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ON THE CLARKE SUBDIFFERENTIAL OF AN INTEGRAL FUNCTIONAL ON L_p, $1 \le p < \infty$

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ABSTRACT. Given an integral functional defined on L_p, $1 \le p < \infty$, under a growth condition we give an upper bound of the Clarke directional derivative and we obtain a nice inclusion between the Clarke subdifferential of the integral functional and the set of selections of the subdifferential of the integrand.

0**. Introduction.** It was firstly for locally Lipschitz numerical functions that F. H. Clarke introduced the generalized subdifferential (see [2]). Then the geometrical notion of the tangent cone was developed following Clarke [4, 9] and this enabled R. T. Rockafellar [7, 8] to study a subdifferential of any numerical function. For integral functionals, *i.e.*, for functionals of the type:

$$x \longmapsto I_f(x) = \int_\Omega^* f\big(\omega, x(\omega)\big)\, d\mu(\omega),$$

where f is the integrand defined on the cartesian product of a measured space Ω by a finite dimensional space E with values in $\mathbb{R} \cup \{\infty\}$, the study of the subdifferential was carried out by R. T. Rockafellar in the convex case [6], and the Lipschitz case is treated, among others, in F. H. Clarke's book [2]. There remains the non-smooth case in general. We fill this gap in the present paper when I_f is defined on a space L_p, $1 \le p < \infty$. We suppose that the integrand f satisfies a growth condition which guarantees that the associated integral functional does not take the value $-\infty$. Under this sole condition we show that the generalized Clarke-Rockafellar derivative of the functional integral is bounded above by the integral functional associated with the generalized derivative of the integrand, *i.e.*, with R. T. Rockafellar's notation (see [7]), at a given point x of the domain of I_f:

$$\forall y \in L_p, \quad (I_f)^{\uparrow}(x; y) \le I_{f^{\uparrow}(x;.)}(y).$$

Finally, with the help of this upper bound, we deduce that the elements of the subdifferential of I_f at x are measurable selections of the subdifferential of the integrand f at x:

$$\partial^{\uparrow} I_f(x) \subset \left\{ x^* \in L_q, \frac{1}{p} + \frac{1}{q} = 1 : x^*(\omega) \in \partial^{\uparrow} f\big(\omega, x(\omega)\big), \mu\text{-a.e.} \right\}.$$

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1. **Preliminaries.** Let $(X, |.|)$ be a normed vector space. Given a filtered family $(M_r)_r$ of subsets of X, we define its lower limit as follows:

$$\liminf M_r = \left\{x \in X : \lim_r d(x, M_r) = 0\right\},$$

with $d(x, M_r) = \inf\{|x - m|, m \in M_r\}$.

For a subset M of X and an element x belonging to its closure, the Clarke tangent cone [2, 4, 5, 7] is defined by:

$$T_x^{\uparrow} M = \liminf_{\substack{r \to 0_+ \\ x' \xrightarrow{M} x}} r^{-1}(M - x'), \tag{1.1}$$

where $x' \xrightarrow{M} x$ means $x' \in M$, $x' \longrightarrow x$.

The Clarke tangent cone is a closed convex cone with apex 0.

To this geometric tangent notion corresponds a generalized derivative for numerical functions. R. T. Rockafellar considers this case in [7]. More explicitly, if f is a numerical function defined on X with values in $\mathbb{R} \cup \{\infty\}$, one considers its epigraph $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ and the relation

$$\operatorname{epi} f^{\uparrow}(x; .) = T^{\uparrow}_{(x, f(x))} \operatorname{epi} f \tag{1.2}$$

enables to define the Clarke-Rockafellar generalized derivative $f^{\uparrow}(x; .)$ of f at the point x. The generalized derivative defined in such a way is convex lower semi-continuous and its subdifferential at 0 in the sense of convex analysis is just the Clarke subdifferential of f at x:

$$\partial^{\uparrow} f(x) = \left\{x^* \in X^* : \langle x^*, .\rangle \leq f^{\uparrow}(x; .)\right\}. \tag{1.3}$$

In the present work we study the subdifferential of integral functionals defined on spaces L_p, $1 \leq p < \infty$.

