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Asymptotically self-similar behaviour of global solutions for semilinear heat equations with algebraically decaying initial data

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We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where $N > 2$, $p > 1$, and u_0 is a bounded continuous non-negative function in $\mathbf{R}^N$. We study the case where $u_0(x)$ decays at the rate $|x|^{-2/(p-1)}$ as $|x| \to \infty$, and investigate the convergence property of the global solutions to the forward self-similar solutions. We first give the precise description of the relationship between the spatial decay of initial data and the large time behaviour of solutions, and then we show the existence of solutions with a time decay rate slower than the one of self-similar solutions. We also show the existence of solutions that behave in a complicated manner.

Keywords: semilinear heat equation; self-similar solution; asymptotic behaviour; Joseph-Lundgren critical exponent

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1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $N > 2$, $p > 1$, and $u_0 \not\equiv 0$ is a given bounded continuous non-negative function in $\mathbf{R}^N$. It is known that there exists $T = T(u_0) > 0$ such that (1.1) has a unique classical solution $u \in C^{2,1}(\mathbf{R}^N \times (0,T)) \cap C(\mathbf{R}^N \times [0,T))$ which is bounded in $\mathbf{R}^N \times [0,T']$ for all $T' < T$, and $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \to \infty$ as $t \to T$ if $T < \infty$. We say that u is global if $T = \infty$, and that u blows up in finite time if $T < \infty$. It is well known by [**12**, **31**] that if $1 < p \leqslant (N+2)/N$ then (1.1) has no global solution. Then the condition $p > (N+2)/N$ is necessary for the existence of global solutions of (1.1).

The equation in (1.1) is invariant under the similarity transformation

$$u(x,t) \mapsto u_\lambda(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t) \quad \text{for all } \lambda > 0.$$

In particular, a forward self-similar solution u has the form $u(x,t) = t^{-1/(p-1)} \phi(x/\sqrt{t})$, where ϕ satisfies the elliptic equation

$$\Delta\phi + \frac{1}{2} x \cdot \nabla\phi + \frac{1}{p-1}\phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

If a solution ϕ of (1.2) is radially symmetric about the origin, then $\phi = \phi(r)$, $r = |x|$, satisfies $\phi'(0) = 0$ and

$$\phi' + \left(\frac{N-1}{r} + \frac{r}{2}\right)\phi' + \frac{1}{p-1}\phi + \phi^p = 0 \quad \text{for } r > 0. \tag{1.3}$$

Let $p > (N+2)/N$, and let ϕ be a positive solution of (1.3). It was shown by [**16**, **28**] that ϕ satisfies

$$\lim_{r\to\infty} r^{2/(p-1)}\phi(r) = \ell \tag{1.4}$$

with some $\ell \geqslant 0$, and that $\phi(r)$ decays exponentially as $r \to \infty$ if $\ell = 0$ in (1.4).

Forward self-similar solutions are global in time and often used to describe the large time behaviour of global solutions to (1.1). In the case where initial data $u_0(x)$ has the exponential decay at $|x| = \infty$ in (1.1), it was shown by Kavian [**19**] and Kawanago [**20**] that certain solutions are asymptotic to the forward self-similar solution whose profile decays exponentially in space. In this paper, we consider the problem (1.1) in the case where $u_0(x)$ decays at $|x|^{-2/(p-1)}$ as $|x| \to \infty$, and investigate the convergence property of the global solutions to the forward self-similar solutions.

For each $\ell > 0$, we denote by S_ℓ the set of all positive solution ϕ of (1.3) satisfying

$$\phi'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}\phi(r) = \ell. \tag{1.5}$$

We call $\underline{\phi}_\ell$ a minimal solution of S_ℓ if $\underline{\phi}_\ell \in S_\ell$ satisfies $\underline{\phi}_\ell \leqslant \phi$ for all $\phi \in S_\ell$. We denote by $\underline{w}_\ell$ a self-similar solution corresponding to the minimal solution $\underline{\phi}_\ell$ of S_ℓ, that is,

$$\underline{w}_\ell(x,t) = t^{-1/(p-1)}\underline{\phi}_\ell(|x|/\sqrt{t}). \tag{1.6}$$

To state the existence and nonexistence of solutions of S_ℓ, we need some notations. Define a constant L by

$$L = \left(\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right)^{1/(p-1)}.$$

Note that $U(r) = Lr^{-2/(p-1)}$, $r = |x|$, is a singular stationary solution of (1.1) when $p > N/(N-2)$. Define p_{JL} by

$$p_{JL} = \begin{cases} \infty, & 3 \leqslant N \leqslant 10, \\ 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geqslant 11. \end{cases}$$

The exponent p_{JL} has appeared in several different studies of (1.1) and related problems, see, for example, [**13**–**15**, **18**, **36**]. The following results were shown by [**22**, **23**].

Proposition A. *Let $p > (N+2)/N$.*

(i) *There exists $\ell^* > 0$ such that $S_\ell \neq 0$ if $0 < \ell < \ell^*$ and $S_\ell = \emptyset$ if $\ell > \ell^*$.*

(ii) *If $S_\ell \neq \emptyset$ then there exists a minimal solution $\underline{\phi}_\ell$ of S_ℓ.*

(iii) *If $(N+2)/N < p < p_{JL}$ then $\ell^* > L$ and there exists a unique solution of S_{ℓ^*}. If $p \geqslant p_{JL}$ then $\ell^* = L$ and $S_{\ell^*} = \emptyset$.*

For the properties (i) and (ii), see lemma 3.1 of [**23**]. (See also lemma 4.1 of [**22**].) Property (iii) is a consequence of theorem 1.1 and corollary 1.2 of [**23**]. For the multiplicity of solutions of S_ℓ, it was shown that, if $(N+2)/N < p < p_{JL}$, then there exists $\ell > 0$ such that S_ℓ has at least two solutions, while if $p \geqslant p_{JL}$, then S_ℓ has a unique solution for all $\ell \in (0, \ell^*)$. (See [**23**, **24**, **33**].)

First, we consider the problem (1.1) in the case where $u_0(x)$ satisfies

$$\lim_{|x|\to\infty} |x|^{2/(p-1)} u_0(x) = \ell \tag{1.7}$$

with $\ell > 0$. We show that the large time behaviour of global solutions is determined by the decay property of initial data at the spatial infinity.

Theorem 1.1. *Let $p > (N+2)/N$, and let $\ell^* > 0$ be the constant in proposition A. Assume that $u_0 \in C(\mathbf{R}^N)$ satisfies*

$$0 \leqslant u_0(x) < \ell^* |x|^{-2/(p-1)} \quad \textit{for } x \in \mathbf{R}^N \setminus \{0\}, \tag{1.8}$$

and (1.7) with some $\ell \in (0, \ell^*)$. Then the solution u of (1.1) is global in time and satisfies

$$\lim_{t\to\infty} t^{1/(p-1)} \|u(\cdot, t) - \underline{w}_\ell(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = 0, \tag{1.9}$$

where $\underline{w}_\ell$ is defined by (1.6).

Remark 1.2.

(i) It was shown by Lee and Ni [**21**] that the decay rate $|x|^{-2/(p-1)}$ of the initial data is a borderline between blow-up and global existence. In fact, by [**25**, theorem 1.3], if (1.7) holds with $\ell > \ell^*$, then the solution u of (1.1) blows up in finite time.

(ii) It was shown by [**3**, **32**] that, if the initial data are small and coincide for large x with a homogeneous function of degree $-2/(p-1)$, then the solution is asymptotically self-similar. See also [**31**, § 20.3]. These results are proved by semigroup techniques and suitable fixed point argument, which are completely different from our approach.

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(iii) In the case where initial data $u_0(x)$ decays at $|x|^{-\sigma}$ as $x = \infty$ with $\sigma > 2/(p-1)$, it is known that global solutions behave like the fundamental solutions of the heat equation up to multiple constants. See [**31**, theorem 20.6].

(iv) It should be mentioned that the exact convergence rate of solutions to the forward self-similar solutions was shown by Fila *et al.* [**11**] in the case where the initial data has a specific decay rate as $x \to \infty$.

We consider the case where $\ell = \ell^*$ in theorem 1.1 when $p \geqslant p_{JL}$. Recall that $\ell^* = L$ and $S_{\ell^*} = \emptyset$ if $p \geqslant p_{JL}$ by proposition A. We obtain the following result, which is a slight improvement of [**15**, theorem 4 (i)].

THEOREM 1.3. *Let $p \geqslant p_{JL}$. Assume that u_0 satisfies*

$$0 \leqslant u_0(x) < L|x|^{-2/(p-1)} \quad \text{for } x \in \mathbf{R}^N \setminus \{0\} \tag{1.10}$$

and

$$\lim_{x\to\infty} |x|^{2/(p-1)} u_0(x) = L. \tag{1.11}$$

Then the solution u of (1.1) *is global and satisfies*

$$\lim_{t\to\infty} t^{1/(p-1)} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = \infty. \tag{1.12}$$

REMARK 1.4. Assume that $p > p_{JL}$. Let λ_1 be a positive constant defined by

$$\lambda_1 = \frac{N-2-2m-\sqrt{(N-2-2m)^2-8(N-2-m)}}{2} \tag{1.13}$$

with $m = 2/(p-1)$. It was shown by Yanagida [**37**, theorem 6.1] that, if u_0 satisfies (1.10) and

$$\lim_{|x|\to\infty} |x|^{\lambda_1}(|x|^{2/(p-1)}u_0(x) - L) = \infty, \tag{1.14}$$

then the solution u of (1.1) is global and satisfies

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = 0. \tag{1.15}$$

Thus, by combining with theorem 1.3, if u_0 satisfies (1.10), (1.11) and (1.14), then the solution u of (1.1) is global and satisfies (1.12) and (1.15).

