STRUCTURE THEORY AND STABLE RANK FOR C*-ALGEBRAS OF FINITE HIGHER-RANK GRAPHS

DAVID PASK ⬤, ADAM SIERAKOWSKI ⬤ AND AIDAN SIMS ⬤

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong NSW 2522, Australia (dpask@uow.edu.au; asierako@uow.edu.au; asims@uow.edu.au)

Abstract   We study the structure and compute the stable rank of C^*-algebras of finite higher-rank graphs. We completely determine the stable rank of the C^*-algebra when the k-graph either contains no cycle with an entrance or is cofinal. We also determine exactly which finite, locally convex k-graphs yield unital stably finite C^*-algebras. We give several examples to illustrate our results.

Introduction

Stable rank is a non-commutative analogue of topological dimension for C^*-algebras introduced by Rieffel in the early 1980s [34], and widely used and studied ever since (see, for example [3, 8, 11, 12, 15, 18, 24, 30, 33, 36, 44, 46]). The condition of having stable rank 1, meaning that the invertible elements are dense in the C^*-algebra, has attracted significant attention, in part due to its relevance to the classification of C^*-algebras. Specifically, all separable, simple, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by their Elliott invariant [6, 13, 19, 29, 40], and stable rank distinguishes two key cases: the stably finite C^*-algebras in this class have stable rank 1 [36, Theorem 6.7], while the remainder are Kirchberg algebras with stable rank infinity [34, Proposition 6.5]. It follows that a simple C^*-algebra whose stable rank is finite but not equal to 1 does not belong to the class of C^*-algebras classified by their Elliott invariants.

Higher rank graphs (or k-graphs) Λ are generalizations of directed graphs. They give rise to an important class of C^*-algebras $C^*(\Lambda)$ due to their simultaneous concreteness of presentation and diversity of structure [9, 22, 26]. They provide good test cases for general theory [5, 43] and have found unexpected applications in general C^*-algebra theory. For example, the first proof that Kirchberg algebras in the UCT class have nuclear dimension 1 proceeded by realizing them as direct limits of 2-graph C^*-algebras [39]. Nevertheless, and despite their deceptively elementary presentation in terms of generators and relations, k-graph C^*-algebras, in general, remain somewhat mysterious – for example, it remains an unanswered question whether all simple k-graph C^*-algebras are $\mathcal{Z}$-stable and hence

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classifiable. This led us to investigate their stable rank. In this paper, we shed some light on how to compute the stable rank of k-graph C^*-algebras; though unfortunately, the simple C^*-algebras to which our results apply all have stable rank either 1 or ∞, so we obtain no new information about $\mathcal{Z}$-stability or classifiability.

This paper focuses on unital k-graph C^*-algebras. For $k = 1$, i.e., for directed graph C^*-algebras (unital or not), a complete characterization of stable rank has been obtained [10, 17, 18]. In this paper, our main contribution is for $k \geq 2$, a characterization of stable rank for C^*-algebras associated with

(1) finite k-graphs that have no cycle with an entrance, and

(2) finite k-graphs that are cofinal.

In the first case (1), we prove that such k-graphs are precisely the ones for which the associated C^*-algebra is stably finite. Partial results on how to characterize stably finite k-graph C^*-algebras have appeared in the past, see [7, 28, 41]. It turns out that in the unital situation, all such C^*-algebras are direct sums of matrix algebras over commutative tori of dimension at most k; the precise dimensions of the tori are determined by the degrees of certain cycles (called initial cycles) in the k-graph. Their C^*-algebraic structure is therefore independent of the factorization property that determines how the one-dimensional subgraphs of a k-graph fit together to give it its k-dimensional nature.

We also settle the second case (2) where the k-graphs are cofinal using our characterization of stable finiteness in combination with a technical argument on the von Neumann equivalence of (direct sums of) vertex projections. We initially obtained this result for selected 2-graphs using Python.

We now give a brief outline of the paper; Figure 1 may also help the reader to navigate. In §1, we introduce terminology, including the notion (and examples) of an initial cycle. In §2, we consider the stably finite case. In Proposition 2.7, we prove that the stable finiteness of $C^*(\Lambda)$ is equivalent to the condition that no cycle in the k-graph Λ has an entrance. In Theorem 2.5, we characterize the structure of $C^*(\Lambda)$ in the stably finite case and compute the stable rank of such algebras in Corollary 2.9. In §3, we characterize which k-graphs yield C^*-algebras with stable rank 1 (Theorem 3.1 and Corollary 3.4) and show how the dimension of the tori that form the components of their spectra can be read off from (the skeleton of) the k-graph, see Proposition 3.3.

In §4, we look at k-graphs which are cofinal. Firstly, in Proposition 4.1, we study the special case when $C^*(\Lambda)$ is simple. Then, in Theorem 4.4, we compute stable rank when Λ is cofinal and contains a cycle with an entrance (so $C^*(\Lambda)$ is not stably finite). In §5, we illustrate the difficulty in the remaining case where Λ is not cofinal and contains a cycle with an entrance by considering 2-vertex 2-graphs with this property. We are able to compute the stable rank exactly for all but three classes of examples, for which the best we can say is that the stable rank is either 2 or 3.

1. Background

In this section, we recall the definition of stable rank, and the notions of stably finite and purely infinite C^*-algebras. We also recall the definitions of k-graphs and their associated C^*-algebras. We discuss the path space of a locally convex k-graph and describe initial

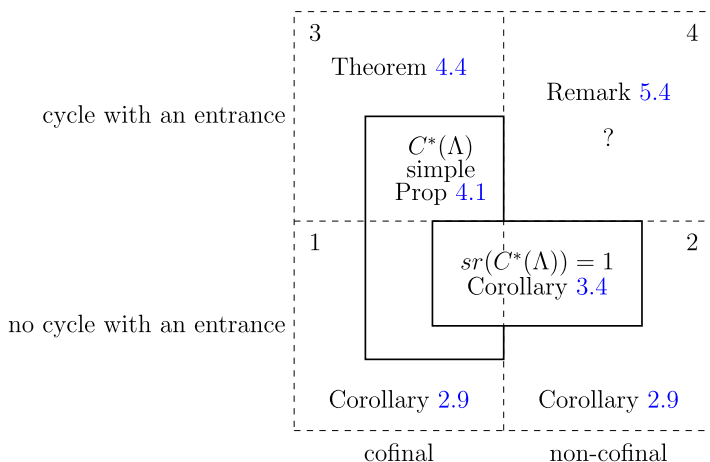Figure 1. Overview of some of our results. The '?' indicates unknown stable rank.

cycles and their periodicity group. The reader familiar with these terms can skim through or skip this section.

1.1. Stable rank of *C**-algebras

Let A be a unital C^*-algebra. Following [2], let

$$Lg_n(A) := \left\{ (x_i)_{i=1}^n \in A^n : \exists (y_i)_{i=1}^n \in A^n \text{ such that } \sum_{i=1}^n y_i x_i = 1 \right\}.$$

The *stable rank* of A, denoted $sr(A)$, is the smallest n such that $Lg_n(A)$ is dense in A^n, or ∞ if there is no such n. For a C^*-algebra A without a unit, we define its stable rank to be that of its minimal unitization $\widetilde{A}$.

A C^*-algebra A has stable rank one if and only if the set A^{-1} of invertible elements in A is dense in A. We will make frequent use of the following key results concerning the stable rank of C^*-algebras of functions on tori, matrix algebras, stable C^*-algebras and direct sums later in the paper:

(1) $sr(C(\mathbb{T}^\ell)) = \lfloor \ell/2 \rfloor + 1$;

(2) $sr(M_n(A)) = \lceil (sr(A) - 1)/n \rceil + 1$;

(3) $sr(A \otimes \mathcal{K}) = 1$ if $sr(A) = 1$, and $sr(A \otimes \mathcal{K}) = 2$ if $sr(A) \neq 1$; and

(4) $sr(A \oplus B) = \max(sr(A), sr(B))$.

Stable rank is in general not preserved under Morita equivalence (unless the stable rank is one). For more details, see [34].

1.2. Stably finite and purely infinite C*-algebras

A projection in a C^*-algebra is said to be *infinite* if it is (von Neumann) equivalent to a proper subprojection of itself. If a projection is not infinite, it is said to be *finite*. A unital C^*-algebra A is said to be *finite* if its unit is a finite projection, and *stably finite* if $M_n(A)$ is finite for each positive integer n [37, Definition 5.1.1]. We refer to [7, 41] for results about stably finite graph C^*-algebras.

A simple C^*-algebra A is *purely infinite* if every non-zero hereditary sub-C^*-algebra of A contains an infinite projection. For the definition when A is non-simple, we refer the reader to [20].

1.3. Higher rank graphs

Following [21, 25, 32], we recall some terminology for k-graphs. For $k \geq 1$, a *k-graph* is a non-empty, countable, small category equipped with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n, and we define $\Lambda^n := d^{-1}(n)$. If $k = 1$, then Λ is isomorphic to the free category generated by the directed graph with edges Λ^1 and vertices Λ^0. The generators of $\mathbb{N}^k$ are denoted $e_1, \ldots, e_k$, and n_i denotes the ith coordinate of $n \in \mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we write $m \leq n$ if $m_i \leq n_i$ for all i, and we write $m \vee n$ for the coordinatewise maximum of m and n, and $m \wedge n$ for the coordinatewise minimum of m and n.

If Λ is a k-graph, its *vertices* are the elements of Λ^0. The factorization property implies that the vertices are precisely the identity morphisms, and so can be identified with the objects. For each $\lambda \in \Lambda$ the *source* $s(\lambda)$ is the domain of λ, and the *range* $r(\lambda)$ is the codomain of λ (strictly speaking, $s(\lambda)$ and $r(\lambda)$ are the identity morphisms associated with the domain and codomain of λ). Given $\lambda, \mu \in \Lambda$, $n \in \mathbb{N}^k$ and $E \subseteq \Lambda$, we define

$$\lambda E := \{\lambda\nu : \nu \in E, r(\nu) = s(\lambda)\},$$

$$E\mu := \{\nu\mu : \nu \in E, s(\nu) = r(\mu)\},$$

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}.$$

We say that a k-graph Λ is *row-finite* if $|v\Lambda^n| < \infty$ is finite for each $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, *finite* if $|\Lambda^n| < \infty$ for all $n \in \mathbb{N}^k$, and *locally convex* if for all distinct $i, j \in \{1, \ldots, k\}$, and all paths $\lambda \in \Lambda^{e_i}$ and $\mu \in \Lambda^{e_j}$ such that $r(\lambda) = r(\mu)$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are non-empty.

Standing Assumption. *We have two standing assumptions. The first is that all of our k-graphs Λ are finite. This implies, in particular, that they are row-finite. The second is that all of our k-graphs Λ are locally convex.*

A vertex v is called a *source* if there exist $i \leq k$ such that $v\Lambda^{e_i} = \emptyset$. The term *cycle*, distinct from 'generalised cycle' [14], will refer to a path $\lambda \in \Lambda \setminus \Lambda^0$ such that $r(\lambda) = s(\lambda)$.

We will occasionally illustrate k-graphs as k-coloured graphs. We refer to [16] for the details, but in short there is a one-to-one correspondence between k-graphs and k-coloured graphs together with factorization rules for bi-coloured paths of length 2 satisfying an associativity condition [16, Equation (3.2)].

1.4. Higher rank graph C^*-algebras

Let Λ be a row-finite, locally convex k-graph. Following [32], a *Cuntz–Krieger Λ-family* in a C^*-algebra B is a function $s : \lambda \mapsto s_\lambda$ from Λ to B such that

(1) $\{s_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;

(2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(4) $s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The C^*-algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz–Krieger Λ-family. It is unital if and only if $|\Lambda^0| < \infty$, in which case $1 = \sum_{v \in \Lambda^0} s_v$. The universal family in $C^*(\Lambda)$ is denoted $\{s_\lambda : \lambda \in \Lambda\}$.

