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ON THE EVOLUTION OF TOPOLOGY IN DYNAMIC CLIQUE COMPLEXES

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Abstract

We consider a time varying analogue of the Erdős–Rényi graph and study the topological variations of its associated clique complex. The dynamics of the graph are stationary and are determined by the edges, which evolve independently as continuous-time Markov chains. Our main result is that when the edge inclusion probability is of the form $p = n^{\alpha}$, where n is the number of vertices and $\alpha \in (-1/k, -1/(k+1))$, then the process of the normalised kth Betti number of these dynamic clique complexes converges weakly to the Ornstein–Uhlenbeck process as $n \to \infty$.

Keywords: Dynamic Erdős–Rényi graph; Betti numbers; Ornstein–Uhlenbeck

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1. Introduction

The classic Erdős–Rényi graph $G(n, p)$ is well known as the random graph on n vertices where each edge appears with probability p, independently of the others. It is ubiquitous in applied literatures dealing with network models and, despite its apparent simplicity, has been of theoretical interest ever since Erdős and Rényi, over half a century ago in [9], established a sharp threshold for its connectivity. They showed that, for fixed $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}\{G(n, p) \text{ is connected}\} \to \begin{cases} 1 & \text{if } p \geq (1+\varepsilon)\log(n)/n, \\ 0 & \text{if } p \leq (1-\varepsilon)\log(n)/n. \end{cases}$$

Allowing for the interpretation that connectedness is a (almost trivial) topological property, their result can be considered as the first result describing a topological phase transition in a random graph. Since 1959, a substantial literature has grown around the properties of the Erdős–Rényi graph, providing much finer detail than the original result. A more recent literature, some of which we shall describe briefly below, has considered more detailed topological information about objects generated by $G(n, p)$.

In this paper we take all of this a step further, applying these richer probabilistic results in the topological setting, to temporally evolving Erdős–Rényi graphs. We need a few definitions, or at least descriptions, in order to define what we mean by this.

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1.1. Background

1.1.1. *Dynamic Erdős–Rényi graphs.* The dynamic Erdős–Rényi graph depends on three parameters: the number of nodes, $n \in \mathbb{N}$, the connectivity probability $p \in [0, 1]$, and a rate, $\lambda > 0$. Denoted by $\{G(n, p, t) \colon t \geq 0\}$, it is a time-varying subgraph of the complete graph on n vertices with the following properties.

(i) The initial value $G(n, p, 0)$ is distributed as the (static) Erdős–Rényi graph $G(n, p)$.

(ii) For $t \geq 0$, each edge independently evolves as a continuous-time on/off Markov chain. The waiting time in the states 'off' and 'on' are exponential with parameters λp and $\lambda(1 - p)$, respectively.

If $e(t)$ denotes the state of one of these edges at time t, then it follows immediately from the above description that, for any t_1, t_2,

$$\mathbb{P}\{e(t_2) = \text{on} \mid e(t_1) = \text{on}\} = p + (1 - p)\mathrm{e}^{-\lambda|t_2 - t_1|}, \tag{1.1}$$

and

$$\mathbb{P}\{e(t_2) = \text{off} \mid e(t_1) = \text{off}\} = (1 - p) + p\mathrm{e}^{-\lambda|t_2 - t_1|}. \tag{1.2}$$

From this it follows that, for any $t \geq 0$,

$$\mathbb{P}\{e(t) = \text{on}\} = p. \tag{1.3}$$

Consequently, $\{G(n, p, t) \colon t \geq 0\}$ is a stationary reversible Markov process and, for each $t \geq 0$, it is a realisation of the (static) Erdős–Rényi graph, $G(n, p)$.

The dynamic Erdős–Rényi graph described here is an example of a continuous-time 'Edge Markovian Evolving Graph' (EMEG), a class of dynamic models that has often been used to model real world dynamic networks. In particular, if one thinks of the static Erdős–Rényi graph as a simple, but generic model for 'faulty connections' between nodes, then the dynamic version is clearly relevant to 'Intermittently Connected Mobile Networks' (ICMNs) [21], [22]. The ICMNs have given rise to many interesting new questions, such as temporal connectivity [3], [6] and dynamic community detection [7], all related, in one way or another, to issues of connectivity. For us, however, the importance of the dynamic Erdős–Rényi graph lies in its relative analytic accessibility for also tackling more sophisticated topological issues. Further, in the same way that results proven for the static case have turned out to be of a 'universal' nature regarding connectivity, in that they hold for far more complicated graphs and networks, we believe that the topological results of the paper have similar extensions.

