Ergod. Th. & Dynam. Sys. (2021), **41**, 578–592
doi:10.1017/etds.2019.60

The numbers of periodic orbits hidden at fixed points of holomorphic maps

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(*Received* 10 *November* 2018 *and accepted in revised form* 12 *June* 2019)

Abstract. Let f be an n-dimensional holomorphic map defined in a neighborhood of the origin such that the origin is an isolated fixed point of all of its iterates, and let $\mathcal{N}_M(f)$ denote the number of periodic orbits of f of period M hidden at the origin. Gorbovickis gives an efficient way of computing $\mathcal{N}_M(f)$ for a large class of holomorphic maps. Inspired by Gorbovickis' work, we establish a similar method for computing $\mathcal{N}_M(f)$ for a much larger class of holomorphic germs, in particular, having arbitrary Jordan matrices as their linear parts. Moreover, we also give another proof of the result of Gorbovickis [On multi-dimensional Fatou bifurcation. *Bull. Sci. Math.* **138**(3)(2014) 356–375] using our method.

Key words: smooth dynamics, periodic orbits, holomorphic maps, fixed-point indices, Dold indices

2010 Mathematics Subject Classification: 32H50, 37C25 (Primary); 37F45, 37F50 (Secondary)

1. *Introduction and main results*

Let $\mathbb{C}^n$ be the n-dimensional complex vector space, let U be an open subset of $\mathbb{C}^n$ and let $g: U \to \mathbb{C}^n$ be a holomorphic map. If $p \in U$ is an isolated zero of g, say, there exists a ball B centered at p with $\overline{B} \subset U$ such that p is the unique solution of the equation $g(x) = 0$ in $\overline{B}$, then we can define the *zero order* of g at p by

$$\pi_g(p) = \#(g^{-1}(v) \cap B) = \#\{x \in B : g(x) = v\},$$

where v is a regular value of g such that $|v|$ is small enough and # denotes the cardinality. $\pi_g(p)$ is a well-defined integer (see [**4**] or [**7**] for the details). If g is regarded as a continuous map of real variables, the zero order is the (local) topological degree.

Denote by $\mathcal{O}(\mathbb{C}^n, 0, 0)$ the space of all germs of holomorphic maps fixing the origin between two neighborhoods of the origin in $\mathbb{C}^n$. For $g \in \mathcal{O}(\mathbb{C}^n, 0, 0)$, we call the origin an *isolated fixed point* of g if the origin is an isolated zero of $g - \mathrm{id}$. When the origin is an isolated fixed point, its *fixed point index* is defined by $\mu_g(0) = \pi_{g-\mathrm{id}}(0)$.

Now we use the fixed point index to define the local Dold index(see [**2**] or [**9**]). Let g be a germ in $\mathcal{O}(\mathbb{C}^n, 0, 0)$, let M be a positive integer and assume that the origin is an isolated fixed point of g^M. Then the *Dold index* of g at the origin is defined by

$$P_M(g, 0) = \sum_{s \subset P(M)} (-1)^{\#s} \mu_{g^{M:s}}(0),$$

where $P(M)$ is the set of all prime factors of M, $\#s$ is the number of elements of s and $M : s = M(\prod_{k\in s} k)^{-1}$(if $s = \emptyset$, set $M : s = M$).

Here is a basic property of the Dold index.

PROPOSITION 1.1. [**9**] *Let g be a germ in $\mathcal{O}(\mathbb{C}^n, 0, 0)$ and let M be a positive integer. If the origin is an isolated fixed point of g^M, then $P_M(g, 0) \geq 0$ and $M \mid P_M(g, 0)$.*

Since $M \mid P_M(g, 0)$, $\mathcal{N}_M(g) = P_M(g, 0)/M$ is a well-defined integer and is called the number of periodic orbits of period M hidden at the fixed point(the origin) in [**9**].

Remark 1. [**9**] $P_M(g, 0)$ can be interpreted to be the number of periodic points of period M of g hidden at the origin: any holomorphic map $f : \Delta^n \to \mathbb{C}^n$ sufficiently close to g has exactly $P_M(g, 0)$ distinct periodic points of period M near the origin, provided that all the fixed points of f^M near the origin are simple. Thus, the number $\mathcal{N}_M(g) = P_M(g, 0)/M$ can be taken as the number of periodic orbits of period M hidden at the origin.

In this paper, Λ denotes a Jordan matrix

$$\Lambda = \mathrm{diag}(A_{k_1}, \ldots, A_{k_m}), \tag{1.1}$$

where

$$A_{k_j} = \begin{pmatrix} \lambda_j & 1 & & & \\ & \lambda_j & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_j & 1 \\ & & & & \lambda_j \end{pmatrix}_{k_j \times k_j}, \quad 1 \leq j \leq m, k_1 + \cdots + k_m = n,$$

$\lambda_j (j = 1, 2, \ldots, m)$ are the d_jth primitive roots of unity and d_j is equal to zero if λ_j is not a root of unity; θ denotes a map from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$ such that $\theta(j) = q$ if $k_1 + k_2 + \cdots + k_{q-1} < j \leq k_1 + k_2 + \cdots + k_q$; d denotes a map from $\{1, 2, \ldots, n\}$ to $\{d_1, d_2, \ldots, d_m\}$ such that $j \mapsto d_{\theta(j)}$; e denotes a map from $\{1, 2, \ldots, n\}$ to $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ such that $j \mapsto \lambda_{\theta(j)}$; and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is

https://doi.org/10.1017/etds.2019.60 Published online by Cambridge University Press

a germ of a holomorphic map

$$f(x) = \Lambda x + o(x), \tag{1.2}$$

where $o(x)$ denotes the higher-order terms.

We recall resonant polynomial normal forms. Let $v_1, \ldots, v_n$ be the standard orthonormal basis in $\mathbb{C}^n$. Then a monomial of degree greater than one proportional to the monomial $x_1^{k_1} \ldots x_n^{k_n} v_s$ is said to be resonant with respect to Λ if

$$e(s) = (e(1))^{k_1} \cdots (e(n))^{k_n}.$$

A map f of the form (1.2) is said to be *in resonant polynomial normal form* if $f(x) = \Lambda x + F(x)$, where $F(x)$ is a sum of finitely many resonant monomials with respect to Λ.

The following result tells us that holomorphic germs of the form (1.2) can be reduced to corresponding resonant polynomial normal forms in terms of the numbers of periodic orbits hidden at the fixed point.

THEOREM 1.1. [**3**] *For a map f of the form (1.2), there exists a map $\tilde{f}(x) = \Lambda x + o(x)$ in resonant polynomial normal form such that, for every positive integer m, if the origin is an isolated fixed point of the mth iterate f^m of f, then $\mathcal{N}_m(f) = \mathcal{N}_m(\tilde{f})$.*

Remark 2. Theorem 1.1 differs slightly from the result in [**3**] and can be easily proved based on it.

