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COMPUTATION OF NILPOTENT ENGEL GROUPS

WERNER NICKEL

Dedicated to M. F. (Mike) Newman on the occasion of his 65th birthday

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Abstract

This paper reports on a facility of the ANU NQ program for computation of nilpotent groups that satisfy an Engel-n identity. The relevant details of the algorithm are presented together with results on Engel-n groups for moderate values of n.

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1. Introduction

A group G is called an *Engel-n group* if $[g, {}_n h] = 1$ for all $g, h \in G$. Here, a commutator $[g, h]$ denotes the expression $g^{-1}h^{-1}gh$ and $[g, {}_n h]$ is defined recursively by $[g, {}_1 h] = [g, h]$ and $[g, {}_{n+1} h] = [[g, {}_n h], h]$. With his solution of the restricted Burnside problem, Zel'manov [11] proved that Engel-n Lie rings are locally nilpotent. This implies that a finitely generated Engel-n group has a largest nilpotent factor group. It is unknown if Engel-n groups are locally nilpotent. For a discussion of recent progress on this question see the introduction of Vaughan-Lee [10]. We denote the largest nilpotent factor group of the free d-generator Engel-n group by $E(d, n)$. In this paper we report computations of $E(d, n)$ for small values of d and n.

In the rest of this introduction the relevant facts about nilpotent groups and their computation as factor groups of finitely presented groups are sketched. Robinson [7, Chapter 5] gives a general introduction into nilpotent and polycyclic groups. For detailed information on polycyclic presentations consult Sims [8, Chapter 9].

A finitely generated nilpotent group G is polycyclic and has a central series

$$G = G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} = \{1\}$$

with cyclic factors. Set

$$I = \{\, i \mid 1 \leq i \leq n,\ G_i/G_{i+1} \text{ finite} \,\}$$

and let m_i be the order of G_i/G_{i+1} for $i \in I$. If one chooses an element $a_i \in G_i$ for $1 \leq i \leq n$ such that

$$G_i/G_{i+1} = \langle a_i G_{i+1} \rangle,$$

then each element of G can be expressed uniquely as a word of the form

$$a_1^{e_1} \cdots a_n^{e_n} \quad \text{with } e_i \in \mathbb{Z} \text{ for } 1 \leq i \leq n \text{ and } 0 \leq e_i < m_i \text{ for } i \in I.$$

Such a word is called *normal*. The sequence $A = (a_1, \ldots, a_n)$ is called a *polycyclic generating sequence for* G.

For $1 \leq i < j \leq n$, the commutator $[a_j, a_i]$ is an element of G_{j+1} and can be expressed as a normal word w_{ij} in the elements $\{a_{j+1}, \ldots, a_n\}$. Likewise, $a_i^{m_i}$ is an element of G_{i+1} for $i \in I$ and can be expressed as a normal word w_{ii} in $\{a_{i+1}, \ldots, a_n\}$.

Let $\gamma_k(G)$ denote the k-th term of the lower central series of G with $G = \gamma_1(G)$. The *weight* $\mathrm{wt}(a_i)$ of a_i is defined to be the smallest positive integer k such that $a_i \in \gamma_k(G)$. If the central series above refines the lower central series, then w_{ij} is a word in generators of weight at least $\mathrm{wt}(a_i) + \mathrm{wt}(a_j)$ for $1 \leq i < j \leq n$. In this case, A is called *weighted*. Note that $w_{ij} = 1$ if $\mathrm{wt}(a_i) + \mathrm{wt}(a_j)$ exceeds the nilpotency class of G.

