

THE STRONG GIANT IN A RANDOM DIGRAPH

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Abstract

Consider a random directed graph on n vertices with independent and identically distributed outdegrees with distribution F having mean μ , and destinations of arcs selected uniformly at random. We show that if $\mu > 1$ then for large n there is very likely to be a unique giant strong component with proportionate size given as the product of two branching process survival probabilities, one with offspring distribution F and the other with Poisson offspring distribution with mean μ . If $\mu \leq 1$ there is very likely to be no giant strong component. We also extend this to allow for F varying with n .

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1. Introduction

Given $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and given a probability distribution F on $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, consider a random directed multigraph $G_{n,F}$ on vertex set $[n] := \{1, \dots, n\}$, defined as follows (a multigraph is a graph with multiple arcs and loops allowed). Let each vertex of $G_{n,F}$ have outdegree independently sampled from distribution F . Given these outdegrees, the arcs have their destinations sampled independently uniformly from $[n]$.

Consider also a random directed graph $\tilde{G}_{n,F}$, similar to $G_{n,F}$ but with loops and multiple arcs excluded. Let the outdegrees be sampled independently from distribution F as before, denoting the respective outdegrees by ξ_1, \dots, ξ_n . Given these outdegrees, let the set of destinations of the arcs from vertex i in $\tilde{G}_{n,F}$ be selected uniformly at random from the collection of all $\binom{n-1}{\xi_i}$ subsets of $[n] \setminus \{i\}$ with ξ_i elements, independently of the arcs from other vertices. If $\xi_i > n - 1$, include all arcs from i in the graph $\tilde{G}_{n,F}$ (so in this case the outdegree of i is $n - 1$, not ξ_i).

In the special case where F is the Dirac distribution at k for some $k \in \mathbb{N}$ (i.e. $F(\{k\}) = 1$), the random directed graph $\tilde{G}_{n,F}$ is also known as $G_{n,k\text{-out}}$ or just $G_{k\text{-out}}$, mentioned in [2] and studied in [8] and elsewhere.

Random graph models with specified degree distributions (e.g. with power-law decay of the tails) are of much recent interest, and directed graphs are often a better model for real-world networks than the undirected ones; see [3], [6], [9], and the references therein. Our model is a simple and natural way to allow for an arbitrary specified outdegree distribution in a random directed graph.

We use the abbreviation ‘digraph’ to mean either a directed graph or a directed multigraph. For vertices i, j of a finite digraph G we write $i \rightsquigarrow j$ if there is a directed path from i to j (or if $i = j$), and $i \leftrightarrow j$ if both $i \rightsquigarrow j$ and $j \rightsquigarrow i$. We say G is *strongly connected* if for any two vertices i, j we have $i \leftrightarrow j$. For $k \in \mathbb{N}$ let $L_k(G)$ denote the number of vertices in the k th

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largest strongly connected component of G (if k exceeds the number of such components, set $L_k(G) = 0$.)

Set $\mu_F := \sum_k kF(\{k\})$, the mean of the distribution F . In Theorem 1 we show that $G_{n,F}$ and $\tilde{G}_{n,F}$ enjoy a ‘giant component’ phenomenon also seen in other random graph models such as the Erdős–Rényi random graph (see, e.g. [2]) if and only if $\mu_F > 1$. In Theorems 2 and 3 we extend this result to allow for F varying with n , and in Theorem 4 we give a further result on the distributional limit of the proportionate number of vertices $j \in [n]$ such that $1 \rightsquigarrow j$.

Related random digraph models are considered (and results analogous to Theorem 1 are derived) in [1] and [5], but they are not the same as ours. In [5], the degrees are imposed globally whereas here they are determined locally. In [1], each vertex has a randomly determined type, and each arc is included at random with probability determined by the type of its endpoints.

The random digraph model considered here seems at least as natural as those in [1] and [5]. In some sense, it is intermediate between the one in [5] which is homogeneous and the one in [1] which is inhomogeneous; loosely speaking, one may say that a random graph is homogeneous if all its vertices have the same status; see [1] and the references therein. Our graphs are *semi-homogeneous* in the sense that they are inhomogeneous with respect to outdegree but homogeneous with respect to indegree. Nevertheless, our model does not appear to be a special case of that in [1], since the status (present/absent) of different arcs from a given vertex are not conditionally independent given the type (i.e. outdegree) of that vertex. Also, in [1] the number of types is assumed to be finite, whereas we allow for F with infinite support.

2. Statement of results

Given a probability distribution F on $\mathbb{Z}_+ \cup \{\infty\}$, let x_F be the smallest solution in $[0, 1]$ of $x = \phi_F(x)$, where we set $\phi_F(x) = \sum_{k=0}^\infty x^k F(\{k\})$, and set $\sigma(F) := 1 - x_F$. It is well known (see, e.g. [7]) that $\sigma(F)$ is the survival probability of a Galton–Watson branching process with offspring distribution F , and that $\sigma(F) > 0$ if and only if $\mu_F > 1$. In the special case where F is a Poisson distribution with parameter $\mu \in [0, \infty)$ (so $\phi_F(x) = e^{\mu(x-1)}$), we write $\sigma'(\mu)$ for $\sigma(F)$, and we set $\sigma'(\infty) = 1$.

Let ‘ $\xrightarrow{\mathbb{P}}$ ’ denote convergence in probability.

Theorem 1. *Given any probability distribution F on \mathbb{Z}_+ , as $n \rightarrow \infty$, we have*

$$\frac{L_1(G_{n,F})}{n} \xrightarrow{\mathbb{P}} \sigma'(\mu_F)\sigma(F), \quad \frac{L_2(G_{n,F})}{n} \xrightarrow{\mathbb{P}} 0, \tag{1}$$

and

$$\frac{L_1(\tilde{G}_{n,F})}{n} \xrightarrow{\mathbb{P}} \sigma'(\mu_F)\sigma(F), \quad \frac{L_2(\tilde{G}_{n,F})}{n} \xrightarrow{\mathbb{P}} 0. \tag{2}$$

It is natural to ask whether the convergence in probability statements of Theorem 1 hold uniformly over all choices of the outdegree distribution F . This amounts to asking whether similar statements hold if we allow F to vary with n . Our next results tell us that this is indeed the case if for each n we impose a deterministic bound b_n on the outdegrees in $G_{n,F}$ satisfying $b_n = o(n)$ (i.e. $b_n/n \rightarrow 0$ as $n \rightarrow \infty$). For $n \in \mathbb{N}$, let \mathcal{M}_n be the class of probability distributions F on \mathbb{Z}_+ which are supported by $\{0, 1, \dots, n\}$, i.e. which satisfy $F(\{0, 1, \dots, n\}) = 1$.

Given a probability distribution F on $\mathbb{Z}_+ \cup \{\infty\}$, and a sequence of probability distributions $(F_n)_{n \geq 1}$ on \mathbb{Z}_+ , we write $F_n \xrightarrow{w} F$ if F_n converges weakly to F as $n \rightarrow \infty$, i.e. if $\lim_{n \rightarrow \infty} F_n(\{k\}) = F(\{k\})$ for all $k \in \mathbb{Z}_+$. We note that if $F_n \xrightarrow{w} F$ then $\sigma(F_n) \rightarrow \sigma(F)$ as $n \rightarrow \infty$. Likewise, $\sigma'(\mu)$ is continuous in μ , including at $\mu = \infty$.