Let $\mathcal{O}_\Lambda^n$ be the set of all germs that are in resonant polynomial normal form. Define a map

$$\tau : \mathcal{O}_\Lambda^n \to \mathcal{O}(\mathbb{C}^n, 0, 0), \quad f(x) = \Lambda x + F(x) \mapsto \tau f(x) = (\Lambda - \tilde{\Lambda})x + F(x),$$

where the elements of the main diagonal of $\tilde{\Lambda}$ are identical to that of Λ and other elements are set to zero. Note that the notation $\tilde{\Lambda}$ will be also used in §3.

Let W_n denote the set comprising all the possible words of length n with each digit of a word taking 0 or 1, and $W_n^* = W_n \backslash \{(0 \cdots 0)\}$. For a subset S of the set $\{1, 2, \ldots, n\}$, we set $W(S) = (w_1 \cdots w_n) \in W_n^*$, where $w_j = 1$ if and only if $j \in S$. Similarly, when $w = (w_1 \cdots w_n) \in W_n$, denote by $S(w)$ the set of all indices j such that $w_j = 1$. For a matrix Λ of the form (1.1) and a positive integer k, we set $w(\Lambda) = (w_1 w_2 \cdots w_n) \in W_n$ with $w_j = 0$ if and only if $d(j) = 0$, and we set $w(\Lambda, k) = (w_{1k} w_{2k} \cdots w_{nk}) \in W_n$ with $w_{jk} = 1$ if and only if $d(j) \mid k$. The notation $w(\Lambda, k)$ is also written as $w(k)$ when there is no ambiguity.

For $w \in W_n^*$, we denote by $|w|$ the sum $|w| = \sum_{j=1}^n w_j$. If $S(w) = \{s_1, \ldots, s_{|w|}\}$ with $s_1 < \cdots < s_{|w|}$, then the subspace of $\mathbb{C}^n$ spanned by the coordinates with indices from $S(w)$ will be denoted by $\mathbb{C}^{|w|}$. Let $p_w : \mathbb{C}^n \to \mathbb{C}^{|w|}$ be the orthogonal projection from $\mathbb{C}^n$ to $\mathbb{C}^{|w|}$: i.e.,

$$p_w(x) = (x_{s_1}, \ldots, x_{s_{|w|}})$$

and let $i_w : \mathbb{C}^{|w|} \to \mathbb{C}^n$ be the natural inclusion of $\mathbb{C}^{|w|}$ into $\mathbb{C}^n$: i.e.,

$$i_w(\tilde{x}_{s_1}, \ldots, \tilde{x}_{s_{|w|}}) = (x_1, \ldots, x_n),$$

with $x_j = \tilde{x}_j$ if $j \in S(w)$ and $x_j = 0$ otherwise. For all $g \in \mathcal{O}(\mathbb{C}^n, 0, 0)$, we set $\pi_{p_{(00\cdots0)} \circ g \circ i_{(00\cdots0)}}(0) = 1$.

Our main result is the following theorem.

THEOREM 1.2. *Let Λ be a matrix of the form (1.1) and let f be a germ in $\mathcal{O}^n_\Lambda$. Then the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$ if and only if the origin in $\mathbb{C}^n$ is an isolated fixed point of f^k for any $k \geq 1$. Moreover, in this case, for any $k \geq 1$,*

$$\mu_{f^k}(0) = \pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0).$$

Remark 3. When $w(\Lambda) = (00 \cdots 0)$, by the inverse function theorem it is clear that the origin in $\mathbb{C}^n$ is an isolated fixed point of f^k for any $k \geq 1$ and $\mu_{f^k}(0) = 1$. So, from now on we assume that $w(\Lambda) \in W^*_n$.

One immediate consequence of Theorem 1.2 is the following corollary.

COROLLARY 1.1. *Let Λ be a matrix of the form (1.1) and let f be a germ in $\mathcal{O}^n_\Lambda$. If the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$, then, for any $M \geq 1$,*

$$\mathcal{N}_M(f) = \frac{1}{M} \sum_{s \subset P(M)} (-1)^{\#s} \pi_{p_{w(M:s)} \circ \tau f \circ i_{w(M:s)}}(0). \tag{1.3}$$

For the particular case in which $k_1 = k_2 = \cdots = k_m = 1$ and $d_1, d_2, \ldots, d_m$ are greater than 1 and are pairwise relatively prime, Corollary 1.1 will be reduced to Corollary 1.2. We will give a new proof of Corollary 1.2; the original proof can be found in [**3**]. In this case, $f(x) = \Lambda x + R(u)x$ for $f \in \mathcal{O}^n_\Lambda$, where $u = (u_1, \ldots, u_n) = (x_1^{d_1}, \ldots, x_n^{d_n})$ and $R(u) = \operatorname{diag}\{r_1(u), \ldots, r_n(u)\}$ with $R(0) = 0$. Given a word $w \in W^*_n$, we define a map

$$R_w : \mathbb{C}^{|w|} \to \mathbb{C}^{|w|}, \quad \tilde{u} \mapsto p_w(r(i_w(\tilde{u}))),$$

where $r(u) = (r_1(u), \ldots, r_n(u))$, and we define a map

$$P_f : W^*_n \to \mathbb{N}, \quad w = (w_1 \cdots w_n) \mapsto \mathcal{N}_{d^w}(f),$$

where $d^w = \prod_{j=1}^n d_j^{w_j}$.

COROLLARY 1.2. *Let $f \in \mathcal{O}^n_\Lambda$. If the origin in $\mathbb{C}^n$ is an isolated fixed point of $f^{d_1 d_2 \cdots d_n}$, then, for any $w \in W^*_n$, the origin in $\mathbb{C}^{|w|}$ is an isolated zero of R_w and $P_f(w) = \pi_{R_w}(0)$.*

2. *Preliminaries*

This section presents several lemmas on zero indices and fixed point indices. They will be used in the proof of Theorem 1.2 in §3.

LEMMA 2.1. *For $g, \tilde{g} \in \mathcal{O}(\mathbb{C}^n, 0, 0)$, let*

$$g(x_1, x_2, \ldots, x_n) = (g_1, g_2, \ldots, g_n)$$

and

$$\tilde{g}(x_1, x_2, \ldots, x_n) = (g_1, \ldots, g_{j-1}, \tilde{g}_j, g_{j+1}, \ldots, g_n).$$

If the origin is their isolated zero with multiplicity N and M, respectively, then the map

$$f : (x_1, x_2, \ldots, x_n) \mapsto (g_1, \ldots, g_{j-1}, \tilde{g}_j g_j, g_{j+1}, \ldots, g_n)$$

has an isolated zero of multiplicity N+M at the origin.

Remark 4. Lemma 2.1 is well known in the theory of zero indices.

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Let $f, g \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. Then f and g are *algebraically equivalent* at the origin if there exists a germ of a holomorphic family of linear non-degenerate maps $A(x) \in GL(n, \mathbb{C})$ such that $f(x) = A(x)g(x)$.

LEMMA 2.2. [**1**] *Let $f, g \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. If f and g are algebraically equivalent at the origin, then the origin is an isolated zero of f if and only if it is an isolated zero of g. In this case, $\pi_f(0) = \pi_g(0)$.*

LEMMA 2.3. [**9**] *Let $f, g \in \mathcal{O}(\mathbb{C}^n, 0, 0)$. If the origin in $\mathbb{C}^n$ is an isolated zero of both f and g with multiplicity N and M, respectively, then the composition $f \circ g$ has an isolated zero with multiplicity NM at the origin.*

The following lemma is from [**5**].