1.5. The path space of a k-graph

Let Λ be a k-graph. For each path $\lambda \in \Lambda$, and $m \leq n \leq d(\lambda)$, we denote by $\lambda(m, n)$ the unique element of Λ^{n-m} such that $\lambda = \lambda'\lambda(m, n)\lambda''$ for some $\lambda', \lambda'' \in \Lambda$ with $d(\lambda') = m$ and $d(\lambda') = d(\lambda) - n$. We abbreviate $\lambda(m, m)$ by $\lambda(m)$. A k-graph morphism between two k-graphs is a degree preserving functor.

Following [14], for each $m \in (\mathbb{N} \cup \{\infty\})^k$, we define a k-graph $\Omega_{k,m}$ by $\Omega_{k,m} = \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$ with range map $r(p, q) = (p, p)$, source map $s(p, q) = (q, q)$, and degree map $d(p, q) = q - p$. We identify $\Omega_{k,m}^0$ with $\{p \in \mathbb{N}^k : p \leq m\}$ via the map $(p, p) \mapsto p$. Given a k-graph and $m \in \mathbb{N}^k$ there is a bijection from Λ^m to the set of morphisms $x : \Omega_{k,m} \to \Lambda$, given by $\lambda \mapsto ((p, q) \mapsto \lambda(p, q))$; the inverse is the map $x \mapsto x(0, m)$. Thus, for each $m \in \mathbb{N}^k$, we may identify the collection of k-graph morphisms from $\Omega_{k,m}$ to Λ with Λ^m. We extend this idea beyond $m \in \mathbb{N}^k$ as follows: Given $m \in (\mathbb{N} \cup \{\infty\})^k \setminus \mathbb{N}^k$, we regard each k-graph morphism $x : \Omega_{k,m} \to \Lambda$ as a path of degree m in Λ and write $d(x):=m$ and $r(x):=x(0)$; we denote the set of all such paths by Λ^m. We denote by W_Λ the collection $\bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^m$ of all paths in Λ; our conventions allow us to regard Λ as a subset of W_Λ. We call W_Λ the *path space* of Λ. We set

$$\Lambda^{\leq \infty} = \big\{x \in W_\Lambda : x(n)\Lambda^{e_i} = \emptyset \text{ whenever } n \leq d(x) \text{ and } n_i = d(x)_i\big\},$$

and for $v \in \Lambda^0$, we define $v\Lambda^{\leq\infty} := \{x \in \Lambda^{\leq\infty} : r(x) = v\}$. Given a cycle τ, we define τ^∞ (informally written as $\tau^\infty := \tau\tau\tau\ldots$) to be the unique element of W_Λ such that $d(\tau^\infty)_i$ is equal to ∞ when $d(\tau)_i > 0$ and equal to 0 when $d(\tau)_i = 0$, and such that $(\tau^\infty)(n \cdot d(\tau), (n + 1) \cdot d(\tau)) = \tau$ for all $n \in \mathbb{N}$.

1.6. Initial cycles and their periodicity group

In this section, we introduce initial cycles and their associated periodicity group. As we will see in Corollary 2.9 and Theorem 3.1, the periodicity group plays an important role in the characterization of stable rank.

Let λ be a cycle in a row-finite, locally convex k-graph Λ. We say λ is a *cycle with an entrance* if there exists $\tau \in r(\lambda)\Lambda$ such that $d(\tau) \leq d(\lambda^\infty)$ and $\tau \neq \lambda^\infty(0, d(\tau))$.

Definition 1.1 (Evans and Sims [14]). Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. We call $\mu \in \Lambda$ an *initial cycle* if $r(\mu) = s(\mu)$ and if $r(\mu)\Lambda^{e_i} = \emptyset$ whenever $d(\mu)_i = 0$.

Remark 1.2. While the wording of Definition 1.1 differs from that of [14], we will show (in Proposition 2.7) that for any k-graph Λ, a path $\mu \in \Lambda$ is an initial cycle in the sense of Definition 1.1 if and only if it is an initial cycle in the sense of [14].

Remark 1.3. An initial cycle may be trivial, in the sense that it has degree 0, so it is in fact a vertex. This vertex must then be a source, as for example, w_4 in Figure 5. It is not true that every source is an initial cycle; for example, w_3 in Figure 5 is a source but not an initial cycle.

As in [14], we let $\mathrm{IC}(\Lambda)$ denote the collection of all initial cycles in Λ; if Λ^0 is finite and Λ has no cycle with an entrance, then $\mathrm{IC}(\Lambda)$ is non-empty – see Lemma 2.2. A vertex $v \in \Lambda^0$ is said to be *on the initial cycle* μ if $v = \mu(p)$ for some $p \le d(\mu)^*$. We let $(\mu^\infty)^0$ denote the collection of all vertices on an initial cycle μ and let $\sim$ be the equivalence relation on $\mathrm{IC}(\Lambda)$ given by $\mu \sim \nu \Leftrightarrow (\mu^\infty)^0 = (\nu^\infty)^0$.

Remark 1.4. For a finite, locally convex k-graph that has no cycle with an entrance, each initial cycle is an 'initial segment' in the following sense:

(1) Every path with range on the initial cycle is in the initial cycle, so paths can not 'enter' an initial cycle (see Lemma 2.1).

Without the assumption that Λ^0 is finite and Λ has no cycle with an entrance property (1) might fail. This is, for example, the case for the 1-graph with one vertex and two edges representing the Cuntz algebra $\mathcal{O}_2$. This suggests that, in general, a different terminology should perhaps be used.

As in [14], we associate a group G_μ to each initial cycle μ. Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. Let μ be an initial cycle in Λ. If μ is not a vertex, we define

$$G_\mu := \{m - n : n, m \le d(\mu^\infty), \mu^\infty(m) = \mu^\infty(n)\}, \tag{1.1}$$

otherwise, we let $G_\mu := \{0\}$.

Definition 1.5. By [14, Lemma 5.8], G_μ is a subgroup of $\mathbb{Z}^k$, and hence isomorphic to $\mathbb{Z}^{\ell_\mu}$ for some $\ell_\mu \in \{0, \ldots, k\}$: we often refer to ℓ_μ as the *rank* of G_μ.

Remark 1.6. It turns out that $\ell_\mu = |\{i \le k : d(\mu)_i > 0\}|$ – see Proposition 3.3.

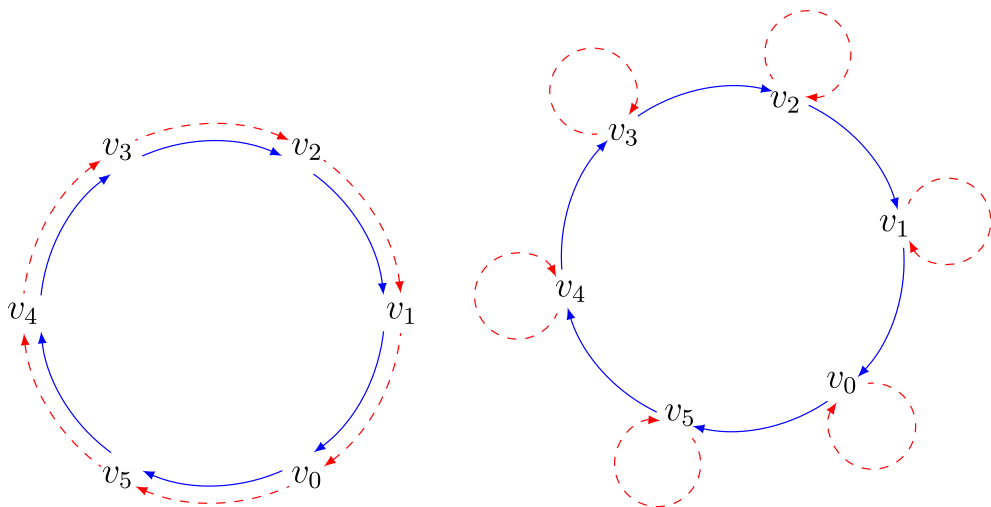* This is not the definition in [14, p. 202], but we expect this was the intended definition.

Figure 2. Two 2-graphs Λ_1, Λ_2, each containing lots of initial cycles, but only one such up to $\sim$ - equivalence.

1.7. Examples of initial cycles

Figure 2 illustrates[†] two examples of 2-graphs Λ_1 and Λ_2 containing lots of initial cycles. In fact, a cycle in either Λ_1 or Λ_2 is an initial cycle precisely if it contains edges of both colours. Each initial cycle in either Λ_i visits every vertex, so any two initial cycles in Λ_i are $\sim$ - equivalent. A computation shows that for each initial cycle μ in either Λ_i, $G_\mu \cong \mathbb{Z}^2$. Notice that each vertex on a cycle in either Λ_i has exactly one red (dashed) and one blue (solid) incoming and outgoing edge and exactly one infinite path with range at that vertex.

Below, we illustrate how the stable rank of each $C^*(\Lambda_i)$ can be computed. For this, we need some definitions and a lemma.

(1) Fix an integer $n \geq 1$, let L_n denote the connected 1-graph with n vertices $v_0, \ldots, v_{n-1}$ and n morphisms $f_0, \ldots, f_{n-1}$ of degree 1 such that $s(f_i) = v_{i+1 \,(\mathrm{mod}\ n)}$ and $r(f_i) = v_i$ for $0 \leq i \leq n-1$.

(2) Let (Λ_1, d_1) and (Λ_2, d_2) be k_1-, k_2-graphs respectively, then $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a $(k_1 + k_2)$-graph where $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \to \mathbb{N}^{k_1+k_2}$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ [21, Proposition 1.8].

(3) Let $f : \mathbb{N}^\ell \to \mathbb{N}^k$ be a monoid morphism. If (Λ, d) is a k-graph we may form the ℓ-graph $f^*(\Lambda)$ as follows: $f^*(\Lambda) = \{(\lambda, n) : d(\lambda) = f(n)\}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$ and $r(\lambda, n) = r(\lambda)$ [21, Example 1.10].

[†] We have illustrated Λ_1 and Λ_2 as 2-coloured graphs, we refer to [16] for details on how to visualize k-graphs as colours graphs.

(4) Let Λ be a k–graph and define $g_i : \mathbb{N} \to \mathbb{N}^k$ by $g_i(n) = n e_i$ for $1 \le i \le k$ (so $\ell = 1$). The 1-graphs $\Lambda_i := g_i^*(\Lambda)$ are called the *coordinate graphs* of Λ.

Lemma 1.7 (Kumjian and Pask [21, Proposition 1.11, Corollary 3.5(iii), Corollary 3.5(iv)]).

(1) *Let (Λ_i, d_i) be k_i-graphs for $i = 1, 2$, then $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ via the map $s_{(\lambda_1, \lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ for $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.*

(2) *Let Λ be a k-graph and $f : \mathbb{N}^\ell \to \mathbb{N}^k$ a surjective monoid morphism. Then $C^*(f^*(\Lambda)) \cong C^*(\Lambda) \otimes C(\mathbb{T}^{\ell-k})$.*

Let Λ be a 1-graph and define $f_1 : \mathbb{N}^2 \to \mathbb{N}$ by $(m_1, m_2) \mapsto m_1 + m_2$. Then $f_1^*(L_6)$ is isomorphic to the 2-graph Λ shown on the left in Figure 2. The 2-graph $L_6 \times L_1$ is isomorphic to the 2-graph shown on the right in Figure 2. Using Lemma 1.7 and that $C^*(L_6) \cong M_6(C(\mathbb{T}))$ [1, Lemma 2.4], we get $C^*(\Lambda_i) \cong C^*(L_6) \otimes C(\mathbb{T}) \cong M_6(C(\mathbb{T}^2))$, $i = 1, 2$, so both have stable rank 2 as discussed in § 1.1.

2. Structure and stable rank in the stably finite case

In this section, we study finite k-graphs whose C^*-algebras are stably finite, corresponding to boxes 1 and 2 in Figure 1. In Proposition 2.7, we show that stable finiteness is equivalent to the lack of infinite projections and provide a characterization in terms of properties of the k-graph. We also provide a structure result and compute stable rank of such C^*-algebras – see Theorems 2.5 and 2.6. We begin with four technical lemmas needed to prove Theorem 2.5.