Theorem 2. Let $(b_n)_{n \geq 1}$ be an \mathbb{N} -valued sequence with $b_n = o(n)$ as $n \rightarrow \infty$. Suppose that $(F_n)_{n \geq 1}$ is a sequence of probability distributions on \mathbb{Z}_+ with $F_n \in \mathcal{M}_{b_n}$ for each n , satisfying $F_n \xrightarrow{w} F$ for some probability distribution F on $\mathbb{Z}_+ \cup \{\infty\}$, and also $\mu_{F_n} \rightarrow \mu_\infty$ as $n \rightarrow \infty$ for some $\mu_\infty \in [0, \infty]$. Then as $n \rightarrow \infty$, we have

$$\frac{L_1(G_n, F_n)}{n} \xrightarrow{\mathbb{P}} \sigma'(\mu_\infty)\sigma(F), \quad \frac{L_2(G_n, F_n)}{n} \xrightarrow{\mathbb{P}} 0, \tag{3}$$

and

$$\frac{L_1(\tilde{G}_n, F_n)}{n} \xrightarrow{\mathbb{P}} \sigma'(\mu_\infty)\sigma(F), \quad \frac{L_2(\tilde{G}_n, F_n)}{n} \xrightarrow{\mathbb{P}} 0. \tag{4}$$

As a corollary, we may deduce a result about uniform convergence. To state this we need a metrization of convergence in probability. Given random variables X, Y on the same probability space, set $d(X, Y) := \sup\{\varepsilon : \mathbb{P}[|X - Y| > \varepsilon] > \varepsilon\}$.

Theorem 3. Let $(b_n)_{n \geq 1}$ be an \mathbb{N} -valued sequence with $b_n = o(n)$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{M}_{b_n}} d\left(\frac{L_1(G_n, F)}{n}, \sigma'(\mu_F)\sigma(F)\right) = 0, \tag{5}$$

$$\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{M}_{b_n}} d\left(\frac{L_1(\tilde{G}_n, F)}{n}, \sigma'(\mu_F)\sigma(F)\right) = 0, \tag{6}$$

and

$$\lim_{n \rightarrow \infty} \sup_{F \in \mathcal{M}_{b_n}} \frac{(d(L_2(G_n, F), 0))}{n} = \lim_{n \rightarrow \infty} \sup_{F \in \mathcal{M}_{b_n}} \frac{(d(L_2(\tilde{G}_n, F), 0))}{n} = 0.$$

Theorem 1 cannot be deduced directly from Theorems 2 or 3 because distribution F in the statement of Theorem 1 could have unbounded support. We shall give an example at the end of this section to show that the condition $b_n = o(n)$ is needed in Theorems 2 and 3; it is too much to expect the convergence in Theorem 3 to be uniform over *all* probability distributions on $[n]$.

Our proof of Theorems 1, 2, and 3 relies heavily on the following result, which is of independent interest. Given a vertex i of a finite digraph G , let $T_i := T_i(G)$ be the number of vertices j such that $i \rightsquigarrow j$ (including i itself). In epidemic modelling, this may be considered as the final size of the epidemic, i.e. the total number of individuals to become infected starting from a single infected individual i . Let ‘ \xrightarrow{D} ’ denote convergence in distribution.

Theorem 4. Suppose that $(F_n)_{n \geq 1}$ is a sequence of probability distributions on \mathbb{N}_0 such that $F_n \xrightarrow{w} F$ for some probability distribution F on $\mathbb{N}_0 \cup \{\infty\}$ and $\mu_{F_n} \rightarrow \mu_\infty$ for some $\mu_\infty \in [0, \infty]$. Suppose that either $F_n = F$ for all n , or that there exists an \mathbb{N} -valued sequence $(b_n)_{n \in \mathbb{N}}$ such that $b_n = o(n)$ as $n \rightarrow \infty$ and $F_n \in \mathcal{M}_{b_n}$ for all n . Then

$$\frac{T_1(G_n, F_n)}{n} \xrightarrow{D} \sigma'(\mu_\infty)\xi, \tag{7}$$

and

$$\frac{T_1(\tilde{G}_n, F_n)}{n} \xrightarrow{D} \sigma'(\mu_\infty)\xi, \tag{8}$$

where ξ is a Bernoulli random variable with parameter $\sigma(F)$.

Theorem 4 extends a recent result of Comets *et al.* [4], who proved (7) in the case with $F_n = F$ for all n . Only the case of (7) with $\mu_\infty < \infty$ (but with F_n possibly varying with n) is used in proving our other results; the rest of Theorem 4 is included for its own sake.

In the rest of this paper we prove the theorems stated above. Before embarking on the detailed proof we introduce further notation and give some intuitive ideas behind the proof of the theorems.

Given F , let $(Z_m)_{m \geq 0} := (Z_m(F))_{m \geq 0}$ be a Galton–Watson branching process with offspring distribution F . If $F(\{\infty\}) > 0$ then we may have $Z_m = \infty$ for some m , in which case we set $Z_n = \infty$ for all $n \geq m$.

Given also $\mu_\infty \in [0, \infty]$, let $(Z'_m)_{m \geq 0} := (Z'_m(\mu_\infty))_{m \geq 0}$ be a branching process with Poisson offspring distribution with mean μ_∞ , independent of $(Z_m)_{m \geq 0}$ (with $Z_0 = Z'_0 = 1$). Let $T := \sum_{m=0}^\infty Z_m$ and $T' := \sum_{m=0}^\infty Z'_m$. If $\mu_\infty = \infty$ then set $Z'_m := +\infty$ for all $m \geq 1$, and set $T' = \infty$. Then $\sigma(F) = \mathbb{P}[T = \infty]$ and $\sigma'(\mu_\infty) = \mathbb{P}[T' = \infty]$.

Given a vertex i of a digraph G , for $m \in \mathbb{N}$ let $S_{i,m}(G)$ denote the size of the m th out-generation starting from i , i.e. the number of vertices j of G such that there is a directed path from i to j and the shortest such path is of length m . Let $S'_{i,m}(G)$ denote the size of the m th in-generation starting from i , i.e. the number of vertices j of G such that there is a directed path from j to i and the shortest such path is of length m . Set $S_{i,0}(G) = 1$ and $S'_{i,0}(G) = 1$. Set $\mathbf{S}_{i,m}(G) := (S_{i,0}(G), \dots, S_{i,m}(G))$ and set $\mathbf{S}'_{i,m}(G) := (S'_{i,0}(G), \dots, S'_{i,m}(G))$. Then $T_i(G) = \sum_{m=0}^\infty S_{i,m}(G)$. Set $T'_i := T'_i(G) := \sum_{m=0}^\infty S'_{i,m}$, the total number of vertices that can be reached by a backwards directed path from vertex i in the graph G . Let $\mathbb{P}_{n,F}$ (respectively $\tilde{\mathbb{P}}_{n,F}$) denote probability with reference to the graph $G_{n,F}$ (respectively $\tilde{G}_{n,F}$). Let $\mathbb{E}_{n,F}$ (respectively $\tilde{\mathbb{E}}_{n,F}$) denote expectation with reference to the graph $G_{n,F}$ (respectively $\tilde{G}_{n,F}$).