LEMMA 2.4. [**5**] *Let $m > 1$ be a positive integer and let f be a germ in $\mathcal{O}(\mathbb{C}^n, 0, 0)$. If the origin is an isolated fixed point of f and for each eigenvalue λ of $Df(0)$ either $\lambda = 1$ or $\lambda^m \neq 1$ holds, then the origin is also an isolated fixed point of f^m and $\mu_f(0) = \mu_{f^m}(0)$.*

3. *The proof of Theorem 1.2*

This section gives the proof of Theorem 1.2. The proof of sufficiency and necessity for Theorem 1.2 needs the following two propositions.

PROPOSITION 3.1. *Let Λ be a matrix with the form (1.1) and let f be a germ in $\mathcal{O}^n_\Lambda$. For any positive integer k, the origin is an isolated zero of τf if and only if the origin is an isolated zero of τf^k, where $\tau f^k = f^k - \tilde{\Lambda}^k$.*

Proof. Firstly, we fix k and assume that all eigenvalues of Λ are non-zero, that is, $\tilde{\Lambda}$ is invertible. Let $g = \tilde{\Lambda}^{-1} f$. Then $g = \tilde{\Lambda}^{-1} f = \tilde{\Lambda}^{-1}(\tilde{\Lambda} + \tau f) = \mathrm{id} + \tilde{\Lambda}^{-1}\tau f$. Lemma 2.2 says that the origin is an isolated fixed point of g if and only if it is an isolated zero of τf. Since f is in resonant polynomial normal form, $\tilde{\Lambda} f = f \circ \tilde{\Lambda}$ and $\tilde{\Lambda}^{-1} f = f \circ \tilde{\Lambda}^{-1}$. Thus

$$g^k = \tilde{\Lambda}^{-k} f^k = \tilde{\Lambda}^{-k}(\tilde{\Lambda}^k + \tau f^k) = \mathrm{id} + \tilde{\Lambda}^{-k}\tau f^k.$$

Again by Lemma 2.2, the origin is an isolated fixed point of g^k if and only if it is an isolated zero of τf^k.

Evidently, the origin being an isolated fixed point of g^k implies that it is also an isolated fixed point of g. Conversely, by Lemma 2.4, the origin being an isolated fixed point of g implies that it is an isolated fixed point of g^k. Consequently, the origin is an isolated zero of τf if and only if it is an isolated zero of τf^k.

For the general case, let $w' = (w'_1 w'_2 \cdots w'_n)$, where $w'_j = 0$ if and only if $e(j) = 0$, and let $\tilde{w}' = (\tilde{w}'_1 \tilde{w}'_2 \cdots \tilde{w}'_n)$, where $\tilde{w}'_j = 1$ if and only if $e(j) = 0$. Without loss of generality, we suppose that $S(w') = \{1, 2, \ldots, |w'|\}$ and $S(\tilde{w}') = \{|w'| + 1, \ldots, n\}$. It is easy to see that

$$\tau f = \begin{pmatrix} p_{w'} \circ \tau f \circ i_{w'} \\ p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'} \end{pmatrix} + \begin{pmatrix} 0 \\ p_{\tilde{w}'} \circ \tau f - p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'} \end{pmatrix}$$

and

$$\tau f^k = \begin{pmatrix} p_{w'} \circ \tau f^k \circ i_{w'} \\ p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'} \end{pmatrix} + \begin{pmatrix} 0 \\ p_{\tilde{w}'} \circ \tau f^k - p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'} \end{pmatrix},$$

where both $p_{\tilde{w}'} \circ \tau f - p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'}$ and $p_{\tilde{w}'} \circ \tau f^k - p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'}$ have no monomial with all of its variable indices completely from $S(\tilde{w}')$ or $S(w')$. Thus, the origin is an isolated zero of τf (τf^k) if and only if zero is an isolated zero of both $p_{w'} \circ \tau f \circ i_{w'}$ ($p_{w'} \circ \tau f^k \circ i_{w'}$) and $p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'}$ ($p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'}$).

Next, we only need to prove that zero is an isolated zero of $p_{w'} \circ \tau f \circ i_{w'}$ if and only if zero is an isolated zero of $p_{w'} \circ \tau f^k \circ i_{w'}$ and zero is an isolated zero of $p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'}$ if and only if zero is an isolated zero of $p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'}$.

On the one hand, since f is in resonant polynomial normal form,

$$f \circ i_{w'} = i_{w'} \circ p_{w'} \circ f \circ i_{w'}$$

and hence $(p_{w'} \circ f \circ i_{w'})^k = p_{w'} \circ f^k \circ i_{w'}$. Similarly,

$$p_{\tilde{w}'} \circ f^k \circ i_{\tilde{w}'} = (p_{\tilde{w}'} \circ f \circ i_{\tilde{w}'})^k.$$

The Jacobian matrix at the origin of $p_{w'} \circ f \circ i_{w'}$ is invertible according to the definition of w'. Thus, by the previous case, zero is an isolated zero of $p_{w'} \circ \tau f \circ i_{w'} = \tau p_{w'} \circ f \circ i_{w'}$ if and only if zero is an isolated zero of

$$p_{w'} \circ \tau f^k \circ i_{w'} = \tau p_{w'} \circ f^k \circ i_{w'} = \tau (p_{w'} \circ f \circ i_{w'})^k.$$

On the other hand, $p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'} = p_{\tilde{w}'} \circ f \circ i_{\tilde{w}'}$ and

$$p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'} = p_{\tilde{w}'} \circ f^k \circ i_{\tilde{w}'} = (p_{\tilde{w}'} \circ f \circ i_{\tilde{w}'})^k.$$

It is easy to check that the origin is an isolated zero of $p_{\tilde{w}'} \circ f \circ i_{\tilde{w}'}$ if and only if it is an isolated zero of $(p_{\tilde{w}'} \circ f \circ i_{\tilde{w}'})^k$. Thus, the origin is an isolated zero of $p_{\tilde{w}'} \circ \tau f \circ i_{\tilde{w}'}$ if and only if it is an isolated zero of $p_{\tilde{w}'} \circ \tau f^k \circ i_{\tilde{w}'}$. □

For $1 \leq K < n$, let $w(0, K) = (w_{0K1} \cdots w_{0Kn}) \in W^n$, where w_{0Kj} is equal to 1 if $1 \leq j \leq K$ and 0 otherwise. Let $w(1, K) = (w_{1K1} \cdots w_{1Kn}) \in W^n$, where w_{1Kj} is equal to 0 if $1 \leq j \leq K$ and 1 otherwise.

