\{X_t, t \in T\}$ *will be called* $\mathcal{X}$*-valued nonhomogeneous Markov chains indexed by a homogeneous tree* T *with the initial distribution* (1.1) *and transition matrices* (1.2)*, or called* T*-indexed nonhomogeneous Markov chains with state-space* $\mathcal{X}$*.*

REMARK 1.1: *If for all* n*,*

$$P_n = (P(x, y)), \quad x, y \in \mathcal{X}, \tag{1.4}$$

then $\{X_t, t \in T\}$ *will be called* $\mathcal{X}$*-valued homogeneous Markov chains indexed by a homogeneous tree.*

REMARK 1.2: *For the Cayley tree* $T_{C,N}$*, when* $N = 1$*, a nonhomogeneous Markov chain indexed by a tree will reduce to a nonhomogeneous Markov chain on line.*

Let T be the homogeneous tree $T_{C,N}$. If $(X_t)_{t\in T}$ is a nonhomogeneous Markov chain indexed by tree T with finite state space $\mathcal{X}$ defined as Definition 1.1. Let $(\alpha_n)_{n=0}^{\infty}$ and $(\phi(n))_{n=0}^{\infty}$ be two sequences of nonnegative integers such that $\lim_{n\to\infty} \phi(n) = \infty$. Denote

$$f_{\alpha_n,\phi(n)}(\omega) = -\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}[\log Pr(X_t, t \in L_{\alpha_n}) + \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \log P_k(X_{1_t}, X_t)], \tag{1.5}$$

where log is the natural logarithm. $f_{\alpha_n,\phi(n)}(\omega)$ will be called generalized entropy density of $(X_t)_{t\in T_{(\alpha_n)}^{(\alpha_n+\phi(n))}}$. If $\alpha_n \equiv 0$ and $\phi(n) = n$, $f_{\alpha_n,\phi(n)}(\omega)$ will reduce to the classical entropy of $(X_t)_{t\in T^{(n)}}$ defined as follows

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}[\log Pr(X_0) + \sum_{k=1}^{n} \sum_{t\in L_k} \log P_k(X_{1_t}, X_t)], \tag{1.6}$$

$f_n(\omega)$ will be called the entropy density of $(X_t)_{t\in T^{(n)}}$. The convergence of $f_n(\omega)$ to a constant in a sense (L_1 convergence, convergence in probability, a.e. convergence) is called the Shannon–McMillan theorem or the entropy theorem or the asymptotic equipartition property (AEP) in information theory.

The subject of tree-indexed processes has been deeply studied and made abundant achievements. Benjamini and Peres [2] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [3] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger (see [16,17]), by using Pemantle's result [10] and a combinatorial approach, have studied the Shannon–McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu [14] and Yang [13] have studied a strong law of large numbers for Markov chains fields on a homogeneous tree (a particular case of tree-indexed Markov chains and PPG-invariant random fields). Yang and Ye [15] have established the Shannon–McMillan theorem with convergence almost surely for nonhomogeneous Markov chains on a homogeneous tree. Huang and Yang (see [8]) have studied the Shannon–McMillan theorem in the sense of almost surely for finite homogeneous Markov chains indexed by a uniformly bounded infinite tree. Recently, Wang and Yang [12] have studied the strong limit theorems of delayed stochastic sums of the functions sequences of two variables for nonhomogeneous Markov chains and a generalized entropy ergodic theorem for nonhomogeneous Markov chains with convergence almost surely.

In this manuscript, we first study a generalized strong limit theorem for nonhomogeneous Markov chains indexed by a homogeneous tree $T_{C,N}$. Then we prove the generalized strong law of large numbers and the generalized AEP for delayed sums of finite nonhomogeneous Markov chains indexed by a homogeneous tree which are the extensions of some results of [15] and [12].

2. SOME LEMMAS

In this section, we at first give a lemma which is very useful for proving our main results. Then as corollaries, we give two useful limit theorems for delayed sums of the frequencies of occurrence of states and the ordered couples of states for nonhomogeneous Markov chains indexed by homogeneous tree $T_{C,N}$.