Throughout, we consider a measured space $(\Omega, \mathcal{S}, \mu)$ where μ is a σ-finite measure, $\mathcal{S}$ a μ-complete tribe.

If E is a finite dimensional space, $\beta(E)$ denotes the Borel tribe in E and $L_p(\Omega, E)$ the Lebesgue space of classes of p-integrable functions (for almost μ-everywhere equality) with $1 \leq p < \infty$. Let $f : \Omega \times E \longrightarrow \mathbb{R} \cup \{\infty\}$ be a $\mathcal{S} \otimes \beta(E)$-measurable integrand, "normal" in R. T. Rockafellar's sense [6], *i.e.*, such that for all $\omega \in \Omega$, $f(\omega, .)$ is lower semicontinuous.

We denote by I_f the integral functional defined on $L_p(\Omega, E)$ by:

$$I_f(x) = \operatorname{Inf}\left\{\int_\Omega u\, d\mu, u \in L_1(\Omega, \mathbb{R}), u(.) \geq f\big(., x(.)\big)\, \mu\text{-a.e.}\right\}.$$

We assume that the integrand satisfies the following growth condition:

(C) There exists $(a, x_0) \in L_1(\Omega, \mathbb{R}) \times L_p(\Omega, E)$, $b \in \mathbb{R}_+$ such that for all $(\omega, e) \in \Omega \times E$:

$$f(\omega, e) \geq -b|e - x_0(\omega)|^p + a(\omega).$$

When this condition is fulfilled the integral functional I_f does not take the value $-\infty$ on $L_p(\Omega, E)$.

Condition (C) holds when I_f is finite at some point x_0 of $L_p(\Omega, E)$ and when the integrand f is Lipschitz, that is:

(L) There exists $k \in L_q(\Omega, \mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that for all $\omega \in \Omega$, $e_1, e_2 \in E$

$$|f(\omega, e_1) - f(\omega, e_2)| \leq k(\omega)|e_1 - e_2|.$$

Indeed, Hölder's inequality implies:

$$f(\omega, e) \geq -k(\omega)|e - x_0(\omega)| + f\big(\omega, x_0(\omega)\big) \geq -\frac{1}{p}|e - x_0(\omega)|^p - \frac{1}{q}k(\omega)^q + f\big(\omega, x_0(\omega)\big).$$

Moreover, in the case where the integrand f is convex and satisfies $f = f^{**}$, with $f^*(\omega, e^*) = \sup_e\big(\langle e^*, e\rangle - f(\omega, e)\big)$, the above condition is satisfied when there exists an element $x^* \in L_q(\Omega, E)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $I_{f^*}(x^*)$ is finite.

Let us prove the latter assertion: x^* being such an element, for all $(\omega, e) \in \Omega \times E$ we have:

$$f(\omega, e) \geq \big\langle x^*(\omega), e\big\rangle - f^*\big(\omega, x^*(\omega)\big);$$

thus Hölder's inequality yields:

$$f(\omega, e) \geq -\frac{1}{p}|e|^p - \frac{1}{q}|x^*(\omega)|^q - f^*\big(\omega, x^*(\omega)\big).$$

In the sequel, if $x: \Omega \longrightarrow E$ is measurable, $f(x)$ will stand for, by abuse of notation, the function $\omega \longrightarrow f\big(\omega, x(\omega)\big)$.

2. **Measurability questions.** In this paragraph, we study the mathematical objects introduced previously, *i.e.*, the generalized derivative and the notion of subdifferential. For that end, we need the following lemma.