PROPOSITION 3.2. *Let g be a germ in $\mathcal{O}(\mathbb{C}^n, 0, 0)$ and K be a positive integer such that $1 \leq K < n$. If the Jacobian matrix of $p_{w(0,K)} \circ g \circ i_{w(0,K)}$ at the origin in $\mathbb{C}^K$ is invertible and $p_{w(0,K)} \circ g \circ i_{w(1,K)} \equiv 0$, then the origin in $\mathbb{C}^n$ is an isolated zero of g if and only if the origin in $\mathbb{C}^{n-K}$ is an isolated zero of $p_{w(1,K)} \circ g \circ i_{w(1,K)}$. In this case, $\pi_g(0) = \pi_{p_{w(1,K)} \circ g \circ i_{w(1,K)}}(0)$.*

Proof. For sufficiently small $x_{K+1}, x_{K+2}, \ldots, x_n$, we define a map $G_{x_{K+1},\ldots,x_n} \in \mathcal{O}(\mathbb{C}^K, 0, 0)$ with $G_{x_{K+1},\ldots,x_n}(x_1, \ldots, x_K) = p_{w(0,K)} \circ g(x_1, x_2, \ldots, x_n)$. It is clear that $G_{0,\ldots,0} = p_{w(0,K)} \circ g \circ i_{w(0,K)}$. By the inverse function theorem and Rouché's theorem (see [**9**, Theorem 2.3]), there exists a neighborhood U_{n-K} of 0 in $\mathbb{C}^{n-K}$ and a neighborhood V_K of 0 in $\mathbb{C}^K$ such that, for any $(x_{K+1}, \ldots, x_n) \in U_{n-K}$, $G_{x_{K+1},\ldots,x_n}$ is a diffeomorphism on V_K. Since $p_{w(0,K)} \circ g \circ i_{w(1,K)} \equiv 0$, $G_{x_{K+1},\ldots,x_n}(0) = 0$ for any $(x_{K+1}, \ldots, x_n) \in U_{n-K}$ and, consequently,

$$g^{-1}\{0\} \cap V_K \times U_{n-K} \subset \{(x_1, \ldots, x_n) \in V_K \times U_{n-K} : x_1 = \cdots = x_K = 0\}. \quad (3.1)$$

Relation (3.1) indicates that the origin in $\mathbb{C}^n$ is an isolated zero of g if and only if the origin in $\mathbb{C}^{n-K}$ is an isolated zero of $p_{w(1,K)} \circ g \circ i_{w(1,K)}$.

If the origin is an isolated zero of $p_{w(1,K)} \circ g \circ i_{w(1,K)}$, then there exists a neighborhood $V_{n-K} \subset U_{n-K}$ of the origin in $\mathbb{C}^{n-K}$ such that

$$(p_{w(1,K)} \circ g \circ i_{w(1,K)})^{-1}\{0\} \cap V_{n-K} = \{0\}$$

and

$$g^{-1}\{0\} \cap V_K \times V_{n-K} = \{0\}.$$

Let $a = (a_{K+1}, a_{K+2}, \ldots, a_n) \in \mathbb{C}^{n-K}$ be a regular value of $p_{w(1,K)} \circ g \circ i_{w(1,K)}$ such that its norm $|a|$ is sufficiently small. The set $(p_{w(1,K)} \circ g \circ i_{w(1,K)})^{-1}\{a\} \cap V_{n-K}$ includes a finite number of points $b_1, \ldots, b_{\pi_{p_{w(1,K)} \circ g \circ i_{w(1,K)}}(0)}$: i.e.,

$$(p_{w(1,K)} \circ g \circ i_{w(1,K)})^{-1}\{a\} \cap V_{n-K} = \{b_1, \ldots, b_{\pi_{p_{w(1,K)} \circ g \circ i_{w(1,K)}}(0)}\}.$$

Let $\tilde{a} = (0, \ldots, 0, a_{K+1}, \ldots, a_n) \in \mathbb{C}^n$. Then, similarly to the process of obtaining (3.1), we can easily have

$$g^{-1}\{\tilde{a}\} \cap V_K \times V_{n-K} = \{i_{w(1,K)}(b_1), \ldots, i_{w(1,K)}(b_{\pi_{p_{w(1,K)} \circ g \circ i_{w(1,K)}}(0)})\}.$$

Then, according to the conditions of the proposition and the definition of a, it is easy to check that $\tilde{a}$ is a regular value of g. Thus $\pi_g(0) = \pi_{p_{w(1,K)} \circ g \circ i_{w(1,K)}}(0)$. □

Proof of Theorem 1.2. We first give the proof of the sufficiency. Assume that the origin in $\mathbb{C}^n$ is an isolated fixed point of f^k for any $k \geq 1$. We need to show that the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$. Let $k_0 = \prod_{d_j \neq 0} d_j$. We firstly prove that

$$\tau(p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)})^{k_0} = p_{w(\Lambda)} \circ (f^{k_0} - \mathrm{id}) \circ i_{w(\Lambda)}. \tag{3.2}$$

To see this, since f is in resonant polynomial normal form, we have

$$f \circ i_{w(\Lambda)} = i_{w(\Lambda)} \circ p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)},$$

and hence

$$(p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)})^{k_0} = p_{w(\Lambda)} \circ f^{k_0} \circ i_{w(\Lambda)}. \tag{3.3}$$

Apply the map τ to both sides of equation (3.3) and we get

$$\tau(p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)})^{k_0} = \tau(p_{w(\Lambda)} \circ f^{k_0} \circ i_{w(\Lambda)}) = p_{w(\Lambda)} \circ (f^{k_0} - \mathrm{id}) \circ i_{w(\Lambda)},$$

and thus (3.2) holds.

Next, we will show that the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $\tau\left(p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)}\right)^{k_0}$. By (3.2), we only need to prove that it is an isolated zero of $p_{w(\Lambda)} \circ (f^{k_0} - \mathrm{id}) \circ i_{w(\Lambda)}$. Since the origin in $\mathbb{C}^n$ is an isolated fixed point of f^k for any $k \geq 1$, it is, particularly, an isolated zero of $f^{k_0} - \mathrm{id}$. Let $S(w(\Lambda)) = \{j_1, \ldots, j_{|w(\Lambda)|}\}$ and $\mathbf{x}^t = (x^t_{j_1}, \ldots, x^t_{j_{|w(\Lambda)|}}) \in \mathbb{C}^{|w(\Lambda)|}$ with the norm $|\mathbf{x}^t|$ being small enough. Then $p_{w(\Lambda)} \circ (f^{k_0} - \mathrm{id}) \circ i_{w(\Lambda)}(\mathbf{x}^t) = 0$ means that

$$(f^{k_0} - \mathrm{id})(i_{w(\Lambda)}(\mathbf{x}^t)) = (f^{k_0} - \mathrm{id})(0, \ldots, 0, x^t_{j_1}, 0, \ldots, 0, x_{j_{S(w(\Lambda))}}, 0, \ldots, 0) = 0.$$

Consequently, the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ (f^{k_0} - \mathrm{id}) \circ i_{w(\Lambda)}$.

The final step is to show that the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$. According to the definition of τ, we have

$$p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)} = \tau p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)}.$$

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Therefore, we only need to prove that the origin is an isolated zero of $\tau p_{w(\Lambda)} \circ f \circ i_{w(\Lambda)}$. This is proved in Proposition 3.1.