LEMMA 2.1: *Let T be the homogeneous tree $T_{C,N}$. If $(X_t)_{t\in T}$ is a nonhomogeneous Markov chain indexed by tree T with finite state space $\mathcal{X}$ defined as Definition 1.1, and $\{(g_n(x,y))_{n=1}^{\infty}\}$ be functions defined on $\mathcal{X}^2$. Let*

$$G_n(\omega)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}E[g_k(X_{1_t},X_t)|X_{1_t}], \tag{2.1}$$

here and thereafter we always let $\gamma>0$, $(\alpha_n)_{n=0}^{\infty}$ and $(\phi(n))_{n=0}^{\infty}$ be two sequences of nonnegative integers such that $\lim_{n\to\infty}\phi(n)=\infty$. Set

$$D(\gamma)=\{\omega:\limsup_{n\to\infty}\frac{1}{a_n}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}E[g_k^2(X_{1_t},X_t)e^{\gamma|g_k(X_{1_t},X_t)|}|X_{1_t}]=M(\gamma,\omega)<\infty\}, \tag{2.2}$$

$$H_n(\omega)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}g_k(X_{1_t},X_t). \tag{2.3}$$

Suppose that

$$a_n=|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|,\quad \sum_{n=1}^{\infty}\exp(-\varepsilon a_n)<\infty, \tag{2.4}$$

for any $\varepsilon>0$, then

$$\lim_{n\to\infty}\frac{H_n(\omega)-G_n(\omega)}{a_n}=0\quad a.e.\quad on\quad D(\gamma). \tag{2.5}$$

REMARK 2.1: *If $\{(g_n(x,y))_{n=1}^{\infty}\}$ are uniformly bounded, then equation (2.2) holds obviously.*

Proof of Lemma 2.1.: Let λ be a nonzero real number, define

$$t_{\alpha_n,\phi(n)}(\lambda,\omega)=\frac{e^{\lambda\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}g_k(X_{1_t},X_t)}}{\prod_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\prod_{t\in L_k}E[e^{\lambda g_k(X_{1_t},X_t)}|X_{1_t}]},n=1,2,\ldots \tag{2.6}$$

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let $\mathcal{F}_n = \sigma(X^{T^{(n)}})$, it is easy to see that

$$\begin{aligned} E[t_{\alpha_n,\phi(n)}(\lambda,\omega)] &= E[E[t_{\alpha_n,\phi(n)}(\lambda,\omega)|\mathcal{F}_{\alpha_n+\phi(n)-1}]] \\ &= E[t_{\alpha_n,\phi(n)-1}(\lambda,\omega)E[\frac{\prod_{t\in L_{\alpha_n+\phi(n)}} e^{\lambda g_{\alpha_n+\phi(n)}(X_{1_t},X_t)}}{\prod_{t\in L_{\alpha_n+\phi(n)}} E[e^{\lambda g_{\alpha_n+\phi(n)}(X_{1_t},X_t)}|X_{1_t}]}|\mathcal{F}_{\alpha_n+\phi(n)-1}] \\ &= E[t_{\alpha_n,\phi(n)-1}(\lambda,\omega)\frac{\prod_{t\in L_{\alpha_n+\phi(n)}} E[e^{\lambda g_{\alpha_n+\phi(n)}(X_{1_t},X_t)}|X_{1_t}]}{\prod_{t\in L_{\alpha_n+\phi(n)}} E[e^{\lambda g_{\alpha_n+\phi(n)}(X_{1_t},X_t)}|X_{1_t}]}] \\ &= E[t_{\alpha_n,\phi(n)-1}(\lambda,\omega)] = \cdots = E[t_{\alpha_n,1}(\lambda,\omega)] = 1. \end{aligned} \quad \textbf{(2.7)}$$

For any $\epsilon > 0$, by using Markov inequality we have

$$\begin{aligned} \sum_{n=1}^{\infty} Pr\left(\frac{\log t_{\alpha_n,\phi(n)}(\lambda,\omega)}{a_n} \geq \varepsilon\right) &= \sum_{n=1}^{\infty} Pr(t_{\alpha_n,\phi(n)}(\lambda,\omega) \geq \exp(\varepsilon a_n)) \\ &\leq \sum_{n=1}^{\infty} \frac{E[t_{\alpha_n,\phi(n)}(\lambda,\omega)]}{\exp(\varepsilon a_n)} = \sum_{n=1}^{\infty} \exp(-\varepsilon a_n) < \infty. \end{aligned} \quad \textbf{(2.8)}$$