1.1.2. *Clique complexes.* The study of the topology of Erdős–Rényi graphs typically revolves around the clique complexes that they generate, which we now define.

We first introduce the notion of an *abstract simplicial complex* which is a purely combinatorial notion. A family $\mathcal{K}$ of non-empty finite subsets of V is an abstract simplicial complex if it is closed under the operation of taking non-empty subsets, i.e., $\mathcal{Y} \subset \mathcal{X} \in \mathcal{K} \implies \mathcal{Y} \in \mathcal{K}$. Elements of $\mathcal{K}$ are called faces or simplices, and the dimension of a face $\mathcal{X}$ is its cardinality $|\mathcal{X}|$ minus 1. Elements of dimension 0 are called vertices. The dimension of $\mathcal{K}$, denoted $\dim(\mathcal{K})$, is the supremum over dimensions of all its faces.

Abstract simplicial complexes also have concrete, geometric realisations in Euclidean space. In particular, if $\mathcal{K}$ is finite, which is the only situation of interest to us, then this is simple. Firstly, embed the vertices of $\mathcal{K}$ as an affinely independent subset in $\mathbb{R}^N$, for sufficiently large N. For example, take N to be the number of vertices, number the vertices $v_1, \ldots, v_N$, write $e_j \in \mathbb{R}^N$

for the vector with a 1 in the jth position and all other entries 0, and map $v_j \to e_j$. Then any face $\mathcal{X} \in \mathcal{K}$ can be identified with the geometric simplex in $\mathbb{R}^N$ spanned by the corresponding embedded vertices. The geometric realisation is then the union of all such simplices.

Consider a (undirected) graph G. Then a *clique* in G is just a subset of vertices in G such that each pair of vertices is joined by an edge. The *clique complex*, $\mathfrak{X}(G)$, is the collection of all subsets of vertices that form a clique in G. Since a subset of a clique is itself a clique, $\mathfrak{X}(G)$ is indeed an abstract simplicial complex. In the corresponding geometric realisation, each clique of k vertices is represented by a simplex of dimension $k-1$. The 1-skeleton of $\mathfrak{X}(G)$ (which is the underlying graph of the complex) is a graph with a vertex for every 1-element set in $\mathfrak{X}(G)$ and an edge for every 2-element set in $\mathfrak{X}(G)$, and so is isomorphic to G itself.

Henceforth, we will study the temporal evolution of the topology of the clique complexes generated from the dynamic Erdős–Rényi graph; namely, the sets

$$\mathfrak{X}(n, p; t) := \mathfrak{X}(G(n, p; t)).$$

In order to do this, we shall study the Betti numbers of these sets.

1.1.3. *Betti numbers.* Throughout this paper we work with reduced Betti numbers and for notational convenience we shall drop the word reduced henceforth. There is really no good way to define Betti numbers in a few, self-contained, paragraphs. Formally, for an integer $k \geq 0$, the kth Betti number $\beta_k \equiv \beta_k(X)$ of a topological space X is the rank of the abelian group $H_k(X, \mathbb{A})$, the reduced kth homology group of X with coefficients from the abelian group $\mathbb{A}$. The reduced homology groups themselves are the quotient groups $H_k = \ker \delta_k / \mathrm{Im}\delta_{k+1}$, where the δ_k are the boundary maps for X. In this paper we assume that $\mathbb{A} = \mathbb{Q}$, the field of rationals, consistent with [14], [16], and [18].

The problem is that, as succinct as this description may be, it is of little help to a reader who has not already worked through one of the standard texts on algebraic geometry such as [12], or perhaps the less standard [8], which is motivated by computational issues and somewhat closer to the specific focus of the current paper.

