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RUIN THEORY IN A DISCRETE TIME RISK MODEL WITH INTEREST INCOME

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ABSTRACT

In this paper we consider a discrete time insurance risk model with interest income. Using the recursive calculation method of De Vylder & Goovaerts (1988), recursive equations for the finite time ruin probabilities and the distribution of the time of ruin are derived. Fredholm type integral equations for the ultimate ruin probability, the distribution of the severity of ruin, the joint distribution of surplus before and after ruin, and the probability of absolute ruin are obtained. Numerical results are included.

KEYWORDS

Ruin Probability; Severity of Ruin; Absolute Ruin Probability; Interest Income; Recursive Calculation; Fredholm Type Integral Equation

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1. INTRODUCTION

The compound Poisson model is the most commonly used in actuarial science. Various methods have been developed to deal with the ruin probability and related problems on this model. Some examples of these are the recursive calculation, upper bounds, asymptotic results and renewal equations. Over the past two decades people in actuarial science have also started paying attention to the severity of ruin. Gerber, Goovaerts & Kaas (1987) considered the distribution of the severity of ruin, and an integral equation for the mentioned distribution was obtained (see also Panjer & Willmot,1992; and Dickson & Egidio dos Reis, 1994). In cases where the claims have an exponential-mixture or Gamma-mixture distribution, closed form solutions for the distribution of severity of ruin were obtained in the same paper. Later, Dufresne & Gerber (1988) introduced the distribution of the surplus immediately prior to ruin in the classical compound Poisson risk model. Similar results to those in Gerber, Goovaerts & Kaas (1987) were obtained in that paper. Gerber & Shiu (1997, 1998) examined the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. They showed that, as a function of the initial surplus, the joint density of the surplus immediately before ruin and the deficit at ruin satisfies a renewal equation. A recent paper by Li & Garrido (2002) can be viewed as the discrete version of Gerber & Shiu (1998). For other references see, for example, Dickson (1989, 1993), Dickson & Waters (1992, 1999), Dickson & Egidio dos Reis (1995), Egidio dos Reis (2000), Lin & Willmot (2000) and Willmot (2000).

In comparison to the huge amount of literature on the compound Poisson model, there are relatively few papers dealing with the discrete time model. Cheng, Gerber & Shiu (2000) considered the severity of ruin under a compound binomial process. In De Vylder & Goovaerts (1988) a simple recursive method was used to derive some recursive formulae for the finite time ruin probability in a discrete model. Yang (1999) considered a discrete time model with interest income, which we will also do in this paper. By using the method in De Vylder & Goovaerts (1988), recursive formulae for the finite time ruin probability and distribution of ruin time are derived. Fredholm type integral equations satisfied by the ultimate ruin probability, severity of ruin, joint distribution of surplus before and after ruin, and probability of absolute ruin are also obtained.

The rest of this paper is constructed as follows. Section 2 considers the finite time ruin probability. Recursive formulae are obtained in the models with interest income. Section 3 derives the recursive formulae for the time of ruin. The Fredholm type equation is derived for the ultimate ruin probability. In Section 4 we obtain the integral equation for the joint distribution of surplus before and after ruin. Section 5 discusses the probability of absolute ruin. Some numerical results are presented in Section 6.

2. Finite Time Ruin Probability

In this section we derive recursive equations for calculating the finite time ruin probabilities in the models with interest income. Let r be the compound interest rate. We will assume that r is a non-negative constant. The dynamic of the surplus is given by:

$$U_n(u) = u(1+r)^n + \sum_{i=1}^{n} X_i(1+r)^{n-i+1} - \sum_{i=1}^{n} Y_i(1+r)^{n-i} \tag{1}$$

where $n = 1, 2, \ldots$ and $U_0(u) = u$ is the initial surplus, X_i is a sequence of independent and identically distributed (i.i.d.) non-negative random variables

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and denotes the premium collected during time interval $[i-1, i)$ or ith year, Y_i is a sequence of i.i.d. non-negative random variables, independent of X_i, and denotes the claim amount during the time interval $[i-1, i)$ or ith year. We assume that the expectations of X_1 and Y_1 are finite, and denote their distribution function by $F_X(x)$ and $F_Y(y)$ respectively. Here we assume that the premium is paid at the beginning of the time period and that the claim is paid at the end. The premium and claim random variables X_i and Y_i can be discrete or not.