Now we prove the necessity and $\mu_{f^k}(0) = \pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0)$ for any $k \geq 1$. We fix a $k \geq 1$ and assume that the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$. It is clear that $S(w(k)) \subset S(w(\Lambda))$, and then

$$p_{w(k)} \circ \tau f \circ i_{w(k)} = p_{w(k)} \circ i_{w(\Lambda)} \circ (p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}) \circ p_{w(\Lambda)} \circ i_{w(k)}. \tag{3.4}$$

This implies that the origin in $\mathbb{C}^{|w(k)|}$ is an isolated zero of $p_{w(k)} \circ \tau f \circ i_{w(k)}$. Indeed, let $\mathbf{x}_k^t \in \mathbb{C}^{|w(k)|}$ with the norm $|\mathbf{x}_k^t|$ being small enough such that $p_{w(k)} \circ \tau f \circ i_{w(k)}(\mathbf{x}_k^t) = 0$. Then, by (3.4),

$$p_{w(k)} \circ i_{w(\Lambda)} \circ (p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}) \circ p_{w(\Lambda)} \circ i_{w(k)}(\mathbf{x}_k^t) = 0.$$

Since f is in resonant polynomial normal form, it follows from the above formula that

$$(p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}) \circ p_{w(\Lambda)} \circ i_{w(k)}(\mathbf{x}_k^t) = 0,$$

that is,

$$(p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)})(p_{w(\Lambda)} \circ i_{w(k)}(\mathbf{x}_k^t)) = 0.$$

Since the origin in $\mathbb{C}^{|w(\Lambda)|}$ is an isolated zero of $p_{w(\Lambda)} \circ \tau f \circ i_{w(\Lambda)}$, we see that $p_{w(\Lambda)} \circ i_{w(k)}(\mathbf{x}_k^t) = 0$ and thus $\mathbf{x}_k^t = 0$. Consequently, the origin in $\mathbb{C}^{|w(k)|}$ is an isolated zero of $p_{w(k)} \circ \tau f \circ i_{w(k)}$.

Let $\tilde{\Lambda}_{w(k)} = p_{w(k)} \circ \tilde{\Lambda} \circ i_{w(k)}$. Since $d(j) \neq 0$ for any $j \in S(w(k))$, $\tilde{\Lambda}_{w(k)}$ is invertible. Let

$$g_{w(k)} = \tilde{\Lambda}_{w(k)}^{-1} \circ p_{w(k)} \circ f \circ i_{w(k)}.$$

We will show that the origin is an isolated fixed point of $g_{w(k)}$. According to the definition of τ,

$$g_{w(k)} = \tilde{\Lambda}_{w(k)}^{-1} \circ p_{w(k)} \circ f \circ i_{w(k)} = \mathrm{id} + \tilde{\Lambda}_{w(k)}^{-1} \circ p_{w(k)} \circ \tau f \circ i_{w(k)}.$$

By Lemma 2.2, the origin is an isolated fixed point of $g_{w(k)}$ and

$$\mu_{g_{w(k)}}(0) = \pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0). \tag{3.5}$$

Lemma 2.4 further tells us that the origin is an isolated fixed point of $g_{w(k)}^k$ and

$$\mu_{g_{w(k)}}(0) = \mu_{g_{w(k)}^k}(0). \tag{3.6}$$

Since f is in resonant polynomial normal form,

$$\tilde{\Lambda}_{w(k)}^{-1} \circ p_{w(k)} \circ f \circ i_{w(k)} = p_{w(k)} \circ f \circ i_{w(k)} \circ \tilde{\Lambda}_{w(k)}^{-1}$$

and

$$f \circ i_{w(k)} = i_{w(k)} \circ p_{w(k)} \circ f \circ i_{w(k)}.$$

Then

$$g_{w(k)}^k = \tilde{\Lambda}_{w(k)}^{-k} \circ p_{w(k)} \circ f^k \circ i_{w(k)} = p_{w(k)} \circ f^k \circ i_{w(k)}. \tag{3.7}$$

Therefore we conclude from (3.7) that the origin is an isolated fixed point of $p_{w(k)} \circ f^k \circ i_{w(k)}$.

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Now, we are ready to prove that, for any $k \geq 1$, the origin is an isolated fixed point of f^k and $\mu_{f^k}(0) = \pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0)$. For this purpose, we firstly observe that (3.5), (3.6) and (3.7) together give $\pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0) = \mu_{p_{w(k)} \circ f^k \circ i_{w(k)}}(0)$: i.e., $\pi_{p_{w(k)} \circ \tau f \circ i_{w(k)}}(0) = \pi_{p_{w(k)} \circ (f^k - \mathrm{id}) \circ i_{w(k)}}(0)$. Then, without loss of generality, we assume that $w(k) = w(1, K)$ with $1 \leq K < n$. It is clear that $p_{w(0,K)} \circ (f^k - \mathrm{id}) \circ i_{w(1,K)} \equiv 0$. By Proposition 3.2, we only need to show that the Jacobian matrix of $p_{w(0,K)} \circ (f^k - \mathrm{id}) \circ i_{w(0,K)}$ at the origin is invertible to complete the proof. To see this, the definitions of $w(1, K)$ and $w(k)$ say that there exists a positive integer $1 \leq j_0 < m$ such that $k_1 + \cdots + k_{j_0} = K$ and $\lambda_j^k - 1 \neq 0$ for $j = 1, 2, \ldots, j_0$, respectively. Thus, the determinant of the Jacobian matrix of $p_{w(0,K)} \circ (f^k - \mathrm{id}) \circ i_{w(0,K)}$ at the origin is equal to $\prod_{j=1}^{j=j_0} (\lambda_j^k - 1)^{k_j} \neq 0$ and therefore the Jacobin matrix is invertible. □

4. *The proof of Corollary 1.2*

Recall that

$$f(x) = \Lambda x + R(u)x, \tag{4.1}$$

where $u = (u_1, \ldots, u_n) = (x_1^{d_1}, \ldots, x_n^{d_n})$ and $R(u) = \mathrm{diag}\{r_1(u), \ldots, r_n(u)\}$ with $R(0) = 0$.

For any $w \in W_n^*$, since the origin in $\mathbb{C}^n$ is an isolated fixed point of $f^{d_1 d_2 \cdots d_n}$, it is an isolated zero of $\tau f^{d_1 d_2 \cdots d_n}$. Therefore it is also an isolated zero of τf by Proposition 3.1. Together with (4.1), we have that the origin in $\mathbb{C}^{|w|}$ is an isolated zero of $p_w \circ R(u) \circ i_w = R_w(p_w \circ u \circ i_w)$, and thus it is also an isolated zero of R_w.