With these relations one obtains the following presentation for G on A:

$$\begin{aligned} \langle\, a_1, \ldots, a_n \mid \quad a_i^{m_i} &= w_{ii} \quad \text{for } i \in I; \\ [a_j, a_i] &= w_{ij} \quad \text{for } 1 \leq i < j \leq n \,\rangle. \end{aligned}$$

A presentation of this form, together with the weight function wt, is called a *weighted polycyclic presentation* and can be used to perform explicit computations in G by using a collection algorithm to transform words in A into normal words in A (see Sims [8, Section 9.4]). If each element of a group defined by a presentation of this form is equal to a *unique* normal word, then the presentation is called *consistent.*

There exist algorithms to compute a weighted polycyclic presentation of $H/\gamma_{c+1}(H)$ for a group H given by a finite presentation. An implementation of such an algorithm is the ANU Nilpotent Quotient program (Nickel [6]). The program computes epimorphisms from H onto $H/\gamma_k(H)$ for $k = 2, 3, \ldots$. The first step of the algorithm calculates a weighted polycyclic presentation for the largest Abelian quotient

$H/H' = H/\gamma_2(H)$ together with an epimorphism $H \twoheadrightarrow H/\gamma_2(H)$. Given an epimorphism $H \twoheadrightarrow H/\gamma_k(H)$ and a consistent weighted polycyclic presentation for $H/\gamma_k(H)$, the k-th step has three stages: The polycyclic presentation is extended to a weighted polycyclic presentation for the largest central (downward) extension of $H/\gamma_k(H)$ which is a homomorphic image of H. Then the resulting presentation is changed into a consistent presentation. Finally, the relations of H are enforced yielding a consistent weighted polycyclic presentation for $H/\gamma_{k+1}(H)$ together with a lifting of the epimorphism onto $H/\gamma_{k+1}(H)$. The whole step can be performed for increasing values of k until the largest nilpotent factor group of H is found or a specified bound on the nilpotency class is reached. A detailed description of the algorithm is given in Nickel [6].

The ANU NQ has the option to enforce an Engel-n identity on the quotient groups it calculates. It does this by evaluating, in addition to the given relations, a finite set of instances of the Engel-n identity in the last stage of each step and adds those instances that do not evaluate to the identity as relations. In particular, it is possible to start with a free group of finite rank d and compute a weighted polycyclic presentation of $E(d, n)$ for a given positive integer n. The set of instances used is described in the next section.

2. Checking identities in nilpotent groups

Let G be a finitely generated nilpotent group of nilpotency class c and $\omega(\xi_1, \dots, \xi_k)$ a k-variable identity. The task of checking if G satisfies the identity $\omega(\xi_1, \dots, \xi_k)$ can be reduced to checking a finite set of instances by an approach based on a result of Higman [3]. For a convenient formulation of this result, let F be the free group on a set $X = \{x_1, \dots, x_l\}$ and, for $Z \subseteq X$, let π_Z be the endomorphism of F which maps each element in Z to the identity in F and fixes every other element of X. The following is a slightly less general version than Higman's result, see Sims [8, Proposition 11.7.3] for the general statement.

LEMMA 1 (Higman's Lemma). *An element w of a free group F on X can be written as*

$$w = u\,v$$

where v is a non-empty product of commutators each involving every element of X and u is a product of words of the form $\pi_Z(w)$ or $\pi_Z(w^{-1})$ with $\emptyset \neq Z \subseteq X$.

Note that $\pi_Z(w)$ is a word in $X \setminus Z$. Lemma 1 is the key step in proving that a nilpotent group satisfies an identity if the identity is satisfied for a certain finite set of

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instances. More precisely, Higman proved that, if E is a generating set for G, then G satisfies $\omega(\xi_1, \ldots, \xi_k)$ if the identity is satisfied for all k-tuples $(h_1, \ldots, h_k)$ of words in E such that the sum of their lengths does not exceed c. Vaughan-Lee [9] used this approach for checking exponent laws in finite p-groups. Here, we will prove the corresponding result in the context of finitely generated nilpotent groups that are given by a weighted polycyclic presentation. The weight of a normal word $a_1^{e_1} \cdots a_n^{e_n}$ is defined as $|e_1| \operatorname{wt}(a_1) + \cdots + |e_n| \operatorname{wt}(a_n)$.