In the model here we assume the premiums to be random. The random premium model has been used in some papers. See, for example, Willmot (1996) and Yang (1999). This model is a natural extension of the classical model where the premium is set as a constant over time. In practice the premium rate will be affected by many factors, such as political events, economic environment changes, inflation rate, etc. Therefore, a random premium is more realistic. Of course the claim random variable cannot be independent of the premium random variable in practice. However, if we let $X_i = E[Y_i] + \delta_i$, where δ_i is i.i.d. and independent with Y_i, then our model makes sense. In the classical model we need to assume that the risk loading is positive in order to exclude the trivial case where the ruin probability is 1. In our model, since the interest effect is included, the ruin probability will not be equal to 1 even where the risk loading is negative, therefore we do not have to make the positive risk loading assumption (by assuming that δ_i are non-negative, the risk loading will be positive). In this case, as will be proved in Section 5, the surplus process may or may not return to positive if ruin occurs.

Write:

$$Z_i = Y_i - (1+r)X_i.$$

$\{Z_i, i \geq 1\}$ are then i.i.d. random variables. We denote the distribution function of Z_i by:

$$\begin{aligned} G(u) &= P\{Z_i \leq u\} \\ &= P\{Y_i - (1+r)X_i \leq u\} \\ &= \int_0^\infty F_Y(u + (1+r)x)\, dF_X(x). \end{aligned}$$

Then we can rewrite model (1) as:

$$U_n(u) = u(1+r)^n - (1+r)^n S_n$$

where $S_n = \sum_{i=1}^n \frac{Z_i}{(1+r)^i}$. Let the stopping time T be the time of ruin:

$$T = \min\{n > 0 : U_n < 0\}.$$

Define the probability of ruin before or at time n as:

$$\psi_n(u) = P\{T \leq n\}.$$

The probability of non-ruin before or at time n is then:

$$\phi_n(u) = 1 - \psi_n(u). \tag{2}$$

By using the recursive method, we have:

$$\begin{aligned}\phi_1(u) &= P\{U_1(u) \geq 0\} = P\{Z_1 \leq u(1+r)\} \\ &= G(u(1+r))\end{aligned} \tag{3}$$

and

$$\phi_n(u) = P\{T > n\} = \int_{-\infty}^{u(1+r)} \phi_{n-1}(u(1+r) - y)dG(y) \quad (n \geq 2). \tag{4}$$

By using a similar argument to that in De Vylder & Goovaerts (1988), we have the recursive equations for the ruin probabilities:

$$\psi_1(u) = 1 - G(u(1+r)) = \bar{G}(u(1+r))$$
$$\psi_n(u) = \bar{G}(u(1+r)) + \int_{-\infty}^{u(1+r)} \psi_{n-1}(u(1+r) - y)dG(y) \quad (n \geq 2).$$

When X and Y are discrete random variables and

$$P\{X = x_i\} = g_i \quad (i = 0, 1, 2, \cdots)$$
$$P\{Y = y_j\} = h_j \quad (j = 0, 1, 2, \cdots)$$

we have:

$$P\{Z = z_k\} = \sum_{y_j - (1+r)x_i = z_k} g_i h_j = p_k.$$

The corresponding results of the non-ruin and ruin probabilities are:

$$\phi_1(u) = \sum_{z_k \le u(1+r)} p_k \tag{5}$$

$$\phi_n(u) = \sum_{z_k \le u(1+r)} \phi_{n-1}(u(1+r) - z_k)p_k \tag{6}$$

and

$$\psi_1(u) = \sum_{z_k > u(1+r)} p_k \tag{7}$$

$$\psi_n(u) = \psi_1(u) + \sum_{z_k \le u(1+r)} \psi_{n-1}(u(1+r) - z_k)p_k \quad (n \ge 2). \tag{8}$$

If we set $r = 0$, the model above becomes the classical discrete insurance risk model and the recursive equations above become the same as those in De Vylder & Goovaerts (1988).