For nonlinear parabolic equations, it is well known that positive global solutions must converge to a single steady state under fairly general assumptions (see, e.g., [**1**, **6**, **7**]). On the other hand, Poláčik and Yanagida [**30**] showed that, when $p \geqslant p_{JL}$, (1.1) has a positive global solution that behaves in a complicated manner. For $\alpha > 0$, we denote by $v_\alpha = v_\alpha(r)$, $r = |x|$, a radially symmetric solution of

$$\Delta v + v^p = 0 \quad \text{in } \mathbf{R}^N \tag{1.16}$$

satisfying $v_\alpha(0) = \alpha$. It was shown by [**30**, theorem 1.1] that, in the case $p \geqslant p_{JL}$, for any infinite sequence $\{(\alpha_i, \xi_i, \varepsilon_i)\}$ with $\alpha_i > 0$, $\xi_i \in \mathbf{R}^N$ and $\varepsilon_i > 0$, there exists u_0 such that the solution u of (1.1) exists globally in time and satisfies the following:

(i) There exists an increasing sequence $\{s_i\}$ tending to ∞ as $i \to \infty$ such that

$$\|u(\cdot, s_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i \quad \text{for each } i = 1, 2, \ldots.$$

(ii) There exists a sequence $\{t_i\}$ with $t_i \in (s_i, s_{i+1})$ such that

$$\|u(\cdot, t_i) - v_{\alpha_i}(\cdot - \xi_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i \quad \text{for each } i = 1, 2, \ldots.$$

Recall that $p_{JL} = \infty$ if $3 \leqslant N \leqslant 10$, and hence the above non-convergent solution can be observed when $N \geqslant 11$. For $N \geqslant 3$, we obtain the following result.

THEOREM 1.5. *Let $p > (N+2)/N$. For any infinite sequence $\{(\ell_i, \xi_i, \varepsilon_i)\}$ with*

$$0 < \ell_i < \ell^*, \quad \xi_i \in \mathbf{R}^N \quad \textit{and} \quad \varepsilon_i > 0, \tag{1.17}$$

there exists $u_0 \in C(\mathbf{R}^N)$ such that the solution u of (1.1) exists globally in time and there exists an increasing sequence $\{t_i\}$ tending to ∞ as $i \to \infty$ such that

$$t_i^{1/(p-1)} \|u(\cdot, t_i) - \underline{w}_{\ell_i}(\cdot - \xi_i, t_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i \quad \textit{for each } i = 1, 2, \ldots, \tag{1.18}$$

where $\underline{w}_\ell(x, t)$ is defined by (1.6).

As a consequence of theorem 1.5, we obtain the analogous phenomena to [**30**] for global solutions tending to 0 as $t \to \infty$.

COROLLARY 1.6. *Let $p > (N+2)/N$. For any infinite sequence $\{(\ell_i, \xi_i, \varepsilon_i)\}$ with (1.17), there exists $u_0 \in C(\mathbf{R}^N)$ such that the solution u of (1.1) exists globally in time and satisfies the following:*

(i) *There exists an increasing sequence $\{s_i\}$ tending to ∞ as $i \to \infty$ such that*

$$s_i^{1/(p-1)} \|u(\cdot, s_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i \quad \textit{for each } i = 1, 2, \ldots.$$

(ii) *There exists an increasing sequence $\{t_i\}$ with $t_i \in (s_i, s_{i+1})$ such that (1.18) holds.*

REMARK 1.7. The existence of global solutions with complicated asymptotic behaviour was also shown by [**4**, **5**] by the different argument. In [**4**, **5**] it was shown that there exist solutions which are asymptotic to many different self-similar solutions along different time sequences. These results are proved by employing the properties of the semiflow on some Banach spaces.

In the proofs of theorems 1.1, 1.3 and 1.5, we will employ the rescaling to the self-similar variables, which have been used by Kavian [**19**]. For a solution u of

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(1.1), define

$$w(y,s) = (t+1)^{1/(p-1)} u(x,t)$$

with

$$y = \frac{x}{\sqrt{t+1}} \quad \text{and} \quad s = \log(t+1).$$

Then w satisfies

$$\begin{cases} w_s = \Delta w + \frac{1}{2} y \cdot \nabla w + \frac{1}{p-1} w + w^p, & y \in \mathbf{R}^N, \ s \geqslant 0, \\ w(y,0) = w_0(y), & y \in \mathbf{R}^N, \end{cases} \tag{1.19}$$

where $w_0 = u_0$. Since (1.2) is the corresponding stationary problem to (1.19), we can expect that the study of the behaviour of solutions of (1.19) is reduced to the analysis of stability properties of solutions of (1.2).

It should be mentioned that, in some cases, the large time behaviour of solutions is determined by the decay property of initial data at the spatial infinity. Recall that $v_\alpha(r)$, $r = |x|$, is the radially symmetric solution of (1.16) satisfying $v_\alpha(0) = \alpha$, and that λ_1 is the positive constant given by (1.13). It was shown by Poláčik and Yanagida [**29**] that, if $p > p_{JL}$, and if u_0 satisfies (1.10) and

$$\lim_{|x| \to \infty} |x|^{m+\lambda_1} |u_0(x) - v_\alpha(x)| = 0,$$

then the solution u of (1.1) is global and satisfies

$$\lim_{|x| \to \infty} \|u(\cdot,t) - v_\alpha(\cdot)\|_{L^\infty(\mathbf{R}^N)} = 0.$$

It was also shown by [**29**] that, if $p > p_{JL}$, and if u_0 satisfies (1.10) and

$$L|x|^{-2/(p-1)} - c_1|x|^{-\ell} \leqslant u_0(x) \leqslant L|x|^{-2/(p-1)} - c_2|x|^{-\ell} \quad \text{for } |x| > R$$

with some $\ell > m + \lambda_1$ and $c_1, c_2, R > 0$, then the solution of (1.1) is global and unbounded as $t \to \infty$. For more precise analysis, we refer to [**9**, **17**, **26**, **27**, **34**, **35**] for the convergence to the steady state, and to [**8**, **10**] for the grow-up rate of solutions. In the proof of theorems 1.1 and 1.3, we will show that analogous results of [**29**] hold for the problem (1.19) and the corresponding stationary problem (1.2). Making use of these results together with the continuous dependence of the initial value, we will obtain theorem 1.5.

The paper is organized as follows: In § 2, we present some preliminary results which will be used in the sequel. We construct continuous weak super and sub-solutions to (1.2) in § 3, and give the proof of theorems 1.1 and 1.3 in § 4. Finally, we prove theorem 1.5 and corollary 1.6 in § 5.

2. Preliminaries

We first recall the definition of continuous weak super and sub-solutions to the problem (1.19). We say that w is a continuous weak supersolution of (1.19) for $0 \leqslant s \leqslant S$ if w is continuous on $\mathbf{R}^N \times [0,S]$, $w(y,0) \geqslant w_0(y)$ and satisfies, for any

$\eta \in C^{2,1}(\mathbf{R}^N \times [0,S])$ with $\eta \geqslant 0$ and supp $\eta(\cdot,s)$ being compact in $\mathbf{R}^N$ for all $s \in [0,S]$,

$$\int_{\mathbf{R}^N} w(y,s)\eta(y,s)\mathrm{d}y\Big|_{s=0}^{s=S'} \geqslant \int_0^{S'}\int_{\mathbf{R}^N} w(y,s)H(y,s) + w(y,s)^p\eta(x,s)\mathrm{d}y\mathrm{d}s \quad (2.1)$$

for all $S' \in [0,S]$, where

$$H(y,s) = \left(\eta_s + \Delta\eta - \frac{1}{2}y\cdot\nabla\eta + \left(\frac{1}{p-1} - \frac{N}{2}\right)\eta\right)(y,s).$$

Continuous weak subsolutions are defined in a similar way by reversing the inequalities. The following result was shown by [**25**, lemma 2.3].

LEMMA 2.1. *If $\overline{w}$ and $\underline{w}$, respectively, are bounded continuous weak super and sub-solutions of* (1.19) *for $0 \leqslant s \leqslant S$. Then $\underline{w} \leqslant \overline{w}$ in $\mathbf{R}^N \times [0,S]$ and* (1.19) *has a unique classical solution w with $\underline{w} \leqslant w \leqslant \overline{w}$ in $\mathbf{R}^N \times [0,S]$.*

We say that v is a continuous weak supersolution (subsolution) of (1.2) if v is continuous in $\mathbf{R}^N$ and satisfies, for any $\eta \in C^2(\mathbf{R}^N)$ with $\eta \geqslant 0$ and compact support in $\mathbf{R}^N$,

$$\int_{\mathbf{R}^N} v(y)\tilde{H}(y) + v(y)^p\eta(x)\mathrm{d}y \leqslant\ (\geqslant)\ 0,$$

where

$$\tilde{H}(y) = \left(\Delta\eta - \frac{1}{2}y\cdot\nabla\eta + \left(\frac{1}{p-1} - \frac{N}{2}\right)\eta\right)(y).$$

The following result was shown by [**25**, lemma 2.5].