For any $\varepsilon > 0$, by using Borel–Cantelli lemma it is easy to get that

$$\limsup_{n\to\infty} \frac{\log t_{\alpha_n,\phi(n)}(\lambda,\omega)}{a_n} \leq 0 \ a.e. \quad \textbf{(2.9)}$$

By equations (2.6), (2.9) and simple computation, we obtain that

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{\lambda g_k(X_{1_t}, X_t) - \log E[e^{\lambda g_k(X_{1_t},X_t)}|X_{1_t}]\right\} \leq 0 \ a.e. \quad \textbf{(2.10)}$$

Letting $0 < \lambda < \gamma$, dividing both sides of above inequality by λ, we arrive at

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{g_k(X_{1_t}, X_t) - \frac{1}{\lambda}\log E[e^{\lambda g_k(X_{1_t},X_t)}|X_{1_t}]\right\} \leq 0 \ a.e. \text{ on } D(\gamma), \quad \textbf{(2.11)}$$

by using the inequalities $\ln x \leq x - 1 (x > 0)$ and $0 \leq e^x - 1 - x \leq 2^{-1}x^2 e^{|x|}$, as $0 < \lambda \leq \gamma$, from (2.11) it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{g_k(X_{1_t}, X_t) - E[g_k(X_{1_t}, X_t)|X_{1_t}]\right\} \\ &\leq \limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{\frac{1}{\lambda}\log E[e^{\lambda g_k(X_{1_t},X_t)}|X_{1_t}] - E[g_k(X_{1_t}, X_t)|X_{1_t}]\right\} \\ &\leq \limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{\frac{E[e^{\lambda g_k(X_{1_t},X_t)} - 1 - \lambda g_k(X_{1_t}, X_t)|X_{1_t}]}{\lambda}\right\} \\ &\leq \frac{\lambda}{2} \limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \left\{E[g_k^2(X_{1_t}, X_t)e^{\lambda|g_k(X_{1_t},X_t)|}|X_{1_t}]\right\} \\ &\leq \frac{\lambda}{2} M(\gamma,\omega) < \infty \ a.e. \text{ on } D(\gamma). \end{aligned} \quad \textbf{(2.12)}$$

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Letting $\lambda \to 0^+$ in (2.12), it follows that

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \{g_k(X_{1_t}, X_t) - E[g_k(X_{1_t}, X_t)|X_{1_t}]\} \le 0 \text{ } a.e. \text{ on } D(\gamma). \quad \textbf{(2.13)}$$

Let $-\gamma \le \lambda < 0$. By (2.10), we similarly get

$$\liminf_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \{g_k(X_{1_t}, X_t) - E[g_k(X_{1_t}, X_t)|X_{1_t}]\} \ge 0 \text{ } a.e. \text{ on } D(\gamma). \quad \textbf{(2.14)}$$

Now, (2.5) can be derived easily from (2.13) and (2.14). The proof of lemma 2.1 is completed. ■

Remark 2.2: *If $\alpha = 0, \phi(n) = n$, $t_{\alpha_n,\phi(n)}(\lambda,\omega) = t_{0,n}(\lambda,\omega)$ will be reduced to a nonnegative martingale, then by Doob' martingale theorem, we can obtain Theorem 1 in reference Yang and Ye [15]. Obviously in our Lemma 2.1, the stochastic sequence $t_{\alpha_n,\phi(n)}(\lambda,\omega)$ is not a martingale, so that we cannot use Doob's martingale theorem directly. That is the main difficulty we need to overcome in our proof.*

Let $S_{\alpha_n,\phi(n)}(i)(i \in \mathcal{X})$ be the number of i in the set of random variables $X^{T^{(\alpha_n+\phi(n))}_{(\alpha_n)}} = \{X_t, t \in T^{(\alpha_n+\phi(n))}_{(\alpha_n)}\}$; $S_{\alpha_n,\phi(n)}(i;j)$ be the number of couple $(i;j)$ in the set of random couples $\{(X_{1_t}, X_t), t \in T^{(\alpha_n+\phi(n))}_{(\alpha_n+1)}\}$, these are

$$S_{\alpha_n,\phi(n)}(i) = \sum_{k=\alpha_n}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \delta_i(X_t), \quad \textbf{(2.15)}$$

$$S_{\alpha_n,\phi(n)}(i;j) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \delta_i(X_{1_t})\delta_j(X_t). \quad \textbf{(2.16)}$$