3. Distribution of the Time of Ruin

In this section we derive a recursive equation for the probability function of the time of ruin. Consider model (1) and denote the probability function of the ruin time T by:

$$Q_n(u) = P\{T = n\}. \tag{9}$$

Then:

$$\begin{aligned} Q_1(u) &= P\{T = 1\} = P\{T > 0\} - P\{T > 1\} \\ &= 1 - G(u(1+r)) = \bar{G}(u(1+r)). \end{aligned} \tag{10}$$

Recursively, for $n \ge 2$, we have:

$$\begin{aligned} Q_n(u) &= E[P\{T = n|Z_1\}] = E[Q_{n-1}(u - Z_1)] \\ &= \int_{-\infty}^{u(1+r)} Q_{n-1}(u(1+r) - y)dG(y). \end{aligned} \tag{11}$$

Remark. From the definition, we know that the probability of ultimate ruin is given by:

$$\psi(u) = p\{T < \infty\} = \sum_{n=1}^{\infty} Q_n(u).$$

It is easy to see that $\psi(u)$ satisfies the following Fredholm type integral equation:

$$\psi(u) = \bar{G}(u(1+r)) + \int_{-\infty}^{u(1+r)} \psi(u(1+r)-y)dG(y),$$
$$= \bar{G}(u(1+r)) - \int_{0}^{\infty} \psi(s)dG(u(1+r)-s). \tag{12}$$

It is well known that, in the classical compound Poisson model, the ultimate ruin probability satisfies a Volterra integral equation of the second kind (see Panjer & Willmot, 1992, p382). However, in the discrete time model we see that the ultimate ruin probability satisfies a Fredholm type equation.

If both X and Y are discrete, we have:

$$Q_1(u) = \sum_{z_k > u(1+r)} p_k \tag{13}$$

$$Q_n(u) = \sum_{z_k \le u(1+r)} Q_{n-1}(u(1+r)-z_k)p_k. \tag{14}$$

4. The Joint Distribution of Surplus Immediately Before and After Ruin

In this section we derive the integral equations satisfied by the joint distribution of the surplus immediately before and after ruin. The joint distribution of the surplus before and after ruin is defined as:

$$W(u, y, x) = P\{U_T \le -y, U_{T-1} > x, T < \infty | U_0 = u\} \tag{15}$$

where $x > 0$ and $y > 0$. It is not difficult to see that:

$$W(u, y, x) = \sum_{n=1}^{\infty} P\{U_T \le -y, U_{T-1} > x, T = n | U_0 = u\}$$
$$= \sum_{n=1}^{\infty} P\left\{S_n \ge u + \frac{y}{(1+r)^n}, S_{n-1} < u - \frac{x}{(1+r)^{n-1}}, S_{n-2} \le u, \ldots, S_1 \le u\right\}$$
$$= \sum_{n=1}^{\infty} A_n(u, y, x). \tag{16}$$

Using the recursive method, we have:

$$A_1(u,y,x) = \begin{cases} \bar{G}^{+}(u(1+r)+y), & \text{if} \quad x \le u \\ 0, & \text{if} \quad x > u \end{cases}$$

$$A_n(u,y,x) = \int_{-\infty}^{u(1+r)} A_{n-1}(u(1+r)-s,y,x)dG(s) \quad (n \ge 2) \tag{17}$$

where $\bar{G}^{+}(x) = P(Z \ge x)$. Therefore, the expression of $W(u,y,x)$ is given when $x \le u$ by:

$$\begin{aligned} W(u,y,x) &= \sum_{n=1}^{\infty} A_n(u,y,x) = A_1(u,y,x) + \sum_{n=2}^{\infty} A_n(u,y,x) \\ &= \bar{G}^{+}(u(1+r)+y) + \sum_{n=2}^{\infty} \int_{-\infty}^{u} A_{n-1}(u(1+r)-s,y,x)dG(s) \\ &= \bar{G}^{+}(u(1+r)+y) + \int_{-\infty}^{u(1+r)} W(u(1+r)-s,y,x)dG(s) \end{aligned} \tag{18}$$

and when $x > u$ by:

$$\begin{aligned} W(u,y,x) &= \sum_{n=1}^{\infty} A_n(u,y,x) \\ &= A_2(u,y,x) + \sum_{n=3}^{\infty} A_n(u,y,x) \\ &= \int_{-\infty}^{u(1+r)} W(u(1+r)-s,y,x)dG(s). \end{aligned} \tag{19}$$

That is, $W(u,y,x)$ is the solution of the following Fredholm type integral equation:

$$W(u,y,x) = H(u(1+r),y,x) - \int_{0}^{\infty} W(s,y,x)dG(u(1+r)-s) \tag{20}$$

where:

$$H(u(1+r),y,x) = \begin{cases} \bar{G}^{+}(u(1+r)+y) & \text{if} \quad x \le u \\ 0 & \text{if} \quad x > u \end{cases}$$

and

$$\bar{G}^{+}(x) = P(Z \geq x).$$

Remark. Using the same notation as in Section 2, the distribution of the severity at ruin is given by:

$$V(u, y) = P\{U_T \leq -y, T < \infty | U_0 = u\}. \tag{21}$$

If we let $x \to -\infty$ in (20), we obtain the integral equation satisfied by $V(u, y)$:

$$V(u, y) = \bar{G}^{+}(u(1+r)+y) - \int_0^\infty V(s, y) dG(u(1+r)-s). \tag{22}$$

5. Absolute Ruin Probability

In this section we consider the following model:

$$U_n = u(1+r)^n + c\sum_{i=1}^{n}(1+r)^{n+1-i} - \sum_{i=1}^{n} Y_i(1+r)^{n-i} \tag{23}$$

where c is a constant which denotes the premium income in one time interval. All other notation is the same as before.

If the surplus falls below $-\frac{c(1+r)}{r}$, then even if there is no claim occurring during the next time interval, the surplus process will not increase. This is due to the fact that the premium income will be less than the interest paid for the debt. In this case, we call the following probability:

$$\Psi(\tilde{u}) = P\left\{\bigcup_{n=1}^{\infty}\left(U_n < -\frac{c(1+r)}{r}\right) | U_0 = u\right\} \tag{24}$$

the probability of absolute ruin, where:

$$\tilde{u} = u + \frac{c(1+r)}{r}.$$

In Dassios & Embrechts (1989), the above probability was defined and studied for a piecewise-deterministic Markov process model. The concept of absolute ruin was introduced there. Define:

$$\Psi_c(u) = P\{\tilde{T} < \infty\}$$

where:

$$\tilde{T} = \min\left\{n > 0 : U_n < -\frac{c(1+r)}{r}\right\}.$$

It is easy to see that $\Psi_c(u) = \Psi_0(u + c\frac{(1+r)}{r})$ and $\Psi_0(u) = \psi(u)$. $\Psi_0(u)$ has been obtained in Section 3. Also define:

$$\Psi_{cn}(u) = P\{\tilde{T} \leq n\}. \tag{25}$$

Then $\Psi_{cn}(u) = \psi_n(u + c\frac{1+r}{r})$, which has been discussed in Section 2. Similarly, we can discuss the distribution of the absolute ruin time.

6. Numerical Examples

In this section we provide some numerical results to illustrate the methods of this paper. First, we give an algorithm to explain how to solve the Fredholm type equation. In the literature there are many works on the numerical methods for solving Fredholm type equations. See, for example, Delves & Mohamed (1985).