LEMMA 2.2. *Let w_0 in* (1.19) *be a bounded continuous weak supersolution* (*subsolution*) *of* (1.2), *and let w be a global solution of* (1.19). *Then the solution $w(y,s)$ is nonincreasing* (*nondecreasing*) *in $s \geqslant 0$.*

We call v is a classical supersolution (subsolution) of (1.2) if $v \in C^2(\mathbf{R}^N)$ and satisfies

$$\Delta v + \frac{1}{2}y\cdot\nabla v + \frac{1}{p-1}v + v^p \leqslant\ (\geqslant)\ 0, \quad y \in \mathbf{R}^N.$$

The following lemma by [**25**, lemma 2.6] gives a way to construct continuous weak super and sub-solutions of (1.2).

LEMMA 2.3.

(i) *Let $v_1 = v_1(r)$ and $v_2 = v_2(r)$, $r = |y|$, be classical radial supersolutions of* (1.2). *Assume that $v_1(R) = v_2(R)$ and $v_1'(R) \geqslant v_2'(R)$ with some $R > 0$. Define*

$$\tilde{v}(r) = \begin{cases} v_1(r), & 0 \leqslant r \leqslant R,\\ v_2(r), & R < r < \infty. \end{cases} \quad (2.2)$$

Then $\tilde{v}(|y|)$ is a continuous weak supersolution of (1.2).

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(ii) *Let* $v_1 = v_1(r)$ *and* $v_2 = v_2(r)$, $r = |y|$, *be classical radial subsolutions of* (1.2). *Assume that* $v_1(R) = v_2(R)$ *and* $v_1'(R) \leqslant v_2'(R)$ *with some* $R > 0$. *Define* $\tilde{v}$ *by* (2.2). *Then* $\tilde{v}(|y|)$ *is a continuous weak subsolution of* (1.2).

The following convergence result was shown by [**25**, proposition 3.1 (iii)].

LEMMA 2.4. *Let* w *be a global solution of* (1.19) *such that* $w = w(r, s)$, $r = |y|$, *is spatially radially symmetric about the origin. Assume that* $w(r, s)$ *is nonincreasing in* $s \geqslant 0$ *for each fixed* $r \geqslant 0$, *and put* $\phi(r) = \lim_{s\to\infty} w(r, s)$ *for* $r \geqslant 0$. *Assume, in addition, that there exists a continuous function* W *on* $[0, \infty)$, *satisfying* $W(r) \to 0$ *as* $r \to \infty$, *such that* $w(r, s) \leqslant W(r)$ *for* $r \geqslant 0$ *and* $s \geqslant 0$. *Then* $\phi \in C^2[0, \infty)$ *and satisfies* (1.3) *with* $\phi'(0) = 0$ *and*

$$\|w(\cdot, s) - \phi(\cdot)\|_{L^\infty([0,\infty))} \to 0 \quad \text{as } s \to \infty.$$

For $\alpha > 0$, we denote by $\phi(r; \alpha)$ a unique solution of (1.3) satisfying $\phi(0) = \alpha$ and $\phi'(0) = 0$. It was shown by [**16**, theorem 5] that $\phi(\cdot, \alpha) \in C^2[0, \infty)$ and the limit

$$\ell(\alpha) = \lim_{r\to\infty} r^{2/(p-1)} \phi(r; \alpha)$$

exists and is locally Lipschitz continuous on $\alpha > 0$. The following result was shown by [**23**, theorem 1.1 and corollary 1.2].

LEMMA 2.5. *Let* $p > (N + 2)/N$. *There exists* $\alpha^* \in (0, \infty]$ *satisfying the following* (i)–(iii).

(i) *If* $\alpha \in (0, \alpha^*)$, *then* $\phi(r; \alpha) > 0$ *for* $r \geqslant 0$ *and* $\ell(\alpha) > 0$, *and* $\phi(r; \alpha)$ *is a minimal solution of* $S_{\ell(\alpha)}$.

(ii) *If* $\underline{\phi}_\ell$ *is a minimal solution of* S_ℓ *with* $\ell \in (0, \ell^*)$, *then there exists* $\alpha \in (0, \alpha^*)$ *such that* $\phi(r; \alpha) \equiv \underline{\phi}_\ell(r)$.

(iii) *If* $p \geqslant p_{JL}$, *then* $\alpha^* = \infty$.

By lemma 2.5, we obtain the following results.

LEMMA 2.6.

(i) *Let* $p > (N + 2)/N$, *and let* $\ell, \hat{\ell} \in (0, \ell^*)$ *with* $\ell < \hat{\ell}$. *If* $\phi \in S_\ell$ *satisfies* $\phi(r) \leqslant \underline{\phi}_{\hat{\ell}}(r)$ *for* $r \geqslant 0$. *Then* $\phi(r) \equiv \underline{\phi}_\ell(r)$.

(ii) *Let* $p > (N + 2)/N$. *For any* $\varepsilon > 0$, *there exists* $\ell_\varepsilon \in (0, \ell^*)$ *such that* $\|\underline{\phi}_{\ell_\varepsilon}\|_{L^\infty([0,\infty))} < \varepsilon$.

(iii) *Let* $p \geqslant p_{JL}$. *For any* $M > 0$, *there exists* $\ell_M \in (0, L)$ *such that* $\|\underline{\phi}_{\ell_M}\|_{L^\infty([0,\infty))} > M$.

Proof.
(i) By lemma 2.5 (ii), we have $\underline{\phi}_{\hat{\ell}}(0) < \alpha^*$, and hence $\phi(0) < \alpha^*$. By lemma 2.5 (i), ϕ is the minimal solution of S_ℓ. This implies that $\phi(r) \equiv \underline{\phi}_\ell(r)$.

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(ii) We may assume that $\varepsilon < \alpha^*$. Take $\alpha \in (0, \varepsilon)$. Then, by lemma 2.5 (i), $\phi(r;\alpha)$ is a minimal solution of $S_{\ell(\alpha)}$ with some $\ell(\alpha) \in (0, \ell^*)$. Note that $\|\phi(\cdot,\alpha)\|_{L^\infty([0,\infty))} = \alpha$. Put $\ell_\varepsilon = \ell(\alpha)$. Then we have $\underline{\phi}_{\ell_\varepsilon}(r) \equiv \phi(r;\alpha)$ and $\|\underline{\phi}_{\ell_\varepsilon}\|_{L^\infty([0,\infty))} = \alpha < \varepsilon$.

(iii) Recall that $\ell^* = L$ if $p \geqslant p_{JL}$ by proposition A (iii). Take $\alpha > M$. By lemma 2.5 (i) and (iii), we see that $\phi(r;\alpha)$ is a minimal solution of $S_{\ell(\alpha)}$ with some $\ell(\alpha) \in (0, L)$. Put $\ell_M = \ell(\alpha)$. Then we have $\underline{\phi}_{\ell_M}(r) \equiv \phi(r;\alpha)$ and $\|\underline{\phi}_{\ell_M}\|_{L^\infty([0,\infty))} = \alpha > M$. □

3. Construction of continuous weak super and sub-solutions

For simplicity, we denote by $\mathcal{L}v$ the differential operator

$$\mathcal{L}v = v' + \left(\frac{N-1}{r} + \frac{r}{2}\right)v' + \frac{1}{p-1}v \tag{3.1}$$

for $v \in C^2(0,\infty)$. Then the equation (1.3) is written as

$$\mathcal{L}\phi + \phi^p = 0 \quad \text{for } r > 0. \tag{3.2}$$

Recall that S_ℓ is the set of all positive solution ϕ of (1.3) satisfying (1.5), and that $\underline{\phi}_\ell$ denotes the minimal solution of S_ℓ. In this section, we will show the following two propositions.

PROPOSITION 3.1. *Suppose that $S_{\hat\ell} \neq \emptyset$ for some $\hat\ell > 0$. Let $u_0(x)$ satisfy*

$$0 \leqslant u_0(x) \leqslant \underline{\phi}_{\hat\ell}(|x|) \quad \textit{for } x \in \mathbf{R}^N.$$

(i) *Assume that there exists $\ell \in (0, \hat\ell)$ such that $u_0(x) \leqslant \underline{\phi}_\ell(|x|)$ for sufficiently large $|x|$. Then there exists a continuous weak supersolution $\overline{v}_0(|x|)$ of (1.2) satisfying*

$$u_0(x) \leqslant \overline{v}_0(|x|) \quad \textit{for } x \in \mathbf{R}^N, \tag{3.3}$$

$$\underline{\phi}_\ell(|x|) \leqslant \overline{v}_0(|x|) \leqslant \underline{\phi}_{\hat\ell}(|x|) \quad \textit{for } x \in \mathbf{R}^N \quad \textit{and} \tag{3.4}$$

$$\lim_{|x|\to\infty} |x|^{2/(p-1)}\overline{v}_0(|x|) = \ell. \tag{3.5}$$

(ii) *Assume that there exists $\ell \in (0, \hat\ell)$ such that $u_0(x) \geqslant \underline{\phi}_\ell(|x|)$ for sufficiently large $|x|$. Then there exists a continuous weak subsolution $\underline{v}_0(|x|)$ of (1.2) satisfying*

$$0 \leqslant \underline{v}_0(|x|) \leqslant u_0(x) \quad \textit{for } x \in \mathbf{R}^N, \tag{3.6}$$

$$\underline{v}_0(|x|) \leqslant \underline{\phi}_\ell(|x|) \quad \textit{for } x \in \mathbf{R}^N \quad \textit{and} \tag{3.7}$$

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$$\lim_{|x|\to\infty} |x|^{2/(p-1)}\underline{v}_0(|x|) = \ell. \tag{3.8}$$

We denote by $w(y, s; w_0)$ the solution of (1.19).