Corollary 2.1: *Let T be the homogeneous tree $T_{C,N}$ and $(X_t)_{t\in T}$ be a nonhomogeneous Markov chain indexed by tree T with finite state space $\mathcal{X}$ defined as Definition 1.1. $S_{\alpha_n,\phi(n)}(i)$ is defined as (2.15), $(\alpha_n)_{n=0}^{\infty}$ and $(\phi(n))_{n=0}^{\infty}$ are two sequences of nonnegative integers such that $\lim_{n\to\infty} \phi(n) = \infty$, letting $a_n = |T^{(\alpha_n+\phi(n))}_{(\alpha_n)}|$, then we have*

$$\lim_{n\to\infty} \frac{1}{a_n}\{S_{\alpha_n,\phi(n)}(i) - \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} P_k(X_{1_t}, i)\} = 0 \text{ } a.e. \quad \textbf{(2.17)}$$

Proof: Let $g_k(x,y) = \delta_i(y)$ in Lemma 2.1. It is easy to see that $\{g_k(x,y), k \ge 1\}$ satisfy the condition (2.2) of Lemma 2.1. Noting that

$$H_n(\omega) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} g_k(X_{1_t}, X_t) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \delta_i(X_t) = S_{\alpha_n,\phi(n)}(i) - \sum_{t\in L_{\alpha_n}} \delta_i(X_t), \quad \textbf{(2.18)}$$

$$G_n(\omega) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} E[g_k(X_{1_t}, X_t)|X_{1_t}] = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} P_k(X_{1_t}, i). \quad \textbf{(2.19)}$$

Noting that

$$\frac{\sum_{t\in L_{\alpha_n}}\delta_i(X_t)}{a_n}\leq \frac{N^{\alpha_n}}{N^{\alpha_n}+N^{\alpha_n+1}+\cdots+N^{\alpha_n+\phi(n)}}\to 0,\ as\ n\to\infty,$$

so that the conclusion (2.17) is derived from Lemma 2.1 directly. ■

Corollary 2.2: *Under the same conditions of Lemma 2.1, let $S_{\alpha_n,\phi(n)}(i,j)$ be defined as (2.16), then*

$$\lim_{n\to\infty}\frac{1}{a_n}\{S_{\alpha_n,\phi(n)}(i,j)-\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1_t})P_k(i,j)\}=0\ a.e. \quad \textbf{(2.20)}$$

Proof: Let $g_k(x,y)=\delta_i(x)\delta_j(y)$ in Lemma 2.1. It is easy to see that $\{g_k(x,y),k\geq 1\}$ satisfy the condition (2.2) of Lemma 2.1. Noting that

$$H_n(\omega)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}g_t(X_{1_t},X_t)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1_t})\delta_j(X_t)=S_{\alpha_n,\phi(n)}(i,j), \quad \textbf{(2.21)}$$

$$G_n(\omega)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}E[g_t(X_{1_t},X_t)|X_{1_t}]=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1_t})P_k(i,j). \quad \textbf{(2.22)}$$

so that equation (2.20) is true because of Lemma 2.1. We complete the proof. ■

3. MAIN RESULTS

In this section, we mainly prove the strong law of large numbers for frequencies of occurrence of states and the pairs of states for delayed sums of nonhomogeneous Markov chain indexed by a homogeneous tree $T_{C,N}$ and the generalized entropy ergodic theorem for the model.

