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DISPERSIVE AND SUPERADDITIVE ORDERING

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Abstract

Recently many authors have established connections between dispersive ordering and some other partial orderings of distributions. This paper presents the connection which superadditive ordering has with dispersive ordering.

1. Introduction

Let F and G be two distribution functions and let F^{-1} and G^{-1} be their corresponding left-continuous inverses. G is said to be more dispersed than F (Lewis and Thompson (1981), written $F \overset{\text{disp}}{<} G$, if

$$(1.1) \qquad G(G^{-1}(\alpha)+a) \leqq F(F^{-1}(\alpha)+a) \quad \text{for all} \quad a>0 \text{ and } \alpha \in (0, 1).$$

Deshpande and Kochar (1983) have shown that (1.1) is equivalent to

$$(1.2) \qquad G^{-1}(\beta)-G^{-1}(\alpha) \geqq F^{-1}(\beta)-F^{-1}(\alpha) \quad \text{whenever} \quad 0<\alpha<\beta<1.$$

The last inequality is equivalent to saying

$$(1.3) \qquad G^{-1}F(x)-x \quad \text{is non-decreasing in } x,$$

(see for example Shaked (1982)).

The support of F will be denoted by S_F. In reliability theory, if F and G are such that $F(0)=G(0)=0$ and G is strictly increasing on S_G, an interval, three well-known orderings of distributions are introduced (see Barlow and Proschan (1975)). F is said to be convex ordered with respect to G, written $\overset{c}{<} G$, if $G^{-1}F(x)$ is a convex function in x on S_F, assumed an interval.

F is star-ordered with respect to G, written $F \overset{*}{<} G$, if $G^{-1}F(x)$ is star-shaped, i.e., $G^{-1}F(x)/x$ is increasing in x for $x \in S_F$.

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F is said to be superadditive (subadditive) with respect to G, written $F \overset{\text{su}}{<} G \, (F \underset{\text{su}}{<} G)$, if

(1.4) $$G^{-1}F(x+y) \geqq (\leqq) G^{-1}F(x) + G^{-1}F(y) \quad \text{for all} \quad x, y \in S_F.$$

Recently, many connections have been established among the convex ordering, the star-shaped ordering and the dispersive ordering. See Barlow and Proschan (1975), Deshpande and Kochar (1983), Sathe (1984), and Bartoszewicz (1985a), (1985b), among others.

The purpose of this paper is to establish connections between the superadditive ordering and the dispersive ordering of distributions. To avoid technical complications in the statements of the results and in the proofs we shall assume throughout that the supports of the underlying distributions are intervals and that the distributions have no atoms. Thus, these distributions will have strictly increasing and continuous inverses on $(0, 1)$.

2. The main results

We first show that superadditive ordering neither implies nor is implied by the dispersive ordering.

Example 2.1. Let $F(x) = G(x+\theta)$, $\theta > 0$. Then $F \overset{\text{disp}}{<} G$ but $F \overset{\text{su}}{\nless} G$. That is, the dispersive ordering does not imply the superadditive ordering.

Example 2.2. Let F have the Weibull distribution function

$$F(x) = 1 - \exp(-x^2), \quad x > 0, \quad \text{and let} \quad G(x) = 1 - \exp(-x), \quad > 0.$$

Since $G^{-1}F(x) - x = x^2 - x$ is not non-decreasing in x for all $x \geqq 0$, then $F \overset{\text{disp}}{\nless} G$. However, it is easy to show that $G^{-1}F$ is superadditive.

Next we establish, through the following set of results, connections between the dispersive and the superadditive orderings.

Theorem 2.3. If $F \overset{\text{su}}{<} G$ and $F \overset{\text{st}}{<} G$, then $F \overset{\text{disp}}{<} G$.