LEMMA 2. *Let G be a group of nilpotency class c given by a weighted polycyclic presentation with generating sequence $A = (a_1, \ldots, a_n)$. Then G satisfies the identity $\omega(\xi_1, \ldots, \xi_k)$ if $\omega(u_1, \ldots, u_k) = 1$ for all normal words $u_1, \ldots, u_k$ in A with $\operatorname{wt}(u_1) + \cdots + \operatorname{wt}(u_k) \leq c$.*

PROOF. Every element of G can be expressed as a normal word in A. Therefore, we need to show that $\omega(u_1, \ldots, u_k) = 1$ for all normal words $u_1, \ldots, u_k$ without the weight restriction. This is done by induction on $w = \operatorname{wt}(u_1) + \cdots + \operatorname{wt}(u_k)$ which can be assumed to be larger than c. The induction hypothesis is that $\omega(v_1, \ldots, v_k) = 1$ for all k-tuples $(v_1, \ldots, v_k)$ of normal words in A with $\operatorname{wt}(v_1) + \cdots + \operatorname{wt}(v_k) < w$.

The concatenation $u_1 \cdots u_k$ of $u_1, \ldots, u_k$, performed without applying free reduction, is a (not necessarily normal) word $b_1 b_2 \cdots b_l$ with $b_i \in A^{\pm 1}$. Let F be the free group on $X = \{x_1, \ldots, x_l\}$ and $t_1, \ldots, t_k$ words in X such that $t_i(b_1, \ldots, b_l) = u_i$. Consider the homomorphism $\varphi : F \to G$ mapping $x_i \mapsto b_i$ and note that $\varphi(\pi_Z(t_i))$, for any non-empty $Z \subseteq X$, is a normal word in A whose weight is less than $\operatorname{wt}(u_i)$.

By Higman's Lemma $\omega(t_1, \ldots, t_k)$ can be written as a product of words of the form $\pi_Z(\omega(t_1, \ldots, t_k))^{\pm 1}$ and of commutators which each involve every x_i. By the induction hypothesis,

$$\varphi(\pi_Z(\omega(t_1, \ldots, t_k))) = \omega(\varphi(\pi_Z(t_1)), \ldots, \varphi(\pi_Z(t_k))) = 1.$$

A commutator that involves every x_i is mapped by φ to a commutator of weight at least $w > c$ in G. This shows that

$$\omega(u_1, \ldots, u_k) = \varphi(\omega(t_1, \ldots, t_k)) = 1. \qquad \square$$

Since there are only finitely many normal words of a given weight, the previous lemma gives a finite set of instances of $\omega(\xi_1, \ldots, \xi_k)$ that need to be checked.

A version of Lemma 2 was used by Havas and Newman [2] to obtain a practical test set for checking the exponent of finite groups of prime power order, see also Sims [8, Section 11.7].

In an infinite nilpotent group, the following observation can be used to reduce the set of instances to be checked further:

LEMMA 3. *In the statement of Lemma 2, it suffices to check the identity* $\omega(\xi_1, \dots, \xi_k)$ *for normal words with non-negative exponents.*

PROOF. It is well known that a finitely generated nilpotent group is residually finite (see Robinson [7, Section 5.4]). Suppose $\omega(u_1, \dots, u_k) = 1$ in G for all normal words $u_1, \dots, u_k$ in A with non-negative exponents.

If $\omega(v_1, \dots, v_k) \neq 1$ for some normal words $v_1, \dots, v_k$ in A, then there is a normal subgroup N of G of finite index such that $\omega(v_1, \dots, v_k) \notin N$. Since G/N is finite, there are normal words $u_1, \dots, u_k$ in A with non-negative exponents such that $v_i N = u_i N$. This gives

$$\omega(v_1, \dots, v_k)N = \omega(u_1, \dots, u_k)N = N$$

because $\omega(u_1, \dots, u_k) = 1$ in G. This contradicts $\omega(v_1, \dots, v_k) \notin N$. □

COROLLARY 4. *Let G be a group of nilpotency class c given by a weighted polycyclic presentation with generating sequence* $A = (a_1, \dots, a_n)$. *Then G is an Engel-n group if and only if* $[u, {}_n v] = 1$ *for all normal words u and v in A with non-negative exponents and* $\mathrm{wt}(u) + \mathrm{wt}(v) \leq c$.