Thus, we shall not attempt to define Betti numbers rigorously, but shall start with three examples and then allow some imprecision. For the following discussion, it is useful to assume that the topological space X is a subset of some finite-dimensional Euclidean space $\mathbb{R}^N$. As for the examples, $\beta_0(X)$ is equal to one less than the number of connected components in X. Then $\beta_1(X)$ counts the number of one-dimensional, or 'topologically circular' holes—think of holes in a two- or three-dimensional object that you could poke a finger through. If X is three-dimensional then $\beta_2(X)$ counts the number of 'voids' within X—think of the interior of a tennis ball, or of a bagel that has an air pocket running around the entire ring. Higher-order Betti numbers are rather harder to describe this way, since everyday language lacks the vocabulary needed to describe high-dimensional objects. Roughly speaking, however, $\beta_k(X)$ counts the number of distinct regions in X which are 'topologically equivalent to' the boundary of a solid, k-dimensional set, something which we refer to as a '$(k-1)$-cycle' below. As such, increasing k increases the qualitative level of topological complexity one is studying, while increasing β_k for a fixed k is an indication of quantitatively more complexity at the given level. This is true only up to a point, since for all $k \geq N$, $\beta_k(X) \equiv 0$. Fortunately, at least in order to understand the thrust of the main results of this paper, these necessarily imprecise descriptions of Betti numbers should suffice.

The results of this paper concentrate on the $n \to \infty$ asymptotic behaviour of stochastic processes describing the normalised Betti numbers of the clique complexes associated with the dynamic Erdős–Rényi graphs; namely,

$$\bar{\beta}_{n,k}(t) := \frac{\beta_{n,k}(t) - \mathbb{E}[\beta_{n,k}(t)]}{\sqrt{\operatorname{var}[\beta_{n,k}(t)]}},$$

where

$$\beta_{n,k}(t) := \beta_k(\mathfrak{X}(n, p; t)) = \beta_k(\mathfrak{X}(G(n, p; t))).$$

1.2. Results

1.2.1. *Erdős–Rényi graphs and associated topology.* The topological study of static random graphs and their associated simplicial complexes, beyond classical issues of connectivity and degree, has seen considerable recent activity, including [1], [13], [15], [16], [18], [19], and [20]. A recent, well-motivated review is [17]. Most of this literature follows the theme that Betti numbers of increasing index are good quantifiers of topological complexity, and so are the appropriate measure to study.

In terms of the (static) Erdős–Rényi graph, heuristics imply that for small p the associated clique complex will, with high probability, be topologically simple, but that the complexity will grow with increasing p. Thinking a little more deeply, as p grows the clique complex changes from a collection of disconnected vertices (so that β_0 is large) to a highly connected object (so that, at full connectivity, β_0 drops to its minimum value of 0). At about the same stage, simple, one-dimensional cycles start forming (so that β_1 grows) until these cycles fill in and then to produce empty tetrahedra-type objects (so that β_1 drops while β_2 grows). The following result, which combines the result from [16, Theorem 1.1] and the discussion below [18, Erratum, Equation (1)] confirms this description.

Theorem 1.1. (See [16] and [18].) *Fix $k \geq 1, M > 0$, and $t \geq 0$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, as $n \to \infty$,*

$$\mathbb{P}\{\beta_{n,k}(t) \neq 0,\ \beta_{n,j}(t) = 0,\ \text{for all } j \neq k\} = 1 - o(n^{-M}).$$

Since $G(n, p, t)$ is distributed as an Erdős–Rényi graph, the above result is a simple rephrasing of the original result given in [16] and [18]. This result shows that there is a sequence of clearly marked phase transitions, and between each of these there is a dominant Betti number, and so a dominant type of homology in the clique complex. Of more interest to us, however, is the following central limit theorem that is a consequence of [18, Theorem 2.4 and Erratum, Theorem 1.1].

Theorem 1.2. (See [18].) *Fix $k \geq 1$, $t \geq 0$, and let p be as in Theorem 1.1. Then, as $n \to \infty$,*

$$\frac{\beta_{n,k}(t) - \mathbb{E}[\beta_{n,k}(t)]}{\sqrt{\operatorname{var}[\beta_{n,k}(t)]}} \xrightarrow{\text{D}} \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ denotes a standard Gaussian and '$\xrightarrow{\text{D}}$' denotes convergence in distribution.

