RADIAL AND ANGULAR LIMITS OF MEROMORPHIC FUNCTIONS

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1. Introduction. Let us say that a function defined in the open unit disk D has the *Montel property* if the set of those points $e^{i\theta}$ on the unit circle C where the radial limit exists coincides with the set where the angular limit exists. By a classical theorem of Montel **(4)**, every bounded holomorphic function has this property. Meromorphic functions omitting at least three values and, more generally, the normal functions recently introduced by Lehto and Virtanen **(3)** also enjoy the Montel property (also see **1**).

In this paper we show that a function of bounded characteristic (which necessarily has finite angular limits almost everywhere) need not have the Montel property.

Since a bounded holomorphic function possesses the Montel property, one might suspect that a function which is "almost" bounded in the sense that it belongs to every Hardy class H_p $(0 < p < \infty)$ would also have the Montel property. Lemma 2 shows that, on the contrary, if $f(z)$ is any meromorphic function which approaches infinity along some radius of D, then there exists a bounded holomorphic function $B(z)$ such that $B(z)f(z)$ fails to have the Montel property.

In § 2 we prove that, corresponding to each countable set $\{\zeta_n\}$ on C, there is a function f which belongs to every Hardy class H_p $(0 < p < \infty)$ and which has, at each point ζ_n, a radial limit but no angular limit.

In § 3 extensions of the above results are established, and in § 4 the results of this paper are compared with those of Lappan **(2)** concerning non-normal sums and products of normal functions.

The author is indebted to Professor W. Rudin, who suggested (oral communication) that Lemma 2, which is the keystone of this paper, might be true. Also, the author would like to thank Professor G. Piranian for supplying the essential link in the proof of Theorem 2.

Finally, we should remark that the analogy (noted in § 4) between the results of Lappan and those of this paper is more than superficial. There is a theory which embraces them both, and which, curiously enough, is related to an interpolation problem in H_∞.

2. The main result. We begin with the construction of a function f that belongs to every Hardy class H_p $(0 < p < \infty)$ and has a radial limit

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but no angular limit at the point 1. In the discussion of Blaschke products, we shall use the notation

$$b(z;a) = \frac{|a|}{a}\frac{a-z}{1-\bar{a}z}, \qquad B(z;\{z_n\}) = \prod b(z;z_n).$$

LEMMA 1. *Let* $a_t = 1 - te^{i\phi}$ *where* ϕ *is fixed* $(0 < |\phi| < \pi/2)$. *Then*

$$\inf\{|b(r;a_t)| : 0 \leqslant r \leqslant 1; 0 < t < \cos\phi\} > 0.$$

The restriction on t in the conclusion serves only to ensure that $|a_t| < 1$. Setting $\cos\phi = c$, we obtain

$$\begin{aligned}|b(r;a_t)|^2 &= \frac{|1-r-te^{i\phi}|^2}{|1-r+tre^{-i\phi}|^2}\\ &= \frac{(1-r)^2 - 2(1-r)tc + t^2}{(1-r)^2 + 2(1-r)trc + t^2r^2}\\ &= \frac{[t-c(1-r)]^2 + (1-c^2)(1-r)^2}{[rt+c(1-r)]^2 + (1-c^2)(1-r)^2}.\end{aligned}$$

For $0 < 1 - r \leqslant t$, we drop the second term in the numerator and conclude that

$$|b(r;a_t)|^2 > \frac{[t(1-c)]^2}{[t(1+c)]^2 + (1-c^2)t^2} = \frac{(1-c)^2}{2(1+c)}.$$

For $t \leqslant 1 - r \leqslant 1$, we drop the first term in the numerator and obtain the inequality

$$|b(r;a_t)|^2 > \frac{1-c^2}{(1+c)^2 + 1 - c^2} = \frac{1-c}{2}.$$

This concludes the proof of Lemma 1.

LEMMA 2. *Let S denote a closed non-radial line segment contained in D except for one end point at* $e^{i\theta}$. *Then there exists a Blaschke product that has infinitely many zeros on S and maps the radius terminating at* $e^{i\theta}$ *onto a curve bounded away from the origin.*

Proof. Without loss of generality, we may suppose that $\theta = 0$. On the radius of D that terminates at $z = 1$, the function $b(z;a_t)$ considered in Lemma 1 approaches a value of modulus 1. It follows from Lemma 1 that if

$$t(n+1)/t(n) \to 0$$

rapidly enough, then $B(z;\{a_{t(n)}\})$ has the desired property.

LEMMA 3. *Let* $\{\zeta_n\}$ *denote a countable subset of C, and for each n let* S_n *denote a closed segment that lies in D except for one end point at* ζ_n, *and that makes a fixed angle* θ *with the radius at* ζ_n. *Then there exists a Blaschke product that has infinitely many zeros on each segment* S_n *and maps the radius at each point* ζ_n *onto a curve which is bounded away from the origin.*

The proof is as obvious as that of Lemma 2.