LEMMA 2.1. *Let M be a subset of E and let e be in the closure of M. Then*

$$T_e^{\uparrow} M = \liminf_{\substack{n \to \infty \\ e' \xrightarrow{M} e}} n(M - e').$$

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PROOF. Let $S = \liminf_{\substack{n\to\infty \\ e' \xrightarrow{M} e}} n(M - e')$; by virtue of 1.1, it is clear that $T_e^{\uparrow}M$ is contained in S. Conversely, if $s \in S$, let us show that for every sequence $(r_i)_i$ of real numbers converging to 0 and for every sequence $(e'_i) \xrightarrow{M} e$, we have $s \in \liminf r_i^{-1}(M - e'_i)$. For that matter, observe first of all that if for any real number r, $0 < r \le 1$, n is the integer part of r^{-1}, we necessarily have:

$$d\big(s, r^{-1}(M - e')\big) = r^{-1} d(rs, M - e') \le r^{-1}|rs - n^{-1}s| + r^{-1} d(n^{-1}s, M - e');$$

hence:

$$d\big(s, r^{-1}(M - e')\big) \le |s|\,|1 - (rn)^{-1}| + (rn)^{-1} d\big(s, n(M - e')\big),$$

and if n_i stands for the integer part of r_i^{-1}, we have

$$d\big(s, r_i^{-1}(M - e'_i)\big) \le |s|\,|1 - (r_i n_i)^{-1}| + (r_i n_i)^{-1} d\big(s, n_i(M - e'_i)\big).$$

Now, since $\lim_i r_i n_i = 1$, $\lim_i d\big(s, n_i(M - e'_i)\big) = 0$, we obtain $\lim_i d\big(s, r_i^{-1}(M - e'_i)\big) = 0$. The proof is therefore complete. ∎

Let M be a measurable multifunction, in the sense of [1], defined on Ω and with closed non-empty values in E.

PROPOSITION 2.2. *If x is a measurable selection of M, then the multifunction $T_{x(.)}^{\uparrow}M(.)$ defined by*

$$\omega \longrightarrow T_{x(\omega)}^{\uparrow}M(\omega) \text{ is measurable.}$$

PROOF. If B is the unit ball of E, we introduce the integrands

$$f_{n,m}(\omega, e) = \sup\left\{ d\Big(e, n\big(M(\omega) - e'\big)\Big), e' \in M(\omega) \cap \left\{x(\omega) + \frac{1}{m}B\right\}\right\}.$$

By applying ([1] III 39), $f_{n,m}(., e)$ is measurable and moreover, for all $\omega \in \Omega$ and e, e' in E, we have

$$f_{n,m}(\omega, e) \le |e - e'| + f_{n,m}(\omega, e')$$

which shows that $f_{n,m}(\omega, .)$ is Lipschitz. So by ([1] III 14), $f_{n,m}$ is $\mathcal{S} \otimes \beta(E)$-measurable. Furthermore, Lemma 2.1 gives

$$\text{graph}\,(T_x^{\uparrow}M) = (\limsup_{n,m} f_{n,m})^{-1}(0)$$

and ([6] 1 E) implies the announced result. ∎

COROLLARY 2.3. *Let f be a normal integrand and let x be a measurable selection of its domain. Then $f^{\uparrow}\big(x(.); .\big)$ is also a normal integrand.*

PROOF. By 1.2 we have:

$$\text{epi}\, f^{\uparrow}(x; .) = T_{(x, f(x))}^{\uparrow}\, \text{epi}\, f$$

and since f is normal, epi f is measurable (by [6] 2 A) and following Proposition 2.2, $T_{(x,f(x))}^{\uparrow}$ epi f is measurable. Using again ([6] 2 A) allows us to complete the proof. ∎

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COROLLARY 2.4. *Let f be a normal integrand and let $x\colon \Omega \longrightarrow E$ be a measurable selection of its domain. Then the multifunction $\partial^{\uparrow} f(x)$ defined by $\omega \longrightarrow \partial^{\uparrow} f\big(\omega, x(\omega)\big)$ is measurable.*