1.2.2. *Dynamic Erdős–Rényi graphs and associated topology.* The main result of this paper is the following extension of Theorem 1.2.

Theorem 1.3. *Fix $k \geq 1$, $\lambda > 0$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, as $n \to \infty$,*

$$\{\bar{\beta}_{n,k}(t) \colon t \geq 0\} \xrightarrow{\text{D}} \{\mathcal{U}_{\lambda} \colon t \geq 0\}$$

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where $\{\mathcal{U}_\lambda(t)\colon t \geq 0\}$ is the stationary, zero-mean, Ornstein–Uhlenbeck process with covariance $\operatorname{cov}[\mathcal{U}_\lambda(t_1), \mathcal{U}_\lambda(t_2)] = \mathrm{e}^{-\lambda|t_1-t_2|}$, and here '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution on the Skorokhod space of functions on $[0, \infty)$.

Although, in view of Theorem 1.2, it is not surprising that the limits of the random processes $\bar{\beta}_{n,k}$ are Gaussian, it is somewhat surprising that, as Ornstein–Uhlenbeck processes, they are Markovian. While the underlying dynamic Erdős–Rényi process is Markovian, this is not the case for the processes $\bar{\beta}_{n,k}$, as shown in Appendix A.

1.2.3. *On proving Theorem 1.3.* Since working directly with Betti numbers is difficult, we adopt the approach of [14] and [18]. Let $f_{n,k}(t)$ denote the number of $(k+1)$-cliques in $G(n, p, t)$ and let

$$\chi_n(t) := \sum_{j=0}^{n-1}(-1)^j f_{n,j}(t) = 1 + \sum_{j=0}^{n-1}(-1)^j \beta_{n,j}(t) \tag{1.4}$$

be the Euler–Poincaré characteristic of $\mathfrak{X}(n, p, t)$; see [8, p. 101]. Define

$$\bar{f}_{n,k}(t) := \frac{f_{n,k}(t) - \mathbb{E}[f_{n,k}(t)]}{\sqrt{\operatorname{var}[f_{n,k}(t)]}} \quad \text{and} \quad \bar{\chi}_n(t) := \frac{\chi_n(t) - \mathbb{E}[\chi_n(t)]}{\sqrt{\operatorname{var}[\chi_n(t)]}}. \tag{1.5}$$

We first establish weak convergence for $\{\bar{f}_{n,k}(t)\colon t \geq 0\}$. Using the first equality in (1.4), we then establish weak convergence for $\{\bar{\chi}_n(t)\colon t \geq 0\}$. Finally, Theorem 1.3 is proven using the second equality in (1.4) and Theorem 1.1.

To carry this out, in Section 2 we quote some results on the convergence of random variables and processes. In Section 3 we discuss some preliminary results concerning the mean and variance of $f_{n,k}(t)$, $\chi_n(t)$, and $\beta_{n,k}(t)$. The covariance functions of the processes $\bar{f}_{n,k}$, $\bar{\chi}_n$, and $\bar{\beta}_{n,k}$ are derived in Section 4 and exploited in Section 5 to establish convergence of the finite-dimensional distributions of the $\bar{\beta}_{n,k}$. In Section 6 we establish tightness for the processes $\bar{\beta}_{n,k}$, and complete the proof of Theorem 1.3.

2. On convergence in distribution

To help the reader and make this paper a little more self-contained, we now quote two theorems about weak convergence. The first, from [2], is a central limit theorem for dissociated random variables (defined formally in the statement of Theorem 2.1). The second, which comes from combining [10, Theorems 7.8, 8.6, and 8.8], is about convergence, in the Skorokhod space, to the stationary Ornstein–Uhlenbeck process.