The intuition for Theorem 1 or 2 is that (with $G_{n,F}$ or $\tilde{G}_{n,F}$ or G_{n,F_n} or \tilde{G}_{n,F_n} now denoted G_n for short) for any fixed m and i , the distribution of the random vector $\mathbf{S}_{i,m}(G_n)$ approximates in the large- n limit to that of the branching process $\mathbf{Z}_m := (Z_0, \dots, Z_m)$. Moreover, the indegree of vertex i in G_n is asymptotically Poisson with mean μ_∞ (where in the setting of Theorem 1 we set $\mu_\infty = \mu_F$), and the random vector $\mathbf{S}'_{i,m}(G_n)$ converges in distribution (as $n \rightarrow \infty$) to the random vector $\mathbf{Z}' := (Z'_0, \dots, Z'_m)$ with $\mathbf{S}_{i,m}(G_n)$, $\mathbf{S}'_{i,m}(G_n)$, $\mathbf{S}_{j,m}(G_n)$, and $\mathbf{S}'_{j,m}(G_n)$ asymptotically independent for fixed i, j, m with $j \neq i$. We justify these assertions in Lemma 2 below.

One might reasonably hope that for large K , the condition that $T_i(G_n) > K$ and $T'_i(G_n) > K$ would be approximately necessary and sufficient for i to lie in a giant strong component. Our argument to demonstrate this (in Lemma 4 below) is based on the branching process approximation combined with Theorem 4.

We now give an example to show what can go wrong if we drop the condition $b_n = o(n)$ in Theorem 2, 3, or 4. Suppose that we take $F_n(\{n-1\}) = 2n^{-1}$ and $F_n(\{2\}) = 1 - 2n^{-1}$. Then the limiting distribution F of F_n is a unit point mass at 2 (with $\sigma(F) = 1$) and the limit of μ_F is 4. If the conclusion of Theorem 2 were still true for this example then the $n^{-1}L_1(\tilde{G}_{n,F_n})$ should approximate to $\sigma'(4)$.

Consider, however, the successive in-generations $S'_{1,m}(\tilde{G}_{n,F_n})$, $m \geq 1$. While $S'_{1,1}(\tilde{G}_{n,F_n})$ does converge in distribution to the first generation Z'_1 of a branching process with Poisson offspring distribution with mean 4, the second generation $S'_{1,2}(\tilde{G}_{n,F_n})$ does not converge in distribution to Z'_2 . This is because the vertices of $S'_{1,1}(\tilde{G}_{n,F_n})$ decompose into two types, namely those of outdegree 2 and those of outdegree $n - 1$ (with an asymptotically Poisson number of each type with mean 2), but in subsequent generations $S'_{1,m}(\tilde{G}_n)$ for $m \geq 2$, there

are no vertices of the second type (because all such vertices would be included in the first generation). Therefore the branching process approximation fails.

In our proofs we shall repeatedly use the fact that for any $n \in \mathbb{N}$ and any probability distribution F on \mathbb{Z}_+ , the random digraph $\tilde{G}_{n,F}$ stochastically dominates $G_{n,F}$, i.e.

$$G_{n,F} \prec_{st} \tilde{G}_{n,F}, \tag{9}$$

in the sense that there exist coupled realizations of these two random digraphs for which $G_{n,F}$ (with loops removed and multiple edges reduced to single edges) is a (directed) subgraph of $\tilde{G}_{n,F}$.

3. Proof of Theorem 4

Throughout this section we assume that F_n, F, μ_∞ , and (if applicable) b_n are as in the statement of Theorem 4. Also we write just \mathbb{P}_n (respectively $\tilde{\mathbb{P}}_n, \mathbb{E}_n, \tilde{\mathbb{E}}_n$) for \mathbb{P}_{n,F_n} (respectively $\tilde{\mathbb{P}}_{n,F_n}, \mathbb{E}_{n,F_n}, \tilde{\mathbb{E}}_{n,F_n}$).

Given a digraph $G = (V, E)$ and given $i, j \in V$, we write $i \rightarrow j$ if there is an arc of G from i to j . Given also $B \subset V$ we write $i \rightarrow B$ if $i \rightarrow j$ for at least one $j \in B$. In the following lemma the notation H stands for ‘hit’ and A stands for ‘avoid’.

Lemma 1. Fix $r, s \in \mathbb{Z}_+$, let $H_{r,s}$ be the event $\{1 \rightarrow \{r + 2, \dots, r + 1 + s\}\}$ and let A_r be the complement of the event $H_{0,r}$. Then

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n,F_n}[A_r] = \lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_{n,F_n}[A_r] = 1, \tag{10}$$

and

$$\lim_{n \rightarrow \infty} (n\mathbb{P}_{n,F_n}[H_{r,s} \mid A_r]) = \lim_{n \rightarrow \infty} (n\tilde{\mathbb{P}}_{n,F_n}[H_{r,s} \mid A_r]) = s\mu_\infty. \tag{11}$$

Proof. For $k \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, set $p_{n,k} := F_n(\{k\})$ and $\tilde{p}_{n,k} := p_{n,k}$ for $k \leq n - 2$ with $\tilde{p}_{n,n-1} := \sum_{k \geq n-1} p_{n,k}$ and $\tilde{p}_{n,k} := 0$ for $k \geq n$. Then

$$\tilde{\mathbb{P}}_n[A_r] = \sum_k \tilde{p}_{n,k} \prod_{i=1}^k \left(\frac{n-i-r}{n-i} \right)$$

with the product interpreted as unity for $k = 0$. By Fatou’s lemma, $\tilde{\mathbb{P}}_n[A_r] \rightarrow 1$ as $n \rightarrow \infty$. By (9), we have $\tilde{\mathbb{P}}_n[A_r] \leq \mathbb{P}_n[A_r]$ so $\mathbb{P}_n[A_r] \rightarrow 1$ as well, which gives us (10).

By the union bound, we have

$$\tilde{\mathbb{P}}_n[H_{r,s}] \leq \sum_k \tilde{p}_{n,k} \left(\frac{ks}{n-1} \right) \leq \frac{1}{n-1} s\mu_{F_n}$$

so that $\limsup(n\tilde{\mathbb{P}}_n[H_{r,s}]) \leq s\mu_\infty$, and, therefore, also $\limsup(n\mathbb{P}_n[H_{r,s}]) \leq s\mu_\infty$. Hence,

$$\limsup(n\mathbb{P}_n[H_{r,s} \cap A_r]) \leq s\mu_\infty, \quad \limsup(n\tilde{\mathbb{P}}_n[H_{r,s} \cap A_r]) \leq s\mu_\infty. \tag{12}$$

By conditioning on the outdegree of vertex 1 and then using the estimate $e^x \geq 1 + x$ for $x \in \mathbb{R}$, we have

$$\mathbb{P}_n[H_{r,s} \cap A_r] = \sum_k p_{n,k} \left(1 - \frac{r}{n} \right)^k \left[1 - \left(1 - \frac{s}{n-r} \right)^k \right] \tag{13}$$

$$\geq \sum_k p_{n,k} \left(1 - \frac{r}{n} \right)^k \left[1 - \exp\left(-\frac{ks}{n-r} \right) \right]. \tag{14}$$

Also

$$\tilde{\mathbb{P}}_n[H_{r,s} \cap A_r] = \sum_k \tilde{p}_{n,k} \left[\prod_{i=1}^k \left(1 - \frac{r}{n-i} \right) - \prod_{i=1}^k \left(1 - \frac{r+s}{n-i} \right) \right]. \tag{15}$$

Suppose that $F_n = F$ for all n . Then both in (14) and (15), the expression inside the square brackets is asymptotic to $n^{-1}ks$, and hence by Fatou’s lemma, $\liminf(n\mathbb{P}_n[H_{r,s} \cap A_r]) \geq s\mu_F$ and $\liminf(n\tilde{\mathbb{P}}_n[H_{r,s} \cap A_r]) \geq s\mu_F$. Combined with (12), this gives us (11) in the case with $F_n = F$ for all n .