Considering a Fredholm type integral equation:

$$f(t) = \lambda \int_a^b K(t, s) f(s) ds + g(t). \tag{26}$$

The method we describe here is called the Nystrom method. This method requires the choice of some approximate quadrature rule:

$$\int_a^b y(s) ds = \sum_{j=1}^{N} w_j y(s_j). \tag{27}$$

Here the set $\{w_j\}$ is the weights of the quadrature rule, while the N points $\{s_j\}$ are the abscissae.

We will see that the numerical calculation involves $O(N^3)$ operations, so an efficient method of solving the Fredholm integral equation is by using a high-order quadrature rule (such as the Gauss quadrature rule) to keep N as small as possible.

If we apply the quadrature rule (27) to equation (26), we obtain:

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$$f(t) = \lambda \sum_{j=1}^{N} w_j K(t, s_j) f(s_j) + g(t). \tag{28}$$

Evaluating equation (28) at the quadrature points:

$$f(t_i) = \lambda \sum_{j=1}^{N} w_j K(t_i, s_j) f(s_j) + g(t_i). \tag{29}$$

Let f be the vector with the ith component being $f(t_i)$, g be the vector with the ith component being $g(t_i)$, K be the matrix with the ij element being $K(t_i, s_j)$, and define:

$$\tilde{K}_{ij} = K(t_i, s_j) w_j.$$

Then, in matrix notation, equation (29) becomes $(1 - \lambda \tilde{K}) \cdot f = g$, where $\tilde{K}$ is a matrix with the ij element being $\tilde{K}_{ij}$. This is a system of N linear algebraic equations with N unknowns. This system can be solved by standard triangular decomposition techniques. From this, we know where the $O(N^3)$ operations come from.

In the following, we will assume that both the claim random variable Y and the premium random variable X are discrete random variables. Table 1 presents the distributions of the premium random variable X and the claim random variable Y. Table 2 provides the ruin time probability function, which was calculated by using equations (10) and (11), when $r = 0$. Table 3 presents the distribution of ruin probability when $r = 0$. Table 4 is calculated from equation (12) with $r = 0$. Tables 5-9 present the corresponding results for the models with interest rate $r > 0$. Tables 10 and 11 were obtained using equation (22). Tables 12 and 13 were obtained by solving equation (20). In Tables 14 and 15 we consider the probability of absolute ruin. The claim random variable Y is the same as before, but we use a constant premium rate of $c = 2.1$ instead of a random premium in this case. The distribution of the time of absolute ruin and the probability of absolute ruin are given in Tables 14 and 15, respectively.

Table 1. The distributions of X and Y

$x_k =$	1.2	1.8	2.1	2.5	3.1	
$g_k =$	0.1	0.2	0.3	0.3	0.1	
$y_k =$	1.5	2.2	2.6	2.8	3	3.2
$h_k =$	0.35	0.3	0.2	0.05	0.05	0.05

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Table 2. Distribution of the ruin time when $r = 0$

$P(T = n)$	$u = 0.8$	$u = 1.1$	$u = 1.4$	$u = 1.7$
$n = 1$	0.125000	0.055000	0.015000	0.005000
$n = 2$	0.090900	0.074450	0.051925	0.028275
$n = 3$	0.056215	0.053237	0.046229	0.033075
$n = 4$	0.037811	0.038756	0.036335	0.029793
$n = 5$	0.027813	0.029630	0.028993	0.025415
$n = 6$	0.021496	0.023468	0.023489	0.021434
$n = 7$	0.017172	0.019035	0.019310	0.018081
$n = 8$	0.014020	0.015695	0.016055	0.015289
$n = 9$	0.011613	0.013085	0.013457	0.012957
$n = 10$	0.009713	0.010991	0.011343	0.011001