The ANU NQ uses the set of instances described by the corollary to enforce the Engel-n identity in each step of computing nilpotent factor groups of a given finitely presented group.

3. Free Engel groups

In this section we present results on the largest nilpotent quotients of free Engel-n groups for small values of n. More detailed information about the groups presented is available from the author's home page on the World Wide Web. All timings were obtained on an Intel Pentium II-333MHz processor with 128 MB running Linux 2.0.36.

3.1. Engel-4 groups The group $E(2, 4)$ is torsion free, has nilpotency class 6 and Hirsch length 11. The terms of the lower centrals series are

$$C_\infty^2, \quad C_\infty, \quad C_\infty^2, \quad C_\infty^3, \quad C_2 \times C_\infty^2, \quad C_\infty.$$

The computation was completed in about 0.2 seconds. The following is a weighted polycyclic presentation for $E(2, 4)$. Trivial commutator relations between two gener-

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ators are omitted.

$$
\begin{array}{llll}
\langle\, a_1, \ldots, a_{12} \mid & a_{11}^2 = a_{12}^3, & & \\
\ [a_2, a_1] = a_3, & & & \\
\ [a_3, a_1] = a_4, & [a_3, a_2] = a_5, & & \\
\ [a_4, a_1] = a_6, & [a_4, a_2] = a_7 a_9^{-3} a_{10}^3 a_{12}^{-6}, & [a_4, a_3] = a_9^3 a_{11} a_{12}^6, & \\
\ [a_5, a_1] = a_7, & [a_5, a_2] = a_8, & [a_5, a_3] = a_{10}^{-3} a_{11}, & [a_5, a_4] = a_{12}^{-3}, \\
 & [a_6, a_2] = a_9^{-2} a_{10} a_{12}^{-6}, & & \\
\ [a_7, a_1] = a_9, & [a_7, a_2] = a_{10}, & & \\
\ [a_8, a_1] = a_{10}^{-2} a_{11}, & & & \\
 & [a_9, a_2] = a_{12}, & & \\
\ [a_{10}, a_1] = a_{12} \,\rangle & & &
\end{array}
$$

The weights of the generators are

generator	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
weight	1	1	2	3	3	4	4	4	5	5	5	6

The power relation in the presentation above can be removed. For this set $b_{11} = a_{11}a_{12}^{-2}$, which implies $a_{12} = b_{11}^{-2}$ and $a_{11} = b_{11}^{-3}$, and apply the obvious sequence of Tietze transformations.

The following is a defining set of instances of the Engel-4 identity for $E(2,4)$ as a nilpotent group.

$$
\begin{array}{lllll}
[a, {}_4b], & [b, {}_4a], & [a^2, {}_4b], & [b^2, {}_4a] & \\
[a^{-1}, {}_4ab], & [a, {}_4ab^{-1}], & [a, {}_4ab^2], & [b, {}_4ab^2], & [a, {}_4a^2b].
\end{array}
$$

The group $E(3,4)$ has nilpotency class 9 and Hirsch length 88. The torsion subgroup of $E(3,4)$ is isomorphic to

$$C_5^{44} \times C_{10}^5 \times C_{30}^4 \times C_{60}^4.$$

Because the Engel-4 identity has weight 5, the first 4 factors of the lower central series of $E(3,4)$ are isomorphic to the corresponding factors of the free nilpotent group of class 4 and have respective free Abelian ranks 3, 3, 8, and 18. The other terms of the lower central series are:

$$
\begin{array}{lll}
C_2^4 \times C_{10}^5 \times C_{30} \times C_\infty^{24}, & C_5^9 \times C_{10}^9 \times C_\infty^{26}, & \\
C_5^{23} \times C_{10} \times C_{30}^3 \times C_\infty^6, & C_5^3 \times C_{30}^3, & C_3.
\end{array}
$$

There is a defining set of 278 instances of the Engel-4 identity for $E(3,4)$ as a nilpotent group. The computation took about 16 hours CPU time and about 9 MB memory.