Before stating the theorems, we remind the reader of the definition of the L_1-Wasserstein metric for real-valued random variables. For two real-valued random variables Y_1 and Y_2, their L_1-Wasserstein distance is

$$d_1(Y_1, Y_2) = \sup_{\psi} |\mathbb{E}[\psi(Y_1)] - \mathbb{E}[\psi(Y_2)]|,$$

where the sup is over all functions $\psi\colon \mathbb{R} \to \mathbb{R}$ with $\sup_{y_1 \neq y_2}|\psi(y_1) - \psi(y_2)|)/|y_1 - y_2| \leq 1$. Recall also that convergence in this metric implies convergence in distribution.

Theorem 2.1. (See [2].) *Let $\{Y_i\colon i \equiv (i_1, \dots, i_r) \in I\}$, for some index set I of r-tuples, be a sequence of dissociated random variables. That is, for any $J, L \subseteq I$, $\{Y_i\colon i \in J\}$ and $\{Y_i\colon i \in L\}$ are independent whenever $(\bigcup_{i\in J}\{i_1, \dots, i_r\}) \cap (\bigcup_{i\in L}\{i_1, \dots, i_r\}) = \varnothing$. Let $\mathfrak{W} = \sum_{i\in I} Y_i$ and, for each $i \in I$, let $\mathcal{Y}(i) := \{k \in I\colon \{k_1, \dots, k_r\} \cap \{i_1, \dots, i_r\} \neq \varnothing\}$ be the*

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dependency neighbourhood of i*. If* $\mathbb{E}[Y_i] \equiv 0$ *and* $\mathrm{var}[\mathfrak{W}] = 1$*, then there exists a universal constant* $\rho > 0$ *such that*

$$d_1(\mathfrak{W}, \mathcal{N}(0,1)) \le \rho \sum_{i \in I} \sum_{j,\ell \in \mathcal{Y}(i)} \mathbb{E}[|Y_i Y_j Y_\ell|] + \mathbb{E}[|Y_i Y_j|]\mathbb{E}[|Y_\ell|]. \tag{2.1}$$

Equation (2.1) is obtained by combining [2, Theorem 1 and Equation (2.7)] (see also the discussion above [2, Equation (2.7)]). Let $D_{\mathbb{R}}[0, \infty)$ denote the (Skorokhod) space of right-continuous functions on $[0, \infty)$ with left limits, and write $\hat{d}$ for the usual (Skorokhod) metric on this space.

Theorem 2.2. (See [10].) *Let* $\{X_n(t) \colon t \ge 0\}$*,* $n \ge 1$*, be a sequence of* $(D_{\mathbb{R}}[0, \infty), \hat{d})$*-valued stochastic processes satisfying the following conditions.*

Convergence of finite-dimensional distributions. For any $t_1, \ldots, t_m \ge 0$*,*

$$(X_n(t_1), \ldots, X_n(t_m)) \xrightarrow{\mathrm{D}} (\mathcal{U}_\lambda(t_1), \ldots, \mathcal{U}_\lambda(t_m)) \quad \text{as } n \to \infty.$$

Tightness. The sequence $\{\{X_n(t) \colon t \ge 0\} \colon n \ge 1\}$ *is tight, for which it is sufficient that the following two conditions hold.*

(C1) *There exists* $\Upsilon > 0$ *such that*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E}|X_n(\delta) - X_n(0)|^{\Upsilon} = 0.$$

(C2) *For each* $T > 0$*, there exist constants* $\Upsilon_1 > 0$*,* $\Upsilon_2 > 1$*, and* $K > 0$ *such that, for all* n*,* $0 \le t \le T + 1$*, and* $0 \le h \le t$*,*

$$\mathbb{E}[|X_n(t+h) - X_n(t)|^{\Upsilon_1} |X_n(t) - X_n(t-h)|^{\Upsilon_1}] \le K h^{\Upsilon_2}.$$

Then $\{X_n(t) \colon t \ge 0\} \xrightarrow{\mathrm{D}} \{\mathcal{U}_{\lambda(t)} \colon t \ge 0\}$ *as* $n \to \infty$*.*

3. Preliminary results

We study here the asymptotic variances of $f_{n,k}(t)$, $\chi_n(t)$, and $\beta_{n,k}(t)$. Due to stationarity, these variances are independent of t. We start with some notation.