Now suppose that F_n varies with n but $F_n \in \mathcal{M}_{b_n}$ for all n with $b_n = o(n)$. By (14),

$$n\mathbb{P}_n[H_{r,s} \cap A_r] \geq \left(1 - \frac{r}{n} \right)^{b_n} \sum_k np_{n,k} \left[1 - \exp\left(-\frac{ks}{n-r} \right) \right]$$

and by Taylor’s theorem, for $k \leq b_n$ we have for some $\theta = \theta(n, k) \in (0, 1)$,

$$1 - \exp\left(-\frac{ks}{n-r} \right) = \frac{ks}{n-r} - \left(\frac{1}{2} \right) \left(\frac{ks}{n-r} \right)^2 \exp\left(-\frac{\theta ks}{n-r} \right)$$

so that

$$\left(\frac{n}{ks} \right) \left(1 - \exp\left(-\frac{ks}{n-r} \right) \right) \geq \frac{n}{n-r} - \frac{nks}{(n-r)^2} \geq 1 - \frac{nb_n s}{(n-r)^2}.$$

Hence,

$$n\mathbb{P}_n[H_{r,s} \cap A_r] \geq \left(1 - \frac{r}{n} \right)^{b_n} \left(1 - \frac{nb_n s}{(n-r)^2} \right) \sum_k ksp_{n,k},$$

so that

$$\liminf(n\mathbb{P}_n[H_{r,s} \cap A_r]) \geq s\mu_\infty. \tag{16}$$

Next we estimate the right-hand side of (15). By Taylor’s theorem, we have

$$\prod_{i=1}^k \left(1 - \frac{r}{n-i} \right) \geq \left(1 - \frac{r}{n-k} \right)^k \geq 1 - \frac{kr}{n-k}$$

and

$$\prod_{i=1}^k \left(1 - \frac{r+s}{n-i} \right) \leq \left(1 - \frac{r+s}{n} \right)^k \leq 1 - \frac{k(r+s)}{n} + \frac{k^2(r+s)^2}{2n^2}.$$

Combining these estimates, we obtain

$$\frac{n}{k} \left[\prod_{i=1}^k \left(1 - \frac{r}{n-i} \right) - \prod_{i=1}^k \left(1 - \frac{r+s}{n-i} \right) \right] \geq s - \frac{rk}{n-k} - \frac{(r+s)^2 k}{n}.$$

Hence, by (15), for $F_n \in \mathcal{M}_{b_n}$, we have

$$n\tilde{\mathbb{P}}_n[H_{r,s} \cap A_r] \geq \left(\sum_k ks\tilde{p}_{n,k} \right) (1 + o(1)),$$

and hence $\liminf(n\tilde{\mathbb{P}}_n[H_{r,s} \cap A_r]) \geq s\mu_\infty$. Combined with (16) and (12) this gives us (11) in the case with $F_n \in \mathcal{M}_{b_n}$, completing the proof. \square

Let $(Z_m)_{m \geq 0} = (Z_m(F))_{m \geq 0}$ and $(Z'_m)_{m \geq 0} = (Z'_m(\mu_\infty))_{m \geq 0}$ be branching processes as described in Section 2. We always assume that these branching processes are independent of each other. For later use, we set $T = \sum_{m=0}^\infty Z_m$ and $T' = \sum_{m=1}^\infty Z'_m$. For $m \in \mathbb{Z}_+$, set $\mathbf{Z}_m := (Z_0, \dots, Z_m)$ and $\mathbf{Z}'_m := (Z'_0, \dots, Z'_m)$. Let $(\tilde{\mathbf{Z}}_m, \tilde{\mathbf{Z}}'_m)$ denote an independent copy of $(\mathbf{Z}_m, \mathbf{Z}'_m)$.

Lemma 2. *Let $m \in \mathbb{N}$. Then as $n \rightarrow \infty$, we have*

$$(\mathbf{S}_{1,m}(G_{n,F_n}), \mathbf{S}_{2,m}(G_{n,F_n}), \mathbf{S}'_{1,m}(G_{n,F_n}), \mathbf{S}'_{2,m}(G_{n,F_n})) \xrightarrow{D} (\mathbf{Z}_m, \tilde{\mathbf{Z}}_m, \mathbf{Z}'_m, \tilde{\mathbf{Z}}'_m). \tag{17}$$

Also, (17) holds with G_{n,F_n} replaced by \tilde{G}_{n,F_n} .

Proof. We give the argument for G_{n,F_n} ; the argument for \tilde{G}_{n,F_n} is just the same.

It is rather obvious that $(\mathbf{S}_{1,m}, \mathbf{S}_{2,m})$ converges in distribution to $(\mathbf{Z}_m, \tilde{\mathbf{Z}}_m)$. Formally, this can be proved by induction on m , using (10).

Suppose that we are given (for fixed m) the values of $(\mathbf{S}_{1,m}, \mathbf{S}_{2,m})$ and consider for $r \in \mathbb{N}$ the conditional distribution of $(S'_{1,r}, S'_{2,r})$. We need to show that this converges to the distribution of (Z'_r, \tilde{Z}'_r) . This is done by induction in r and we consider the inductive step, so suppose we also fix for some r the values of $(S'_{1,r}, S'_{2,r})$. Then the value of $S'_{1,r+1}$ is the number of vertices $j \in [n] \setminus \bigcup_{s \leq r} S'_{1,s}$ such that $j \rightarrow S'_{1,r}$ (where we use the notation $S'_{1,s}$ to mean either a set of vertices or its cardinality).

Given $(\mathbf{S}_{1,m}, \mathbf{S}_{2,m}, S'_{1,r}, S'_{2,r})$, the number of $j \in \bigcup_{i \leq m} (S_{1,i} \cup S_{2,i})$ is fixed and the (conditional) probability that any of these has $j \rightarrow S'_{1,r}$ tends to 0. We need to consider the other j s, i.e. with $j \notin \bigcup_{r' \leq r} S'_{1,r'}$ and $j \notin \bigcup_{i \leq m} (S_{1,i} \cup S_{2,i})$.

For these values of j the conditioning means we know there are no arcs from j to the set $\bigcup_{r' < r} S'_{1,r'}$, a fixed number of vertices. Therefore, by Lemma 1, the conditional probability that there is an arc from j to one of the vertices in $S'_{1,r}$ is asymptotic to $n^{-1} S'_{1,r} \mu_\infty$. Hence, by standard binomial-Poisson convergence, the (conditional) distribution of the number of such vertices j such that $j \rightarrow S'_{1,r}$ is asymptotically Poisson with parameter $S'_{1,r} \mu_\infty$, which is the same as the conditional distribution of the next value of the branching process Z'_{r+1} .

We can then apply a similar argument for $S'_{2,r}$ to complete the induction. □

We now prove a part of Theorem 4.