Table 3. Finite time ruin probabilities when $r = 0$

$P(T \leq n)$	$u = 0.8$	$u = 1.1$	$u = 1.4$	$u = 1.7$
$n = 1$	0.125000	0.055000	0.015000	0.005000
$n = 2$	0.215900	0.129450	0.066925	0.033275
$n = 3$	0.272115	0.182687	0.113154	0.066350
$n = 4$	0.309926	0.221443	0.149489	0.096143
$n = 5$	0.337739	0.251073	0.178482	0.121558
$n = 6$	0.359235	0.274541	0.201971	0.142992
$n = 7$	0.376407	0.293576	0.221281	0.161073
$n = 8$	0.390427	0.309271	0.237336	0.176362
$n = 9$	0.402040	0.322356	0.250792	0.189319
$n = 10$	0.411753	0.342266	0.262135	0.200320

Table 4. Ultimate ruin probabilities when $r = 0$

$u = 0.8$	$\psi(0.8) = 0.691289$
$u = 1.1$	$\psi(1.1) = 0.620652$
$u = 1.4$	$\psi(1.4) = 0.582348$
$u = 1.7$	$\psi(1.7) = 0.514321$

From Table 2 we can see that, unlike the ruin probabilities and finite time ruin probabilities (see Tables 3 and 4), the probability of $P(T = n)$ is not a decreasing function of u.

Table 3 tells us that the finite time ruin probabilities are decreasing as the initial surplus increases, as expected.

Similar to before, as shown in Table 4, the ultimate ruin probabilities decrease as the initial surplus increases.

If we compare Tables 2 to 5, we can see that when interest r increases from 0 to 0.03, the probability of ruin is decreasing most of the time. This is reasonable, since, when interest is included, the initial surplus will grow as time goes on. Economically, this means that the investment incomes will help

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Table 5. Distribution of the ruin time when $r = 0.03$

$P(T = n)$	$u = 0.5$	$u = 0.8$	$u = 1.1$	$u = 1.4$
$n = 1$	0.180000	0.110000	0.045000	0.010000
$n = 2$	0.098900	0.088825	0.069600	0.041475
$n = 3$	0.057224	0.056409	0.051429	0.038299
$n = 4$	0.037745	0.038737	0.037462	0.031097
$n = 5$	0.026851	0.028288	0.028229	0.024954
$n = 6$	0.020095	0.021543	0.021946	0.020186
$n = 7$	0.015579	0.016898	0.017462	0.016487
$n = 8$	0.012379	0.013535	0.014127	0.013575
$n = 9$	0.010008	0.011002	0.011563	0.011245
$n = 10$	0.008189	0.009036	0.009542	0.009355

Table 6. Finite time ruin probabilities when $r = 0.03$

$P(T \leq n)$	$u = 0.5$	$u = 0.8$	$u = 1.1$	$u = 1.4$
$n = 1$	0.180000	0.110000	0.045000	0.010000
$n = 2$	0.278900	0.198825	0.114600	0.051475
$n = 3$	0.336124	0.255234	0.166029	0.089704
$n = 4$	0.373869	0.293971	0.203491	0.120801
$n = 5$	0.400720	0.322259	0.231720	0.145755
$n = 6$	0.420815	0.343802	0.253666	0.165941
$n = 7$	0.436394	0.360700	0.271128	0.182428
$n = 8$	0.448773	0.374235	0.285255	0.196003
$n = 9$	0.458781	0.385237	0.296818	0.207248
$n = 10$	0.466970	0.394273	0.306360	0.216603