We write $[n] := \{1, \ldots, n\}$ for the vertex set of the dynamic Erdős–Rényi graph. This is not dependent on t. We write $\binom{[n]}{j+1}$ to denote the collection of all subsets of $[n]$ of size $j + 1$, while $\binom{n}{j+1}$ is the usual binomial coefficient. For $A \in \binom{[n]}{j+1}$, let $\mathbf{1}_A(t)$ be the indicator function for A being a $(j+1)$-clique in $G(n, p, t)$. We can now write

$$f_{n,j}(t) = \sum_{A \in \binom{[n]}{j+1}} \mathbf{1}_A(t), \tag{3.1}$$

from which it immediately follows that

$$\mathbb{E}[f_{n,j}(t)] = \binom{n}{j+1} p^{\binom{j+1}{2}} \tag{3.2}$$

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and

$$\begin{aligned}\mathbb{E}[f_{n,j}^2(t)] &= \sum_{A_1 \in \binom{[n]}{j+1}} \sum_{A_2 \in \binom{[n]}{j+1}} \mathbb{E}[\mathbf{1}_{A_1}(t)\,\mathbf{1}_{A_2}(t)] \\ &= \binom{n}{j+1} \sum_{A_2 \in \binom{[n]}{j+1}} \mathbb{E}[\mathbf{1}_{A_1}(t)\,\mathbf{1}_{A_2}(t)] \\ &= \binom{n}{j+1} \sum_{i=0}^{j+1} \binom{j+1}{i} \binom{n-j-1}{j+1-i} \frac{p^{2\binom{j+1}{2}}}{p^{\binom{i}{2}}},\end{aligned}$$

where in the second equality A_1 is an arbitrary but fixed $(j+1)$-face. The second equality follows because the inner sum on the right-hand side is the same for each A_1, and the third equality follows by combining all faces A_2 that share i vertices with A_1. Hence,

$$\operatorname{var}[f_{n,j}(t)] = \binom{n}{j+1} \sum_{i=0}^{j+1} \binom{j+1}{i} \binom{n-j-1}{j+1-i} \frac{p^{2\binom{j+1}{2}}}{p^{\binom{i}{2}}} - \binom{n}{j+1}^2 p^{2\binom{j+1}{2}}. \tag{3.3}$$

In the next result we show the behaviour of $\operatorname{var}[f_{n,j}(t)]$ as $n \to \infty$ for different j.

Lemma 3.1. *Fix $k \geq 1$, $j \geq 0$, and $t \geq 0$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1(k+1))$. Then*

(i) $\operatorname{var}[f_{n,0}(t)] \equiv 0$;

(ii) *if $j = 2k-1$ and $\alpha \in [-1/(k+0.5), -1/(k+1))$, or if $1 \leq j \leq 2k-2$, then*

$$\operatorname{var}[f_{n,j}(t)] \leq 2^{j+1} n^{2j} p^{2\binom{j+1}{2}-1};$$

(iii) *if $j = 2k-1$ and $\alpha \in (-1/k, -1/(k+0.5)]$, or if $j \geq 2k$, then*

$$\operatorname{var}[f_{n,j}(t)] \leq 2^{j+1} n^{j+1} p^{\binom{j+1}{2}}.$$

Proof. The first claim is trivial since $f_{n,0}(t) \equiv n$. So we prove only the other two. Since $\binom{n}{j+1} = \sum_{i=0}^{j+1} \binom{j+1}{i}\binom{n-j-1}{j+1-i}$, from (3.3), it follows that

$$\operatorname{var}[f_{n,j}(t)] = \sum_{i=2}^{j+1} \binom{j+1}{i} \binom{n}{j+1} \binom{n-j-1}{j+1-i} \big[p^{2\binom{j+1}{2}-\binom{i}{2}} - p^{2\binom{j+1}{2}}\big]. \tag{3.4}$$

The summation starts from 2 because the term in the square brackets above is 0 for $i = 0, 1$. Note that $\binom{n}{j+1}\binom{n-j-1}{j+1-i} \leq n^{2j+2-i}$. Further, $p = n^{\alpha}$ with $\alpha < 0$. Hence, the term inside the square bracket is positive for each i, and bounded from above by $p^{2\binom{j+1}{2}-\binom{i}{2}}$. Hence, to prove the desired result, it suffices to obtain bounds for $\sum_{i=2}^{j+1} \binom{j+1}{i} n^{\zeta_j(i)}$, where $\zeta_j(i) = 2j+2-i+\alpha[2\binom{j+1}{2} - \binom{i}{2}]$.