PROPOSITION 3.2. *Suppose that $S_{\hat{\ell}} \neq \emptyset$ for some $\hat{\ell} > 0$. Let $\ell \in (0, \hat{\ell})$.*

(i) *Assume that $\overline{v}_0(|x|)$ is a continuous weak supersolution of (1.2) satisfying (3.4) and (3.5). Then $w(y, s; \overline{v}_0) \geqslant \underline{\phi}_\ell(|y|)$ for all $y \in \mathbf{R}^N$, $s \geqslant 0$ and $\|w(\cdot, s; \overline{v}_0) - \underline{\phi}_\ell(|\cdot|)\|_{L^\infty(\mathbf{R}^N)}$ as $s \to \infty$.*

(ii) *Assume that $\underline{v}_0(|x|)$ is a continuous weak subsolution of (1.2) satisfying (3.7) and (3.8). Then $w(y, s; \underline{v}_0) \leqslant \underline{\phi}_\ell(|y|)$ for all $y \in \mathbf{R}^N$, $s \geqslant 0$ and $\|w(\cdot, s; \underline{v}_0) - \underline{\phi}_\ell(|\cdot|)\|_{L^\infty(\mathbf{R}^N)}$ as $s \to \infty$.*

In the proof of proposition 3.1, we put $\psi(r) = r^{-\mu}$ with $\mu > 2/(p-1)$. Then, by a direct calculation, we obtain

$$\mathcal{L}\psi = \left(-\frac{\mu}{2} + \frac{1}{p-1}\right) r^{-\mu} + O(r^{-\mu-2}) \quad \text{as } r \to \infty.$$

Note here that

$$p\underline{\phi}_{\hat{\ell}}(r)^{p-1}\psi(r) = O(r^{-\mu-2}) \quad \text{as } r \to \infty.$$

Then, from $\mu > 2/(p-1)$, there exists $R_0 > 0$ such that

$$\mathcal{L}\psi + p\underline{\phi}_{\hat{\ell}}^{p-1}\psi \leqslant 0 \quad \text{for } r > R_0. \tag{3.9}$$

The following results hold.

LEMMA 3.3. *Suppose that $S_{\hat{\ell}} \neq \emptyset$ for some $\hat{\ell} > 0$. Let $\ell \in (0, \hat{\ell})$.*

(i) *Define v_0 by*

$$v_0(r) = \underline{\phi}_\ell(r) + C\psi(r) \quad \textit{for } r > 0, \tag{3.10}$$

where $C > 0$ is a constant. Assume that there exists $R \geqslant R_0$ such that

$$v_0(r) \leqslant \phi_{\hat{\ell}}(r) \quad \textit{for } r > R. \tag{3.11}$$

Then $\mathcal{L}v_0 + v_0^p \leqslant 0$ for $r > R$.

(ii) *Define v_0 by*

$$v_0(r) = \underline{\phi}_\ell(r) - C\psi(r) \quad \textit{for } r > 0, \tag{3.12}$$

where $C > 0$ is a constant. Assume that there exists $R \geqslant R_0$ such that $v_0(r) \geqslant 0$ for $r > R$. Then $\mathcal{L}v_0 + v_0^p \geqslant 0$ for $r > R$.

Proof.

(i) From (3.9) the function v_0, defined by (3.10), satisfies

$$\mathcal{L}v_0 + v_0^p = \mathcal{L}\underline{\phi}_\ell + C\mathcal{L}\psi + v_0^p \leqslant -\underline{\phi}_\ell^p - Cp\underline{\phi}_{\hat{\ell}}^{p-1}\psi + v_0^p$$

for $r > R_0$. By the mean value theorem and (3.11), for $r > R$, we obtain

$$v_0^p - \underline{\phi}_\ell^p = (\underline{\phi}_\ell + C\psi)^p - \underline{\phi}_\ell^p \leqslant pv_0^{p-1}C\psi \leqslant Cp\underline{\phi}_{\hat{\ell}}^{p-1}\psi.$$

Thus we obtain $\mathcal{L}v_0 + v_0^p \leqslant 0$ for $r > R$.

(ii) From (3.9) the function v_0, defined by (3.12), satisfies

$$\mathcal{L}v_0 + v_0^p = \mathcal{L}\underline{\phi}_\ell - C\mathcal{L}\psi + v_0^p \geqslant -\underline{\phi}_\ell^p + Cp\underline{\phi}_{\hat{\ell}}^{p-1}\psi + v_0^p$$

for $r > R_0$. By the mean value theorem, for $r > R$, we obtain

$$\underline{\phi}_\ell^p - v_0^p = \underline{\phi}_\ell^p - (\underline{\phi}_\ell - C\psi)^p \leqslant p\underline{\phi}_\ell^{p-1}C\psi \leqslant Cp\underline{\phi}_{\hat{\ell}}^{p-1}\psi.$$

Thus we obtain $\mathcal{L}v_0 + v_0^p \geqslant 0$ for $r > R$. □

Proof of proposition 3.1.

(i) Take $R_1 \geqslant R_0$ such that

$$u_0(x) \leqslant \underline{\phi}_\ell(|x|) \quad \text{for } |x| \geqslant R_1. \tag{3.13}$$

Choose $C > 0$ such that $\underline{\phi}_\ell(R_1) + C\psi(R_1) \geqslant \phi_{\hat{\ell}}(R_1)$. Since we have

$$r^{2/(p-1)}\left(\underline{\phi}_\ell(r) + C\psi(r)\right) \to \ell \quad \text{and} \quad r^{2/(p-1)}\phi_{\hat{\ell}}(r) \to \hat{\ell} \quad \text{as } r \to \infty$$

with $\ell < \hat{\ell}$, there exists $R_2 \geqslant R_1$ such that

$$\underline{\phi}_\ell(R_2) + C\psi(R_2) = \phi_{\hat{\ell}}(R_2) \quad \text{and} \quad \underline{\phi}_\ell(r) + C\psi(r) \leqslant \phi_{\hat{\ell}}(r) \quad \text{for } r > R_2. \tag{3.14}$$

Define $\overline{v}_0(r)$ by

$$\overline{v}_0(r) = \begin{cases} \underline{\phi}_{\hat{\ell}}(r), & 0 < r < R_2, \\ \underline{\phi}_\ell(r) + C\psi(r), & r \geqslant R_2. \end{cases}$$

By the right-hand side of (3.14), we have $\overline{v}_0(r) \leqslant \underline{\phi}_{\hat{\ell}}(r)$ for $r > R_2$. By lemma 3.3, $\overline{v}_0(|x|)$ is a classical radial supersolution of (1.2) for $|x| > R_2$. By lemma 2.3, $\overline{v}_0(|x|)$ is a continuous weak supersolution of (1.2). It is clear that (3.5) holds. Note that $\overline{v}_0(|x|) \geqslant \underline{\phi}_\ell(|x|) \geqslant u_0(x)$ for $|x| \geqslant R_2$ by (3.13). Thus (3.3) and (3.4) hold.

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(ii) Take $R_1 \geqslant R_0$ such that

$$u_0(x) \geqslant \underline{\phi}_\ell(|x|) \quad \text{for } |x| \geqslant R_1. \tag{3.15}$$

Choose $C > 0$ such that $\underline{\phi}_\ell(R_1) - C\psi(R_1) \leqslant 0$. Since we have

$$r^{2/(p-1)} \left(\underline{\phi}_\ell(r) + C\psi(r) \right) \to \ell > 0 \quad \text{as } r \to \infty,$$

there exists $R_2 \geqslant R_1$ such that

$$\underline{\phi}_\ell(R_2) - C\psi(R_2) = 0 \quad \text{and} \quad \underline{\phi}_\ell(r) - C\psi(r) \geqslant 0 \quad \text{for } r > R_2.$$

Define $\underline{v}_0(r)$ by

$$\underline{v}_0(r) = \begin{cases} 0, & 0 < r < R_2, \\ \underline{\phi}_\ell(r) - C\psi(r), & r \geqslant R_2. \end{cases}$$

By lemma 3.3, $\underline{v}_0(|x|)$ is a classical radial subsolution of (1.2) for $|x| > R_2$. By lemma 2.3, $\underline{v}_0(|x|)$ is a continuous weak subsolution of (1.2). It is clear that (3.8) holds. Note that $\underline{v}_0(|x|) \leqslant \underline{\phi}_\ell(|x|) \leqslant u_0(x)$ for $|x| \geqslant R_2$ by (3.15). Thus (3.6) and (3.7) hold. □