Lemma 3. *Under the assumptions of Theorem 4, the first conclusion (7) holds in the case where $\mu_\infty < \infty$.*

Proof. First consider the graphs G_{n,F_n} with $F_n = F$ for all n and some fixed distribution F on \mathbb{Z}_+ . In this case, we can obtain (7) from a result from [4]. The model in [4] is not described there in terms of a random digraph, but it is not hard to see that it can be interpreted that way. In particular, the random variable $N_n(\tau_n)$ in [4, Theorem 2.2] can be interpreted as being the same as our $T_1(G_{n,F})$. Therefore, by [4, Theorem 2.2], there exists a coupling of the branching process $(Z_m)_{m \geq 0}$ and the sequence of random digraphs $(G_{n,F})_{n \geq 0}$ such that

$$\frac{T_1(G_{n,F})}{n} \xrightarrow{P} \sigma'(\mu_F) \mathbf{1}_{\{T=\infty\}} \quad \text{as } n \rightarrow \infty. \tag{18}$$

Note that our $\sigma'(\mu_F)$ is the p of [4]. The distributional convergence (7) is immediate from (18).

Next, we consider G_{n,F_n} in the case with F_n varying with n , assuming also that $\mu_\infty < \infty$. The proof for this case involves adapting the proof of [4, Theorem 2.2].

The argument in [4] (for fixed F) involves considering an exploration process of the random graph starting from vertex 1, where at each step one of the currently unassigned arcs out of one of the vertices currently being considered is assigned its destination (uniformly at random over $[n]$), and if this destination is a previously unconsidered vertex then this vertex is added to those currently being considered at the next stage. If there are no unassigned arcs out of the current set of vertices under consideration, the exploration process terminates.

Let $(K_{n,i})_{i \in \mathbb{N}}$ (respectively $(K_i)_{i \in \mathbb{N}}$) be a sequence of independent and identically distributed random variables with the distribution F_n (respectively F). For $t \geq 0$ set $R_n(t) := \sum_{i=1}^{\lfloor t \rfloor} K_{n,i}$ and $R(t) := \sum_{i=1}^{\lfloor t \rfloor} K_i$, as in [4, Equation (25)].

Let surv_n denote the event that $1 + R_n(t) - t > 0$ for all $t \in \mathbb{N}$ (or, equivalently, that $R_n(u + 1) - u > 0$ for all $u \in \mathbb{Z}_+$), and let surv denote the event that $1 + R(t) - t > 0$ for all $t \in \mathbb{N}$. Note that $\mathbb{P}[\text{surv}_n] = \sigma(F_n)$ and $\mathbb{P}[\text{surv}] = \sigma(F)$ because the exploration process of a branching process with offspring distribution F can be interpreted as a random walk with successive steps having the distribution of $K_1 - 1$.

For $t \in \mathbb{Z}_+$, let $N_n(t)$ denote the number of coupons collected after n attempts in a coupon collector process with n coupons (starting with $N_n(0) = 1$; see [4] for a formal description), running independently of the random walk $R_n(\cdot)$. For $t \geq 0$, set $S_n(t) := R_n(N_n(\lfloor t \rfloor)) - t$. As described in [4], there is a coupling in which $S_n(t)$ (for $t \in \mathbb{N}$) can be viewed as the total number of unassigned out-arcs from the current set of vertices after t stages of the exploration process up to time τ_n , where τ_n denotes the first t such that $S_n(t) \leq 0$.

We claim that there exists $\varepsilon > 0$ such that

$$\lim_{n \rightarrow \infty} (\mathbb{P}[\tau_n \geq n\varepsilon, \text{surv}_n]) = \sigma(F). \tag{19}$$

This is proved by following the proof of [4, Lemma 4.2] (the notation σ^{GW} in [4] denotes an extinction probability, whereas our $\sigma(F)$ is a survival probability!) Most of the proof of [4, Lemma 4.2] carries over easily to the present setting. We just elaborate on the assertion in that proof that $z^{-1}G(z)^{1-2\varepsilon} < 1$ for some $z < 1$. Here the G of [4] is a probability generating function which we denote by ϕ_n with $\phi_n(z) := \sum_k z^k F_n(\{k\})$. Also set $\phi(z) := \sum_k z^k F(\{k\})$, and note that $\lim_{n \rightarrow \infty} \phi_n(z) = \phi(z)$ for $z \in (0, 1)$.

By Fatou’s lemma, $\liminf_{a \downarrow 0} a^{-1}(1 - \phi(1 - a)) \geq \mu_F$. Assuming $\mu_F > 1$, taking $\varepsilon > 0$ and $\delta > 0$ with $(1 - 2\varepsilon)(\mu_F - \delta) > 1$, and then $a \in (0, 1)$ (close to 0) with $(1 - a(\mu_F - \delta))^{1-2\varepsilon} < (1 - a)$ and also $a^{-1}(1 - \phi(1 - a)) > \mu_F - \delta$, it follows that $\phi(1 - a)^{1-2\varepsilon} < (1 - a)$, and hence for large n that $\phi_n(1 - a)^{1-2\varepsilon} < 1 - a$.

For $q > 0$, we have $n^{-1}\mathbb{E}[R_n(nq)] \rightarrow q\mu_\infty$ as $n \rightarrow \infty$, and since we are assuming $\mu_\infty < \infty$ and $b_n = o(n)$, we have

$$\text{var} \left[\frac{R_n(nq)}{n} \right] = \frac{\lfloor nq \rfloor \text{var}[K_{n,1}]}{n^2} \leq \left(\frac{q}{n} \right) \mathbb{E}K_{n,1}^2 \leq \left(\frac{q}{n} \right) \mathbb{E}[b_n K_{n,1}] \rightarrow 0.$$

By following the proof of [4, Equation (26)], for each positive s , we have

$$\frac{S_n(ns)}{n} \rightarrow (1 - e^{-s})\mu_\infty - s \tag{20}$$

in probability. This weaker version of [4, Equation (26)] suffices to give us [4, Equation (27)].

At the end of the three-line display just after [4, Equation (27)], there are three terms which we wish to show tend to 0. The first term tends to 0 by (19) and the fact that $\sigma(F_n) \rightarrow \sigma(F)$ as $n \rightarrow \infty$. The third term can be shown to tend to 0 using the same fact. To show that the second

term tends to 0, we use the next three-line display of [4]; we need to check that for $\delta > 0$, we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}[\inf S_n(ns), s \in [\varepsilon, \theta - \delta]] = 0, \tag{21}$$

where θ is the solution in $(0, \infty)$ to $(1 - e^{-\theta})/\theta = 1/\mu_\infty$. To see this, set $h := \inf_{s \in [\varepsilon, \theta - \delta]} ((1 - e^{-s})\mu_\infty - s) > 0$ and take $s_1, \dots, s_\ell \in [\varepsilon, \theta - \delta]$ with $s_1 = \varepsilon$, $s_\ell = \theta - \delta$, and $0 < s_{i+1} - s_i < h/4$ for $1 \leq i \leq \ell - 1$. By (20), with probability tending to 1, we have for each i that $n^{-1}S_n(ns_i) > h/2$, and then using the fact that $S_n(t) = R_n(N_n(\lfloor t \rfloor) - t)$, we have $n^{-1}S_n(ns) \geq h/4$ for all $s \in [s_i, s_{i+1}]$ which gives us $S_n(ns) > 0$ for $s \in [\varepsilon, \theta - \delta]$, and hence (21). We can then follow the rest of the argument in [4] to obtain (7) in the case where $\mu_\infty < \infty$. \square

Lemma 4. *It is the case that*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n[1 \rightsquigarrow 2] = \sigma'(\mu_\infty)\sigma(F) \tag{22}$$

and

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[1 \rightsquigarrow 2] = \sigma'(\mu_\infty)\sigma(F).$$