As $\alpha < 0$, ζ_j is a convex function. Hence, either $\zeta_j(2)$ or $\zeta_j(j+1)$ maximises $\zeta_j(i)$ for $i \in \{2, \ldots, j+1\}$. When the conditions of (ii) hold, $\zeta_j(2) \geq \zeta_j(j+1)$. Similarly, when the conditions of (iii) hold, $\zeta_j(j+1) \geq \zeta_j(2)$. At $\alpha = 1/(k+0.5)$ and $j = 2k-1$, $\zeta_j(2) = \zeta_j(j+1)$. Since $\sum_{i=2}^{j+1} \binom{j+1}{i} \leq 2^{j+1}$, the desired result is now easy to see. □

Let p be as in Lemma 3.1. In the next result we compute the exact order of $\operatorname{var}[f_{n,k}(t)]$.

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Lemma 3.2. *Fix $k \geq 1$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, for each $t \geq 0$,*

$$\operatorname{var}[f_{n,k}(t)] = \Theta\big(n^{2k} p^{2\binom{k+1}{2}-1}\big).$$

Proof. From (3.4), recall that

$$\operatorname{var}[f_{n,k}(t)] = \sum_{i=2}^{k+1} \binom{k+1}{i}\binom{n}{k+1}\binom{n-k-1}{k+1-i}\big[p^{2\binom{k+1}{2}-\binom{i}{2}} - p^{2\binom{k+1}{2}}\big].$$

Observe that $2\binom{k+1}{2} - \binom{i}{2} < 2\binom{k+1}{2}$ for each $i \in \{2, \dots, k+1\}$. Hence to prove the desired result, it suffices to show that

$$\sum_{i=2}^{k+1} \binom{k+1}{i}\binom{n}{k+1}\binom{n-k-1}{k+1-i}\big[p^{2\binom{k+1}{2}-\binom{i}{2}}\big] = \Theta\big(n^{2k} p^{2\binom{k+1}{2}-1}\big).$$

Since $\binom{n}{k+1}\binom{n-k-1}{k+1-i} = \Theta(n^{2k+2-i})$, arguing as in the proof of Lemma 3.1, the above claim is easy to see, and the result follows. □

The following result is now immediate from Lemmas 3.1 and 3.2.

Corollary 3.1. *Fix $k \geq 1$, $j \geq 0$, and $t \geq 0$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then,*

$$\lim_{n\to\infty} \frac{\operatorname{var}[f_{n,j}(t)]}{\operatorname{var}[f_{n,k}(t)]} = 0 \quad \textit{whenever } j \neq k.$$

We next compare $\operatorname{var}[(-1)^k \chi_n(t) - f_{n,k}(t)]$ and $\operatorname{var}[\chi_n(t)]$ with $\operatorname{var}[f_{n,k}(t)]$ as $n \to \infty$.

Lemma 3.3. *Fix $k \geq 1$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, for each $t \geq 0$,*

$$\lim_{n\to\infty} \frac{\operatorname{var}[(-1)^k \chi_n(t) - f_{n,k}(t)]}{\operatorname{var}[f_{n,k}(t)]} = 0.$$

