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ON PROPERTY B OF FAMILIES OF SETS

BY
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1. Introduction. A family $\mathcal{F}$ of sets is said to have property B if there exists a set S such that $S \cap F \neq \phi$ and $S \not\supset F$ for all $F \subset \mathcal{F}$. S is called a B-set for $\mathcal{F}$. Let $n \geq 2$ and $N \geq 2n-1$. Let $V = \{1, 2, \ldots, N\}$ and let $\mathcal{G} = \{G : G \subset V, |G| = n\}$. Erdös [3] defined $m_N(n)$ to be the size of a smallest subfamily of $\mathcal{G}$ which does not have property B and proved the following results:

$$m_{2q-1}(n) \geq m_{2q}(n) \geq 2^{n-1} \prod_{i=1}^{n-1} \left(1 + \frac{i}{2q-2i}\right) \tag{1}$$

and

$$m_{2q+1}(n) \leq m_{2q}(n) \leq q2^n \prod_{i=1}^{n-1} \left(1 - \frac{i}{2q-i}\right)^{-1}. \tag{2}$$

Erdös also pointed out that

$$m_{2n-1}(n) = m_{2n}(n) = \binom{2n-1}{n} \tag{3}$$

and remarked "I cannot compute $m_{2n+1}(n)$ and in fact do not know the value of $m_9(4)$".

In a recent paper [2] de Vries proved that

$$m_{2q-1}(n) \geq \binom{2q-1}{n-1} \Big/ \binom{q-1}{n-1} \tag{4}$$

which improves the lower bound for $m_{2q-1}(n)$ afforded by (1) by a factor $q-n+1$. From (4) it follows that

$$m_{2n+1}(n) \geq \frac{1}{n} \binom{2n+1}{n-1} \sim \frac{1}{n} \binom{2n+1}{n}. \tag{5}$$

The main result of [2] is the following theorem.

THEOREM (de Vries). *Equality holds in* (5) *if and only if there exists a Steiner system* $S(n-1, n, 2n+1)$.

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The only values of n for which such Steiner systems are known to exist are $n=3$ and $n=5$ so that $m_7(3)=7$, (this was known earlier) and $m_{11}(5)=66$.

We shall be concerned mainly with getting upper bounds for $m_{2n+1}(n)$. We remark that (2) is effective only when q is large compared to n, say $q>cn$ for some constant $c>1$. In fact, one may readily check that the right side of (2) reduces to $2n\binom{2n-1}{n}$ when $q=n$. Our main result is contained in the following theorem:

THEOREM 1. *For every $\varepsilon>0$ and all sufficiently large n,*

$$m_{2n+1}(n)<\binom{2n+1}{n}\frac{(1+\varepsilon)\log n}{n}. \tag{6}$$

Thus our upper bound differs from the lower estimate given by (5) *by a factor of* log n.

The upper bound given by (6) holds only for all sufficiently large n. It is natural to ask whether the trivial upper bound $m_{2n+1}(n)\le\binom{2n-1}{n}$, which follows from (3) and the fact that $m_N(n)$ is non-increasing in N, can be improved for all $n\ge3$. We shall show that this is indeed the case.

THEOREM 2. *There exists a constant $c<1$ such that for all $n\ge3$*

$$m_{2n+1}(n)<c\binom{2n-1}{n}.$$

Our final result concerns $m_9(4)$. It is implicit in [1] that $m_9(4)\le27$, and it follows from (4) that $m_9(4)\ge21$. Moreover, since it is known that there is no Steiner system $S(3,4,9)$, we have, from the theorem of de Vries, $m_9(4)\ge22$. We shall narrow the gap slightly by proving the following result:

THEOREM 3. $24\le m_9(4)\le26$.

2. Proof of theorem 1. We first state a lemma which will be our main tool in the proof.

LEMMA. *Let A and B be non-empty finite sets. Let R be a relation from A to B such that for each $a\in A$ there exists $b\in B$ such that aRb and for each $b\in B$, there exists $a\in A$ such that aRb. For $a\in A$ let $B_a=\{b:b\in B, aRb\}$ and for $b\in B$ let $A_b=\{a:a\in A, aRb\}$. Let $u=\min\{|A_b|:b\in B\}$ and $v=\max\{|B_a|:a\in A\}$. Then there exists a set $A'\subset A$ such that for all $b\in B$, $b\in B_a$ for some $a\in A'$ and such that*

$$|A'|\le\frac{|A|}{u}(1+\log v).$$

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We do not prove this lemma since various formulations of it, which are essentially equivalent to that given above, have appeared recently in the literature (see, for example [4] or [5]). We only remark that the set A' is obtained by a type of greedy algorithm; that is, the elements of A' are selected one at a time by making the most efficient choice at each stage. Thus the set A' is not explicitly given.