Proof. Write just σ' for $\sigma'(\mu_\infty)$, σ for $\sigma(F)$, and T_1 for $T_1(G)$. We first prove (22). By symmetry, we have $\mathbb{P}_n[1 \rightsquigarrow 2 \mid T_1] = (T_1 - 1)/n$, and therefore by Lemma 3, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}_n[1 \rightsquigarrow 2] = \lim_{n \rightarrow \infty} \mathbb{E}_n \left[\frac{(T_1 - 1)}{n} \right] = \sigma'(\mu_\infty)\sigma(F) \quad \text{if } \mu_\infty < \infty. \tag{23}$$

Now suppose that $\mu_\infty = \infty$. Given $\varepsilon > 0$, we can choose $K \in \mathbb{N}$ such that $\mathbb{P}[T > K] < \sigma + \varepsilon$. By the branching process approximation (Lemma 2), $\lim_{n \rightarrow \infty} \mathbb{P}_n[T_1 > K] = \mathbb{P}[T > K] < \sigma + \varepsilon$, and also by symmetry $\mathbb{P}_n[\{1 \rightsquigarrow 2\} \cap \{T_1 \leq K\}] \leq (K/(n - 1))\mathbb{P}_n[T_1 \leq K]$, which tends to 0, so

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n[1 \rightsquigarrow 2] \leq \sigma + \varepsilon. \tag{24}$$

Given $h \in \mathbb{N}$, let F^h (respectively F_n^h) denote the distribution of a random variable $\min(\xi, h)$ (respectively $\min(\xi^n, h)$), where ξ (respectively ξ^n) is a random variable with distribution F (respectively F_n). Pick $h \in \mathbb{N}$ with $\sigma'(h) > 1 - \varepsilon$ and $\sigma(F^h) \geq \sigma(F)(1 - \varepsilon)$. Here we are using the continuity of the branching process survival probability in the offspring distribution.

Given n , choose $a_n \in \mathbb{N}$ with $\mu_{F_n^{a_n}} \in [h, h + 1]$ (this is possible for all large enough n because $\mu_\infty = \infty$). Note that $a_n \geq h$. Let \mathbb{P}_n^* denote probability for a random digraph of the form of $G_{n, F_n^{a_n}}$.

Suppose first that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Then $F_n^{a_n}$ converges weakly to F , so by monotonicity and the case already proved, we have

$$\liminf \mathbb{P}_n[1 \rightsquigarrow 2] \geq \liminf \mathbb{P}_n^*[1 \rightsquigarrow 2] \geq \sigma'(h)\sigma(F) \geq (1 - \varepsilon)\sigma(F).$$

Suppose instead that a_n is bounded. For any subsequence of n we can take a further subsequence such that along this subsequence a_n tends to a finite limit a so that $F_n^{a_n}$ converges weakly to F^a , and also $\mu_{F_n^{a_n}}$ tends to a limit y (between h and $h + 1$). Also $a \geq h$ so $\sigma(F^a) \geq \sigma(F^h) \geq \sigma(F)(1 - \varepsilon)$. Then by monotonicity and the case already proved, as $n \rightarrow \infty$ along this further subsequence, we have

$$\liminf \mathbb{P}_n[1 \rightsquigarrow 2] \geq \liminf \mathbb{P}_n^*[1 \rightsquigarrow 2] = \sigma'(y)\sigma(F^a) \geq (1 - \varepsilon)^2\sigma(F),$$

and since ε is arbitrary, combined with (24) this gives us (22) for the $\mu_\infty = \infty$ case. Combined with (23) this gives us (22) in full generality.

Now consider \tilde{G}_{n, F_n} . By (9) and (22), we have

$$\liminf_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[1 \rightsquigarrow 2] \geq \liminf_{n \rightarrow \infty} \mathbb{P}_n[1 \rightsquigarrow 2] = \sigma'(\mu_\infty)\sigma(F). \tag{25}$$

On the other hand, given $\varepsilon > 0$, we can choose $K \in \mathbb{N}$ such that $\mathbb{P}[\{T > K\} \cap \{T' > K\}] < \sigma'(\mu_\infty)\sigma(F) + \varepsilon$. By the branching process approximation (Lemma 2),

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[\{T_1 \leq K\} \cup \{T'_2 \leq K\}] = \mathbb{P}[\{T \leq K\} \cup \{T' \leq K\}] > 1 - (\sigma'\sigma + \varepsilon),$$

and also by symmetry $\tilde{\mathbb{P}}_n[\{1 \rightsquigarrow 2\} \cap \{T_1 \leq K\}] \leq (K/(n-1))\tilde{\mathbb{P}}_n[T_1 \leq K]$ which tends to 0, and similarly $\tilde{\mathbb{P}}_n[\{1 \rightsquigarrow 2\} \cap \{T'_2 \leq K\}] \rightarrow 0$, so by the union bound

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[\{1 \rightsquigarrow 2\} \cap (\{T_1 \leq K\} \cup \{T'_2 \leq K\})] = 0.$$

Therefore, $\limsup_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[1 \rightsquigarrow 2] \leq \sigma'\sigma + \varepsilon$. Combined with (25) this shows that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[1 \rightsquigarrow 2] = \sigma'\sigma. \tag{□}$$

Proof of Theorem 4. Set $X_n = n^{-1}T_1(G_{n, F_n})$ and $\tilde{X}_n := n^{-1}T_1(\tilde{G}_{n, F_n})$. Given $\varepsilon > 0$, we may choose finite K such that $\mathbb{P}[T > K] \leq \sigma(F) + \varepsilon/2$. Then by the branching process approximation (Lemma 2), for large enough n we have $\mathbb{P}[\tilde{X}_n \leq \varepsilon] \geq \mathbb{P}[T_1(\tilde{G}_{n, F_n}) \leq K] \geq 1 - \sigma(F) - \varepsilon$. Also \tilde{X}_n stochastically dominates X_n by (9). Hence,

$$\liminf_{n \rightarrow \infty} \mathbb{P}[X_n \leq t] \geq \liminf_{n \rightarrow \infty} \mathbb{P}[\tilde{X}_n \leq t] \geq 1 - \sigma(F), \quad t > 0. \tag{26}$$

In view of Lemma 3, to prove (7) we need to consider only the case with F_n varying with n and $\mu_\infty = \infty$, so we assume that $\mu_\infty = \infty$ for a while. Then $\sigma'(\mu_\infty) = 1$. By Lemma 4, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}X_n = \sigma(F). \tag{27}$$

In the $\sigma(F) = 0$ case, this gives us (7) at once, so we now assume that $\sigma(F) > 0$ too. Let $\varepsilon \in (0, \sigma(F))$. Since $X_n \leq 1$, we have $\mathbb{E}X_n \leq \varepsilon\mathbb{P}[X_n \leq \varepsilon] + (1 - \mathbb{P}[X_n \leq \varepsilon])$ so that $(1 - \varepsilon)\mathbb{P}[X_n \leq \varepsilon] \leq 1 - \mathbb{E}X_n$ and using (27), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}[X_n \leq \varepsilon] \leq \frac{1 - \sigma(F)}{1 - \varepsilon}$$

so that

$$\liminf_{n \rightarrow \infty} \mathbb{P}[X_n > \varepsilon] \geq \frac{\sigma(F) - \varepsilon}{1 - \varepsilon} \geq \sigma(F) - \varepsilon. \tag{28}$$