PROOF. Set $\Omega_1 = \big\{\omega \in \Omega : f_\omega^{\uparrow}\big(x(\omega); 0\big) = -\infty\big\}$. Then by Corollary 2.3, Ω_1 is measurable and for all $\omega \in \Omega_1$ we have $\partial^{\uparrow} f\big(\omega, x(\omega)\big) = \emptyset$. For all $\omega \in \Omega \setminus \Omega_1$, $f_\omega^{\uparrow}\big(x(\omega); .\big)$ is convex lower semicontinuous, proper and using (1.3) and ([6] 2 X) we conclude that $\partial^{\uparrow} f(x)$ is measurable on $\Omega \setminus \Omega_1$. ∎

3. **Epi-derivative and subdifferential of an integral functional defined on L_p, $1 \le p < \infty$.** The main result of the present work is:

THEOREM. *Let $f\colon \Omega \times E \longrightarrow \mathbb{R} \cup \{\infty\}$ be a normal integrand satisfying the growth condition (C). If $x \in L_p(\Omega, E)$ is such that I_f is finite at x, then for all y in $L_p(\Omega, E)$:*

$$(I_f)^{\uparrow}(x; y) \le 1_{f^{\uparrow}(x;.)}(y).$$

Furthermore, we always have the inclusion:

$$\partial^{\uparrow}(I_f)(x) \subset \left\{x^* \in L_q(\Omega, E), \frac{1}{p} + \frac{1}{q} = 1 : x^*(\omega) \in \partial^{\uparrow} f_\omega\big(x(\omega)\big)\, \mu\text{-}a.e.\right\}.$$

We shall prove this result after having verified some intermediate results.

If M is a measurable multifunction defined on Ω with non-empty closed values in $E \times \mathbb{R}$, $L_p \times L_1(M)$ will stand for the set of measurable selections which are in $L_p(\Omega, E) \times L_1(\Omega, \mathbb{R})$. With this notation, we have the following result:

PROPOSITION 3.1. *For all $x \in L_p \times L_1(M)$ the following inclusion always holds:*

$$L_p \times L_1(T_x^{\uparrow} M) \subset T_x^{\uparrow}\big(L_p \times L_1(M)\big).$$

PROOF. For $z = (e, r)$, and $z' = (e', r')$ in $E \times \mathbb{R}$, set $\rho(z, z') = |e - e'|^p + |r - r'|$. Then in $E \times \mathbb{R}$, $\lim_n z_n = z \Leftrightarrow \lim_n \rho(z, z_n) = 0$, so that if for a subset A of $E \times \mathbb{R}$, we set

$$\rho(z, A) = \inf\big\{\rho(z, a), a \in A\big\},$$

then

$$z \in T_a^{\uparrow} A \Longleftrightarrow \lim_{\substack{r \to 0_+ \\ a' \xrightarrow{A} a}} \rho\big(z, r^{-1}(A - a')\big) = 0.$$

Likewise, in $L_p \times L_1$ we have: $(x_n) \longrightarrow x \Leftrightarrow \lim_n \int_\Omega \rho(x, x_n)\, d\mu = 0$, and if we denote for a subset N of $L_p \times L_1$:

$$d_\rho(x, N) = \inf_{u \in N} \int_\Omega \rho(x, u)\, d\mu$$

we have:

$$y \in T_x N \Longleftrightarrow \lim_{\substack{r \to 0_+ \\ x' \xrightarrow{N} x}} d_\rho\big(y, r^{-1}(N - x')\big) = 0.$$

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By virtue of ([6] 3 A), we have:

$$d_\rho\big(y, L_p \times L_1(M)\big) = \int_\Omega \inf_{e \in M(\omega)} \rho\big(y(\omega), e\big)\, d\mu = \int_\Omega \rho\big(y(\omega), M(\omega)\big)\, d\mu.$$

Let $y \in L_p \times L_1(T_x^{\uparrow} M)$ and let $(r_n)_n$ converge to 0_+ and $(x_n) \xrightarrow{L_p \times L_1(M)} x$. We can always extract from $(x_n)_n$ a subsequence $(x_{n_k})_k$ converging μ-almost everywhere. Then we have for all $\omega \in \Omega$:

$$\lim_k \rho\Big(y(\omega), r_{n_k}^{-1}\big(M(\omega) - x_{n_k}(\omega)\big)\Big) = 0.$$