We return now to the proof of Theorem 1. Let $V=\{1,2,\ldots,2n+1\}$, $\mathcal{G}=\{G:G\subset V,|G|=n\}$ and $\mathcal{H}=\{H:H\subset V,|H|=n-1\}$. In the lemma take $A=B=\mathcal{G}$ and let aRb mean $a=b$ or $a\cap b=\phi$. Then $u=v=n+2$ and denoting A' in this case by $\mathcal{F}_n$ we have

$$|\mathcal{F}_n|\le\binom{2n+1}{n}\frac{1+\log(n+2)}{n+2}. \tag{7}$$

Next, in the lemma, take $A=\mathcal{G}$, $B=\mathcal{H}$ and let aRb mean $a\cap b=\phi$. Then $u=\binom{n+2}{2}$ and $v=\binom{n+1}{2}$ and denoting A' in this case by $\mathcal{F}_{n-1}$ we get

$$|\mathcal{F}_{n-1}|\le\binom{2n+1}{n}\frac{\left(1+\log\binom{n+2}{2}\right)}{\binom{n+2}{2}}. \tag{8}$$

Let $\mathcal{F}=\mathcal{F}_n\cup\mathcal{F}_{n-1}$. We claim that $\mathcal{F}$ does not have property B. Suppose, to the contrary, $\mathcal{F}$ has a B-set S. We may assume $|S|\le n$, since otherwise we take its complement. If $|S|=n$ then $S\in\mathcal{G}$ and $S=F$ or $S\cap F=\phi$ for some $F\in\mathcal{F}_n$. If $|S|=n-1$, then $S\in\mathcal{H}$ and $S\cap F=\phi$ for some $F\in\mathcal{F}_{n-1}$. Finally if $|S|<n-1$ then $S\subset H$ for some $H\in\mathcal{H}$ and consequently $|S\cap F|=\phi$ for some $F\in\mathcal{F}_{n-1}$. Thus we get a contradiction in all cases. Hence $\mathcal{F}$ does not have property B. It is an immediate consequence of (7) and (8) that

$$|\mathcal{F}|<\binom{2n+1}{n}\frac{(1+\varepsilon)\log n}{n}$$

for every $\varepsilon>0$, provided $n\ge n_0(\varepsilon)$. This completes the proof of Theorem 1.

Remark. It is clear from (7) and (8) that $\mathcal{F}_{n-1}$ forms a very small part of $\mathcal{F}$. It would be of interest to know whether $\mathcal{F}_n$ may be chosen so that it does not have property B. We have not been able to decide this.

3. Proof of theorem 2. We shall prove that for $n\ge 6$,

$$m_{2n+1}(n)\le\binom{2n-1}{n}\frac{4n^2-10n+3}{4n^2-8n+3}. \tag{9}$$

It follows from (9), Theorem 1, and the fact that $m_7(3)=7<\binom{5}{3}$, $m_9(4)\le 26\le\binom{7}{4}$ and $m_{11}(5)=66<\binom{9}{5}$ that Theorem 2 holds.

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Let $A=\{1,2,\ldots,2n-5\}$, $B=\{2n-4,2n-3,2n-2\}$ and $C=\{2n-1,2n,2n+1\}$. Let $V=A\cup B\cup C$ and let $\mathscr{G}=\{G:G\subset V,|G|=n\}$. A set $G\subset V$ is said to be of type (x, y, z) if $|G\cap A|=x$, $|G\cap B|=y$ and $|G\cap C|=z$. The collection of all $G\in\mathscr{G}$ of type (x, y, z) will be denoted by $[x, y, z]$. It will be convenient to describe matters graph theoretically. Let $\hat{\mathscr{G}}$ be the graph (Fig. 1) whose vertices are the triples $[x, y, z]$, two distinct vertices $[x, y, z]$ and $[x', y', z']$ being joined by an edge if $x+x'\le 2n-5$, $y+y'\le 3$, and $z+z'\le 3$. Note then that if $G\in[x, y, z]$, there exists $G'\in[x', y', z']$ such that $G\cap G'=\phi$.