Let $\delta \in (0, \frac{1}{2})$ and set $\varepsilon = \sigma(F)\delta^2/2$. Suppose that $\mathbb{P}[X_n \leq 1 - \delta \mid X_n > \varepsilon] > \delta$ for infinitely many n . Then for such n , we have

$$\mathbb{E}[X_n \mid X_n > \varepsilon] \leq (1 - \delta)\delta + (1 - \delta) = 1 - \delta^2$$

and, hence, by (26), along this subsequence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{E}X_n &= \limsup_{n \rightarrow \infty} (\mathbb{E}[X_n \mathbf{1}_{\{X_n \leq \varepsilon\}}] + \mathbb{P}[X_n > \varepsilon]\mathbb{E}[X_n \mid X_n > \varepsilon]) \\ &\leq \varepsilon + \sigma(F)(1 - \delta^2) \\ &= \sigma(F)\left(\frac{1 - \delta^2}{2}\right), \end{aligned}$$

which contradicts (27). Hence, $\mathbb{P}[X_n \leq 1 - \delta \mid X_n > \varepsilon] \leq \delta$ for all but finitely many n , and using (28), we have

$$\liminf_{n \rightarrow \infty} \mathbb{P}[X_n > 1 - \delta] \geq (1 - \delta) \left(\frac{1 - \delta^2}{2} \right) \sigma(F).$$

Hence, for $t \in (0, 1)$, we have $\liminf_{n \rightarrow \infty} \mathbb{P}[X_n > t] \geq \sigma(F)$, and with (26) this shows that $\mathbb{P}[X_n \leq t] \rightarrow 1 - \sigma(F)$. This gives us (7).

We still need to prove (8), not only under the assumption $\mu_\infty = \infty$ so now relax this assumption. By (7), for $t < \sigma'(\mu_\infty)$, we have

$$\limsup \mathbb{P}[\tilde{X}_n \leq t] \leq \limsup \mathbb{P}[X_n \leq t] \leq 1 - \sigma(F).$$

Hence, by (26),

$$\lim_{n \rightarrow \infty} \mathbb{P}[\tilde{X}_n \leq t] = 1 - \sigma(F) \quad \text{for } 0 < t < \sigma'(\mu_\infty). \tag{29}$$

Next, let $T' := \sum_{m \geq 0} Z'_m(\mu_\infty)$ as before. Given $\varepsilon > 0$, choose K with $\mathbb{P}[T' > K] \leq \sigma'(\mu_\infty) + \varepsilon/2$. Let $N_{\text{small}} = \sum_{i=1}^n \mathbf{1}_{\{T'_i(\tilde{G}_{n, F_n}) \leq K\}}$. Using Lemma 2, it follows that

$$\tilde{\mathbb{P}}_{n, F_n}[T'_1 \leq K] \rightarrow \mathbb{P}[T' \leq K], \quad \tilde{\mathbb{P}}_{n, F_n}[T'_1 \leq K, T'_2 \leq K] \rightarrow (\mathbb{P}[T' \leq K])^2,$$

and hence $\mathbb{E}[n^{-1} N_{\text{small}}] \rightarrow \mathbb{P}[T' \leq K]$ and $\text{var}[n^{-1} N_{\text{small}}] \rightarrow 0$. Hence,

$$\mathbb{P}[N_{\text{small}} \leq n(1 - \sigma'(\mu_\infty) - \varepsilon)] \leq \mathbb{P}\left[N_{\text{small}} \leq n\left(\mathbb{P}[T' \leq K] - \frac{\varepsilon}{2}\right)\right] \rightarrow 0.$$

Given n , let I_{small} be the set of indices $j \in [n]$ such that $T'_j(\tilde{G}_{n, F_n}) \leq K$. Then

$$\sum_{i=1}^n \mathbf{1}_{\{\sum_{j \in I_{\text{small}}} \mathbf{1}_{\{i \rightsquigarrow j \text{ in } \tilde{G}_{n, F_n}\}} > \varepsilon n\}} \leq \frac{K}{\varepsilon}$$

so that by symmetry

$$\tilde{\mathbb{P}}_{n, F_n} \left[\sum_{j \in I_{\text{small}}} \mathbf{1}_{\{1 \rightsquigarrow j\}} > \varepsilon n \right] \leq \frac{K}{n\varepsilon},$$

and hence setting $\sigma' = \sigma'(\mu_\infty)$, by the union bound we have

$$\mathbb{P}[\tilde{X}_n \geq \sigma' + 2\varepsilon] \leq \mathbb{P}[N_{\text{small}} \leq n(1 - \sigma' - \varepsilon)] + \tilde{\mathbb{P}}_{n, F_n} \left[\sum_{j \in I_{\text{small}}} \mathbf{1}_{\{1 \rightsquigarrow j\}} > \varepsilon n \right] \rightarrow 0.$$

Combined with (29) this gives us (8). □

4. Proof of Theorems 1, 2, and 3

In this section we make the same assumptions about F, F_n, μ_∞ , and (if applicable) b_n , and use the same notation $\mathbb{P}_n, \tilde{\mathbb{P}}_n, \mathbb{E}_n, \tilde{\mathbb{E}}_n$, as we did in the previous section. Also T and T' are as in the previous section, and we set $\sigma := \sigma(F)$ and $\sigma' := \sigma'(\mu_\infty)$.

Lemma 5. *It is the case that*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n[1 \rightsquigarrow 2] = (\sigma' \sigma)^2, \tag{30}$$

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[1 \rightsquigarrow 2] = (\sigma' \sigma)^2. \tag{31}$$

Proof. We prove (30); the proof of (31) is just the same but with \mathbb{P}_n replaced by $\tilde{\mathbb{P}}_n$ throughout.

If $\mu_F \leq 1$ then $\sigma' \sigma = 0$ and (30) follows from Lemma 4, so now assume that $\mu_F > 1$. Then $\mu_\infty \geq \mu_F > 1$ by Fatou’s lemma, so $\sigma' \sigma > 0$. Choose K such that $\mathbb{P}[\{T > K\} \cap \{T' > K\}] \approx \sigma' \sigma$. The \approx means the approximation can be made arbitrarily close to 0 by the choice of K (but we now view K as fixed).

Let T_i, T'_i be as in Section 2. Then by symmetry, $\mathbb{P}_n[\{1 \rightsquigarrow 2\} \cap \{T_1 \leq K\}] \leq (K/(n - 1))\mathbb{P}_n[T_1 \leq K]$, which tends to 0 as $n \rightarrow \infty$. Similarly, $\lim_{n \rightarrow \infty} \mathbb{P}_n[\{1 \rightsquigarrow 2\} \cap \{T'_2 \leq K\}] = 0$ so that

$$\lim_{n \rightarrow \infty} \mathbb{P}_n[\{1 \rightsquigarrow 2\} \setminus (\{T_1 > K\} \cap \{T'_2 > K\})] = 0. \tag{32}$$

Also, by the branching process approximation (Lemma 2),

$$\lim_{n \rightarrow \infty} \mathbb{P}_n[\{T_1 > K\} \cap \{T'_2 > K\}] = \mathbb{P}[\{T > K\} \cap \{T' > K\}] \approx \sigma' \sigma.$$

Therefore, by (32) and Lemma 4, with Δ denoting the symmetric difference of two sets,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n[\{1 \rightsquigarrow 2\} \Delta (\{T_1 > K\} \cap \{T'_2 > K\})] \approx 0.$$