Lemma 2.1. *Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. Let $v \in \Lambda^0$ be a vertex on an initial cycle $\mu \in \Lambda$. Then*

(1) *there exist paths $\iota_v, \tau_v \in \Lambda$ such that $\mu = \iota_v \tau_v$ and $s(\iota_v) = v = r(\tau_v)$;*

(2) *the path $\mu_v := \tau_v \iota_v$ satisfies $r(\mu_v) = s(\mu_v)$, and $r(\mu_v)\Lambda^{e_i} = \emptyset$ whenever $d(\mu_v)_i = 0$;*

(3) *if $f \in \Lambda^{e_i}$ is an edge with range v on μ, then $f = \mu_v(0, e_i)$ and $\mu = \nu' f \nu''$ for some $\nu', \nu'' \in \Lambda$;*

(4) *if $n \le d(\mu)$ and $\lambda \in v\Lambda^{\le n}$, then $\lambda = \mu_v(0, d(\lambda))$; and*

(5) *$s_{\tau_v} s_{\tau_v}^* = s_v$ and $s_{\tau_v}^* s_{\tau_v} = s_{s(\mu)}$.*

Proof. (1). Since v is a vertex on μ, we have $v = \mu(p)$ for some $p \le d(\mu)$. Set $\iota_v := \mu(0, p)$ and $\tau_v := \mu(p, d(\mu))$. Then $\mu = \iota_v \tau_v$ and $s(\iota_v) = v = r(\tau_v)$.

(2). By property (1), $s(\iota_v) = v = r(\tau_v)$, so the path $\mu_v := \tau_v \iota_v \in \Lambda$ satisfies $r(\mu_v) = s(\mu_v)$. Suppose $r(\mu_v)\Lambda^{e_i}$ is non-empty, say $\alpha \in r(\mu_v)\Lambda^{e_i}$. Then $(\iota_v \alpha)(0, e_i) \in r(\mu)\Lambda^{e_i}$. Since μ is an initial cycle, it follows that $d(\mu)_i \ne 0$.

(3). Suppose $f \in \Lambda^{e_i}$ is an edge with range v on μ. Since $r(f) = v = r(\mu_v)$, we have $f \in r(\mu_v)\Lambda^{e_i}$. Now property (2) ensures that $d(\mu_v)_i \ne 0$. Since $d(\mu_v)_i > 0$, there exists a path $\lambda \in s(f)\Lambda^{\le d(\mu_v)-e_i}$. Hence $f\lambda \in \Lambda^{\le e_i}\Lambda^{\le d(\mu_v)-e_i} = \Lambda^{\le d(\mu_v)}$. Now using that $\mu_v \in v\Lambda$

is a cycle and that Λ has no cycle with an entrance, it follows that $v\Lambda^{\leq d(\mu_v)} = \{\mu_v\}$. Hence $\mu_v = f\lambda$. Since $f\lambda = \mu_v = \tau_v\iota_v$, we get $f = \tau_v(0, e_i)$ if $d(\tau_v)_i > 0$ and $f = \iota_v(0, e_i)$ if $d(\tau_v)_i = 0$.

(4). Fix $n \leq d(\mu)$ and $\lambda \in v\Lambda^{\leq n}$. If $d(\lambda) = 0$ then $\lambda = v$ and the statement is trivial, so we may assume that $\lambda \notin \Lambda^0$. Write $d(\lambda) = e_{i_1} + \ldots + e_{i_m}$ where $i_1 \ldots, i_m \in \{1, \ldots, k\}$. By the factorization property $\lambda = \lambda_1 \ldots \lambda_m$ for some $\lambda_j \in \Lambda^{e_{i_j}}$. Repeated applications of part (3) give $\lambda_j = \mu_{r(\lambda_j)}(0, e_{i_j})$ for $j \leq m$. Since $d(\lambda) \leq d(\mu)$, it follows that $\lambda = \mu_v(0, d(\lambda))$.

(5). Since $s(\tau_v) = s(\mu)$ we have $s^*_{\tau_v}s_{\tau_v} = s_{s(\mu)}$. For $n:=d(\tau_v)$ notice that $n \leq d(\mu)$. Fix $\lambda \in v\Lambda^{\leq n}$. Using part (4) we have $\lambda = \mu_v(0, d(\lambda))$. Since $\tau_v \in v\Lambda^{\leq n}$ and since $d(\lambda) \leq n$, we have $\tau_v = \mu_v(0, n) = \lambda\mu_v(d(\lambda), n)$. But both τ_v and λ belong to $\Lambda^{\leq n}$, so $\tau_v = \lambda$. Consequently $v\Lambda^{\leq n} = \{\tau_v\}$ and $s_{\tau_v}s^*_{\tau_v} = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s^*_\lambda = s_v$. $\qquad\square$

Lemma 2.2. *Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. Let $N:=(|\Lambda^0|, \ldots, |\Lambda^0|) \in \mathbb{N}^k$. Then*

(1) $\sum_{\lambda \in \Lambda^{\leq N}} s_\lambda s^*_\lambda = 1_{C^*(\Lambda)}$; *and*

(2) *for every $\lambda \in \Lambda^{\leq N}$, $s(\lambda)$ is a vertex on an initial cycle.*

Proof. For part (1) use $1 = \sum_{v \in \Lambda^0} s_v = \sum_{v \in \Lambda^0} \sum_{\lambda \in v\Lambda^{\leq N}} s_\lambda s^*_\lambda = \sum_{\lambda \in \Lambda^{\leq N}} s_\lambda s^*_\lambda$. For part (2), we refer to the second paragraph of the proof of [14, Proposition 5.9]. $\qquad\square$

Recall that $(\mu^\infty)^0$ denotes the collection of all vertices on an initial cycle μ. For the terminology τ_v, $v \in \Lambda^0$ in the following lemma, see Lemma 2.1.

Lemma 2.3. *Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. Fix an initial cycle $\mu \in \Lambda$. Let $N:=(|\Lambda^0|, \ldots, |\Lambda^0|) \in \mathbb{N}^k$ and for each $\lambda, \nu \in \Lambda^{\leq N}(\mu^\infty)^0$ set (using Lemma 2.1)*

$$\theta_{\lambda,\nu}:=s_{\lambda\tau_{s(\lambda)}}s^*_{\nu\tau_{s(\nu)}}.$$

Then the $\theta_{\lambda,\nu}$ are matrix units, i.e., $\theta^*_{\lambda,\nu} = \theta_{\nu,\lambda}$ and $\theta_{\lambda,\nu}\theta_{\gamma,\eta} = \delta_{\nu,\gamma}\theta_{\lambda,\eta}$ in $C^*(\Lambda)$.

Proof. Firstly, note that each $\theta_{\lambda,\nu}$ makes sense because the source of $\lambda \in \Lambda^{\leq N}(\mu^\infty)^0$ is a vertex on μ and $\tau_{s(\lambda)}$ is a path on the initial cycle μ with range $s(\lambda)$.

Fix $\lambda, \nu, \gamma, \eta \in \Lambda^{\leq N}(\mu^\infty)^0$. Clearly, $\theta^*_{\lambda,\nu} = \theta_{\nu,\lambda}$. We claim that $\theta_{\lambda,\nu}\theta_{\gamma,\eta} = \delta_{\nu,\gamma}\theta_{\lambda,\eta}$. To see this let $v:=r(\mu)$. Then

$$s^*_{\nu\tau_{s(\nu)}}s_{\gamma\tau_{s(\gamma)}} = s^*_{\tau_{s(\nu)}}s^*_\nu s_\gamma s_{\tau_{s(\gamma)}} \qquad (2.1)$$

is non-zero only if $r(\nu) = r(\gamma)$. Since $s_{r(\nu)} = \sum_{\alpha \in r(\nu)\Lambda^{\leq N}} s_\alpha s^*_\alpha$, we have $(s_\nu s^*_\nu)(s_\gamma s^*_\gamma) = \delta_{\nu,\gamma}s_\nu s^*_\nu$, so if (2.1) is non-zero, then $\nu = \gamma$ and then

$$s^*_{\nu\tau_{s(\nu)}}s_{\gamma\tau_{s(\gamma)}} = \delta_{\nu,\gamma}s^*_{\nu\tau_{s(\nu)}}s_{\gamma\tau_{s(\gamma)}} = \delta_{\nu,\gamma}s_{s(\nu\tau_{s(\nu)})} = \delta_{\nu,\gamma}s_{s(\tau_{s(\nu)})} = \delta_{\nu,\gamma}s_v.$$

Hence $\theta_{\lambda,\nu}\theta_{\gamma,\eta} = s_{\lambda\tau_{s(\lambda)}}(\delta_{\nu,\gamma}s_v)s^*_{\eta\tau_{s(\eta)}} = \delta_{\nu,\gamma}\theta_{\lambda,\eta}$ as claimed. $\qquad\square$

The next lemma is of general nature. See [22, Lemma 3.3] for a special case of this result.

Lemma 2.4. *Suppose that* $\{e_{ij}^{(k)} : 1 \leq k \leq r,\ 1 \leq i,\ j \leq n_k\}$ *is a system of matrix units in a unital C^*-algebra A, in the sense that*

(1) $e_{ij}^{(k)} e_{jl}^{(k)} = e_{il}^{(k)}$;

(2) $e_{ij}^{(k)} e_{mn}^{(l)} = 0$ *if* $k \neq l$ *or if* $j \neq m$;

(3) $(e_{ij}^{(k)})^* = e_{ji}^{(k)}$; *and*

(4) $\sum_{k=1}^{r} \sum_{i=1}^{n_k} e_{ii}^{(k)} = 1$.

For $k \leq r$ let $p^{(k)} := \sum_{i=1}^{n_k} e_{ii}^{(k)}$. Suppose that for each $a \in A$, $a = \sum_{k=1}^{r} p^{(k)} a p^{(k)}$. Then, for $1 \leq k \leq r$, each $e_{11}^{(k)}$ is a projection and $A \cong \bigoplus_{k=1}^{r} M_{n_k}(e_{11}^{(k)} A e_{11}^{(k)})$.

Proof. Clearly $A \cong \bigoplus_{k=1}^{r} p^{(k)} A p^{(k)}$ via $a \mapsto (p^{(1)} a p^{(1)},\ \ldots,\ p^{(r)} a p^{(r)})$ and inverse $(a^{(1)},\ \ldots,\ a^{(r)}) \mapsto \sum_{k=1}^{r} a^{(k)}$. Routine calculations show that for each $k \in \{1,\ \ldots,\ r\}$, the elements $v_i := e_{i1}^{(k)}$, $i = 1,\ \ldots,\ n_k$, satisfy

$$v_i^* v_j = \delta_{i,j} e_{11}^{(k)} \text{ for } 1 \leq i, j \leq n_k, \quad \text{and} \quad p^{(k)} = \sum_{i=1}^{n_k} v_i v_i^*.$$

By [22, Lemma 3.3], $p^{(k)} A p^{(k)} \cong M_{n_k}(e_{11}^{(k)} A e_{11}^{(k)})$ completing the proof. $\qquad \square$

We now characterize the structure of k-graph C^*-algebras $C^*(\Lambda)$ such that Λ is finite and has no cycle with an entrance. For the notions of IC(Λ), $\sim$, $(\mu^\infty)^0$, and ℓ_μ see § 1.6. We note that in Theorem 2.5, IC$(\Lambda) \neq \emptyset$ and $\ell_\mu = |\{i \leq k : d(\mu)_i > 0\}|$ (see Lemma 2.2(2) and Proposition 3.3).

Theorem 2.5 (Structure theorem). *Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. For $N := (|\Lambda^0|,\ \ldots,\ |\Lambda^0|) \in \mathbb{N}^k$,*

$$C^*(\Lambda) \cong \bigoplus_{[\mu] \in \mathrm{IC}(\Lambda)/\sim} M_{\Lambda^{\leq N}(\mu^\infty)^0}\big(s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}\big),$$

and each $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)} \cong C(\mathbb{T}^{\ell_\mu})$.