Proof of proposition 3.2. We will show (i) only since we can show (ii) by the similar argument. Since $\overline{v}_0 = \overline{v}_0(|x|)$ is radially symmetric, a solution w of (1.19) with $w_0 = \overline{v}_0$ is spatially symmetric and hence it is written as $w(r, s; \overline{v}_0)$, $r = |y|$. By lemma 2.2, $w(r, s; \overline{v}_0)$ is nonincreasing in $s \geqslant 0$ for each fixed $r \geqslant 0$. Put $\phi(r) = \lim_{s\to\infty} w(r, s; \overline{v}_0)$. By applying lemma 2.1 with $\underline{w} = \underline{\phi}_\ell$, we obtain

$$\underline{\phi}_\ell(r) \leqslant w(r, s; \overline{v}_0) \leqslant \overline{v}_0(r) \tag{3.16}$$

for $r \geqslant 0$ and $s \geqslant 0$. By lemma 2.4, ϕ satisfies (1.3) with $\phi'(0) = 0$ and

$$\|w(\cdot, s; \overline{v}_0) - \phi\|_{L^\infty([0,\infty))} \to 0 \quad \text{as } s \to \infty.$$

Then it follows from (3.16) that

$$\underline{\phi}_\ell(r) \leqslant \phi(r) \leqslant \overline{v}_0(r) \leqslant \underline{\phi}_{\hat{\ell}}(r) \quad \text{for } r \geqslant 0.$$

From (3.5), we have $\phi \in S_\ell$. lemma 2.6 (i) implies that $\phi(r) \equiv \underline{\phi}_\ell(r)$. Thus we obtain $\|w(\cdot, s; \overline{v}_0) - \underline{\phi}_\ell\|_{L^\infty([0,\infty))}$ as $s \to \infty$. From (3.16), we obtain $w(y, s; \overline{v}_0) \geqslant \underline{\phi}_\ell(|y|)$ for all $y \in \mathbf{R}^N$ and $s \geqslant 0$. □

4. Proof of theorems 1.1 and 1.3

In this section, we show the following theorem.

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THEOREM 4.1. *Let $p > (N+2)/N$, and let $\ell \in (0, \ell^*)$. Assume that u_0 satisfies*

$$0 \leqslant u_0(x) < \ell^* |x|^{-2/(p-1)} \quad \textit{for } x \in \mathbf{R}^N \setminus \{0\}. \tag{4.1}$$

Then, for any $\varepsilon > 0$, there exists $\delta \in (0, \ell^* - \ell)$ satisfying that, if

$$\limsup_{x \to \infty} \big| |x|^{2/(p-1)} u_0(x) - \ell \big| < \delta, \tag{4.2}$$

then the solution u of (1.1) is global in time and satisfies

$$\limsup_{t \to \infty} t^{1/(p-1)} \| u(\cdot, t) - \underline{w}_\ell(\cdot, t) \|_{L^\infty(\mathbf{R}^N)} < \varepsilon. \tag{4.3}$$

Theorem 1.1 follows from theorem 4.1 immediately. In order to prove theorem 4.1, we first show the following proposition.

PROPOSITION 4.2. *Let $p > (N+2)/N$, and suppose that $S_{\hat{\ell}} \neq \emptyset$ for some $\hat{\ell} > 0$. Put $\ell \in (0, \hat{\ell})$. Assume that u_0 satisfies*

$$0 \leqslant u_0(x) \leqslant \underline{\phi}_{\hat{\ell}}(x) \quad \textit{for } x \in \mathbf{R}^N.$$

Then, for any $\varepsilon > 0$, there exists $\delta \in (0, \hat{\ell} - \ell)$ such that, if (4.2) holds, then the solution w of (1.19) with $w_0 = u_0$ is global in time and satisfies

$$\limsup_{s \to \infty} \| w(\cdot, s) - \underline{\phi}_\ell(|\cdot|) \|_{L^\infty(\mathbf{R}^N)} < \varepsilon. \tag{4.4}$$

To show proposition 4.2, we need the following lemma.

LEMMA 4.3. *Let $p > (N+2)/N$, and let $\ell \in (0, \hat{\ell})$.*

(i) *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \ell) > 0$ such that*

$$\| \underline{\phi}_{\ell+\delta} - \underline{\phi}_\ell \|_{L^\infty([0,\infty))} < \varepsilon \quad \textit{and} \quad \| \underline{\phi}_{\ell-\delta} - \underline{\phi}_\ell \|_{L^\infty([0,\infty))} < \varepsilon. \tag{4.5}$$

(ii) *Let $\delta \in (0, \hat{\ell} - \ell)$. If $u_0 \in C(\mathbf{R}^N)$ satisfies (4.2), then there exists $R > 0$ such that*

$$\underline{\phi}_{\ell-\delta}(|x|) \leqslant u_0(x) \leqslant \underline{\phi}_{\ell+\delta}(|x|) \quad \textit{for } |x| > R. \tag{4.6}$$

Proof.

(i) By [**25**, proposition 4.2], we obtain $\| \underline{\phi}_{\ell_2} - \underline{\phi}_{\ell_1} \|_{L^\infty([0,\infty))} \to 0$ as $\ell_2 \to \ell_1$. Then, for any $\varepsilon > 0$ and $\ell > 0$, there exists $\delta = \delta(\varepsilon, \ell) > 0$ such that (4.5) holds.

(ii) If u_0 satisfies (4.2), then there exists $\delta' \in (0, \delta)$ and $R_1 > 0$ such that

$$(\ell - \delta')|x|^{-2/(p-1)} \leqslant u_0(x) \leqslant (\ell + \delta')|x|^{-2/(p-1)} \quad \text{for } |x| > R_1. \tag{4.7}$$

Since $\delta' \in (0, \delta)$, there exists $R_2 > 0$ such that

$$\begin{aligned} \underline{\phi}_{\ell-\delta}(|x|) \leqslant (\ell - \delta')|x|^{-2/(p-1)} \quad \text{and} \quad & \underline{\phi}_{\ell+\delta}(|x|) \\ & \geqslant (\ell + \delta')|x|^{-2/(p-1)} \quad \text{for } |x| > R_2. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain (4.6) with $R = \max\{R_1, R_2\}$. □

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Proof of proposition 4.2. By lemma 4.3, for any $\varepsilon > 0$, there exist $\delta \in (0, \hat{\ell} - \ell)$ and $R > 0$ such that (4.5) and (4.6) hold. Applying proposition 3.1 (i) and (ii) with $\ell = \ell + \delta$ and $\ell = \ell - \delta$, respectively, there exist continuous weak super and sub-solutions $\overline{v}_0$ and $\underline{v}_0$ of (1.2) satisfying

$$\begin{cases} 0 \leqslant \underline{v}_0(x) \leqslant u_0(x) \leqslant \overline{v}_0(x) \leqslant \underline{\phi}_{\hat{\ell}}(|x|) \quad \text{for } x \in \mathbf{R}^N, \\ \underline{v}_0(x) \leqslant \underline{\phi}_{\ell-\delta}(|x|) \quad \text{and} \quad \underline{\phi}_{\ell+\delta}(|x|) \leqslant \overline{v}_0(x) \quad \text{for } x \in \mathbf{R}^N, \text{ and} \\ \lim_{|x|\to\infty} |x|^{2/(p-1)}\underline{v}_0(x) = \ell - \delta \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{2/(p-1)}\overline{v}_0(x) = \ell + \delta. \end{cases}$$

proposition 3.2 implies that $w(y, t, \overline{v}_0) \geqslant \underline{\phi}_{\ell+\delta}(|y|)$ and $w(y, t, \underline{v}_0) \leqslant \underline{\phi}_{\ell-\delta}(|y|)$ for all $y \in \mathbf{R}^N$ and $s \geqslant 0$, and that

$$\|w(\cdot, t, \overline{v}_0) - \underline{\phi}_{\ell+\delta}\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{and} \quad \|w(\cdot, t, \underline{v}_0) - \underline{\phi}_{\ell-\delta}\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } t \to \infty. \tag{4.9}$$

By lemma 2.1, we obtain

$$w(x, t, \underline{v}_0) \leqslant w(x, t, u_0) \leqslant w(x, t, \overline{v}_0) \quad \text{for } (x, t) \in \mathbf{R}^N \times (0, \infty).$$

Then we have

$$\begin{aligned} \|(w(\cdot, t, u_0) - \underline{\phi}_\ell)_+\|_{L^\infty(\mathbf{R}^N)} &\leqslant \|w(\cdot, t, \overline{v}_0) - \underline{\phi}_\ell\|_{L^\infty(\mathbf{R}^N)} \\ &\leqslant \|w(\cdot, t, \overline{v}_0) - \underline{\phi}_{\ell+\delta}\|_{L^\infty(\mathbf{R}^N)} + \|\underline{\phi}_{\ell+\delta} - \underline{\phi}_\ell\|_{L^\infty(\mathbf{R}^N)}, \end{aligned}$$

where $a_+ = \max\{a, 0\}$. Thus, from the left-hand side of (4.5) and (4.9), we obtain

$$\limsup_{t\to\infty} \|(w(\cdot, t, u_0) - \underline{\phi}_\ell)_+\|_{L^\infty(\mathbf{R}^N)} \leqslant \|\underline{\phi}_{\ell+\delta} - \underline{\phi}_\ell\|_{L^\infty(\mathbf{R}^N)} < \varepsilon.$$

By the similar argument, we obtain

$$\limsup_{t\to\infty} \|(w(\cdot, t, u_0) - \underline{\phi}_\ell)_-\|_{L^\infty(\mathbf{R}^N)} \leqslant \|\underline{\phi}_\ell - \underline{\phi}_{\ell-\eta}\|_{L^\infty(\mathbf{R}^N)} < \varepsilon,$$

where $a_- = \max\{-a, 0\}$. Thus we obtain (4.4). □

To prove theorem 4.1, we also need the following proposition. Recall that $\underline{w}_\ell$ is the self-similar solution defined by (1.6) with the minimal solution $\underline{\phi}_\ell$ of S_ℓ.