THEOREM 1. *Let $\{\zeta_n\}$ be a countable subset of C. Then there exists a function f that belongs to every Hardy class H_p $(0 < p < \infty)$ and has, at each point ζ_n, the radial limit ∞ but no angular limit.*

Proof. Suppose that some function F has the radial limit ∞ at each point ζ_n and belongs to every Hardy class H_p. If B denotes the Blaschke product constructed in Lemma 3, then the function $f(z) = B(z)\,F(z)$ has the required properties.

Now let $F_n(z) = \log\{1/(1 - \bar{\zeta}_n z)\}$ denote the branch of the corresponding function that vanishes at the origin, and let

$$F(z) = \sum_{1}^{\infty} 2^{-n} F_n(z).$$

Since F_1 belongs to every Hardy class H_p, and since each F_n has the form $F_n(z) = F_1(\zeta_1 \bar{\zeta}_n z)$, we see immediately that F belongs to every Hardy class. Moreover, the real part of $F_n(z)$ is greater than $\log(1/2)$ for all z in D and tends to ∞ as z tends to ζ_n, and therefore $F(z) \to \infty$ as $z \to \zeta_n$. This concludes the proof of Theorem 1.

Remark. In our theorem the radial limit at ζ_n is ∞. We can make the limit finite, provided we are willing to accept a function f that is only meromorphic and of bounded characteristic. Example:

$$f(z) = \{\exp \sum 2^{-n}(z + \zeta_n)/(z - \zeta_n)\}/B(z),$$

where B is the Blaschke product in Lemma 3.

3. Extensions and refinements. The pathology exhibited in Theorem 1 can be extended and intensified in a number of ways.

First of all, the function exhibiting the pathology can be "almost" bounded in a more stringent sense than that of belonging to every Hardy class H_p $(0 < p < \infty)$. Indeed, as Professor G. Piranian has pointed out to the author, corresponding to each positive, non-decreasing function h defined on $[0, \infty)$ and each set E of measure zero on C, there is a holomorphic function F in D which has the limit ∞ at each point of E and for which

$$\sup\left\{\frac{1}{2\pi}\int_0^{2\pi} h(|F(re^{i\theta})|)\,d\theta : 0 < r < 1\right\} < \infty.$$

A proof of this result will be given in a subsequent paper, where it will be used in another connection. If h increases rapidly enough, for example, if $h(x) \geqslant e^x$, then F belongs to every Hardy class H_p $(0 < p < \infty)$. Thus we have the following extension of Theorem 1.

THEOREM 2. *Let $\{\zeta_n\}$ be a countable subset of C, and let h be a positive, non-decreasing function defined on $[0, \infty)$. Then there exists a holomorphic function f in D which satisfies the growth condition*

$$\sup\left\{\frac{1}{2\pi}\int_0^{2\pi} h(|f(re^{i\theta})|)\,d\theta : 0 < r < 1\right\} < \infty,$$

and which has, at each point ζ_n, *the radial limit* ∞ *but no angular limit.*

Next, let us turn to the question of intensifying the pathology locally. Any continuous function in D which has the radial limit ∞ at a point $e^{i\theta}$ will approach ∞ as z approaches $e^{i\theta}$ from within a certain region containing the radius in question; by modifying our previous arguments, one can preclude the region from containing any symmetric triangular neighbourhood of $e^{i\theta}$. More precisely, we have the following theorem.

THEOREM 3. *Suppose that* f *is holomorphic in* D *and has the radial limit* ∞ *at* $e^{i\theta}$. *Then there exists a Blaschke product* B *such that* $B(z)f(z)$ *has the radial limit* ∞ *at* $e^{i\theta}$ *and does not approach a limit as* z *approaches* $e^{i\theta}$ *from within any triangle which contains the radius terminating at* $e^{i\theta}$.

4. Non-normal products and sums of normal functions. In their paper concerning normal meromorphic functions, Lehto and Virtanen **(3)** remark that the sum of a normal function and a bounded function (which is necessarily normal) is a normal function. In contradistinction to this, Lappan **(2)** proves the following theorem.

THEOREM (Lappan). *Let* f *be an unbounded normal holomorphic function in* D. *Then there exists a Blaschke product* B_f *and a normal holomorphic function* g *in* D *such that* $B_f(z)f(z)$ *and* $f(z) + g(z)$ *are not normal in* D.

It is easy to see that, if in Lappan's theorem we impose on f the additional requirement that f has infinity as an asymptotic value, then the conclusion can be strengthened: $B_f(z)f(z)$ and $f(z) + g(z)$ do not have the Montel property.

In conclusion, let us note that non-normal functions, for example,

$$(1 - z)\exp\{(1 + z)/(1 - z)\},$$

may possess the Montel property.

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