Proof. Evidently, IC$(\Lambda)/\sim$ is finite since Λ^0 is finite. Let I be a maximal collection of initial cycles satisfying $(\mu^\infty)^0 \cap (\nu^\infty)^0 = \emptyset$ for any $\mu \neq \nu \in I$. For each initial cycle $\mu \in I$ let $\{\theta_{\lambda,\nu}^{(\mu)}\}$ be the matrix units of Lemma 2.3. We first prove that $\{\theta_{\lambda,\nu}^{(\mu)} : \mu \in I\}$ is a system of matrix units, i.e.,

(1) $\theta_{\lambda\lambda'}^{(\mu)} \theta_{\lambda'\lambda''}^{(\mu)} = \theta_{\lambda\lambda''}^{(\mu)}$;

(2) $\theta_{\lambda\lambda'}^{(\mu)} \theta_{\eta\eta'}^{(\nu)} = 0$ if $\mu \neq \nu$ or if $\lambda' \neq \eta$;

(3) $(\theta_{\lambda\lambda'}^{(\mu)})^* = \theta_{\lambda'\lambda}^{(\mu)}$; and

(4) $\sum_{\mu\in I}\sum_{\lambda\in\Lambda^{\leq N}(\mu^\infty)^0}\theta_{\lambda\lambda}^{(\mu)} = 1.$

We start with property (4). Fix $\mu \in I$ and $\lambda \in \Lambda^{\leq N}(\mu^\infty)^0$. Using Lemma 2.1(5), we have $s_{\tau_{s(\lambda)}}s_{\tau_{s(\lambda)}}^* = s_{s(\lambda)}$. Hence $\theta_{\lambda,\lambda}^{(\mu)} = s_{\lambda\tau_{s(\lambda)}}s_{\lambda\tau_{s(\lambda)}}^* = s_\lambda s_\lambda^*$. Lemma 2.2(1) now gives

$$1 = \sum_{\lambda\in\Lambda^{\leq N}} s_\lambda s_\lambda^* = \sum_{\mu\in I}\sum_{\lambda\in\Lambda^{\leq N}(\mu^\infty)^0} s_\lambda s_\lambda^* = \sum_{\mu\in I}\sum_{\lambda\in\Lambda^{\leq N}(\mu^\infty)^0}\theta_{\lambda\lambda}^{(\mu)}. \tag{2.2}$$

Properties (1)–(3) follow from Lemma 2.2 and that $\theta_{\lambda\lambda}^{(\mu)}\theta_{\lambda\lambda}^{(\nu)} = 0$ whenever $\mu \neq \nu$ in I (the latter is a consequence of property (4)).

For each $\mu \in I$, define $p^{(\mu)} := \sum_{\lambda\in\Lambda^{\leq N}(\mu^\infty)^0}\theta_{\lambda\lambda}^{(\mu)}$. Then (2.2) gives $\sum_{\mu\in I} p^{(\mu)} = 1$. We claim that

$$A := \left\{ a \in C^*(\Lambda) : a = \sum_{\mu\in I} p^{(\mu)} a p^{(\mu)} \right\}$$

is all of $C^*(\Lambda)$. Clearly A is a closed linear subspace of $C^*(\Lambda)$. Fix $\alpha,\ \beta \in \Lambda^{\leq N}$ such that $s(\alpha) = s(\beta)$. Using Lemma 2.2(2), it follows that $s(\alpha) \in (\mu^\infty)^0$ for some $\mu \in I$. Since $s_\alpha s_\alpha^* \leq p^{(\mu)}$, we get

$$s_\alpha = p^{(\mu)} s_\alpha s_\alpha^* s_\alpha = p^{(\mu)} s_\alpha, \quad \text{and} \quad s_\alpha s_\beta^* = p^{(\mu)} s_\alpha s_\beta^* p^{(\mu)},$$

so $s_\alpha s_\beta^* \in A$. Using that $\mathrm{span}\{s_\alpha s_\beta^* : \alpha,\ \beta \in \Lambda^{\leq N},\ s(\alpha) = s(\beta)\}$ is dense in $C^*(\Lambda)$, we get $A = C^*(\Lambda)$ as claimed.

For each $\mu \in I$, Lemma 2.1(3)–(4) implies that $r(\mu)\Lambda^{\leq N}$ contains exactly one path which we denote by λ_μ. As in the proof of (4), we have $\theta_{\lambda_\mu,\lambda_\mu}^{(\mu)} = s_{\lambda_\mu}s_{\lambda_\mu}^*$, so $s_{r(\mu)} = \sum_{\lambda\in r(\mu)\Lambda^{\leq N}} s_\lambda s_\lambda^* = \theta_{\lambda_\mu,\lambda_\mu}^{(\mu)}$. Identifying I with $\mathrm{IC}(\Lambda)/\sim$ via the map $\mu \mapsto [\mu]$, Lemma 2.4 provides an isomorphism

$$C^*(\Lambda) \cong \bigoplus_{\mu\in\mathrm{IC}(\Lambda)/\sim} M_{\Lambda^{\leq N}(\mu^\infty)^0}\big(s_{r(\mu)}C^*(\Lambda)s_{r(\mu)}\big).$$

To see that each $s_{r(\mu)}C^*(\Lambda)s_{r(\mu)} \cong C(\mathbb{T}^{\ell_\mu})$, see the proof of [14, Proposition 5.9]. $\quad\square$

The main result of this section is the characterization of stable rank for k-graph C^*-algebras $C^*(\Lambda)$ such that Λ is finite and has no cycle with an entrance. Recall the notion of the floor and ceiling functions: for $x \in \mathbb{R}$, we write $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ and $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\}$.

Theorem 2.6. *Let Λ be a finite, locally convex k-graph that has no cycle with an entrance. For $N := (|\Lambda^0|, \ldots, |\Lambda^0|) \in \mathbb{N}^k$,*

$$sr(C^*(\Lambda)) = \max_{[\mu]\in\mathrm{IC}(\Lambda)/\sim} \left\lceil \frac{\left\lfloor \frac{\ell_\mu}{2} \right\rfloor}{|\Lambda^{\leq N}(\mu^\infty)^0|} \right\rceil + 1.$$

Proof. By Theorem 2.5 and property (4) from §1.1,

$$sr(C^*(\Lambda)) = \max_{[\mu] \in \mathrm{IC}(\Lambda)/\sim} M_{\Lambda^{\leq N}(\mu^\infty)^0}(s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}),$$

and each $s_{r(\mu)} C^*(\Lambda) s_{r(\mu)} \cong C(\mathbb{T}^{\ell_\mu})$. Now using property (2) from §1.1, we get

$$sr(C^*(\Lambda)) = \max_{[\mu] \in \mathrm{IC}(\Lambda)/\sim} \left\lceil \frac{sr(s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}) - 1}{\Lambda^{\leq N}(\mu^\infty)^0} \right\rceil + 1.$$

Finally, property (1) from §1.1, gives $sr(s_{r(\mu)} C^*(\Lambda) s_{r(\mu)}) - 1 = \lfloor \ell_\mu/2 \rfloor$ for each $[\mu] \in \mathrm{IC}(\Lambda)/\sim$, completing the proof. $\square$

Our next Proposition 2.7 characterizes stable finiteness of C^*-algebras of finite, locally convex k-graphs. For other results on stable finiteness of C^*-algebras associated with row-finite k-graphs with no sources, see [7, 14]. Note that the C^*-algebras satisfying the hypotheses of Proposition 2.7 are exactly those shown in Figure 1 in boxes 3 and 4.

We briefly introduce relevant terminology. Following [31], we write $\mathrm{MCE}(\mu, \nu) := \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu) \vee d(\nu)}$ for the set of all minimal common extensions of $\mu, \nu \in \Lambda$. The cycle λ is a *cycle with an entrance in the sense of [14, Definition 3.5]* if there exists a path $\tau \in r(\lambda)\Lambda$ such that $\mathrm{MCE}(\tau, \lambda) = \emptyset^\ddagger$.

Proposition 2.7. *Let Λ be a finite, locally convex k-graph. With notation as above, the following are equivalent:*

(1) Λ *has a cycle μ with an entrance;*

(2) Λ *has a cycle μ with an entrance in the sense of [14, Definition 3.5];*

(3) $C^*(\Lambda)$ *contains an infinite projection; and*

(4) $C^*(\Lambda)$ *is not stably finite.*

Proof. To prove (1)$\Rightarrow$(2) let μ be a cycle with an entrance τ, so $\tau \in r(\mu)\Lambda$ satisfies $d(\tau) \leq d(\mu^\infty)$ and $\tau \neq \mu^\infty(0, d(\tau))$. Fix $n \geq 1$ such that $nd(\mu) \geq d(\tau)$. Clearly $\tau \in r(\mu^n)\Lambda$. Then $\tau \neq \mu^\infty(0, d(\tau)) = \mu^n(0, d(\tau))$, so $\mathrm{MCE}(\mu^n, \tau) = (\mu^n \Lambda \cap \Lambda^{d(\mu^n) \vee d(\tau)}) \cap \tau\Lambda \subseteq \{\mu^n\} \cap \tau\Lambda = \emptyset$. For the proof of (2)$\Rightarrow$(3), see [14, Corollary 3.8].

The implication (3)$\Rightarrow$(4) follows from [37, Lemma 5.1.2]. It remains to prove (4)$\Rightarrow$(1). We establish the contrapositive. Suppose that condition (1) does not hold, that is Λ has no cycle with an entrance. Theorem 2.5 gives that $C^*(\Lambda)$ is isomorphic to a direct sum of matrix algebras over commutative C^*-algebras, hence stably finite, so condition (4) does not hold. $\square$

Remark 2.8. Our main results are for finite k-graphs so Proposition 2.7 is stated in that context, but some of the implications hold more generally.

(1) Only the proof of (4) $\implies$ (1) uses that Λ is finite. The proofs of the implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are valid for any locally convex row-finite k-graph.

$\ddagger$ Formally, if λ is a cycle, then $(\lambda, r(\lambda))$ is a generalized cycle in the sense of [14, Definition 3.1], and an entrance to $(\lambda, r(\lambda))$ is a path $\tau \in s(r(\lambda))\Lambda$ such that $\mathrm{MCE}(r(\lambda)\tau, \lambda) = \emptyset$.
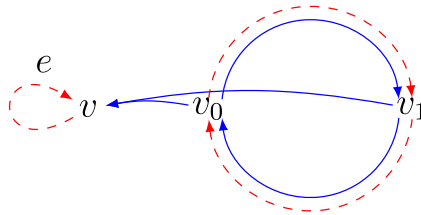
Figure 3. An example of a 2-graph Λ with $C^*(\Lambda)$ of stable rank 2.

(2) Similarly, while (1) and (2) are not equivalent for arbitrary k-graphs, they *are* equivalent for locally convex k-graphs, whether finite or not. To see this, suppose that λ is a cycle in a locally convex k-graph and $\tau \in r(\lambda)\Lambda$ satisfies $\mathrm{MCE}(\lambda, \tau) = \emptyset$. Let $I = \{i \leq k : d(\lambda)_i > 0\}$, let $m_I = \sum_{i \in I} d(\tau)_i e_i$ and $m' := d(\tau) - m_I$, and factorize $\tau = \tau_I \tau'$ with $d(\tau_I) = m_I$. If $\tau_I \neq (\lambda^\infty)(0, m_I)$, then λ is a cycle with an entrance as required, so we may assume that $\tau_I = (\lambda^\infty)(0, m_I)$. So replacing λ with $(\lambda^\infty)(m_I, m_I + d(\lambda))$ and τ with τ' we may assume that $d(\tau) \wedge d(\lambda) = 0$. Since Λ is locally convex, a quick inductive argument shows that there exists $\mu \in s(\tau)\Lambda^{d(\lambda)} \neq \emptyset$. Factorise $\tau\mu = \alpha\beta$ with $d(\alpha) = d(\mu) = d(\lambda)$. Since $\mathrm{MCE}(\tau, \lambda) = \emptyset$, we must have $\alpha \neq \lambda$ and in particular $d(\alpha) = d(\lambda) < d(\lambda^\infty)$ and $\alpha \neq (\lambda^\infty)(0, d(\alpha))$. So once again λ is a cycle with an entrance.