Theorem 3.1: *Let T be the homogeneous tree $T_{C,N}$ and $(X_t)_{t\in T}$ be a nonhomogeneous Markov chain indexed by tree T with finite state space $\mathcal{X}$ defined as Definition 1.1. $(\alpha_n)_{n=0}^{\infty}$ and $(\phi(n))_{n=0}^{\infty}$ are two sequences of nonnegative integers such that $\lim_{n\to\infty}\phi(n)=\infty$. Let $S_{\alpha_n,\phi(n)}(i)$ and $S_{\alpha_n,\phi(n)}(i,j)$ be defined as before, $P=(P(i,j))_{i;j\in\mathcal{X}}$ be another finite transition matrix and be ergodic. If*

$$\lim_{n\to\infty}P_n(i,j)=P(i,j),\ \forall i,\ j\in\mathcal{X}, \quad \textbf{(3.1)}$$

then

$$(i)\quad \lim_{n\to\infty}\frac{S_{\alpha_n,\phi(n)}(i)}{a_n}=\pi_i,\ a.e.\ \forall i\in\mathcal{X}, \quad \textbf{(3.2)}$$

$$(ii)\quad \lim_{n\to\infty}\frac{S_{\alpha_n,\phi(n)}(i,j)}{a_n}=\pi_iP(i,j)\ a.e.\ \forall i,\ j\in\mathcal{X}, \quad \textbf{(3.3)}$$

where π is the unique stationary distribution determined by the transition matrix P.

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Proof: Proof of (i). Obviously it is easy to see that

$$\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}P_k(X_{1t},j)=\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i\in\mathcal{X}}\delta_i(X_{1t})P_k(i,j) \tag{3.4}$$

$$\begin{aligned}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i\in\mathcal{X}}\delta_i(X_{1t})P(i,j)&=\sum_{i\in\mathcal{X}}P(i,j)\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1t})\\&=N\sum_{i\in\mathcal{X}}P(i,j)\sum_{k=\alpha_n}^{\alpha_n+\phi(n)-1}\sum_{t\in L_k}\delta_i(X_t)\\&=N\sum_{i\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)P(i,j).\end{aligned} \tag{3.5}$$

From (3.1), there is no difficulty to derive that

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}[P_k(X_{1t},j)-P(X_{1t},j)]=0, \tag{3.6}$$

where $a_n=|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|$. Then we have by (2.17) and (3.6)

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{a_n}\{S_{\alpha_n,\phi(n)}(j)-\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}P(X_{1t},j)\}\\&=\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i\in\mathcal{X}}\delta_i(X_{1t})[P_k(i,j)-P(i,j)]\\&=0\ a.e.\forall j\in\mathcal{X}.\end{aligned} \tag{3.7}$$

That is, $\forall j\in\mathcal{X}$

$$\lim_{n\to\infty}\left\{\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}S_{\alpha_n,\phi(n)}(j)-\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}\sum_{i\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)P(i,j)\right\}=0\ a.e. \tag{3.8}$$

Multiplying both sides of equation (3.8) by $P(j,k)$, and adding them together for $j\in\mathcal{X}$ and using (3.8) again, we have

$$\begin{aligned}\lim_{n\to\infty}&\left\{\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{j\in\mathcal{X}}S_{\alpha_n,\phi(n)}(j)P(j,k)\right.\\&\left.-\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}\sum_{i,j\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)P(i,j)P(j,k)\right\}\end{aligned}$$

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$$= \lim_{n\to\infty}\left\{\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{j\in\mathcal{X}}S_{\alpha_n,\phi(n)}(j)P(j,k)-\frac{S_{\alpha_n,\phi(n)+1}(k)}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)+1)}|}\right\}$$
$$+\lim_{n\to\infty}\left\{\frac{S_{\alpha_n,\phi(n)+1}(k)}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)+1)}|}-\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}\sum_{i\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)P^{(2)}(i,k)\right\}=0 \ a.e., \tag{3.9}$$

where $P^{(l)}(i,k)$ (l is a positive integer) is the l-step transition probability determined by the transition matrix P. By induction, for all $M\geq 1$, we have

$$\lim_{n\to\infty}\left\{\frac{S_{\alpha_n,\phi(n)+M}(k)}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)+M)}|}-\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}\sum_{i\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)P^{(M+1)}(i,k)\right\}=0 \ a.e. \tag{3.10}$$

Since

$$\lim_{M\to\infty}P^{(M+1)}(i,k)=\pi_k,\quad k\in\mathcal{X}, \tag{3.11}$$

and $\sum_{i\in\mathcal{X}}S_{\alpha_n,\phi(n)-1}(i)=|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|$, thus (3.2) can be obtained from (3.11) and (3.10).