Next, let $S(w) = \{s_1, s_2, \ldots, s_{|w|}\}$ with $s_1 < s_2 < \cdots < s_{|w|}$, $M = d_{s_1} d_{s_2} \ldots d_{s_{|w|}}$ and we will prove that

$$\pi_{R_w(p_w \circ u \circ i_w)}(0) = \sum_{s \subset \{d_{s_1}, d_{s_2}, \ldots, d_{s_{|w|}}\}} (-1)^{\#s} \mu_{f^{M:s}}(0). \tag{4.2}$$

Let $q(m_1, m_2, \ldots, m_{|w|}) = (q_1^{(m_1)}, q_2^{(m_2)}, \ldots, q_{|w|}^{(m_{|w|})})$, where $m_j \in \{1, 2\}$, $q_j^{(1)} = x_{s_j}$, and $q_j^{(2)} = r_{s_j}(u \circ i_w)$ for $j = 1, 2, \ldots, |w|$. Since the origin in $\mathbb{C}^n$ is an isolated zero of τf, the origin in $\mathbb{C}^{|w|}$ is an isolated zero of $q(m_1, m_2, \ldots, m_{|w|})$ for $m_1, m_2, \ldots, m_{|w|} \in \{1, 2\}$, and by Lemma 2.1,

$$\pi_{p_w \circ \tau f \circ i_w}(0) = \sum \{\pi_{q(m_1, m_2, \ldots, m_{|w|})}(0) : \forall j \in \{1, 2, \ldots, |w|\}, m_j \in \{1, 2\}\}$$

and

$$\pi_{R_w(p_w \circ u \circ i_w)}(0) = \pi_{q(2,2,\ldots,2)}(0).$$

In what follows, we will use $q(m_1, m_2, \ldots, m_{|w|})$ to define a probability space to expand $\pi_{R_w(p_w \circ u \circ i_w)}(0)$ to complete the proof of (4.2).

Let $A_\emptyset = \{q(m_1, m_2, \ldots, m_{|w|}) : \forall j \in \{1, 2, \ldots, |w|\}, m_j \in \{1, 2\}\}$, and, for $1 \leq i_1 < i_2 < \cdots < i_t \leq |w|$, let $A_{\{i_1, i_2, \ldots, i_t\}}$ be a set consisting of $q(m_1, m_2, \ldots, m_{|w|}) \in A_\emptyset$ with

$$m_j = \begin{cases} 1, & j \in \{i_1, i_2, \ldots, i_t\}, \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases}$$

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Let the sample space be $\Omega = A_\emptyset$ and let the σ-algebra be the family $\mathcal{F}$ of all subsets of Ω. For $A \in \mathcal{F}$, we define

$$P(A) = \frac{\sum \{\pi_{q(m_1,m_2,\ldots,m_{|w|})}(0) : q(m_1, m_2, \ldots, m_{|w|}) \in A\}}{\sum \{\pi_{q(m_1,m_2,\ldots,m_{|w|})}(0) : q(m_1, m_2, \ldots, m_{|w|}) \in A_\emptyset\}}.$$

It is easy to check that the triple $(\Omega, \mathcal{F}, P)$ is a probability space on Ω, $P(A_\emptyset) = 1$ and that

$$P(\overline{A_{\{1\}}} \cap \overline{A_{\{2\}}} \cap \cdots \cap \overline{A_{\{|w|\}}}) = \frac{\pi_{R_w(p_w \circ u \circ i_w)}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)}. \tag{4.3}$$

By Proposition 3.2, we have, for $1 \le i_1 < i_2 < \cdots < i_t \le |w|$,

$$P(A_{\{i_1,i_2,\ldots,i_t\}}) = \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)}. \tag{4.4}$$

By the addition property of probability measures (see [**6**]),

$$\begin{aligned}
&P(A_{\{1\}} \cup A_{\{2\}} \cup \cdots \cup A_{\{|w|\}}) \\
&\quad = \sum_{j=1}^{|w|} P(A_{\{j\}}) + \cdots + (-1)^{t-1} \sum_{1 \le i_1 < i_2 < \cdots < i_t \le |w|} P(A_{\{i_1\}} \cap A_{\{i_2\}} \cap \cdots \cap A_{\{i_t\}}) \\
&\qquad + \cdots + (-1)^{|w|-1} P(A_{\{1\}} \cap A_{\{2\}} \cap \cdots \cap A_{\{|w|\}}) \\
&\quad = \sum_{j=1}^{|w|} P(A_{\{j\}}) + \cdots + (-1)^{t-1} \sum_{1 \le i_1 < i_2 < \cdots < i_t \le |w|} P(A_{\{i_1,i_2,\ldots,i_t\}}) \\
&\qquad + \cdots + (-1)^{|w|-1} P(A_{\{1,2,\ldots,|w|\}}).
\end{aligned} \tag{4.5}$$

Substituting each term on the right-hand side of (4.5) with (4.4) gives

$$\begin{aligned}
&P(A_{\{1\}} \cup A_{\{2\}} \cup \cdots \cup A_{\{|w|\}}) \\
&= \sum_{j=1}^{|w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad - \sum_{1 \le i_1 < i_2 \le |w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad + \cdots + (-1)^{t-1} \sum_{1 \le i_1 < i_2 < \cdots < i_t \le |w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad + \cdots + (-1)^{|w|-1} \frac{\pi_{p_{W\emptyset} \circ \tau f \circ i_{W\emptyset}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)}.
\end{aligned} \tag{4.6}$$

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Equations (4.3) and (4.6) and the equality $P(\overline{A_{\{1\}}} \cap \overline{A_{\{2\}}} \cap \cdots \cap \overline{A_{\{|w|\}}}) = 1 - P(A_{\{1\}} \cup A_{\{2\}} \cup \cdots \cup A_{\{|w|\}})$ together give

$$\begin{aligned}
&\frac{\pi_{R_w(p_w \circ u \circ i_w)}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad = 1 - \sum_{j=1}^{|w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad + \sum_{1 \le i_1 < i_2 \le |w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad + \cdots + (-1)^t \sum_{1 \le i_1 < i_2 < \cdots < i_t \le |w|} \frac{\pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)} \\
&\quad + \cdots + (-1)^{|w|} \frac{\pi_{p_{W\emptyset} \circ \tau f \circ i_{W\emptyset}}(0)}{\pi_{p_w \circ \tau f \circ i_w}(0)}.
\end{aligned} \tag{4.7}$$

Multiplying $\pi_{p_w \circ \tau f \circ i_w}(0)$ on both sides of (4.7) gives

$$\begin{aligned}
&\pi_{R_w(p_w \circ u \circ i_w)}(0) \\
&\quad = \pi_{p_w \circ \tau f \circ i_w}(0) - \sum_{j=1}^{|w|} \pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_j\}}}(0) \\
&\quad + \sum_{1 \le i_1 < i_2 \le |w|} \pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2}\}}}(0) \\
&\quad + \cdots + (-1)^t \sum_{1 \le i_1 < i_2 < \cdots < i_t \le |w|} \pi_{p_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}} \circ \tau f \circ i_{W\{s_1,s_2,\ldots,s_{|w|}\}\setminus\{s_{i_1},s_{i_2},\ldots,s_{i_t}\}}}(0) \\
&\quad + \cdots + (-1)^{|w|} \pi_{p_{W\emptyset} \circ \tau f \circ i_{W\emptyset}}(0).
\end{aligned}$$

This completes the proof of (4.2) with Theorem 1.2.