Corollary 2.9. *Let Λ be a finite, locally convex k-graph. Suppose that Λ has no cycle with an entrance (i.e., $C^*(\Lambda)$ is stably finite). For $N := (|\Lambda^0|, \ldots, |\Lambda^0|) \in \mathbb{N}^k$,*

$$sr(C^*(\Lambda)) = \max_{[\mu] \in \mathrm{IC}(\Lambda)/\sim} \left\lceil \frac{\left\lfloor \frac{\ell_\mu}{2} \right\rfloor}{|\Lambda^{\leq N}(\mu^\infty)0|} \right\rceil + 1.$$

Remark 2.10. A cycle with an incoming edge may fail to be a cycle with an entrance. This is, for example, the case for any of the red (dashed) cycles in Figure 3.

Example 2.11. In this example, we consider the 2-graph Λ in Figure 3. As before, we refer to [16] for details on how to illustrate 2-graphs as a 2-coloured graph. Here we use blue (solid) and red (dashed) as the first and second colour. We use our results to compute the structure and stable rank of $C^*(\Lambda)$. Firstly, notice that red the cycle $\nu \in v\Lambda^{e_2}$ based at v is not an initial cycle because $d(\nu)_1 = 0$ but $r(\nu)\Lambda^{e_1} \neq \emptyset$. However, the cycle $\mu \in v_0\Lambda^{e_1 + e_2}$ is an initial cycle. There are many other initial cycles, but they are all $\sim$-equivalent to μ. So $\mathrm{IC}(\Lambda)/\sim = \{[\mu]\}$. Since the vertices on a path $\lambda \in v_0\Lambda$ alternate between v_0 and v_1 as we move along the path, it follows that $\mu^\infty(m) = v_{m_1 + m_2 \pmod 2}$ for each $m \in \mathbb{N}^2$. Hence

$$G_\mu = \{m - n : n, m \leq d(\mu^\infty), \mu^\infty(m) = \mu^\infty(n)\}$$
$$= \{m - n : n, m \in \mathbb{N}^2, v_{m_1 + m_2 \pmod 2} = v_{n_1 + n_2 \pmod 2}\}$$
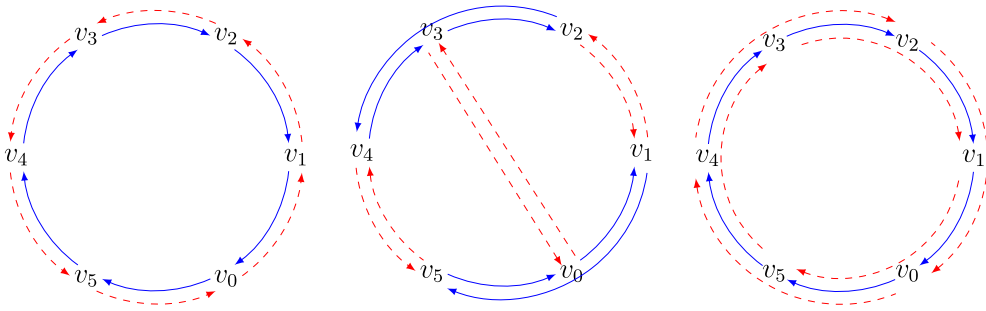
Figure 4. Another three examples of 2-graphs with lots of initial cycles, but only one such up to $\sim$ - equivalence.

$$= \{(m_1 - n_1, m_2 - n_2) : n_i, m_i \in \mathbb{N}, m_1 - n_1 = -(m_2 - n_2) \pmod 2\}$$
$$= \{(k_1, k_2) : k_i \in \mathbb{Z}, k_1 = -k_2 \pmod 2\}$$
$$= \{k \in \mathbb{Z}^2 : k_1 + k_2 \text{ is even}\}$$
$$= \mathbb{Z}(1,1) + \mathbb{Z}(0,2),$$
$$\cong \mathbb{Z}^2.$$

We deduce that $\ell_\mu = \operatorname{rank}(G_\mu) = 2$. Now set $N = (|\Lambda^0|, |\Lambda^0|) = (3,3)$. As mentioned, modulo $\sim$, there is only one initial cycle μ, so any path in $\Lambda^{\leq N}$ has its source on μ. Hence $\Lambda^{\leq N}(\mu^\infty)^0 = \Lambda^{\leq N} = v\Lambda^{\leq N} \sqcup v_0\Lambda^{\leq N} \sqcup v_1\Lambda^{\leq N}$. By Lemma 2.1, $|v_0\Lambda^{\leq N}| = |v_1\Lambda^{\leq N}| = 1$. Using the factorization property to push the red edges to the start of a path and uniqueness of such paths on μ, we have $|v\Lambda^{\leq N}| = |v\Lambda^{(3,3)}| = |v\Lambda^{(3,0)}| = |v\Lambda^{(1,0)}| = 2$. Hence $C^*(\Lambda) \cong M_4(C(\mathbb{T}^2))$ and $sr(C^*(\Lambda)) = \lceil \lfloor \frac{2}{2} \rfloor / 4 \rceil + 1 = 2$ by Theorem 2.6.

Example 2.12. In the following let Λ_1, Λ_2 and Λ_3, Λ_4, Λ_5 be the 2-graphs in Figure 2 and Figure 4 respectively. Up to a swap of the colours these five examples make up all the examples of 2-graphs on 6 vertices with only one initial cycle up to $\sim$ - equivalence and with all vertices on that initial cycle. Let μ_i denote such an initial cycle in Λ_i. For $0 \leq j \leq n - 1$ define $f_j : \mathbb{N}^2 \to \mathbb{N}$ by $f_j(m_1, m_2) = m_1 + jm_2$. With the terminology of §1.7 one can show that $\Lambda_3 = f_5^*(L_6)$, that $\Lambda_4 = L_2 \times L_3$, and that $\Lambda_5 = f_2^*(L_6)$. Hence

$$C^*(\Lambda_i) \cong M_6(C(\mathbb{T}^{\ell_{\mu_i}})) \text{ and } sr(C^*(\Lambda)) = \ell_{\mu_i} = 2 \text{ for each } i = 1, \ldots, 5.$$

In particular $C^*(\Lambda_i) \cong C^*(\Lambda_j)$ for all i, j. These five examples indicate how the number of colours and vertices impacts the structure of the corresponding C^*-algebras. The next Proposition 2.13 verifies this.

In the following, 'the number of vertices' on an initial cycle μ means $|(\mu^\infty)^0|$, and 'the number of colours' means $|\{i \leq k : d(\mu)_i > 0\}|$. The proof borrows material from an independent result (Proposition 3.3).

Proposition 2.13. *Let Λ be a finite, locally convex k-graph on $n = |\Lambda^0|$ vertices. Suppose that Λ has no cycle with an entrance and Λ has exactly one initial cycle, up to*

$\sim$ *- equivalence, with n vertices and ℓ colours. Then*

$$C^*(\Lambda) \cong M_n(C(\mathbb{T}^\ell)), \ \text{and} \ \ sr(C^*(\Lambda)) = \lceil \lfloor \ell/2 \rfloor / n \rceil + 1.$$

Proof. Let μ be an initial cycle in Λ. By Theorem 2.5 and Theorem 2.6, we have

$$sr(C^*(\Lambda)) = \left\lceil \frac{\left\lfloor \frac{\ell_\mu}{2} \right\rfloor}{|\Lambda^{\leq(n,\ldots,n)}|} \right\rceil + 1, \quad C^*(\Lambda) \cong M_{\Lambda^{\leq(n,\ldots,n)}}(C(\mathbb{T}^{\ell_\mu})),$$

where ℓ_μ is the rank of the periodicity group associated with μ (Definition 1.5). By the factorization property Λ has no sources, so $\Lambda^{\leq(n,\ldots,n)} = \Lambda^{(n,\ldots,n)}$. Lemma 2.1(3) implies that $|\Lambda^{(n,\ldots,n)}| = n$. By Proposition 3.3, $\ell_\mu = \ell$. Combining these results gives $C^*(\Lambda) \cong M_n(C(\mathbb{T}^\ell))$, and $sr(C^*(\Lambda)) = \lceil \lfloor \ell/2 \rfloor / n \rceil + 1$. $\qquad\square$

Remark 2.14. To keep the statement of Proposition 2.13 short and clean, we insisted that $(\mu^\infty)^0 = \Lambda^0$, but more general results can be obtained using Theorem 2.5 and Theorem 2.6.

3. Stable rank one

In this section, we characterize which finite k-graphs have C^*-algebras of stable rank 1 – see Theorem 3.1 and Corollary 3.4. We note that Theorem 3.1 is in large contained in [14] and we have structured the proof accordingly.

Theorem 3.1. *Let Λ be a finite, locally convex k-graph. Then $sr(C^*(\Lambda)) = 1$ if and only if $C^*(\Lambda)$ is (stably) finite and $\max_{\mu \in IC(\Lambda)} \ell_\mu = 1$.*

Proof. Suppose that $sr(C^*(\Lambda)) = 1$. Then $sr(M_n(C^*(\Lambda))) = 1$ for each positive integer n [34, Theorem 6.1]. Hence each $M_n(C^*(\Lambda))$ is finite [2, V.3.1.5], which implies that $C^*(\Lambda)$ is stably finite. Since $C^*(\Lambda)$ is finite, it does not contain any infinite projections [37, Lemma 5.1.2]. Hence by [14, Proposition 5.9], there exist $n \geq 1$ and $l_1, \ldots, l_n \in \{0, \ldots, k\}$ such that $C^*(\Lambda)$ is stably isomorphic to $\bigoplus_{i=1}^n C(\mathbb{T}^{l_i})$. Since $sr(C^*(\Lambda)) = 1$, we deduce that $sr(\bigoplus_{i=1}^n C(\mathbb{T}^{l_i})) = 1$, because stable rank 1 for unital C^*-algebras is preserved my stable isomorphism [34, Theorem 3.6]. By property 4 in §1.1, we have $sr(\bigoplus_{i=1}^n C(\mathbb{T}^{l_i})) = \max_{i=1}^n C(\mathbb{T}^{l_i})$. For each $i = 1, \ldots, n$ we use [34, Proposition 1.7] to deduce that $sr(C(\mathbb{T}^{l_i})) = \lfloor l_i/2 \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes 'integer part of'. Hence $\max_{i=1}^n l_i = 1$.

By inspection of the proof of [14, Proposition 5.9], it is clear that each of the integers l_i is the rank of μ for some $\mu \in IC(\Lambda)$, so $\max_{\mu \in IC(\Lambda)} \ell_\mu \geq \max_{i=1}^n l_i = 1$. For each $\mu, \nu \in IC(\Lambda)$ define $P_\nu := \sum_{v \in (\nu^\infty)^0} s_v$ and $\mu \sim \nu \Leftrightarrow (\mu^\infty)^0 = (\nu^\infty)^0$. Since $P_\nu = P_\mu$ whenever $\mu \sim \nu$, the proof of [14, Proposition 5.9] implies that for each $\mu \in IC(\Lambda)$, we have $\ell_\mu = l_i$ for some $i \in \{1, \ldots, n\}$. Consequently, $\max_{\mu \in IC(\Lambda)} \ell_\mu = \max_{i=1}^n l_i$.