Similarly,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n[\{2 \rightsquigarrow 1\} \Delta (\{T_2 > K\} \cap \{T'_1 > K\})] \approx 0,$$

and, therefore,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n[\{1 \rightsquigarrow 2\} \Delta (\{T_1 > K\} \cap \{T'_1 > K\} \cap \{T_2 > K\} \cap \{T'_2 > K\})] \approx 0.$$

By the branching process approximation (Lemma 2), and the inclusion–exclusion formula,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}[\{T_1 > K\} \cap \{T'_1 > K\} \cap \{T_2 > K\} \cap \{T'_2 > K\}] &= (\mathbb{P}[T > K])^2 (\mathbb{P}[T' > K])^2 \\ &\approx (\sigma' \sigma)^2, \end{aligned}$$

and (30) follows. □

Proof of Theorems 1 and 2. We simultaneously prove (1) and (3); the proof of (2) and (4) is just the same (with \mathbb{P}_n replaced by $\tilde{\mathbb{P}}_n$ throughout). Choose a large constant K such that

$$\mathbb{P}[T > K] \approx \sigma, \quad \mathbb{P}[T' > K] \approx \sigma'$$

with \approx interpreted as in the preceding proof. Given n , define the events

$$E_i := \{T_i \leq K\} \cup \{T'_i \leq K\}, \quad i \in [n].$$

By the branching process approximation (Lemma 2), as $n \rightarrow \infty$, we have

$$\mathbb{P}_n[E_1] \rightarrow \mathbb{P}[\{T \leq K\} \cup \{T' \leq K\}], \quad \mathbb{P}_n[E_1 \cap E_2] \rightarrow (\mathbb{P}[\{T \leq K\} \cup \{T' \leq K\}])^2.$$

Therefore, setting $N_{\text{small}} := \sum_{i=1}^n \mathbf{1}_{E_i}$, we have $\text{var}(N_{\text{small}}/n) \rightarrow 0$ and

$$\frac{N_{\text{small}}}{n} \xrightarrow{\mathbb{P}} \mathbb{P}[\{T \leq K\} \cup \{T' \leq K\}] \approx 1 - \sigma' \sigma. \tag{33}$$

Suppose first that $\mu_F \leq 1$. Then $\sigma' \sigma = 0$, and given any $\varepsilon > 0$, we may choose K such that if $n > K/\varepsilon$, by (33), we have

$$\mathbb{P}_n[L_1(G) > \varepsilon n] \leq \mathbb{P}_n[N_{\text{small}} < (1 - \varepsilon)n] \rightarrow 0.$$

This gives us (1) and (3) in the case where $\mu_F \leq 1$.

Now suppose that $\mu_F > 1$. Then $\sigma = \sigma(F) > 0$, and by Fatou's inequality $\mu_\infty \geq \mu_F > 1$ so $\sigma' = \sigma'(\mu_\infty) > 0$. Let $N_{>K}$ be the number of vertices of G_n lying in strongly connected components of order greater than K . For $i \geq 1$, let us write just L_i for $L_i(G)$. Let l be the last i such that $L_i > K$. Then

$$\begin{aligned} \mathbb{P}_n[1 \rightsquigarrow 2 \mid (L_1, L_2, \dots)] &\leq \sum_i \left(\frac{L_i}{n}\right)^2 \leq \sum_{i \leq l} \left(\frac{L_i}{n}\right)^2 + \sum_{l < i \leq n} \left(\frac{K}{n}\right)^2 \\ &\leq \frac{L_1}{n} \left(\sum_{i \leq l} \frac{L_i}{n}\right) + \frac{K^2}{n} \\ &= \frac{L_1}{n} \frac{N_{>K}}{n} + \frac{K^2}{n}. \end{aligned} \tag{34}$$

Note that $N_{>K}$ is determined by (L_1, L_2, \dots) . Let $\varepsilon \in (0, 1)$. By (34), we have

$$\mathbb{P}_n[1 \rightsquigarrow 2 \mid N_{>K} < (1 + \varepsilon^2)\sigma' \sigma n, L_1 \leq (1 - \varepsilon)\sigma' \sigma n] \leq (1 - \varepsilon)(1 + \varepsilon^2)(\sigma' \sigma)^2 + o(1) \tag{35}$$

and (using $L_1 \leq \max(N_{>K}, K)$) also

$$\mathbb{P}_n[1 \rightsquigarrow 2 \mid N_{>K} < (1 + \varepsilon^2)\sigma' \sigma n, L_1 > (1 - \varepsilon)\sigma' \sigma n] \leq (1 + \varepsilon^2)^2(\sigma' \sigma)^2 + o(1). \tag{36}$$

Now $N_{>K} \leq n - N_{\text{small}}$ so by (33), given $\varepsilon > 0$, we can choose K so that

$$\mathbb{P}_n\left[\frac{N_{>K}}{n} < (1 + \varepsilon^2)\sigma' \sigma\right] \rightarrow 1. \tag{37}$$

Then by (35) and (36), we have

$$\begin{aligned} \mathbb{P}_n[1 \rightsquigarrow 2] &\leq (\sigma' \sigma)^2((1 - \varepsilon)(1 + \varepsilon^2)\mathbb{P}_n[L_1 \leq (1 - \varepsilon)\sigma' \sigma n] \\ &\quad + (1 + \varepsilon^2)^2(1 - \mathbb{P}_n[L_1 \leq (1 - \varepsilon)\sigma' \sigma n])) + o(1) \\ &\leq (\sigma' \sigma)^2(1 + \varepsilon(4\varepsilon - \mathbb{P}_n[L_1 \leq (1 - \varepsilon)\sigma' \sigma n])) + o(1) \end{aligned}$$

and by comparison with (30) this shows that

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n[L_1 \leq (1 - \varepsilon)\sigma' \sigma n] \leq 4\varepsilon.$$

Together with (37) and the fact that $L_1 \leq \max(N_{>K}, K)$, this gives us the first part of (1) and of (3).

Since $L_2 \leq \max(N_{>K} - L_1, K)$, for $n > K/(2\varepsilon^2\sigma'\sigma)$ with ε, K as in (37), we have

$$\begin{aligned} \mathbb{P}_n \left[\frac{L_2}{n} > 2\varepsilon^2\sigma'\sigma \right] &\leq \mathbb{P}_n \left[\frac{N_{>K} - L_1}{n} > 2\varepsilon^2\sigma'\sigma \right] \\ &\leq \mathbb{P}_n \left[\frac{N_{>K}}{n} > (1 + \varepsilon^2)\sigma'\sigma \right] + \mathbb{P}_n \left[\frac{L_1}{n} < (1 - \varepsilon^2)\sigma'\sigma \right], \end{aligned}$$

which tends to 0 by (37) and the first part of (1) or (3). This shows that $L_2/n \xrightarrow{\mathbb{P}} 0$, which is the second part of (1) and of (3). \square

Proof of Theorem 3. Suppose that (5) fails. Then we can find a sequence of distributions $F_n \in \mathcal{M}_{b_n}$ such that $\limsup_{n \rightarrow \infty} \sup_{F \in \mathcal{M}_{b_n}} d(n^{-1}L_1(G_{n,F_n}), \sigma'(\mu_{F_n})\sigma(F_n)) > 0$. By taking a subsequence, we may assume that F_n converges to a limiting distribution F on $\mathbb{Z}_+ \cup \{\infty\}$ and μ_{F_n} converges to a (possibly infinite) limit μ_∞ . But then we would have a contradiction of Theorem 2.

This gives us (5). The proof of (6), and of the stated results for $L_2(G_n)$ and $L_2(\tilde{G}_n)$, is similar. \square

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