Furthermore, since $0 \in r^{-1}\big(M(\omega) - x_{n_k}(\omega)\big)$, we have the following bounds:

$$0 \le \rho\Big(y(\omega), r_{n_k}^{-1}\big(M(\omega) - x_{n_k}(\omega)\big)\Big) \le |y_1(\omega)|^p + |y_2(\omega)| \in L_1(\Omega, E \times \mathbb{R}).$$

Using the Dominated Convergence Theorem we then deduce

$$\lim_k d_\rho\Big(y, r_{n_k}^{-1}\big(L_p \times L_1(M - x_{n_k})\big)\Big) = \lim_k \int_\Omega \rho\Big(y(\omega), r_{n_k}^{-1}\big(M(\omega) - x_{n_k}(\omega)\big)\Big)\, d\mu = 0.$$

Since this is true for all sequences $(x_n)_n$ and $(r_n)_n$, we get

$$\lim_{\substack{r \to 0_+ \\ (x_n) \xrightarrow{L_p \times L_1(M)} x}} d_\rho\Big(y, r^{-1}\big(L_p \times L_1(M - x')\big)\Big) = 0,$$

that is, $y \in T_x^{\uparrow}\big(L_p \times L_1(M)\big)$.

PROPOSITION 3.2. *Let f be an integrand. If A is the mapping on $L_p \times L_1$, with values on $L_p \times \mathbb{R}$ defined by $A(x, u) = (x, \int_\Omega u\, d\mu)$, then:*

$$\operatorname{epi} I_f = A\big(L_p \times L_1(\operatorname{epi} f)\big).$$

PROOF. The inclusion $A\big(L_p \times L_1(\operatorname{epi} f)\big) \subset \operatorname{epi} I_f$ is trivial. Conversely let $(x, r) \in \operatorname{epi} I_f$; then $I_f(x) \le r$. We consider two cases. Either $f(x)$ is integrable or $f(x)$ is not integrable with $I_f(x) = -\infty$. In both cases, we can find $u \in L_1(\Omega, \mathbb{R})$ such that $f(x) \le u$, μ-almost everywhere with $\int u\, d\mu \le r$. We can choose then a positive integrable function α such that $\int \alpha\, d\mu = r - \int_\Omega u\, d\mu$, and we set $v = u + \alpha$. Then we have $(x, v) \in L_p \times L_1(\operatorname{epi} f)$ and $A(x, v) = (x, r)$. ∎

DEFINITION. Let $A: X \longrightarrow Y$ be a continuous linear mapping between normed spaces X and Y, C a subset of X. We say that A is open on C at the point x if:

$$\forall (y_n)_n, \quad (y_n) \xrightarrow{A(C)} A(x), \quad \exists x_n \in C \cap A^{-1}(y_n), \quad (x_n) \longrightarrow x.$$

The interest of this definition lies in the following two results.

LEMMA 3.3. *If A is open on C at the point x, then:*

$$A(T_x^{\uparrow} C) \subset T_{A(x)}^{\uparrow} A(C).$$

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PROPOSITION 3.4. *The mapping A defined in Proposition 3.2 is open on $L_p \times L_1(\operatorname{epi} f)$ at every point $\big(x, f(x)\big)$ where $I_f(x) < \infty$.*

Using the characterization of Clarke's tangent cone

$$y \in T_x^{\uparrow} M \iff \forall (r_n) \longrightarrow 0_+, \quad \forall (x_n) \xrightarrow{M} x \quad \exists (y_n) \longrightarrow y : \forall n, x_n + r_n y_n \in M.$$

The proof of Lemma 3.3 is easy and is left to the reader.