Proposition 4.4. *Assume that* $u_0 \in C(\mathbf{R}^N)$ *satisfies* (4.1) *and*

$$\limsup_{|x|\to\infty} |x|^{2/(p-1)} u_0(x) < \ell^*. \tag{4.10}$$

Then there exist $\hat{\ell} \in (0, \ell^*)$ *and* $\tau_0 > 0$ *such that, for any* $\tau \in (0, \tau_0]$,

$$u_0(x) \leqslant \underline{w}_{\hat{\ell}}(x, \tau) \quad \textit{for } x \in R^N. \tag{4.11}$$

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LEMMA 4.5. *Let $M > 0$ and $\ell \in (0, \ell^*)$. Put*

$$\Phi_{M,\ell}(x) = \min\{M, \ell|x|^{-1/(p-2)}\} \quad \text{for } x \in \mathbf{R}^N.$$

Take $\hat{\ell} \in (\ell, \ell^*)$. Then there exists $\tau_0 > 0$ such that, for any $\tau \in (0, \tau_0]$,

$$\underline{w}_{\hat{\ell}}(x, \tau) > \Phi_{M,\ell}(x) \quad \text{for } x \in \mathbf{R}^N. \tag{4.12}$$

Proof. Since $\lim_{s\to\infty} s^{2/(p-1)}\underline{\phi}_{\hat{\ell}}(s) = \hat{\ell} > \ell$, there exists $s_0 > 0$ such that

$$s^{2/(p-1)}\underline{\phi}_{\hat{\ell}}(s) \geqslant \ell \quad \text{for } s \geqslant s_0.$$

Put $r_0 > 0$ such that $M = \ell r_0^{-2/(p-1)}$, and take $\tau_0 > 0$ so that $r_0/\sqrt{\tau_0} \geqslant s_0$. Then, for any $\tau \in (0, \tau_0]$, we have

$$|x|^{2/(p-1)}\underline{w}_{\hat{\ell}}(x, \tau) = (|x|/\sqrt{\tau})^{2/(p-1)}\underline{\phi}_{\hat{\ell}}(|x|/\sqrt{\tau}) \geqslant \ell \quad \text{for } |x| \geqslant r_0,$$

which implies that

$$\underline{w}_{\hat{\ell}}(x, \tau) \geqslant \ell|x|^{-2/(p-1)} \quad \text{for } |x| \geqslant r_0. \tag{4.13}$$

In particular, $\underline{w}_{\hat{\ell}}(x, \tau) \geqslant \ell r_0^{-2/(p-1)} = M$ on $|x| = r_0$. Since $\underline{\phi}_{\hat{\ell}}(s)$ is decreasing for $s > 0$, we have $\underline{w}_{\hat{\ell}}(x, \tau) \geqslant \underline{w}_{\hat{\ell}}(y, \tau)$ if $|x| \leqslant |y|$. Then it follows that

$$\underline{w}_{\hat{\ell}}(x, \tau) \geqslant \ell r_0^{-2/(p-1)} = M \quad \text{for } 0 \leqslant |x| \leqslant r_0. \tag{4.14}$$

From (4.13) and (4.14), we obtain (4.12). □

Proof of proposition 4.4. Put $M = \max\{u_0(x) : x \in \mathbf{R}^N\}$. From (4.1) and (4.11), there exists $\ell_0 \in (0, \ell^*)$ such that

$$u_0(x) \leqslant \ell_0|x|^{-2/(p-1)} \quad \text{for } x \in \mathbf{R}^N \setminus \{0\}.$$

Then we obtain $u_0(x) \leqslant \Phi_{M,\ell_0}(x)$ for $x \in \mathbf{R}^N$. Take $\hat{\ell} \in (\ell_0, \ell^*)$. Then, by lemma 4.5, there exists $\tau_0 > 0$ such that, for any $\tau \in (0, \tau_0]$, we have $\underline{w}_{\hat{\ell}}(x, \tau) \geqslant \Phi_{M,\ell_0}(x)$ for $x \in \mathbf{R}^N$. Thus we obtain $u_0(x) \leqslant \underline{w}_{\hat{\ell}}(x, \tau)$ for $x \in \mathbf{R}^N$. □

Proof of theorem 4.1. Since $\delta \in (0, \ell^* - \ell)$, we have (4.10) if u_0 satisfies (4.2). By proposition 4.4, there exist $\hat{\ell} \in (0, \ell^*)$ and $\tau_0 > 0$ such that $u_0(x) \leqslant \underline{w}_{\hat{\ell}}(x, \tau_0)$ for

$x \in \mathbf{R}^N$, which implies that

$$\tau_0^{1/(p-1)} u_0(\sqrt{\tau_0} x) \leqslant \underline{\phi}_{\hat{\ell}}(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

Put $\tilde{u}(x,t) = \tau_0^{1/(p-1)} u(\sqrt{\tau_0} x, \tau_0 t)$. Then $\tilde{u}$ satisfies $\tilde{u}_t = \Delta \tilde{u} + \tilde{u}^p$ in $\mathbf{R}^N \times (0,\infty)$ and

$$\tilde{u}(x,0) = \tau_0^{1/(p-1)} u_0(\sqrt{\tau_0} x) \leqslant \underline{\phi}_{\hat{\ell}}(|x|) \quad \text{for } x \in \mathbf{R}^N. \tag{4.15}$$

Define $w = w(y,s)$ by

$$w(y,s) = (t+1)^{1/(p-1)} \tilde{u}(x,t) \tag{4.16}$$

with

$$y = \frac{x}{\sqrt{t+1}} \quad \text{and} \quad s = \log(t+1). \tag{4.17}$$

Then w satisfies (1.19) with $w_0(y) = \tilde{u}(y,0)$ for $y \in \mathbf{R}^N$. From (4.15), we have $w_0(y) \leqslant \underline{\phi}_{\hat{\ell}}(y)$ for $y \in \mathbf{R}^N$. For any $\varepsilon > 0$, take $\delta \in (0, \hat{\ell} - \ell)$ as in proposition 4.2. Then (4.4) holds. From (4.16) and (1.6) with (4.17), we have

$$w(y,s) - \underline{\phi}_{\ell}(|y|) = (t+1)^{1/(p-1)} (\tilde{u}(x,t) - \underline{w}_{\ell}(x,t+1)).$$

Note that $(t+1)/t \to 1$ as $t \to \infty$. Thus (4.4) can be written as

$$\limsup_{t\to\infty} t^{1/(p-1)} \|\tilde{u}(\cdot,t) - \underline{w}_{\ell}(\cdot,t+1)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon.$$

By lemma D.1 in [**25**], we see that

$$\lim_{t\to\infty} t^{1/(p-1)} \|\underline{w}_{\ell}(\cdot,t) - \underline{w}_{\ell}(\cdot,t+t_0)\|_{L^\infty(\mathbf{R}^N)} = 0$$

for any $t_0 \in \mathbf{R}$. Then we obtain

$$\limsup_{t\to\infty} t^{1/(p-1)} \|\tilde{u}(\cdot,t) - \underline{w}_{\ell}(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon. \tag{4.18}$$

Note here that

$$\tau_0^{1/(p-1)} \underline{w}_{\ell}(\sqrt{\tau_0}|x|, \tau_0 t) = t^{-1/(p-1)} \underline{\phi}_{\ell}(|x|/\sqrt{t}) = \underline{w}_{\ell}(|x|,t).$$

Then it follows that

$$\tilde{u}(x,t) - \underline{w}_{\ell}(x,t) = \tau_0^{1/(p-1)} \left(u(\sqrt{\tau_0} x, \tau_0 t) - \underline{w}_{\ell}(\sqrt{\tau_0} x, \tau_0 t) \right).$$

Then we have

$$t^{1/(p-1)} \|\tilde{u}(\cdot,t) - \underline{w}_{\ell}(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = (t\tau_0)^{1/(p-1)} \|u(\cdot,\tau_0 t) - \underline{w}_{\ell}(\cdot,\tau_0 t)\|_{L^\infty(\mathbf{R}^N)}.$$

From (4.18), we obtain (4.3). □

We denote by $u(x,t;u_0)$ a solution of (1.1).