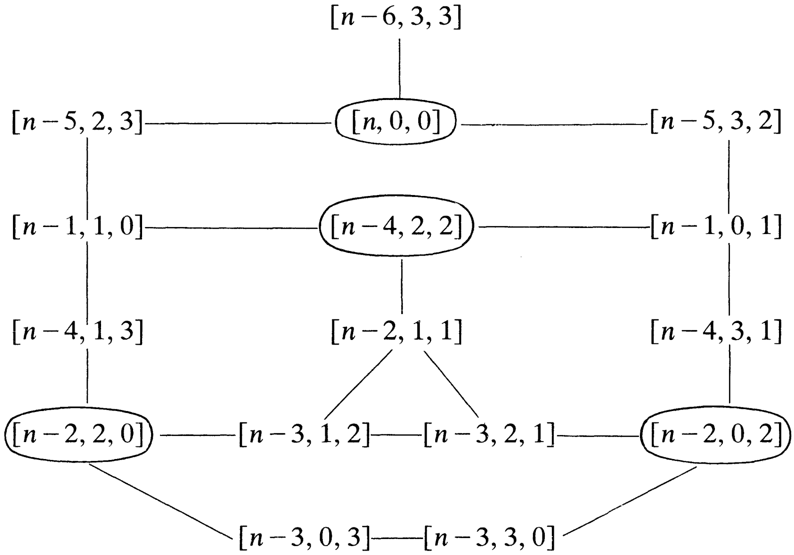

Figure 1.

Let $\mathscr{F}=[n,0,0]\cup[n-4,2,2]\cup[n-2,2,0]\cup[n-2,0,2]$ and let $\hat{\mathscr{F}}$ be the corresponding vertices of $\hat{\mathscr{G}}$. $\hat{\mathscr{F}}$ thus consists of the circled vertices in Fig. 1. Note that every other vertex of $\hat{\mathscr{G}}$ is adjacent to a vertex in $\hat{\mathscr{F}}$.

We claim that $\mathscr{F}$ does not have property B. Suppose, to the contrary, that $\mathscr{F}$ has a B-set S. We may assume $|S|=n$ or $|S|=n-1$.

Suppose $|S|=n$. Then $S\in[x, y, z]$ for some vertx $[x, y, z]$ of $\hat{\mathscr{G}}$. Then it is clear from the graph theoretic set up that either $S\in\mathscr{F}$ or $S\cap F=\phi$ for some $F\in\mathscr{F}$.

Suppose now that $|S|=n-1$. Then S is of type $(x-1, y, z)$ where $[x, y, z]$ is a vertex of $\hat{\mathscr{G}}$. If $[x, y, z]$ does not belong to $\hat{\mathscr{F}}$ then $S\subset G$ for some $G\in[x, y, z]$. Since $G\cap F=\phi$ for some $F\in\mathscr{F}$ we have $S\cap F=\phi$ also. If $[x, y, z]$ belongs to $\hat{\mathscr{F}}$ then S is of one of the types $(n-1,0,0)$, $(n-5,2,2)$, $(n-3,2,0)$ or $(n-3,0,2)$. Here we have $S\cap F=\phi$ for some F in $[n-4,2,2]$, $[n,0,0]$, $[n-2,0,2]$ or $[n-2,2,0]$ respectively. Thus $\mathscr{F}$ does not have property B.

Now

$$|\mathscr{F}|=|[n,0,0]|+|[n-4,2,2]|+|[n-2,2,0]|+|[n-2,0,2]|$$
$$=\binom{2n-5}{n}+9\binom{2n-5}{n-4}+6\binom{2n-5}{n-2}$$
$$=\binom{2n-1}{n}\frac{4n^2-10n+3}{4n^2-8n+3}.$$

This comples the proof of (9) and hence, at the same time, the proof of Theorem 2.

REMARK. We have a number of similar constructions which result from taking different partitions of V. However, we do not present these here since, for large values of n, the bounds obtained are much inferior to (6). In fact, the only merit of Theorem 2 is that it improves the trivial bound for all $n \geq 3$.