Proof. Assume $F \overset{\text{su}}{<} G$. Then

(2.1) $$G^{-1}F(x+y) \geqq G^{-1}F(x) + G^{-1}F(y) \quad \text{for all} \quad x, y \in S_F.$$

Since $F \overset{\text{st}}{<} G$ implies $G^{-1}F(x) \geqq x$, then by (2.1),

$$G^{-1}F(x+y) \geqq G^{-1}F(y) + x, \qquad x, y \in S_F.$$

Equivalently,

$$G^{-1}F(x+y) - (x+y) \geqq G^{-1}F(y) - y, \qquad x, y \in S_F,$$

which holds if and only if $G^{-1}F(x) - x$ is non-decreasing in $x \in S_F$, i.e., if and only if $F \overset{\text{disp}}{<} G$.

Denote by

(2.2) $$\bar{F}(x \mid t) = \frac{1 - F(x+t)}{1 - F(t)}, \qquad x \geqq 0, \qquad t \geqq 0,$$

the conditional reliability of a unit of age t if F is the life distribution of the unit.

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. If $\bar{F}(x \mid t) \leqq \bar{G}(x \mid t)$ for all $x \geqq 0$ and $t \geqq 0$ and $F \overset{\text{su}}{<} G$, then $F \overset{\text{disp}}{<} G$.

Remark 1. Theorem 2.3 can be used to improve Bartoszewicz's (1985a) result which says that if $F \overset{\text{c}}{<} G$ and $F \overset{\text{st}}{<} G$ then $F \overset{\text{disp}}{<} G$, and Bartoszewicz's (1985b) observation which states that if $F \overset{*}{<} G$ and $F \overset{\text{st}}{<} G$ then $F \overset{\text{disp}}{<} G$. This is so since it is well known that $F \overset{\text{c}}{<} G \rightarrow F \overset{*}{<} G \rightarrow F \overset{\text{su}}{<} G$ (see Barlow and Proschan (1975), pp. 107 and 109).

Sathe (1984) has pointed out that if $\lim_{x \to 0+} (G^{-1}F(x)/x) \geqq 1$ and $F \overset{*}{<} G$, then $F \overset{\text{disp}}{<} G$. In our next result, it is shown that the limit condition arises naturally under the superadditive ordering and that Sathe's conclusion still holds under the weaker condition $F \overset{\text{su}}{<} G$. This is contained in the following.

Lemma 2.5. If $F \overset{\text{su}}{<} G$, then $(d/dx)G^{-1}F(x) \geqq \lim_{y \to 0+} (G^{-1}F(y)/y)$ for all $x \in S_F$.

Proof. Note that as $F \overset{\text{su}}{<} G$, then by (1.4),

$$G^{-1}F(x+y) \geqq G^{-1}F(x) + G^{-1}F(y) \quad \text{for all} \quad x, y \in S_F,$$

so that

$$\frac{G^{-1}F(x+y) - G^{-1}F(x)}{y} \geqq \frac{G^{-1}F(y)}{y}. \tag{2.3}$$

Taking limits of both sides of (2.3) as $y \to 0$, the conclusion of the lemma follows.

Theorem 2.6. If $\lim_{x \to 0+} (G^{-1}F(x)/x) \geqq 1$ and $F \overset{\text{su}}{<} G$, then $F \overset{\text{disp}}{<} G$.

Proof. We claim that, under the assumptions of the proposition

$$\frac{d}{dx}\{G^{-1}F(x) - x\} \quad \text{is non-negative for all} \quad x \in S_F. \tag{2.3}$$

To see it, note that, as $F \overset{\text{su}}{<} G$ and $\lim_{x \to 0+} (G^{-1}F(x)/x) \geqq 1$, then by Lemma 2.5,

$$\frac{d}{dx} G^{-1}F(x) \geqq 1 \quad \text{whenever} \quad x \in S_F. \tag{2.5}$$

It follows that $G^{-1}F(x) - x$ is non-decreasing in x for all $x \in S_F$. Hence by (1.3), $F \overset{\text{disp}}{<} G$.

Remark 2. As a byproduct of Theorem 2.6 we have the following improved version of Desphande and Kochar's (1983) observation: If F and G are absolutely continuous with $F(0) = G(0) = 0$, and their corresponding densities are such that $f(0) \geqq g(0) > 0$, then $F \overset{\text{su}}{<} G$ implies $F \overset{\text{disp}}{<} G$.

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