Conversely, suppose $C^*(\Lambda)$ is finite and $\max_{\mu \in IC(\Lambda)} \ell_\mu = 1$. By [37, Lemma 5.1.2], $C^*(\Lambda)$ has no infinite projections. So [14, Corollary 5.7] implies that $C^*(\Lambda)$ is stably isomorphic to $\bigoplus_{i=1}^n C(\mathbb{T}^{l_i})$ for some $n \geq 1$ and $l_1, \ldots, l_n \in \{0, \ldots, k\}$ such that
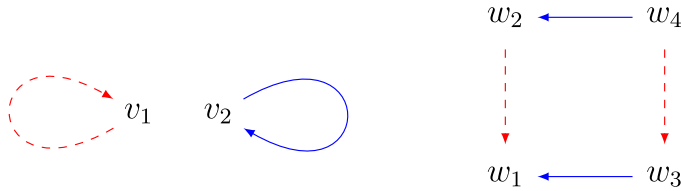
Figure 5. Two examples of 2-graphs with a C^*-algebra of stable rank one.

$\max_{\mu \in IC(\Lambda)} \ell_\mu = \max_{i=1}^n l_i$. By the properties in §1.1, it follows that

$$sr(C^*(\Lambda)) = sr\left(\bigoplus_{i=1}^n C(\mathbb{T}^{l_i})\right) = \max_{i=1,\ldots,n} \lfloor l_i/2 \rfloor + 1 = 1.$$

$\square$

Remark 3.2. It turns out that C^*-algebras of finite k-graphs with $k > 1$ rarely have stable rank one: the condition $\max_{\mu \in IC(\Lambda)} \ell_\mu = 1$ is rather strict. As Proposition 3.3 indicates, if Λ^0 is finite and $sr(C^*(\Lambda)) = 1$ (hence stably finite), then any initial cycle in Λ has at most one colour. Using Lemma 2.2(2) and the factorization property, it follows that any cycle in Λ has at most one colour.

Figure 5 illustrates two examples of 2-graphs Λ with $C^*(\Lambda)$ of stable rank one. The first example, illustrated on the left, has two vertices v_1, v_2, a single edge red (dashed) loop based at v_1, and single edge blue (solid) loop based at v_2. The second example, shown on the right in Figure 5, is different in that it is connected and contains no loops.

Following [14], for $n \in \mathbb{N}^k$ there is a shift map $\sigma^n : \{x \in W_\Lambda : n \le d(x)\} \to W_\Lambda$ such that $d(\sigma^n(x)) = d(x) - n$ and $\sigma^n(x)(p, q) = x(n + p, n + q)$ for $0 \le p \le q \le d(x) - n$ where we use the convention $\infty - a = \infty$ for $a \in \mathbb{N}$. For $x \in W_\Lambda$ and $n \le d(x)$, we then have $x(0, n)\sigma^n(x) = x$. We now show an easy way to compute ℓ_μ, using only the degree of μ.

Proposition 3.3. *Let Λ be a finite, locally convex k-graph such that Λ has no cycle with an entrance. Then for each $\mu \in IC(\Lambda)$,*

$$\ell_\mu = |\{i \le k : d(\mu)_i > 0\}||.$$

Proof. Let $I = \{i \le k : d(\mu)_i > 0\}$. We must show that $\ell_\mu = |I|$. If $I = \emptyset$ then $\ell_\mu = 0 = |I|$, so assume that I is non-empty. By (1.1),

$$G_\mu = \{m - n : n, m \le d(\mu^\infty), \mu^\infty(m) = \mu^\infty(n)\}.$$

Since each $n, m \le d(\mu^\infty)$ satisfy $n, m \in \text{span}_\mathbb{N}\{e_i : i \in I\}$, the rank of G_μ is at most $|I|$. Consequently, it suffices to show G_μ contains a subgroup of rank $|I|$.

Let $v := \mu^\infty(0)$. We claim that for each colour $i \in I$, there exists a positive integer m_i such that $\mu^\infty(0, m_i e_i) = v$. Indeed, since Λ^0 is finite there exists $m < n$ such that $\mu^\infty(m e_i) = \mu^\infty(n e_i)$. Now using that $\mu^\infty \in \Lambda^{\le \infty}$ (see Lemma 2.1) and that for every

vertex w on μ there is a unique path in $w\Lambda^{\leq\infty}$ (see §1.7) we get that $\sigma^{me_i}(\mu^\infty) = \sigma^{ne_i}(\mu^\infty)$. Now for $N:=md(\mu)$ it follows that $\sigma^N(\mu^\infty) = \mu^\infty$. Since

$$\mu^\infty = \sigma^N(\mu^\infty) = \sigma^{N-me_i+me_i}(\mu^\infty) = \sigma^{N-me_i+ne_i}(\mu^\infty) = \sigma^{(n-m)e_i}(\mu^\infty),$$

we get $(\mu^\infty)((n-m)e_i) = v$. Hence μ^∞ contains a cycle of degree $(n-m)e_i$ based at v. In particular, we can use $m_i:=n-m$.

By the preceding paragraph, $\{m_ie_i : i \in I\} \subseteq G_\mu$ is a $\mathbb{Z}$-linearly independent set generating a rank-$|I|$ subgroup of G_μ. So the rank of G_μ is $|I|$. ∎

Corollary 3.4. *Let Λ be a finite, locally convex k-graph. Then $sr(C^*(\Lambda)) = 1$ if and only if Λ has no cycle with an entrance and no initial cycle with more than one colour.*

Proof. Combine Proposition 2.7, Theorem 3.1, and Proposition 3.3. ∎

Remark 3.5. A *graph trace* on a locally convex row-finite k-graph Λ is a function $g\colon \Lambda^0 \to \mathbb{R}^+$ satisfying the graph trace property, $g(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. It is *faithful* if it is non-zero on every vertex in Λ [27, 45].

It can be shown that Corollary 3.4 remains valid if we replace 'has no cycle with an entrance' by 'admits a faithful graph trace'. Indeed, the C^*-algebra of a row-finite and cofinal k-graph Λ with no sources is stably finite if and only if Λ admits a faithful graph trace [7, Theorem 1.1], and for Λ^0 finite, this remains true without 'cofinal' and with 'locally convex' instead of 'no sources' (by virtue of Theorem 2.5 and [28, Lemma 7.1]).

4. Stable rank in the simple and cofinal case

In this section, we focus on stable rank of k-graph C^*-algebras for which the k-graph is cofinal, corresponding to boxes 1 and 3 in Figure 1. Since simple k-graph C^*-algebras constitute a sub-case of this situation (as illustrated below), we consider those first.

Let Λ be a row-finite, locally convex k-graph. Following [42], Λ is *cofinal* if for all pairs $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $s(w\Lambda^{\leq n}) \subseteq s(v\Lambda)$. Following [35], Λ has *local periodicity* m, n at v if for every $x \in v\Lambda^{\leq\infty}$, we have $m - (m \wedge d(x)) = n - (n \wedge d(x))$ and $\sigma^{m\wedge d(x)}(x) = \sigma^{n\wedge d(x)}(x)$. If Λ fails to have local periodicity m, n at v for all $m \neq n \in \mathbb{N}^k$ and $v \in \Lambda^0$, we say that Λ has *no local periodicity*. By [35, Theorem 3.4],

Λ is cofinal and has no local periodicity if and only if $C^*(\Lambda)$ is simple.

The stable rank of 1-graph C^*-algebras is well understood (see [10, Theorem 3.4], [18, Theorem 3.3] and [17, Theorem 3.1]), but the following is new for $k > 1$. Recall that a *cycle* is a path $\lambda \in \Lambda \setminus \Lambda^0$ such that $r(\lambda) = s(\lambda)$.

Proposition 4.1. *Let Λ be a finite, locally convex k-graph. Suppose that Λ is cofinal and has no local periodicity (i.e., $C^*(\Lambda)$ is simple). Then*

$$sr(C^*(\Lambda)) = \begin{cases} 1 & \text{if } \Lambda \text{ contains no cycles} \\ \infty & \text{otherwise.} \end{cases}$$
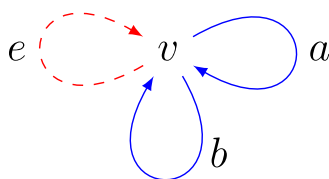
Figure 6. Example of 2-graph with a C^*-algebra of stable rank infinity.

Proof. If Λ contains no cycles then [14, Corollary 5.7] gives $C^*(\Lambda) \cong M_{\Lambda v}(\mathbb{C})$ for some vertex $v \in \Lambda^0$. Using [34, Proposition 1.7 and Theorem 3.6], we obtain that $sr(C^*(\Lambda)) = 1$.

If Λ contains a cycle, then another application of [14, Corollary 5.7] (see also [4, Remark 5.8]) gives that $C^*(\Lambda)$ is purely infinite. Since $C^*(\Lambda)$ is unital, simple and purely infinite, it contains two isometries with orthogonal ranges, so [34, Proposition 6.5] gives $sr(C^*(\Lambda)) = \infty$. □

In conclusion, the stable rank of a unital simple k-graph C^*-algebra is completely determined by the presence or absence of a cycle in the k-graph.

4.1. The cofinal case

We now consider the cofinal case. We start by recalling a result of Jeong, Park and Shin about directed graphs (or 1-graphs). We refer to [18] for the terminology involved.

Proposition 4.2 (Jeong et al. [18, Proposition 3.7]). *Let E be a locally finite directed graph. If E is cofinal then either $sr(C^*(E)) = 1$ or $C^*(E)$ is purely infinite simple.*

Remark 4.3. We illustrate why for k-graphs we can not hope for a result similar to Proposition 4.2. Consider the 2-graph Λ in Figure 6 with two blue edges $a, b \in \Lambda^{e_1}$ and one red edge $e \in \Lambda^{e_2}$ and the factorization property $ae = ea$, $be = eb$. Since Λ has only one vertex, it is automatically cofinal. However, $C^*(\Lambda)$ neither has stable rank one nor is purely infinite simple as the following discussion shows:

The C^*-algebra $C^*(\Lambda)$ fails to have stable rank one because it is not stably finite (containing a cycle with an entrance). It is not simple, so, in particular, not purely infinite simple because

$$\text{for every } x \in v\Lambda^{\leq\infty} \text{ we have } \sigma^{e_2}(x) = x, \tag{4.1}$$

so Λ has local periodicity $p = e_2$, $q = 0$ at v and $C^*(\Lambda)$ is non-simple.

Because of our particular choice of factorization rules $ae = ea$, $be = eb$ Lemma 1.7 implies that $C^*(\Lambda) \cong \mathcal{O}_2 \otimes C(\mathbb{T})$. If we instead used the factorization $ae = eb$, $be = ea$, then Lemma 1.7 would not apply but we would still have $\sigma^{2e_2}(x) = x$ for each $x \in v\Lambda^{\leq\infty}$ making $C^*(\Lambda)$ non-simple.

Remark 4.3 notwithstanding, we are able to provide a characterization of stable rank in the cofinal case. Given a C^*-algebra A, we write $a \oplus b$ for the diagonal matrix $\text{diag}(a, b)$ in $M_2(A)$ and write $\sim$ for the von Neumann equivalence relation between elements in

matrix algebras over A. A unital C^*-algebra A is *properly infinite* if $1 \oplus 1 \oplus r \sim 1$ for some projection r in some matrix algebra over A (for more details, see [38]).