Table 7. Distribution of the ruin time when $r = 0.04$

$P(T = n)$	$u = 0.8$	$u = 1.1$	$u = 1.4$	$u = 1.7$
$n = 1$	0.110000	0.045000	0.010000	0.005000
$n = 2$	0.087075	0.065925	0.036500	0.022800
$n = 3$	0.054655	0.048829	0.034096	0.027309
$n = 4$	0.036774	0.035003	0.027987	0.024270
$n = 5$	0.026424	0.026062	0.022502	0.020305
$n = 6$	0.019859	0.020063	0.018133	0.016781
$n = 7$	0.015407	0.015821	0.014724	0.013854
$n = 8$	0.012224	0.012692	0.012045	0.011458
$n = 9$	0.009852	0.010306	0.009911	0.009498
$n = 10$	0.008027	0.008440	0.008189	0.007888

the insurance company. However, we cannot say that the probability of $P(T = n)$ is a decreasing function of r. There are some exceptions in Tables 2 and 5. For example, when $n = 3$ and $u = 0.8$, the probability of ruin occurring at time 3 with $r = 0.03$ is larger than the probability of ruin occurring at time 3 with $r = 0$.

Table 8. Finite time ruin probabilities when $r = 0.04$

$P(T \leq n)$	$u = 0.8$	$u = 1.1$	$u = 1.4$	$u = 1.7$
$n = 1$	0.110000	0.045000	0.010000	0.050000
$n = 2$	0.197075	0.110925	0.046500	0.027800
$n = 3$	0.251730	0.159754	0.080596	0.055100
$n = 4$	0.288504	0.194757	0.108583	0.079370
$n = 5$	0.314928	0.220819	0.131085	0.099675
$n = 6$	0.334787	0.240822	0.149218	0.116456
$n = 7$	0.350194	0.256703	0.163942	0.130310
$n = 8$	0.362418	0.269395	0.175987	0.141768
$n = 9$	0.372270	0.279701	0.185898	0.151266
$n = 10$	0.380297	0.288141	0.194087	0.159154

Table 9. Ultimate ruin probabilities when $r = 0.04$

$u = 0.8$	$\psi(0.8) = 0.402938$
$u = 1.1$	$\psi(1.1) = 0.322146$
$u = 1.4$	$\psi(1.4) = 0.308977$
$u = 1.7$	$\psi(1.7) = 0.269881$

Table 10. Solution of the integral equation (22) when $r = 0$

$V(u, y)$	$u = 0.5$	$u = 0.6$	$u = 0.8$	$u = 1.0$	$u = 1.1$
$y = 0.6$	0.22592	0.22763	0.21518	0.19611	0.20120
$y = 0.8$	0.14227	0.13315	0.12121	0.11310	0.11576
$y = 0.9$	0.10027	0.09717	0.09453	0.08649	0.08324
$y = 1.0$	0.07016	0.07043	0.06699	0.05917	0.06209
$y = 1.1$	0.04726	0.05214	0.04783	0.04141	0.04144
$y = 1.2$	0.03816	0.03455	0.03039	0.02899	0.03037
$y = 1.5$	0.00749	0.00735	0.00877	0.00976	0.00811

Table 6 indicates that, when interest r increases from 0 to 0.03, the finite time ruin probability becomes smaller. As explained above, this is reasonable.

We have some similar observations from Table 7 as those from Table 2.

Similar to the above, Table 8 further shows that the finite time ruin probabilities decrease as the interest rate increases.

Table 9 indicates that the ultimate ruin probabilities decrease as the interest rate increases.

Table 10 indicates that, when u is small, the probability of severity being great or equal to a certain amount decreases as the initial surplus increases. When the initial surplus is small, the interest has an obvious effect on the

Table 11. Solution of the integral equation (22) when $r = 0.04$

$V(u, y)$	$u = 0.5$	$u = 0.6$	$u = 0.8$	$u = 1.0$	$u = 1.1$
$y = 0.6$	0.18897	0.17773	0.14219	0.14027	0.13430
$y = 0.8$	0.12745	0.08942	0.08684	0.07896	0.07817
$y = 0.9$	0.07036	0.07265	0.06589	0.06214	0.05294
$y = 1.0$	0.05631	0.05092	0.04313	0.03918	0.03686
$y = 1.1$	0.04120	0.04223	0.03507	0.02902	0.02781
$y = 1.2$	0.03403	0.02502	0.02104	0.02089	0.02033
$y = 1.5$	0.00603	0.00653	0.00592	0.00602	0.00559