Proof of theorem 1.3. By lemma 2.6 (iii), for any $M > 0$, there exists $\ell_M \in (0, L)$ such that $\|\underline{\phi}_{\ell_M}\|_{L^\infty([0,\infty))} > M$. Define $\tilde{u}_0(x) = \min\{\underline{\phi}_{\ell_M}(|x|), u_0(x)\}$. From (1.10)

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and (1.11), we have $\tilde{u}_0(x) < L|x|^{-2/(p-1)}$ for $x \in \mathbf{R}^N \setminus \{0\}$ and $\tilde{u}_0(x) \equiv \underline{\phi}_{\ell_M}(|x|)$ for $|x|$ sufficient large. By applying theorem 1.1, we obtain

$$\lim_{t\to\infty} t^{1/(p-1)} \|u(\cdot, t; \tilde{u}_0) - \underline{w}_{\ell_M}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = 0.$$

We note here that $t^{1/(p-1)}\|\underline{w}_{\ell_M}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = \|\underline{\psi}_{\ell_M}(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} > M$ for all $t > 0$. By the comparison principle, $u(x, t; u_0) \geqslant u(x, t; \tilde{u}_0)$ for $x \in \mathbf{R}^N$, $t > 0$. Then we obtain

$$\limsup_{t\to\infty} t^{1/(p-1)}\|u(\cdot, t; u_0)\|_{L^\infty(\mathbf{R}^N)} \geqslant \limsup_{t\to\infty} t^{1/(p-1)}\|u(\cdot, t; \tilde{u}_0)\|_{L^\infty(\mathbf{R}^N)} > M.$$

Since $M > 0$ is arbitrary, we obtain (1.12). □

5. Proof of theorem 1.5 and corollary 1.6

First, we show the following lemma.

LEMMA 5.1. *Let $p > (N + 2)/N$. Assume that $u_0 \in C(\mathbf{R}^N)$ satisfies (4.1). Then the solution u of (1.1) exists globally in time.*

Proof. In the case $p \geqslant p_{JL}$, we have $\ell^* = L$ by proposition A (iii). Then we obtain the global existence of the solution by the argument as in the proof of theorem 6.1 in [**29**].

In the case where $(N + 2)/N < p < p_{JL}$, we put $\mu \in (0, 1)$. By proposition 4.4, there exist $\hat{\ell} \in (0, \ell^*)$ and $\tau_0 > 0$ such that, for any $\tau \in (0, \tau_0]$, $\mu u_0(x) \leqslant \underline{w}_{\hat{\ell}}(x, \tau)$ for $x \in \mathbf{R}^N$. Recall that $S_{\ell^*} \neq \emptyset$ if $(N + 2)/N < p < p_{JL}$ by proposition A (iii). By [**23**, lemma 3.1 (iii)], we have $\underline{\phi}_{\hat{\ell}}(r) < \underline{\phi}_{\ell^*}(r)$ for each $r \geqslant 0$. Then, for all $\tau \in (0, \tau_0]$, we obtain

$$\mu u_0(x) < \underline{w}_{\ell^*}(x, \tau) \quad \text{for } x \in \mathbf{R}^N.$$

By the comparison principle, it follows that

$$u(x, t; \mu u_0) < \underline{w}_{\ell^*}(x, t + \tau) \quad \text{for } x \in \mathbf{R}^N,\ t > 0. \tag{5.1}$$

Since (5.1) holds for any $\tau \in (0, \tau_0]$, we have

$$u(x, t; \mu u_0) \leqslant \underline{w}_{\ell^*}(x, t) \quad \text{for } x \in \mathbf{R}^N,\ t > 0. \tag{5.2}$$

Letting $\mu \to 1$ in (5.2), we obtain $u(x, t; u_0) \leqslant \underline{w}_{\ell^*}(x, t)$ for all $x \in \mathbf{R}^N$ and $t > 0$. Thus the solution of (1.1) exists globally in time. □

We define some special functions which will be used in the proof of theorem 1.5. Let $h(r; a, b)$, $1 < a < b \leqslant \infty$, be a smooth function of $r \geqslant 0$ such that

(i) $0 \leqslant h \leqslant 1$ for $r \geqslant 0$,

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(ii) $h(r; a, b) \equiv 0$ for $r \in [0, a-1] \cup [b+1, \infty)$ if $b < \infty$, and $h(r; a, \infty) \equiv 0$ for $r \in [0, a-1]$,

(iii) $h(r; a, b) \equiv 1$ for $r \in [a, b]$ if $b < \infty$, and $h(r; a, \infty) \equiv 1$ for $r \in [a, \infty)$.

We set $\rho(r; \ell, a, b) = \ell h(r; a, b) r^{-2/(p-1)}$, and define

$$u_\infty^0(r) = \sum_{i=1}^{\infty} \rho(r; \ell_i, a_i, b_i), \tag{5.3}$$

where $\{\ell_i\}$ is a sequence in theorem 1.5 and $\{(a_i, b_i)\}$ is a sequence to be determined later. We choose $\{(a_i, b_i)\}$ such that

$$a_1 > 1, \quad a_i < b_i \quad \text{and} \quad b_i + 2 < a_{i+1} \quad \text{for } i = 1, 2, \ldots. \tag{5.4}$$

Then

$$\text{supp } \rho(r; \ell_i, a_i, b_i) \cap \text{supp } \rho(r; \ell_j, a_j, b_j) = \emptyset \quad \text{if } i \neq j. \tag{5.5}$$

Define auxiliary functions $u_1^0(r) = \rho(r; \ell_1, a_1, \infty)$ and

$$u_k^0(r) = \sum_{i=1}^{k-1} \rho(r; \ell_i, a_i, b_i) + \rho(r; \ell_k, a_k, \infty) \quad \text{for } k = 2, 3, \ldots.$$

Since (5.5) holds and $0 < \ell_i < \ell^*$ for $i = 1, 2, \ldots$, we have

$$u_\infty^0(r) < \ell^* r^{-2/(p-1)} \quad \text{and} \quad u_k^0(r) < \ell^* r^{-2/(p-1)} \quad \text{for } r \geqslant 0. \tag{5.6}$$

By lemma 5.1, solutions $u(x, t; u_\infty^0)$ and $u(x, t; u_\infty^0)$ exist globally in time.

Lemma 5.2.

(i) *For each $k = 1, 2, \ldots$, the solution $u(x, t; u_k^0)$ satisfies*

$$\lim_{t \to \infty} t^{1/(p-1)} \| u(\cdot, t; u_k^0) - \underline{w}_{\ell_k}(\cdot, t) \|_{L^\infty(\mathbf{R}^N)} = 0. \tag{5.7}$$

(ii) *For each $k = 1, 2, \ldots$, one has $\| u_k^0(|\cdot|) - u_\infty^0(|\cdot|) \|_{L^\infty(\mathbf{R}^N)} \leqslant \ell^* b_k^{-2/(p-1)}$.*

Proof.

(i) Note that u_k^0 satisfies (5.6) and $|x|^{2/(p-1)} u_k^0(x) \to \ell_k$ as $|x| \to \infty$. By theorem 1.1, we obtain (5.7).

(ii) We see that $u_k^0(r) - u_\infty^0(r) \equiv 0$ for $0 \leqslant r \leqslant b_k$ and

$$|u_k^0(r) - u_\infty^0(r)| \leqslant \left| \ell_k r^{-2/(p-1)} - \sum_{i=k}^{\infty} \psi(r; \ell_i, a_i, b_i) \right| \quad \text{for } b_k \leqslant r < \infty.$$

Since (5.5) holds and $0 < \ell_i < \ell^*$ for $i = k, k+1, k+2, \ldots$, we obtain

$$|u_k^0(r) - u_\infty^0(r)| \leqslant \ell^* r^{-2/(p-1)} \leqslant \ell^* b_k^{-2/(p-1)} \quad \text{for } b_k \leqslant r < \infty.$$

Thus we obtain $\| u_k^0(|\cdot|) - u_\infty^0(|\cdot|) \|_{L^\infty(\mathbf{R}^N)} \leqslant \ell^* b_k^{-2/(p-1)}$. □

For $\xi \in \mathbf{R}^N$ and $\ell \in (0, \ell^*)$, we obtain

$$\lim_{t\to\infty} t^{2/(p-1)} \|\underline{w}_\ell(\cdot - \xi, t) - \underline{w}_\ell(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = 0. \tag{5.8}$$

In fact, for each $x \in \mathbf{R}^N$, we have

$$t^{2/(p-1)} |\underline{w}_\ell(x - \xi, t) - \underline{w}_\ell(x, t)| = |\underline{\phi}_\ell(|x - \xi|/\sqrt{t}) - \underline{\phi}_\ell(|x|/\sqrt{t})|.$$

Since $\underline{\phi}_\ell(r)$ is uniformly continuous on $[0, \infty)$, we obtain (5.8).

Assume that solutions $u(x, t; u_0)$ and $u(x, t; \tilde{u}_0)$ exist globally in time. By the continuous dependence of solution of (1.1) on the initial value (see, e.g., [**2**, proposition 4.3.7]), for any $\varepsilon > 0$ and $T > 0$, there exists $\delta = \delta(\varepsilon, T) > 0$ such that, if $\|u_0(\cdot) - \tilde{u}_0(\cdot)\|_{L^\infty(\mathbf{R}^N)} < \delta$, then

$$\max_{0\leqslant t\leqslant T} T^{2/(p-1)} \|u(\cdot, t; u_0) - u(\cdot, t; \tilde{u}_0)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon.$$

We carry out the proof of theorem 1.5 by constructing a sequence $\{(a_i, b_i)\}$ satisfying the requirements recursively.