4. Proof of theorem 3. We do not give all of the details, but we give an outline of the method used.

Let $V=\{1,2,3,4,5,6,7,8,9\}$ and let $\mathscr{G}$ be the family of 4-subsets of V. Suppose there exists a subfamily $\mathscr{F}$ and $\mathscr{G}$ which does not have property B and suppose $|\mathscr{F}|=23$. Each member of $\mathscr{F}$ is disjoint from five members of $\mathscr{G}$. If every member of $\mathscr{F}$ were disjoint from some other member of $\mathscr{F}$, there would be at most $4|\mathscr{F}|=92$ members of $\mathscr{G}-\mathscr{F}$ which are disjoint from some member of $\mathscr{F}$. Since $|\mathscr{G}-\mathscr{F}|=126-23>92$ there would be a member of $\mathscr{G}-\mathscr{F}$ which meets each member of $\mathscr{F}$ and is thus a B-set for $\mathscr{F}$. Thus we may suppose that there is a set $A \in \mathscr{F}$ which meets all other members of $\mathscr{F}$. Without loss of generality we may take $A=\{1,2,3,4\}$.

We say that a member G of $\mathscr{G}$ is of type r, $0 \leq r \leq 4$, if $|G \cap A|=r$. Let $\mathscr{G}_r$ be the set of members of $\mathscr{G}$ of type r. One may readily verify that

$$|\mathscr{G}_0|=5,\ |\mathscr{G}_1|=40,\ |\mathscr{G}_2|=60,\ |\mathscr{G}_3|=20,\ |\mathscr{G}_4|=1.$$

We make the following observations:

(a) Each member of $\mathscr{G}_1$ is disjoint from two members of $\mathscr{G}_3$ and three members of $\mathscr{G}_2$.

(b) Each member of $\mathscr{G}_2$ is disjoint from two members of $\mathscr{G}_1$ and three other members of $\mathscr{G}_3$.

(c) Each member of $\mathscr{G}_3$ is disjoint from one member of $\mathscr{G}_0$ and four members of $\mathscr{G}_1$.

For $0 \leq r \leq 4$ let

$$h_r=|\mathscr{F} \cap \mathscr{G}_r|.$$

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We have $h_4=1$ and $h_0=0$ so that

$$h_1+h_2+h_3=22. \tag{10}$$

In order to account for all of the sets in $\mathcal{G}_1$ we must have, by observations (b) and (c),

$$h_1+2h_2+4h_3 \geq |\mathcal{G}_1|=40. \tag{11}$$

In order to account for all of the sets in $\mathcal{G}_2$ we must have, by observations (a) and (b)

$$3h_1+4h_2 \geq |\mathcal{G}_2|=60. \tag{12}$$

Similarly, in order to account for all of the sets in $\mathcal{G}_3$ we must have, by observation (a)

$$2h_1+h_3 \geq |\mathcal{G}_3|=20. \tag{13}$$

One may readily check that the only triples (h_1, h_2, h_3) which satisfy (10), (11), (12), and (13) are $(12, 6, 4)$, $(11, 7, 4)$, $(10, 9, 3)$, $(10, 8, 4)$, $(9, 10, 3)$, $(9, 9, 4)$, $(8, 10, 4)$ and $(8, 9, 5)$.

In the case $(h_1, h_2, h_3)=(12, 6, 4)$ we note that (12) holds with equality. Since $h_1=12>\binom{5}{3}=10$ there must be a 3-subset of $\{5, 6, 7, 8, 9\}$, say $\{5, 6, 7\}$, such that at least two of $\{1, 5, 6, 7\}$, $\{2, 5, 6, 7\}$, $\{3, 5, 6, 7\}$ and $\{4, 5, 6, 7\}$ are in $\mathcal{G}_1 \cap \mathcal{F}$. Without loss of generality, we may take these to be $\{1, 5, 6, 7\}$ and $\{2, 5, 6, 7\}$. However both of these sets are disjoint from $\{3, 4, 8, 9\} \in \mathcal{G}_2$. This contradicts the fact that (12) must hold with equality. Thus $(h_1, h_2, h_3)=(12, 6, 4)$ is ruled out.

Similar, although sometimes more complicated, arguments may be used to rule out the remaining seven cases. We suppress these details.

We have not been able to carry out the above analysis in the cases $|\mathcal{F}|=24$ or 25. However, in trying to apply it to the case $|\mathcal{F}|=26$, we found the following family of 26 sets which does not have property B, thus showing $m_9(4) \leq 26$.

$$\begin{gathered}\{1234\}\{1235\}\{1236\}\{1237\}\{1247\}\{1256\}\{1289\}\{1347\}\\ \{1356\}\{1389\}\{1458\}\{1469\}\{1579\}\{1678\}\{2347\}\{2356\}\\ \{2389\}\{2458\}\{2469\}\{2579\}\{2678\}\{3458\}\{3469\}\{3579\}\\ \{3678\}\{4567\}.\end{gathered}$$

This completes the proof of Theorem 3.

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