Theorem 4.4. *Let Λ be a cofinal, finite, locally convex k-graph. Suppose that Λ contains a cycle with an entrance. Then $C^*(\Lambda)$ is properly infinite and has stable rank ∞.*

Proof. Let μ be a cycle with an entrance τ, that is

$$\tau \in r(\mu)\Lambda, \quad d(\tau) \leq d(\mu^\infty), \text{and } \tau \neq \mu^\infty(0, d(\tau)).$$

Fix $n \geq 1$ such that $m := d(\mu)n \geq d(\tau)$. Since $\tau \neq \mu^\infty(0, d(\tau)) = \mu^n(0, d(\tau))$, there exists $\beta \in s(\tau)\Lambda$ such that μ^n and $\tau\beta$ are distinct elements of $r(\mu)\Lambda^{\leq m}$. Write $r(\mu)\Lambda^{\leq m} = \{\nu_1, \ldots, \nu_N\}$ with $\nu_1 = \mu^n$ and $\nu_2 = \tau\beta$. For each $i = 1, \ldots, N$ set $v_i = s(\nu_i)$ and let $x = (s_{\nu_1}, \ldots, s_{\nu_N})$. Then $xx^* = \sum_{\lambda \in v_1\Lambda^{\leq m}} s_\lambda s_\lambda^* = s_{v_1}$. Moreover for $i \neq j$, $s_{\nu_i}^* s_{\nu_j} = 0$, so

$$s_{v_1} = xx^* \sim x^*x = \mathrm{diag}(s_{\nu_1}^* s_{\nu_1}, \ldots, s_{\nu_N}^* s_{\nu_N}) = s_{v_1} \oplus \cdots \oplus s_{v_N}.$$

We claim that for any pair of vertices $u, v \in \Lambda^0$ there exist a constant $M_{u,v}$ and a projection $p_{u,v}$ in some matrix algebra over $C^*(\Lambda)$ such that

$$\left(\bigoplus_{l=1}^{M_{u,v}} s_u \right) \sim s_v \oplus p_{u,v}. \tag{4.2}$$

To see this, fix $u, v \in \Lambda^0$. Since Λ is cofinal there exists $n \in \mathbb{N}^k$ such that $s(v\Lambda^{\leq n}) \subseteq s(u\Lambda)$. Writing $v\Lambda^{\leq n} = \{\mu_1, \ldots, \mu_{M_{u,v}}\}$ and $u_i = s(\mu_i)$, we have $s_v \sim s_{u_1} \oplus \cdots \oplus s_{u_{M_{u,v}}}$. Since $s(v\Lambda^{\leq n}) = \{u_i : i \leq M_{u,v}\} \subseteq s(u\Lambda)$, for each $i \leq M_{u,v}$ there exists $\lambda_i \in u\Lambda$ such that $s(\lambda_i) = u_i$. Let $m_i = d(\lambda_i)$ for each i. Then for each $i = 1, \ldots, M_{u,v}$,

$$s_u = \sum_{\lambda \in u\Lambda^{\leq m_i}} s_\lambda s_\lambda^* \sim s_{u_i} \oplus p_i$$

for some projection p_i in a matrix algebra over $C^*(\Lambda)$. With $p_{u,v} = \bigoplus_i p_i$, we obtain

$$\left(\bigoplus_{l=1}^{M_{u,v}} s_u \right) \sim \left(\bigoplus_{i=1}^{M_{u,v}} s_{u_i} \right) \oplus p_{u,v} \sim s_v \oplus p_{u,v},$$

which establishes the claim.

Applying (4.2) to $u = v_2$ and $v = v_1$, we get

$$\left(\bigoplus_{l=1}^{M_{v_2,v_1}} s_{v_2} \right) \sim s_{v_1} \oplus p_{v_2,v_1}.$$

Recall that $s_{v_1} \sim s_{v_1} \oplus s_{v_2} \oplus (\bigoplus_{i=3}^N s_{v_i})$. Let $q := p_{v_2,v_1} \oplus (\bigoplus_{l=1}^{M_{v_2,v_1}} \bigoplus_{i=3}^N s_{v_i})$, meaning that if $N = 2$ then $q = p_{v_2,v_1} \oplus 0$. Then

$$s_{v_1} \sim s_{v_1} \oplus \left(\bigoplus_{l=1}^{M_{v_2,v_1}} s_{v_2} \right) \oplus \left(\bigoplus_{l=1}^{M_{v_2,v_1}} \bigoplus_{i=3}^N s_{v_i} \right) \sim s_{v_1} \oplus s_{v_1} \oplus q. \tag{4.3}$$

Applying (4.2) to $u = v_1$ and to each $v \in \Lambda^0 \setminus \{v_1\}$ at the second equality, and putting $L := 2 + \sum_{v \in \Lambda^0 \setminus \{v_1\}} M_{v_1, v}$, we calculate:

$$1 \oplus 1 \oplus \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} p_{v_1, v} \right) \sim 1 \oplus s_{v_1} \oplus \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} (s_v \oplus p_{v_1, v}) \right)$$

$$\sim 1 \oplus s_{v_1} \oplus \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} \left(\bigoplus_{i=1}^{M_{v_1, v}} s_{v_1} \right) \right)$$

$$\sim \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} s_v \right) \oplus s_{v_1} \oplus s_{v_1} \oplus \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} \left(\bigoplus_{i=1}^{M_{v_1, v}} s_{v_1} \right) \right)$$

$$\sim \left(\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} s_v \right) \oplus \left(\bigoplus_{j=1}^{L} s_{v_1} \right).$$

Using (4.3), we have $s_{v_1} \sim s_{v_1} \oplus (\bigoplus_{j=1}^{L-1} s_{v_1}) \oplus (\bigoplus_{j=1}^{L-1} q)$, so $r = (\bigoplus_{v \in \Lambda^0 \setminus \{v_1\}} p_{v_1, v}) \oplus (\bigoplus_{j=1}^{L-1} q)$ satisfies

$$1 \oplus 1 \oplus r \sim 1.$$

Hence 1 is properly infinite. Now [34, Proposition 6.5] gives $sr(C^*(\Lambda)) = \infty$. □

With Proposition 4.2 in mind, the following is a dichotomy for the C^*-algebras associated with cofinal finite k-graphs.

Corollary 4.5. *Let Λ be a cofinal, finite, locally convex k-graph. Then either $C^*(\Lambda)$ is stably finite and $sr(C^*(\Lambda))$ is given by Corollary 2.9, or $C^*(\Lambda)$ is properly infinite and $sr(C^*(\Lambda)) = \infty$.*
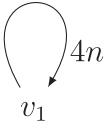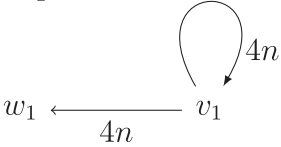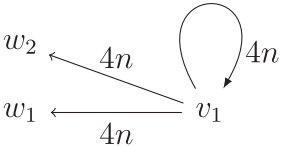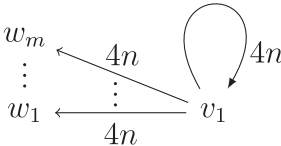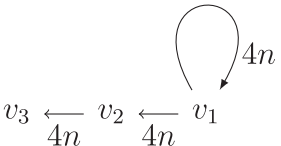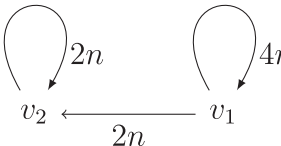
Proof. If $C^*(\Lambda)$ is not stably finite then Λ contains a cycle with an entrance by Proposition 2.7. Hence $C^*(\Lambda)$ is properly infinite and $sr(C^*(\Lambda)) = \infty$ by Theorem 4.4. Conversely, if $C^*(\Lambda)$ is properly infinite, then it is also infinite, so $C^*(\Lambda)$ is not finite and hence not stably finite. If $C^*(\Lambda)$ is stably finite then Corollary 2.9 applies. □

Remark 4.6. Theorem 4.4 includes cases not covered by any of our preceding results. Consider, for example, the 2-graph Λ in Figure 6. By Theorem 4.4, the associated C^*-algebra has stable rank infinity.

Example 4.7. By Corollary 4.5, we can compute the stable rank of k-graph C^*-algebras in boxes 1 to 3 in Figure 1. It, therefore, makes sense to consider the range of stable rank achieve by these C^*-algebra. In box 3, stable rank infinity can be obtained as in Remark 4.3. For finite stable rank, Table 1 lists a few $4n$-graphs Λ together with their associated C^*-algebra and its stable rank (we use a multiple of 4 because it makes the formulas for the stable rank simpler).

Except for the last $4n$-graph, each black edge represents exactly $4n$ edges of different colours, one of each colour; the last $4n$-graph has $2n$ loops at v_2, one each of the first $2n$

Table 1. *A few examples of $4n$-graphs.*

$4n$-graph Λ	$|\Lambda^{\leq N}|$	$C^*(\Lambda)$	$sr(C^*(\Lambda))$
loop $4n$ at v_1	1	$C(\mathbb{T}^{4n})$	$2n+1$
$w_1 \xleftarrow{4n} v_1$ (loop $4n$)	2	$M_2(C(\mathbb{T}^{4n}))$	$n+1$
$w_2 \xleftarrow{4n}$, $w_1 \xleftarrow{4n} v_1$ (loop $4n$)	3	$M_3(C(\mathbb{T}^{4n}))$	$\left\lceil \dfrac{2n}{3} \right\rceil + 1$
$w_m \xleftarrow{4n}$, $\vdots$, $w_1 \xleftarrow{4n} v_1$ (loop $4n$)	$m+1$	$M_{m+1}(C(\mathbb{T}^{4n}))$	$\left\lceil \dfrac{2n}{m+1} \right\rceil + 1$
$v_3 \xleftarrow{4n} v_2 \xleftarrow{4n} v_1$ (loop $4n$)	$\binom{4n}{2}$	$M_{\binom{4n}{2}}(C(\mathbb{T}^{4n}))$	$\left\lceil \dfrac{2n}{\binom{4n}{2}} \right\rceil + 1$
v_2 (loop $2n$) $\xleftarrow{2n} v_1$ (loop $4n$)	2	$M_2(C(\mathbb{T}^{4n}))$,	$n+1$

colours and $2n$ edges from v_1 to v_2, one each of the remaining $2n$ colours. Each example admits a unique factorization rule, so each illustration in Table 1 represents a unique $4n$-graph.

5. Stable rank in the non-stably finite, non-cofinal case

So far we have looked at the stably finite case (including stable rank one) and the cofinal case (including the simple case). Here, we study the remaining case corresponding to box 4 in Figure 1.

We start by revisiting the cofinality condition for row-finite locally convex k-graphs. Following [35], a subset $H \subseteq \Lambda^0$ is *hereditary* if $s(H\Lambda) \subseteq H$. We say H is *saturated* if for all $v \in \Lambda^0$,

$$\{s(\lambda) : \lambda \in v\Lambda^{\leq e_i}\} \subseteq H \text{ for some } i \in \{1, \ldots, k\} \implies v \in H.$$

or equivalently, if $v \notin H$ implies that for each $n \in \mathbb{N}^k$, $s(v\Lambda^{\leq n}) \not\subseteq H$ (see Lemma 5.1). The relevant characterization of cofinal is included in Lemma 5.2 below with a short proof based on [23, 42]. Since this paper focuses on unital k-graph C^*-algebras, it is worth pointing out that Lemmas 5.1 and 5.2 do not assume that $|\Lambda^0| < \infty$.

Lemma 5.1. *Let Λ be a row-finite locally convex k-graph. Then $H \subseteq \Lambda^0$ is saturated if and only if for all $v \in \Lambda^0$, $v \notin H$ implies that for each $n \in \mathbb{N}^k$, $s(v\Lambda^{\leq n}) \not\subseteq H$.*

Proof. Fix $v \in \Lambda^0$. Suppose $v \notin H$. Since Λ is saturated, for all $i \leq k$, $\{s(\lambda) : \lambda \in v\Lambda^{\leq e_i}\} \not\subseteq H$. Clearly $s(v\Lambda^{\leq m}) \not\subseteq H$ for $m = 0$. Fix any $m \in \mathbb{N}^k \setminus \{0\}$. Set $(n^{(0)}, v^{(0)}, \lambda^{(0)}) = (m, v, v)$. Choose i such that $n_i^{(0)} \neq 0$. Since $v^{(0)} \notin H$, there exists $\mu^{(1)} \in v^{(0)}\Lambda^{\leq e_i} \setminus \Lambda H$. Set $(n^{(1)}, v^{(1)}, \lambda^{(1)}) = (n^{(0)} - e_i, s(\mu^{(1)}), \lambda^{(0)}\mu^{(1)})$. Choose i such that $n_i^{(1)} \neq 0$. Since $v^{(1)} \notin H$, there exists $\mu^{(2)} \in v^{(1)}\Lambda^{\leq e_i} \setminus \Lambda H$. Set $(n^{(2)}, v^{(2)}, \lambda^{(2)}) = (n^{(1)} - e_i, s(\mu^{(2)}), \lambda^{(1)}\mu^{(2)})$.