Proof of (ii). Similarly to (3.5), we have

$$\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1^t})P(i,j)=NS_{\alpha_n,\phi(n)-1}(i)P(i,j) \tag{3.12}$$

From (3.1), we can see that

$$\lim_{n\to\infty}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1^t})[P_k(i,j)-P(i,j)]=0.\ a.e. \tag{3.13}$$

(3.3) can be derived from equations (2.20), (3.12), and (3.13). In fact, it is easy to see that

$$\lim_{n\to\infty}\frac{S_{\alpha_n,\phi(n)}(i,j)}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}=\lim_{n\to\infty}\frac{S_{\alpha_n,\phi(n)-1}(i)P(i,j)}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|}=\pi_iP(i,j).\ a.e.$$

The proof of Theorem 3.1 is completed. ■

THEOREM 3.2: *Under the same conditions of Theorem 3.1, let* $|\mathcal{X}|=b$, $a_n=|T_{(\alpha_n)}^{(\alpha_n+\phi(n)-1)}|$, $(\alpha_n)_{n=0}^{\infty}$ *and* $(\phi(n))_{n=0}^{\infty}$ *be two sequences of nonnegative integers such that* $\lim_{n\to\infty}\phi(n)=\infty$, $f_{\alpha_n,\phi(n)}(\omega)$ *be defined as before. If for any* $\varepsilon>0$, *suppose that the following condition is satisfied,*

$$\sum_{n=1}^{\infty}b^{N^{\alpha_n}}\exp(-\varepsilon a_n)<\infty, \tag{3.14}$$

then

$$\lim_{n\to\infty}f_{\alpha_n,\phi(n)}(\omega)=-\sum_{i\in\mathcal{X}}\sum_{j\in\mathcal{X}}\pi_iP(i,j)\log P(i,j)\ a.e. \tag{3.15}$$

where π *is the unique stationary distribution determined by the transition matrix* P.

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PROOF: Here L_{α_n} denotes the set of vertices on level α_n of homogeneous tree T. Since

$$E[\exp(|\log Pr\{X_t, t \in L_{\alpha_n})\}|] = \sum_{x^{L_{\alpha_n}}} \exp(-\log Pr(X^{L_{\alpha_n}} = x^{L_{\alpha_n}}))Pr(X^{L_{\alpha_n}} = x^{L_{\alpha_n}})$$
$$= b^{|L_{\alpha_n}|} = b^{N^{\alpha_n}}, \tag{3.16}$$

for any $\varepsilon > 0$, by Markov inequality and (3.14), we have

$$\sum_{n=1}^{\infty} Pr\{\frac{|\log Pr\{X_t, t \in L_{\alpha_n})\}|}{a_n} \geq \varepsilon\}$$
$$\leq \sum_{n=1}^{\infty} \frac{E[\exp(|\log Pr\{X_t, t \in L_{\alpha_n})\}|]}{\exp(\varepsilon a_n)}$$
$$= \sum_{n=1}^{\infty} \frac{b^{N^{\alpha_n}}}{\exp(\varepsilon a_n)} < \infty. \tag{3.17}$$

By Borel–Cantelli lemma, we get

$$\lim_{n\to\infty} \frac{\log Pr\{X_t, t \in L_{\alpha_n})\}}{a_n} = 0, \ a.e. \tag{3.18}$$

Letting $g_k(x, y) = \log P_k(x, y)$ and $\gamma = 1/2$ in lemma 2.1, and noting that

$$E[(\log P_k(X_{1_t}, X_t))^2 e^{\frac{1}{2}|\log P_k(X_{1_t}, X_t)|}|X_{1_t}]$$
$$= \sum_{j\in\mathcal{X}} P_k^{-\frac{1}{2}}(X_{1_t}, j)(\log P_k(X_{1_t}, X_t))^2 P_k(X_{1_t}, j)$$
$$= \sum_{j\in\mathcal{X}} P_k^{\frac{1}{2}}(X_{1_t}, j)(\log P_k(X_{1_t}, X_t))^2 \leq 16be^{-2}, \tag{3.19}$$