Proof of theorem 1.5. Let $\{(\ell_i, \xi_i, \varepsilon_i)\}$ be a sequence in theorem 1.5. Put u_∞^0 by (5.3) with $\{(a_i, b_i)\}$ satisfying (5.4). Since the left-hand side of (5.6) holds, by lemma 5.1, the solution $u(x, t; u_\infty^0)$ exists globally in time. We will show that one can choose a sequence $\{(a_i, b_i)\}$ recursively such that there exists an increasing sequence $\{t_i\}$ satisfying $t_{i+1} > t_i + 1$ and

$$t_i^{2/(p-1)} \|u(\cdot, t_i; u_\infty^0) - \underline{u}_{\ell_i}(\cdot - \xi_i, t_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i \tag{5.9}$$

for each $i = 1, 2, \ldots$.

Step 1. Take $a_1 > 1$ arbitrarily. We show that one can choose $b_1 > a_1$ such that, for any $\{(a_i, b_i)\}_{i\geqslant 2}$ satisfying (5.4), there exists $t_1 > 1$ satisfying (5.9) with $i = 1$.

By lemma 5.2 (i), there exists $\tilde{t}_1 > 1$ such that

$$t^{2/(p-1)} \|u(\cdot, t; u_1^0) - \underline{w}_{\ell_1}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_1}{4} \quad \text{for } t \geqslant \tilde{t}_1. \tag{5.10}$$

From (5.8) there exists $t_1 \geqslant \tilde{t}_1$ such that

$$t^{2/(p-1)} \|\underline{w}_{\ell_1}(\cdot - \xi_1, t) - \underline{w}_{\ell_1}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_1}{4} \quad \text{for } t \geqslant t_1. \tag{5.11}$$

Combining (5.10) and (5.11), we obtain

$$t^{2/(p-1)} \|u(\cdot, t; u_1^0) - \underline{w}_{\ell_1}(\cdot - \xi_1, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_1}{2} \quad \text{for } t \geqslant t_1. \tag{5.12}$$

By the continuous dependence of initial value, there exists $\delta_1 = \delta_1(\varepsilon_1/2, t_1) > 0$ such that, if $\|u_0(\cdot) - u_1^0(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \delta_1$, then

$$\max_{0\leqslant t\leqslant t_1} t_1^{2/(p-1)} \|u(\cdot, t; u_0) - u(\cdot, t; u_1^0)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_1}{2}. \tag{5.13}$$

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It follows from (5.12) and (5.13) that, if $\|u^0_\infty(\cdot) - u^0_1(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \delta_1$, then

$$t_1^{2/(p-1)}\|u(\cdot, t_1; u^0_\infty) - \underline{w}_{\ell_1}(\cdot - \xi, t_1)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_1.$$

Take $b_1 > a_1$ such that $\ell^* b_1^{-2/(p-1)} < \delta_1$. By lemma 5.2 (ii), for any $\{(a_i, b_i)\}_{i\geqslant 2}$ satisfying (5.4), we obtain

$$\|u^0_\infty(|\cdot|) - u^0_1(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \ell^* b_1^{-2/(p-1)} < \delta_1.$$

Thus (5.9) with $i = 1$ holds for any $\{(a_i, b_i)\}_{i\geqslant 2}$ satisfying (5.4).

Step 2. Let $k \geqslant 2$, and suppose that $\{(a_i, b_i)\}_{i=1}^{k-1}$ satisfy (5.4), and that there exist $\{t_i\}_{i=1}^{k-1}$ such that (5.9) holds for each $i = 1, 2, \ldots, k-1$. Take $a_k > b_{k-1} + 2$ arbitrarily. We will show that one can choose $b_k > a_k$ such that there exists $t_k > t_{k-1} + 1$ satisfying (5.9) with $i = k$ for any $\{(a_i, b_i)\}_{i\geqslant k+1}$ satisfying (5.4).

By lemma 5.2 (i), there exists $\tilde{t}_k > t_{k-1} + 1$ such that

$$t^{2/(p-1)}\|u(\cdot, t : u^0_k) - \underline{w}_{\ell_k}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_k}{4} \quad \text{for } t \geqslant \tilde{t}_k. \tag{5.14}$$

From (5.8) there exists $t_k \geqslant \tilde{t}_k$ such that

$$t^{2/(p-1)}\|\underline{w}_{\ell_k}(\cdot - \xi_k, t) - \underline{w}_{\ell_k}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_k}{4} \quad \text{for } t \geqslant t_k. \tag{5.15}$$

Combining (5.14) and (5.15), we obtain

$$t^{2/(p-1)}\|u(\cdot, t : u^0_k) - \underline{w}_{\ell_k}(\cdot - \xi_k, t)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_k}{2} \quad \text{for } t \geqslant t_k. \tag{5.16}$$

By the continuous dependence of initial value, there exists $\delta_k = \delta_k(\varepsilon_k/2, t_k) > 0$ such that, if $\|u_0(\cdot) - u^0_k(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \delta_k$, then

$$\max_{0\leqslant t\leqslant t_k} t_k^{2/(p-1)}\|u(\cdot, t; u_0) - u(\cdot, t; u^0_k)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_k}{2}. \tag{5.17}$$

It follows from (5.16) and (5.17) that, if $\|u^0_\infty(\cdot) - u^0_k(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \delta_k$, then

$$t_k^{2/(p-1)}\|u(\cdot, t_k : u^0_\infty) - \underline{w}_{\ell_k}(\cdot - \xi_k, t_k)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_k.$$

Take $b_k > a_k + 1$ such that $\ell^* b_k^{-2/(p-1)} < \delta_k$. Then, by lemma 5.2 (ii), we obtain

$$\|u^0_\infty(|\cdot|) - u^0_k(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \ell^* b_k^{-2/(p-1)} < \delta_k.$$

Thus (5.9) with $i = k$ holds for any $\{(a_i, b_i)\}_{i\geqslant k+1}$ satisfying (5.4).

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First we take (a_1, b_1) by Step 1, and then (a_2, b_2) by Step 2 with $k = 2$. Then there exists $t_2 > t_1 + 1$ such that (5.9) with $i = 1, 2$ hold for any $\{(a_k, b_k)\}_{k\geqslant 3}$ satisfying (5.4). Next we take (a_3, b_3) by Step 2 with $k = 3$. Then there exists $t_3 > t_2 + 1$ such that (5.9) with $i = 3$ holds for any $\{(a_k, b_k)\}_{k\geqslant 4}$ satisfying (5.4). Repeating this argument, we can choose a sequence $\{(a_k, b_k)\}$ recursively such that (5.9) holds for each $i = 1, 2, \ldots$. Let u be a solution of (1.1) with $u_0 = u^0_\infty$. Then u exists globally in time and there exists an increasing sequence $\{t_i\}$ such that (1.18) holds for each $i = 1, 2, \ldots$. □

Remark 5.3. By the proof of theorem 1.5, we find that the solution $u(x, t; u^0_\infty)$ is radially symmetric about the origin even if $\xi_k \neq 0$ for $k = 1, 2, \ldots$.

Proof of corollary 1.6. For the sequence $\{(\ell_i, \xi_i, \varepsilon_i)\}$, put $\{(\tilde{\ell}_i, \tilde{\xi}_i, \tilde{\varepsilon}_i)\}$ as follows. For $i = 1, 2, \ldots$,

$$\tilde{\ell}_{2i} = \ell_i, \quad \tilde{\xi}_{2i-1} = \tilde{\xi}_{2i} = \xi_i \quad \text{and} \quad 2\tilde{\varepsilon}_{2i-1} = \tilde{\varepsilon}_{2i} = \varepsilon_i.$$

By lemma 2.6 (ii), for any $\varepsilon > 0$, there exists $\ell_\varepsilon > 0$ such that $\|\underline{\phi}_{\ell_\varepsilon}\|_{L^\infty([0,\infty))} < \varepsilon$. Then, for $i = 1, 2, \ldots$, put $\tilde{\ell}_{2i-1}$ such that $\|\underline{\phi}_{\tilde{\ell}_{2i-1}}(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i/2$. Then it follows that

$$\sup_{t>0} t^{1/(p-1)}\|\underline{w}_{\tilde{\ell}_{2i-1}}(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} = \|\underline{\phi}_{\tilde{\ell}_{2i-1}}(|\cdot|)\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_i}{2}. \tag{5.18}$$

By applying theorem 1.5 with $\{(\tilde{\ell}_i, \tilde{\xi}_i, \tilde{\varepsilon}_i)\}$, there exists u_0 such that the solution u of (1.1) exists globally in time and there exists an increasing sequence $\{\tau_i\}$ with $\tau_i \to \infty$ such that

$$\tau_{2i-1}^{1/(p-1)}\|u(\cdot, \tau_{2i-1}) - \underline{w}_{\tilde{\ell}_{2i-1}}(\cdot - \xi_i, \tau_{2i-1})\|_{L^\infty(\mathbf{R}^N)} < \frac{\varepsilon_i}{2} \tag{5.19}$$

and

$$\tau_{2i}^{1/(p-1)}\|u(\cdot, \tau_{2i}) - \underline{w}_{\tilde{\ell}_{2i}}(\cdot - \xi_i, \tau_{2i})\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i$$

for $i = 1, 2, \ldots$. From (5.18) and (5.19), we obtain

$$\tau_{2i-1}^{1/(p-1)}\|u(\cdot, \tau_{2i-1})\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

Putting $s_i = \tau_{2i-1}$ and $t_i = \tau_{2i}$, we obtain (i) and (ii) in corollary 1.6. □

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