For each step, $|n^{(i)}| = |m| - i$, so $l = |m|$ satisfies $n^{(l)} = 0$. Notice that $\lambda^{(0)} \in v\Lambda^{\leq (m-n^{(0)})}, \lambda^{(1)} \in v\Lambda^{\leq (m-n^{(1)})}, \ldots, \lambda^{(l)} \in v\Lambda^{\leq (m-n^{(l)})}$. Hence $\lambda^{(l)} \in v\Lambda^{\leq m}$ and $s(\lambda^{(l)}) \notin H$ so $s(v\Lambda^{\leq m}) \not\subseteq H$. $\square$

Lemma 5.2 ([23, 42]). *Let Λ be a row-finite locally convex k-graph. Then the following are equivalent:*

(1) Λ is cofinal;

(2) for all $v \in \Lambda^0$, and (λ_i) with $\lambda_i \in \Lambda^{\leq (1, \ldots, 1)}$, and $s(\lambda_i) = r(\lambda_{i+1})$ there exist $i \in \mathbb{N}$ and $n \leq d(\lambda_i)$ such that $v\Lambda\lambda_i(n) \neq \emptyset$; and

(3) Λ^0 contains no non-trivial hereditary saturated subsets.

Proof. Firstly, we show that $(1) \Rightarrow (3)$. Suppose (1) and suppose that $H \subseteq \Lambda^0$ is a non-empty hereditary, saturated set. We show that $H = \Lambda^0$. Fix $v \in \Lambda^0$. Since H is non-empty, there exists $w \in H$. By (1) there exists $n \in \mathbb{N}^k$ such that $s(v\Lambda^{\leq n}) \subseteq s(w\Lambda)$. Since H is hereditary, $s(v\Lambda^{\leq n}) \subseteq s(H\Lambda) \subseteq H$. Hence Lemma 5.1 gives $v \in H$.

Now we show that $(3) \Rightarrow (2)$. Suppose that (2) fails, that is, there exist $v \in \Lambda^0$, and a sequence (λ_i) with $\lambda_i \in \Lambda^{\leq (1, \ldots, 1)}$, $s(\lambda_i) = r(\lambda_{i+1})$ for all i such that for all $i \in \mathbb{N}$ and all $n \leq d(\lambda_i)$, we have $v\Lambda\lambda_i(n) = \emptyset$. Let

$$H = \{w \in \Lambda^0 : w\Lambda\lambda_i(n) = \emptyset \text{ for all } i \in \mathbb{N} \text{ and } n \leq d(\lambda_i)\}.$$

Then H is non-trivial as $v \in H$ and hereditary because if $u\Lambda w \neq \emptyset$ then $s(w\Lambda) \subseteq s(u\Lambda)$. To show that H is saturated take $u \in \Lambda^0$ and $j \leq k$ such that $s(u\Lambda^{\leq e_j}) \subseteq H$. We must show that $u \in H$. Assume otherwise for contradiction. We have $u \notin s(u\Lambda^{\leq e_j})$ because otherwise $u = s(u)$ belongs to H, so $u\Lambda^{e_j} \neq \emptyset$. Since $u \notin H$, there exists $\lambda \in u\Lambda$ such

that $s(\lambda) = \lambda_i(n)$ for some i, n. We claim that $d(\lambda)_j = 0$. Indeed, if not, then $\lambda = \mu\mu'$ for some $\mu \in u\Lambda^{e_j}$. We then have $s(\mu) \in s(u\Lambda^{e_j}) \subseteq H$, so $s(\mu)\Lambda\lambda_i(n) = \emptyset$ contradicting $\mu' \in s(\mu)\Lambda\lambda_i(n)$. Since Λ is locally convex and $d(\lambda)_j = 0$ and since $u\Lambda^{e_j} \neq \emptyset$, we have $\lambda_i(n)\Lambda^{e_j} = s(\lambda)\Lambda^{e_j} \neq \emptyset$. Let $\beta = \lambda_i(n, d(\lambda_i))$. Since Λ is locally convex, either $d(\beta)_j \neq 0$ or $s(\lambda_i)\Lambda^{e_j} \neq \emptyset$. Since $\lambda_{i+1} \in \Lambda^{\leq(1,\ldots,1)}$ it follows that $d(\beta\lambda_{i+1}) \geq e_j$. Now $\lambda' := \lambda\beta\lambda_{i+1} \in u\Lambda$ satisfies $s(\lambda') = \lambda_{i'}(n')$ for some i', n'. But then, just as we got $d(\lambda)_j = 0$, we deduce $d(\lambda')_j = 0$, a contradiction. So H is saturated, so (3) does not hold.

Finally, we prove (2)$\Rightarrow$(1). Given (2), we suppose that (1) fails, and we derive a contradiction. Since (1) fails, there exist $v, w \in \Lambda^0$ such that for all $n \in \mathbb{N}^k$, we have $s(w\Lambda^{\leq n}) \nsubseteq s(v\Lambda)$. Set

$$K = \{u \in \Lambda^0 : s(u\Lambda^{\leq n}) \nsubseteq s(v\Lambda) \text{ for all } n \in \mathbb{N}^k\}.$$

Fix $u \in K$ and $j \leq k$. We claim that there exists $\mu \in u\Lambda^{\leq e_j}$ such that $s(\mu) \in K$. Indeed if $s(u\Lambda^{\leq e_j}) \subseteq \Lambda^0 \setminus K$, then for each $\mu \in u\Lambda^{\leq e_j}$ there exists $n_\mu \in \mathbb{N}^k$ such that $s(\mu\Lambda^{n_\mu}) \subseteq s(v\Lambda)$. Since $s(v\Lambda)$ is hereditary, it follows that $n = \bigvee_{\mu \in u\Lambda^{\leq e_j}} n_\mu$ satisfies $s(u\Lambda^{\leq n+e_j}) = \bigcup_{\mu \in u\Lambda^{\leq e_j}} s(\mu\Lambda^{\leq n}) \subseteq s(v\Lambda)$, contradicting $u \in K$.

Since $w \in K$, we can construct a sequence (λ_i) such that each $\lambda_i \in \Lambda^{\leq(1,\ldots,1)}$, each $s(\lambda_i) = r(\lambda_{i+1})$, and for each $n \leq d(\lambda_i)$, we have $\lambda_i(n) \in K$. By (2), there exist i and $n \leq d(\lambda_i)$ such that $v\Lambda\lambda_i(n) \neq \emptyset$, i.e., such that $s(\lambda_i(n)\Lambda^{\leq 0}) \subseteq s(v\Lambda)$. So $\lambda_i(n) \notin K$, a contradiction. $\qquad\square$

Remark 5.3. When a k-graph Λ has only one vertex, it is automatically cofinal, and we deduce that the stable rank of $C^*(\Lambda)$ is infinite if there exists $j \leq k$ such that $|\Lambda^{e_j}| \geq 2$, and is equal to $\lfloor k/2 \rfloor + 1$ if each $|\Lambda^{e_j}| = 1$.

Remark 5.4. We now present all the 2-graphs Λ with $|\Lambda^0| = 2$ for which we have been unable to compute the stable rank of the associated C^*-algebra $C^*(\Lambda)$ (see Figure 7). In each case, the 2-graph Λ fails to be cofinal, because Λ^0 contains one non-trivial hereditary saturated subset, denoted H.

In Figure 7, for each 2-graph Λ the C^*-algebra $C^*(\Lambda)$ is non-simple with $H = \{u\}$. In the first case, we have $C^*(H\Lambda) \cong C(\mathbb{T})$, which has stable rank 1, and so I_H has stable rank 1 because stable rank 1 is preserved by stable isomorphism. In the remaining two cases, if there is one loop of each colour at u then $C^*(H\Lambda) \cong C(\mathbb{T}^2)$ has stable rank 2, and otherwise, Theorem 4.4 implies that $C^*(H\Lambda)$ has stable rank ∞; either way, since $I_H \cong C^*(H\Lambda) \otimes \mathcal{K}^{\S}$, we have $sr(I_H) = 2$ as discussed in §1.1.

In all three cases, the quotient of $C^*(\Lambda)$ by I_H is $C^*(\Lambda)/I_H \cong C^*(\Lambda \setminus \Lambda H) \cong C(\mathbb{T}^2)$. Hence, by [2, V.3.1.21], we deduce that $sr(C^*(\Lambda)) \in \{2, 3\}$, but we have been unable to determine the exact value in any of these cases.

Perhaps the easiest-looking case is the 2-graph (top left) with one red (dashed) edge from u to v. In this case $C^*(\Lambda) \cong \mathcal{T} \otimes C(\mathbb{T})$, where $\mathcal{T}$ is the Toeplitz algebra generated by

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$\S$ To see this, let X be the set $\{u\} \cup \{\mu f : f \in v\Lambda^{e_2}u \text{ and } \mu \in \Lambda^{\mathbb{N}e_2}v\}$. Use the factorization property and the Cuntz–Krieger relations to see that $I_H = \overline{\text{span}}\{s_\mu a s_\nu^* : \mu, \nu \in X \text{ and } a \in s_u C^*(\Lambda)s_u\}$. It is routine that for any finite subset $F \subseteq X$, the set $\{s_\mu s_\nu^* : \mu, \nu \in F\}$ is a system of matrix units. So Lemma 2.4 gives $\overline{\text{span}}\{s_\mu a s_\nu^* : \mu, \nu \in F \text{ and } a \in s_u C^*(\Lambda)s_u\} \cong s_u C^*(\Lambda)s_u \otimes M_{|F|}(\mathbb{C})$. Taking the direct limit gives $I_H \cong s_u C^*(\Lambda)s_u \otimes \mathcal{K}(\ell^2(X))$.
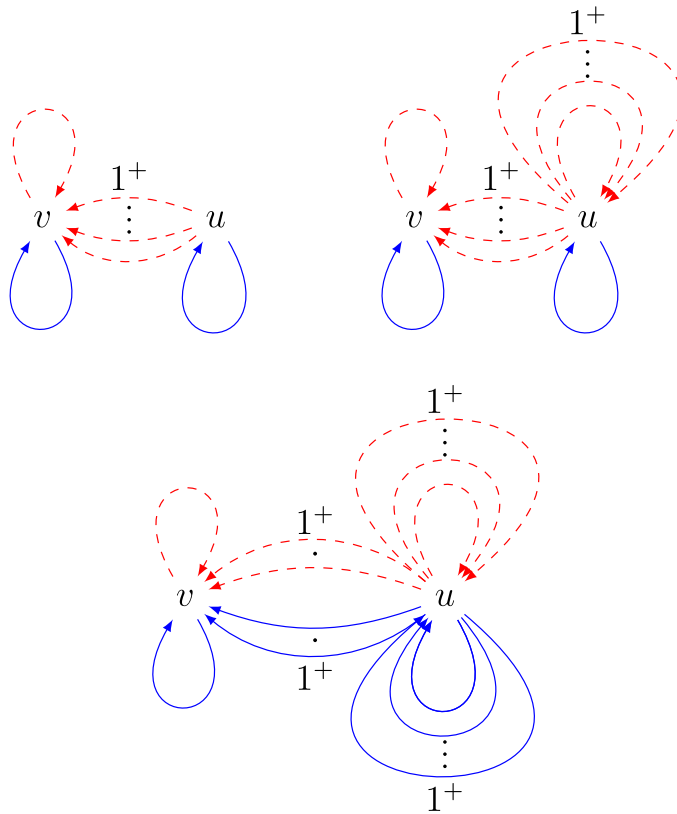
Figure 7. Example of 2-graphs Λ with $C^*(\Lambda)$ of stable rank two or three.

the unilateral shift. Despite knowing the stable rank of each of the components ($sr(\mathcal{T}) = 2$ and $sr(C(\mathbb{T})) = 1$), the stable rank of the tensor product is not known (there is no general formula for stable rank of tensor products).

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