With (4.2) and Lemma 2.3, for the purpose of completing Corollary 1.2, we only need to show that

$$P_M(f, 0) = \sum_{s \subset \{d_{s_1}, d_{s_2}, \ldots, d_{s_{|w|}}\}} (-1)^{\# s} \mu_{f^{M:s}}(0). \tag{4.8}$$

Let $d_{s_j} = q_{s_j 1}^{\alpha_{j1}} \cdots q_{s_j t_j}^{\alpha_{j t_j}}$, where $q_{s_j t}$ is a prime number and α_{jt} is a positive integer for $t = 1, \ldots, t_j$, $t_j \ge 1$ and $j = 1, 2, \ldots, |w|$. If $t_1 = t_2 = \cdots = t_{|w|} = 1$, (4.8) holds immediately by Lemma 2.4. If there exists $j \in \{1, 2, \ldots, |w|\}$ such that $t_j > 1$, from the definition of local Dold indices it can be verified that

$$P_M(f, 0) = P_{\frac{M}{q_{s_j t_j}^{\alpha_{j t_j}}}}\big(f^{q_{s_j t_j}^{\alpha_{j t_j}}}, 0\big) - P_{\frac{M}{q_{s_j t_j}^{\alpha_{j t_j}}}}\big(f^{q_{s_j t_j}^{\alpha_{j t_j} - 1}}, 0\big).$$

Since $\frac{M}{q_{s_jt_j}^{\alpha_{jt_j}}}$ is not a period of the linear part of $f^{q_{s_jt_j}^{\alpha_{jt_j}-1}}$ at the origin, we have, by [**3**, Theorem 1.4],

$$P_{\frac{M}{q_{s_jt_j}^{\alpha_{jt_j}}}}(f^{q_{s_jt_j}^{\alpha_{jt_j}-1}}, 0) = 0,$$

and thus

$$P_M(f, 0) = P_{\frac{M}{q_{s_jt_j}^{\alpha_{jt_j}}}}(f^{q_{s_jt_j}^{\alpha_{jt_j}}}, 0).$$

Repeating the above process ends with

$$P_{\frac{M}{q_{s_jt_j}^{\alpha_{jt_j}}}}(f^{q_{s_jt_j}^{\alpha_{jt_j}}}, 0) = P_{\frac{M}{q_{s_jt_j}^{\alpha_{jt_j}} q_{s_j(t_j-1)}^{\alpha_{j(t_j-1)}}}}(f^{q_{s_jt_j}^{\alpha_{jt_j}} q_{s_j(t_j-1)}^{\alpha_{j(t_j-1)}}}, 0)$$

$$= \cdots = P_{q_{s_11}^{\alpha_{11}} \cdots q_{s_{|w|}1}^{\alpha_{|w|1}}}(f^{\frac{M}{q_{s_11}^{\alpha_{11}} \cdots q_{s_{|w|}1}^{\alpha_{|w|1}}}}, 0).$$

Thus

$$P_M(f, 0) = P_{q_{s_11}^{\alpha_{11}} \cdots q_{s_{|w|}1}^{\alpha_{|w|1}}}(f^{\frac{M}{q_{s_11}^{\alpha_{11}} \cdots q_{s_{|w|}1}^{\alpha_{|w|1}}}}, 0) = \sum_{s \subset \{q_{s_1 1}, \ldots, q_{s_{|w|}1}\}} (-1)^{\#s} \mu_{f^{M:s}}(0).$$

The second equality follows from the definition of local Dold indices. By Lemma 2.4, we also have

$$\sum_{s \subset \{q_{s_1 1}, \ldots, q_{s_{|w|}1}\}} (-1)^{\#s} \mu_{f^{M:s}}(0) = \sum_{s \subset \{d_{s_1}, d_{s_2}, \ldots, d_{s_{|w|}}\}} (-1)^{\#s} \mu_{f^{M:s}}(0),$$

and thus

$$P_M(f, 0) = \sum_{s \subset \{d_{s_1}, d_{s_2}, \ldots, d_{s_{|w|}}\}} (-1)^{\#s} \mu_{f^{M:s}}(0).$$

This completes the proof of Corollary 1.2.

5. *Applications of Theorem 1.2*

Let f be of the form (1.2) and assume that zero is an isolated fixed point of all iterates of f. We consider the sequence of numbers $\mathcal{N}_1(f), \mathcal{N}_2(f), \ldots$.

Zhang [**8**] proved that the linear part of f determines some natural restrictions to the sequence. Specifically, when $m > 1$, $\mathcal{N}_m(f) > 0$ if and only if the map $x \mapsto \Lambda x$ has a periodic orbit of minimal period m, and when $m = 1$, $\mathcal{N}_1(f) > 1$ if and only if the map $x \mapsto \Lambda x$ has a fixed point other than zero. Assume that the $n \times n$ matrix Λ is diagonalizable and that all its eigenvalues are roots of unity of pairwise relatively prime degrees greater than one. Then, when $n \leq 2$, Gorbovickis [**3**] proved that any non-negative integer sequence subject only to the restrictions in [**8**] can be realized on the sequence of the numbers of periodic orbits hidden at the fixed point zero of the germ of some holomorphic map with linear part Λ. But for the case in which $n \geq 3$, this does not hold unless more restrictions are applied to the non-negative integer sequences. We will give two examples for $n \geq 3$ in which a Λ can be found such that this holds subject only to the restrictions in [**8**], without the necessity of making the assumption on Λ.

Example 5.1. The first example is a resonant polynomial normal form

$$f(x) = \begin{pmatrix} \lambda_1 x_1 + x_1^{r_1 d_1 + 1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ \lambda_2 x_2 + x_2 (x_1^{r_2 d_1} - x_3^{r_3 d_3} x_1^{(r_2 - 1) d_1} + x_3^{r_{23} d_3}) \\ \lambda_3 x_3 + x_3 (x_1^{d_1} - x_3^{r_3 d_3}) \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ are roots of unity with degrees $d_1, d_2, d_3 > 1$ such that $d_1 \mid d_2$, $(d_3, d_2) = 1$, $d = d_2/d_1$ and $\lambda_1 = \lambda_2^d$.

Now we compute the numbers of periodic orbits of every positive period hidden at the origin. Since

$$\tau f(x) = \begin{pmatrix} x_1^{r_1 d_1 + 1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_2 (x_1^{r_2 d_1} - x_3^{r_3 d_3} x_1^{(r_2 - 1) d_1} + x_3^{r_{23} d_3}) \\ x_3 (x_1^{d_1} - x_3^{r_3 d_3}) \end{pmatrix},$$

we have

$$\begin{aligned}
\mu_{f^{d_1}}(0) &= \pi_{p_{(100)} \circ \tau f \circ i_{(100)}}(0) = r_1 d_1 + 1, \\
\mu_{f^{d_2}}(0) &= \pi_{p_{(110)} \circ \tau f \circ i_{(110)}}(0) = r_2 d_2 + r_1 d_1 + 1, \\
\mu_{f^{d_3}}(0) &= \pi_{p_{(001)} \circ \tau f \circ i_{(001)}}(0) = r_3 d_3 + 1, \\
\mu_{f^{d_1 d_3}}(0) &= \pi_{p_{(101)} \circ \tau f \circ i_{(101)}}(0) = r_{13} d_1 d_3 + r_3 d_3 + r_1 d_1 + 1, \\
\mu_{f^{d_2 d_3}}(0) &= \pi_{p_{(111)} \circ \tau f \circ i_{(111)}}(0) = r_{23} d_2 d_3 + r_{13} d_1 d_3 + r_3 d_3 + r_1 d_1 + 1, \\
\mu_{f^k}(0) &= \pi_{p_{(000)} \circ \tau f \circ i_{(000)}}(0) = 1,
\end{aligned}$$

where $k \nmid d_2 d_3$. Only the process of computing $\mu_{f^{d_2 d_3}}(0)$ is given here. We divide τf into three parts τf_1, τf_2 and τf_3, as follows.