Vaughan-Lee [10] has proved that the 2- and 3-generator exponent-5 Engel-4 groups are finite of order 5^{11} and 5^{145}, respectively. This corresponds to the fact that the Hirsch length of $E(2,4)$ is 11 and that the sum of the Hirsch length of $E(3,4)$ and the number of occurrences of the prime 5 in its torsion subgroup is 145.

3.2. Engel-5 groups The group $E(2,5)$ has nilpotency class 9 and Hirsch length 23. The torsion subgroup is isomorphic to

$$C_3^8 \times C_{30}^3 \times C_{180}^2.$$

The (non-free) terms of the lower central series are

$$C_2 \times C_6 \times C_\infty^4, \quad C_6^2 \times C_{18}^2 \times C_\infty^4, \quad C_2 \times C_{30}^3 \times C_\infty, \quad C_3^4 \times C_{15}^2.$$

The computation took 388 seconds of CPU time and used 456 kB of memory. There is a defining set of 32 instances of the Engel-5 identity for $E(2,5)$ as a nilpotent group.

3.3. Engel-6 groups The nilpotency class of $E(2,6)$ is 12 and the Hirsch length is 70. The computation was performed in two parts.

First the class-10 quotient of $E(2,6)$ was computed taking about 21 hours of CPU time and 1.3 MB of memory. This yielded a defining set of 113 instances of the Engel-6 identity for the class-10 quotient. This computation was continued in an attempt to complete the computation of the class-11 quotient. However, checking the Engel identity turned out to be very time consuming. Therefore, this computation was not be expected to finish within a reasonable amount of time. During this computation, 3 further necessary instances of the Engel-6 identity were found. The group defined by this set of 116 instances has a largest nilpotent quotient G of class 12. The computation of this quotient took about 100 hours of CPU time and used about 10 MB of memory.

In a second step, it was checked if G satisfies the Engel-6 identity. From the weighted polycyclic presentation for G the Hall polynomials were computed in GAP 4 (cf. [1]) using Merkwitz' [5] implementation of Deep Thought (Leedham-Green & Soicher [4]). This took about 19 hours with about 110 MB of GAP workspace. Using Hall polynomials, arithmetic in a nilpotent group can be performed much more rapidly than by collection. With the help of these polynomials, it was possible to check the Engel-6 identity in G. It turned out that one further instance was necessary in order to satisfy the Engel-6 identity. The effect of this instance was to force a central generator in G to be trivial. The total time for checking the Engel-6 identity was about 7 hours.

The torsion subgroup of $E(2,6)$ is isomorphic to:

$$C_7^5 \times C_{14}^{15} \times C_{84}^{10} \times C_{168}^3 \times C_{840}^2 \times C_{2520} \times C_{12600}^3 \times C_{321564600}^2.$$

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The prime factorisation of 321564600 is $2^3\ 3^3\ 5^2\ 7\ 47\ 181$. The (non-free) factors of the lower central series of $E(2, 6)$ are

$$C_2 \times C_6^2 \times C_\infty^{12},$$
$$C_2^3 \times C_{42}^2 \times C_{84}^3 \times C_\infty^{13},$$
$$C_2^9 \times C_{42}^6 \times C_{126}^2 \times C_\infty^{14},$$
$$C_2^{13} \times C_{14}^3 \times C_{42}^8 \times C_{1050} \times C_{6300}^2 \times C_\infty^8,$$
$$C_2^7 \times C_{14}^{10} \times C_{210}^2 \times C_{53594100}^2,$$
$$C_2^2 \times C_{10},$$

where $53\,594\,100 = 2^2\ 3^2\ 5^2\ 7\ 47\ 181$.

It is surprising to see that the 11th factor of the lower central series of $E(2, 6)$ involves the, in this context rather large, primes 47 and 181 and it would be interesting to obtain an explanation why these primes play a role here.

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Fachbereich Mathematik AG 2
TU Darmstadt
Schloßgartenstraße 7
D-64289 Darmstadt
Germany
URL: www.mathematik.tu-darmstadt.de/~nickel
e-mail: nickel@mathematik.tu-darmstadt.de