$$G_n(\omega) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} E[g_k(X_{1_t}, X_t)|X_{1_t}]$$
$$= \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \sum_{j\in\mathcal{X}} P_k(X_{1_t}, j) \log P_k(X_{1_t}, j), \tag{3.20}$$

$$H_n(\omega) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} g_k(X_{1_t}, X_t) = \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \log P_k(X_{1_t}, X_t), \tag{3.21}$$

combining equations (3.20) with (3.21), it follows from Lemma 2.1 that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \{\log P_k(X_{1_t}, X_t) - \sum_{j\in\mathcal{X}} P_k(X_{1_t}, j) \log P_k(X_{1_t}, j)\} = 0 \ a.e. \tag{3.22}$$

Now we can derive that

$$
\begin{aligned}
&|\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{j\in\mathcal{X}}P_k(X_{1_t},j)\log P_k(X_{1_t},j)-\sum_{i\in\mathcal{X}}\sum_{j\in\mathcal{X}}\pi_iP(i,j)\log P(i,j)|\\
&\le|\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i,j\in\mathcal{X}}\delta_i(X_{1_t})P_k(i,j)\log P_k(i,j)\\
&-\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i,j\in\mathcal{X}}\delta_i(X_{1_t})P(i,j)\log P(i,j)|\\
&+|\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{i,j\in\mathcal{X}}\delta_i(X_{1_t})P(i,j)\log P(i,j)-\sum_{i,j\in\mathcal{X}}\pi_iP(i,j)\log P(i,j)|\\
&\le\sum_{i,j\in\mathcal{X}}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}|P_k(i,j)\log P_k(i,j)-P(i,j)\log(i,j)|\\
&+\sum_{i,j\in\mathcal{X}}|P(i,j)\log P(i,j)|\cdot|\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1_t})-\pi_i|. \qquad \textbf{(3.23)}
\end{aligned}
$$

By the continuity of the function $f(x)=x\log x$ and equation (3.1), for any $i,j\in\mathcal{X}$, it follows that

$$\lim_{n\to\infty}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}|P_k(i,j)\log P_k(i,j)-P(i,j)\log P(i,j)|=0.$$

Since $\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\delta_i(X_{1_t})=NS_{\alpha_n,\varphi(n)-1}(i)$, it is also easy to see that the second term of last inequality of (3.23) tends to zero as n tends to infinity. Thus, the left-hand side in first inequality (3.23) comes to zero a.e. as n tends to infinity, i.e.,

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\sum_{j\in\mathcal{X}}P_k(X_{1_t},j)\log P_k(X_{1_t},j)\\
&=\sum_{i\in\mathcal{X}}\sum_{j\in\mathcal{X}}\pi_iP(i,j)\log P(i,j).\ a.e. \qquad \textbf{(3.24)}
\end{aligned}
$$

Combining with (1.4), (3.18), (3.22), (3.24), we arrive at

$$
\begin{aligned}
\lim_{n\to\infty}f_{\alpha_n,\phi(n)}(\omega)&=-\lim_{n\to\infty}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}[\log Pr(X_t,t\in L_{\alpha_n})+\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\log P_k(X_{1_t},X_t)]\\
&=-\lim_{n\to\infty}\frac{1}{|T_{(\alpha_n)}^{(\alpha_n+\phi(n))}|}\sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)}\sum_{t\in L_k}\log P_k(X_{1_t},X_t)
\end{aligned}
$$

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$$= -\lim_{n\to\infty} \frac{1}{|T^{(\alpha_n+\phi(n))}_{(\alpha_n)}|} \sum_{k=\alpha_n+1}^{\alpha_n+\phi(n)} \sum_{t\in L_k} \sum_{j\in\mathcal{X}} P_k(X_{1_t}, j) \log P_k(X_{1_t}, j)$$

$$= -\sum_{i\in\mathcal{X}} \sum_{j\in\mathcal{X}} \pi_i P(i,j) \log P(i,j) \ a.e., \quad \textbf{(3.25)}$$

where the second equation holds because of (3.18), the third equation can be obtained by (3.22), and the last equation holds by using (3.24). The proof of Theorem 3.2 is completed. ■

Remark 3.1: *In proof of Theorem 3.2, the condition (3.14) is new and necessary to proof the equation (3.18) which is a new prerequisite for delay version of AEP from zero-delay version. For case $\alpha_n \equiv 0$, we do not need to put the condition (3.14) on.*

Let $\alpha_n \equiv 0$ and $\phi(n) = n$, we can easily get the following results which have been proved in reference [15].