Table 12. Solution of the integral equation (20) when $r = 0$ and $u = 0.8$

	$W(u, y, x)$	$y = 0.6$	$y = 0.9$	$y = 1.2$	$y = 1.5$	$y = 1.8$
	$x = 0.1$	0.17885	0.07404	0.02201	0.00597	0.00068
	$x = 0.2$	0.15434	0.06042	0.01588	0.00461	0.00321
$x \leq u$	$x = 0.4$	0.10605	0.03976	0.00742	0.00140	0.00000
	$x = 0.6$	0.06799	0.02258	0.00279	0.00000	0.00000
	$x = 0.8$	0.04289	0.01543	0.00000	0.00000	0.00000
$x > u$	$x = 1.0$	0.00882	0.00201	0.00000	0.00000	0.00000
	$x = 1.2$	0.00261	0.00000	0.00000	0.00000	0.00000

Table 13. Solution of the integral equation (20) when $r = 0.04$ and $u = 0.8$

	$W(u, y, x)$	$y = 0.6$	$y = 0.9$	$y = 1.2$	$y = 1.5$	$y = 1.8$
	$x = 0.1$	0.12316	0.05426	0.01628	0.00434	0.00076
	$x = 0.2$	0.09805	0.04360	0.00943	0.00282	0.00212
$x \leq u$	$x = 0.4$	0.07317	0.03289	0.00632	0.00108	0.00000
	$x = 0.6$	0.04350	0.01920	0.00152	0.00000	0.00000
	$x = 0.8$	0.02984	0.01533	0.00000	0.00000	0.00000
$x > u$	$x = 1.0$	0.00645	0.00067	0.00000	0.00000	0.00000
	$x = 1.2$	0.00322	0.00000	0.00000	0.00000	0.00000

distribution of the severity of ruin. When the initial surplus becomes large, the effect of interest on the distribution of severity becomes complex.

Table 11 indicates that, when the interest increases, the probabilities of the severity of ruin decrease.

Note that Table 12 does not indicate a clear, regular relation when the variables x, y and u change. This is because we are considering the conditional joint distribution here. Therefore, the overall effect of the variables on the probabilities on the left-hand side of equation (20) is rather complex.

Table 14. Distribution of the times of absolute ruin

	$u = 0.8$	$u = 1.1$	$u = 1.4$	$u = 1.8$
$P(\tilde{T} = n)$	$\tilde{u} = 55.4$	$\tilde{u} = 55.7$	$\tilde{u} = 56$	$\tilde{u} = 56.3$
$\vdots$	0	0	0	0
$n = 55$	0	0	0	0
$n = 60$	0.0000005	0.0000002	0.0000001	0
$n = 70$	0.0000144	0.0000060	0.0000034	0.0000013
$n = 80$	0.0000677	0.0000323	0.0000199	0.0000086
$n = 90$	0.0001320	0.0000687	0.0000447	0.0000212
$n = 100$	0.0001627	0.0000897	0.0000604	0.0000306
$n = 110$	0.0001561	0.0000895	0.0000616	0.0000325
$n = 120$	0.0001302	0.0000768	0.0000537	0.0000291

Table 15. Probability of absolute ruin

$u = 0.8$	$\tilde{u} = 55.4$	$\Psi(0.8) = 0.00992690$
$u = 1.1$	$\tilde{u} = 55.7$	$\Psi(1.1) = 0.00566240$
$u = 1.4$	$\tilde{u} = 56.0$	$\Psi(1.4) = 0.00388470$
$u = 1.7$	$\tilde{u} = 56.3$	$\Psi(1.7) = 0.00204760$

In Table 13 we have some similar observations as in Table 12.

Table 14 indicates that, as the initial surplus increases, the absolute ruin probabilities decrease. We also see that, at any time, the probability of absolute ruin is much smaller than the probability of ruin.

Again, as shown in Table 15, the absolute ruin probability is much smaller than the corresponding ruin probability.

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