Let us prove Proposition 3.4. Let $(x_n, r_n)_n$ be a sequence of elements of $\operatorname{epi} I_f$ converging to $\big(x, I_f(x)\big) = A\big(x, f(x)\big)$ where $I_f(x) < \infty$.

By Proposition 3.2, one can find a sequence $(v_n)_n$ of integrable functions such that for all $n \in \mathbb{N}$:

$$(x_n, v_n) \in L_p \times L_1(\operatorname{epi} f); \; A(x_n, v_n) = (x_n, r_n).$$

If we extract from $(x_n)_n$ a subsequence $(x_{n_k})_k$ converging μ-almost everywhere to x, using the normality of the integrand f, we deduce that $\big(\inf\{v_{n_k}, f(x)\}\big)_k$ converges μ-almost everywhere to $f(x)$. Since this last result is true for every subsequence of $\big(\inf\{v_n, f(x)\}\big)_n$, we conclude that this sequence converges in measure to $f(x)$. But, by the growth condition (C) the last sequence is uniformly integrable and consequently the convergence holds true in $L_1(\Omega, \mathbb{R})$.

Since $\lim_n r_n = I_f(x)$, the sequence of positive functions

$$v_n - f(x) - \Big(\inf\big\{v_n, f(x)\big\} - f(x)\Big)$$

converges to 0 in $L_1(\Omega, \mathbb{R})$. This shows also that $(v_n)_n$ converges to $f(x)$ in $L_1(\Omega, \mathbb{R})$. Hence we have shown that $(x_n, v_n) \in A^{-1}(x_n, r_n)$ converges to $\big(x, f(x)\big)$ in $L_p \times L_1(\operatorname{epi} f)$. The proof of Proposition 3.4 is therefore complete. ■

Now we are in a position to prove the main Theorem.

PROOF OF THE THEOREM. We show that if $I_f(x) < \infty$, then $(I_f)^{\uparrow}(x; .)$ is bounded above by $I_{f^{\uparrow}(x;.)}(.)$, which is equivalent to the inclusion:

$$\operatorname{epi} I_{f^{\uparrow}(x;.)}(.) \subset \operatorname{epi}(I_f)^{\uparrow}(x; .).$$

But we have the string of relations:

$$\begin{aligned}
\operatorname{epi} I_{f^{\uparrow}(x;.)}(.) &= A\Big[L_p \times L_1\big(\operatorname{epi} f^{\uparrow}(x; .)\big)\Big] \quad \text{(Proposition 3.2)} \\
&= A\Big[L_p \times L_1(T^{\uparrow}_{(x, f(x))} \operatorname{epi} f)\Big] \quad \text{(see (1.2))} \\
&\subset A\Big[T^{\uparrow}_{(x, f(x))} L_p \times L_1(\operatorname{epi} f)\Big] \quad \text{(Proposition 3.1)} \\
&\subset T^{\uparrow}_{A(x, f(x))} A\Big[L_p \times L_1(\operatorname{epi} f)\Big] \quad \text{(Lemma 3.3 and Proposition 3.4)} \\
&= T^{\uparrow}_{(x, I_f(x))} \operatorname{epi} I_f \quad \text{(Proposition 3.2)} \\
&= \operatorname{epi}(I_f)^{\uparrow}(x; .) \quad \text{(see 1.2)}
\end{aligned}$$

which shows that $\operatorname{epi} I_{f^{\uparrow}(x;.)} \subset \operatorname{epi}(I_f)^{\uparrow}(x; .)$.

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Let $x^* \in \partial^{\uparrow} I_f(x)$; then in the sense of convex analysis we have $x^* \in \partial I_{f^{\uparrow}(x;.)}(0)$, and since $f^{\uparrow}(x;0) \leq 0$ μ-a.e., the normal convex integrand $f^{\uparrow}(x;.)$ verifies the hypotheses of ([6] 3 E) and, consequently, for almost every $\omega \in \Omega$, we have

$$x^*(\omega) \in \partial^{\uparrow} f\big(\omega, x(\omega)\big)$$

which shows the last inclusion of the Theorem. ∎

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