Corollary 3.1 (See [15]): *Let T be the homogeneous tree $T_{C,N}$, and $(X_t)_{t\in T}$ be a non-homogeneous Markov chain indexed by tree T with finite state space $\mathcal{X}$ defined as Definition 1.1, let*

$$S_n(i) = \sum_{t\in T^{(n)}} \delta_i(X_t), \quad \textbf{(3.26)}$$

$$S_n(i,j) = \sum_{t\in T^{(n)}/\{0\}} \delta_i(X_{1_t})\delta_j(X_t), \quad \textbf{(3.27)}$$

and $f_n(\omega)$ be defined as (1.5). Let $P = (P(i,j))_{i;j\in\mathcal{X}}$ be another finite transition matrix and be ergodic. If

$$\lim_{n\to\infty} P_n(i,j) = P(i,j), \quad \forall i,j \in \mathcal{X}, \quad \textbf{(3.28)}$$

then

$$(i) \quad \lim_{n\to\infty} \frac{S_n(i)}{|T^{(n)}|} = \pi_i, \ a.e. \ \forall i \in \mathcal{X}, \quad \textbf{(3.29)}$$

$$(ii) \quad \lim_{n\to\infty} \frac{S_n(i,j)}{|T^{(n)}|} = \pi_i P(i,j) \ a.e. \ \forall i,j \in \mathcal{X}, \quad \textbf{(3.30)}$$

$$(iii) \quad \lim_{n\to\infty} f_n(\omega) = -\sum_{i\in\mathcal{X}} \sum_{j\in\mathcal{X}} \pi_i P(i,j) \log P(i,j) \ a.e., \quad \textbf{(3.31)}$$

where π is the unique stationary distribution determined by the transition matrix P.

Now we give two examples which imply our conditions (3.1) and (3.14) respectively.

Example 3.1: *Condition* (3.1) *can be easily satisfied. For example, let $\mathcal{X} = \{1,2\}$, we consider the 2×2 transition matrices on S, for $n = 1,2,3,4,5\ldots$, let P_n and P are stochastic*

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matrices defined as follows

$$P_n = \begin{pmatrix} \frac{1}{2} - \frac{1}{2^n} & \frac{1}{2} + \frac{1}{2^n} \\ \frac{1}{2} + \frac{1}{2^n} & \frac{1}{2} - \frac{1}{2^n} \end{pmatrix},$$

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Obviously, P is ergodic, and its unique stationary distribution is $\pi = (\pi(1), \pi(2)) = (\frac{1}{2}, \frac{1}{2})$. It is obviously that $\lim_{n\to\infty} P_n(i,j) = P(i,j)$ for any $i, j \in \mathcal{X}$.

EXAMPLE 3.2: *If $\alpha_n \equiv 0$, condition (3.14) is satisfied naturally. Suppose that $\alpha_n \neq 0$ and $N = 1$, as we all have known that the nonhomogeneous chains indexed by trees reduce to nonhomogeneous Markov chains on line. Now let us come to construct a example of positive term series which satisfies condition (3.14). Letting $\mathcal{X} = \{1, 2, 3, \ldots, b\}$, for any positive integer $n \geq 1$, if $\alpha_n = \phi(n) = n$, then $a_n = n + 1$. By simple computation for any $\varepsilon > 0$, we have*

$$\frac{b^{N^{\alpha_{n+1}}}}{\exp(\varepsilon a_{n+1})} \Big/ \frac{b^{N^{\alpha_n}}}{\exp(\varepsilon a_n)} = \frac{1}{\exp(\varepsilon)} < 1, \qquad \textbf{(3.32)}$$

thus our condition (3.14) can be easily derived by D'Alembert convergence criteria for positive series.

REMARK 3.2: *For case $\alpha_n \neq 0$ and $N = 1$, our model is reduced to nonhomogeneous Markov chains on line. At the same time, our condition (3.14) is equivalent to condition (2.1) of Lemma 1 in reference [12].*

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