$$\tau f_1(x) = \begin{pmatrix} x_1^{r_1 d_1 + 1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_2 (x_1^{r_2 d_1} - x_3^{r_3 d_3} x_1^{(r_2 - 1) d_1} + x_3^{r_{23} d_3}) \\ x_3 \end{pmatrix},$$

$$\tau f_2(x) = \begin{pmatrix} x_1^{r_1 d_1 + 1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_2 \\ x_1^{d_1} - x_3^{r_3 d_3} \end{pmatrix}$$

and

$$\tau f_3(x) = \begin{pmatrix} x_1^{r_1 d_1 + 1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_1^{r_2 d_1} - x_3^{r_3 d_3} x_1^{(r_2 - 1) d_1} + x_3^{r_{23} d_3} \\ x_1^{d_1} - x_3^{r_3 d_3} \end{pmatrix}.$$

https://doi.org/10.1017/etds.2019.60 Published online by Cambridge University Press

It is easy to see that $\pi_{\tau f_1}(0) = \pi_{p_{(110)}\circ\tau f\circ i_{(110)}}(0) = r_2 d_2 + r_1 d_1 + 1$ and $\pi_{\tau f_2}(0) = \pi_{p_{(101)}\circ\tau f\circ i_{(101)}}(0) - \pi_{p_{(100)}\circ\tau f\circ i_{(100)}}(0) = r_{13} d_1 d_3 + r_3 d_3$.

By Lemma 2.2,

$$\begin{pmatrix} x_1^{r_1 d_1+1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_1^{r_2 d_1} - x_3^{r_3 d_3} x_1^{(r_2-1)d_1} + x_3^{r_{23} d_3} \\ x_1^{d_1} - x_3^{r_3 d_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1^{r_1 d_1+1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_3^{r_{23} d_3} \\ x_1^{d_1} - x_3^{r_3 d_3} \end{pmatrix}$$

have the same multiplicity at the origin. Similarly, By Lemmas 2.1 and 2.2,

$$\begin{pmatrix} x_1^{r_1 d_1+1} - x_1 x_3^{r_3 r_1 d_3} + x_1 x_3^{r_{13} d_3} + x_2^d \\ x_3^{r_{23} d_3} \\ x_1^{d_1} - x_3^{r_3 d_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2^d \\ x_3^{r_{23} d_3} \\ x_1^{d_1} \end{pmatrix}$$

have the same multiplicity at the origin. This means that $\pi_{\tau f_3}(0) = r_{23} d_3 d_1 d = r_{23} d_3 d_2$, so

$$\begin{aligned} \mu_{f^{d_2 d_3}}(0) = \pi_{\tau f}(0) &= \pi_{\tau f_1}(0) + \pi_{\tau f_2}(0) + \pi_{\tau f_3}(0) \\ &= r_{23} d_2 d_3 + r_{13} d_1 d_3 + r_3 d_3 + r_1 d_1 + 1. \end{aligned}$$

To complete the counting, we introduce the following lemma given in [**8**].

LEMMA 5.1. [**8**] *Let* $f \in \mathcal{O}(\mathbb{C}^n, 0, 0)$ *and let*

$$\mathcal{M}_f = \{m \in N : \textit{the linear part of } f \textit{ at 0 at has a periodic point of period } m\}.$$

Then the following hold.

(1) *For each* $m \in N \setminus \mathcal{M}_f$ *such that the origin is an isolated fixed point of* f^m,

$$\mathcal{N}_m(f) = 0.$$

(2) *For each positive integer* M *such that the origin is an isolated fixed point of* f^M,

$$\mu_{f^M}(0) = \sum_{m\in\mathcal{M}_f, m|M} m\mathcal{N}_m(f).$$

Since $\mathcal{M}_f = \{1, d_1, d_2, d_3, d_1 d_3, d_2 d_3\}$, by Lemma 5.1,

$$\mathcal{N}_1(f) = \mu_f(0) = 1, \quad \mathcal{N}_{d_1}(f) = \frac{\mu_{f^{d_1}}(0) - 1}{d_1} = r_1,$$

$$\mathcal{N}_{d_2}(f) = \frac{\mu_{f^{d_2}}(0) - 1}{d_2} = r_2, \quad \mathcal{N}_{d_3}(f) = \frac{\mu_{f^{d_3}}(0) - 1}{d_3} = r_3,$$

$$\mathcal{N}_{d_1 d_3}(f) = \frac{\mu_{f^{d_1 d_3}}(0) - d_1\mathcal{N}_{d_1}(f) - d_3\mathcal{N}_{d_3}(f) - 1}{d_1 d_3} = r_{13},$$

$$\mathcal{N}_{d_2 d_3}(f) = \frac{\mu_{f^{d_2 d_3}}(0) - d_1 d_3\mathcal{N}_{d_1 d_3}(f) - d_2\mathcal{N}_{d_2}(f) - d_3\mathcal{N}_{d_3}(f) - 1}{d_2 d_3} = r_{23}$$

and $\mathcal{N}_k(f) = 0$ for $k \nmid d_2 d_3$.

Example 5.2. The other example is a resonant polynomial normal form

$$f(x)=\begin{pmatrix} \lambda_1 x_1 + x_1^{r_1 d_1+1} + x_2^{d_2/d_1} \\ \lambda_2 x_2 + x_1^{r_2 d_1} x_2 + x_3^{d_3/d_2} \\ \lambda_3 x_3 + x_1^{r_3 d_1} x_3 + x_4^{d_4/d_3} \\ \cdots \\ \lambda_{n-1} x_{n-1} + x_1^{r_{n-1} d_1} x_{n-1} + x_n^{d_n/d_{n-1}} \\ \lambda_n x_n + x_1^{r_n d_1} x_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots of unity with degrees $d_1, d_2, \ldots, d_n > 1$ such that $d_1 |_{\neq} d_2 |_{\neq} \cdots |_{\neq} d_n$ and $\lambda_j = \lambda_{j+1}^{d_{j+1}/d_j}$ for $j = 1, 2, \ldots, n-1$.

As in Example 5.1, we have $\mathcal{N}_1(f) = 1$ and $\mathcal{N}_{d_j}(f) = r_j$ for $j = 1, 2, \ldots, n$, and $\mathcal{N}_k(f) = 0$ for $k \nmid d_n$.

Acknowledgements. The authors would like to thank the anonymous referee for useful comments and suggestions. The research work was supported by the National Natural Science Foundation of China (Grant No.: 11231009 and 11531007).

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