Proof. From (1.4), we have

$$\begin{aligned}
&\operatorname{var}[(-1)^k \chi_n(t) - f_{n,k}(t)] \\
&\quad \leq \sum_{0 \leq j \leq n-1,\, j \neq k} \operatorname{var}[f_{n,j}(t)] + 2 \sum_{0 \leq i < j \leq n-1;\, i,j \neq k} |\operatorname{cov}[f_{n,i}(t), f_{n,j}(t)]| \\
&\quad = 2 \sum_{0 \leq i \leq j \leq (n-1);\, i,j \neq k} \sqrt{\operatorname{var}[f_{n,i}(t)] \operatorname{var}[f_{n,j}(t)]}, \\
&\quad \leq 2 \sum_{0 \leq i \leq j \leq 4k+4;\, i,j \neq k} \sqrt{\operatorname{var}[f_{n,i}(t)] \operatorname{var}[f_{n,j}(t)]} \\
&\qquad + 2 \sum_{0 \leq i \leq (n-1),\, i \neq k} \sum_{4k+5 \leq j \leq (n-1)} \sqrt{\operatorname{var}[f_{n,i}(t)] \operatorname{var}[f_{n,j}(t)]}.
\end{aligned} \tag{3.5}$$

Let n be sufficiently large. From Lemma 3.1, note that, for $j \geq 2k$,

$$\operatorname{var}[f_{n,j}(t)] \leq 2^{j+1} n^{j+1} p^{\binom{j+1}{2}} \leq 2n^{2j+1+\alpha\binom{j+1}{2}}.$$

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But, for all $j \geq 2k+1$, $2j+1+\alpha\binom{j+1}{2}$ monotonically decreases with j. Hence,

$$\operatorname{var}[f_{n,j}(t)] \leq 2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}} \quad \text{for all } j \geq 4k+5.$$

This implies that

$$\begin{aligned}
&\sum_{0\leq i\leq (n-1),\, i\neq k} \; \sum_{4k+5\leq j\leq (n-1)} \sqrt{\operatorname{var}[f_{n,i}(t)]}\sqrt{\operatorname{var}[f_{n,j}(t)]} \\
&\quad \leq n\sqrt{2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}}} \sum_{0\leq i\leq (n-1),\, i\neq k} \sqrt{\operatorname{var}[f_{n,i}(t)]} \\
&\quad \leq n\sqrt{2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}}} \sum_{0\leq i\leq 4k+4,\, i\neq k} \sqrt{\operatorname{var}[f_{n,i}(t)]} \\
&\qquad + n\sqrt{2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}}} \sum_{4k+5\leq i\leq (n-1)} \sqrt{\operatorname{var}[f_{n,i}(t)]} \\
&\quad \leq n\sqrt{2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}}} \sum_{0\leq i\leq 4k+4,\, i\neq k} \sqrt{\operatorname{var}[f_{n,i}(t)]} \\
&\qquad + n^2\Big[2n^{2(4k+5)+1+\alpha\binom{(4k+5)+1}{2}}\Big]. \qquad (3.6)
\end{aligned}$$

Observe that

$$\lim_{n\to\infty} n^2\left[\frac{n^{2(4k+5)+1}p^{\binom{(4k+5)+1}{2}}}{n^{2k}p^{2\binom{k+1}{2}-1}}\right] = 0.$$

Combining this, (3.6), Lemma 3.2, and Corollary 3.1, it follows that

$$\lim_{n\to\infty} \frac{\sum_{0\leq i\leq (n-1),\, i\neq k} \sum_{4k+5\leq j\leq (n-1)} \sqrt{\operatorname{var}[f_{n,i}(t)]}\sqrt{\operatorname{var}[f_{n,j}(t)]}}{\operatorname{var}[f_{n,k}(t)]} = 0.$$

Similarly, from Corollary 3.1, we have

$$\lim_{n\to\infty} \frac{\sum_{0\leq i\leq j\leq 4k+4;\, i,j\neq k} \sqrt{\operatorname{var}[f_{n,i}(t)]}\sqrt{\operatorname{var}[f_{n,j}(t)]}}{\operatorname{var}[f_{n,k}(t)]} = 0.$$

Combining the above two relations with (3.5), the desired result is easy to see. □

Lemma 3.4. *Fix* $k \geq 1$. *Let* $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. *Then, for each* $t \geq 0$,

$$\lim_{n\to\infty} \frac{\operatorname{var}[\chi_n(t)]}{\operatorname{var}[f_{n,k}(t)]} = 1.$$

Proof. By adding and subtracting $f_{n,k}(t)$, we have

$$\begin{aligned}
&\operatorname{var}[\chi_n(t)] \\
&\quad = \operatorname{var}[f_{n,k}(t)] + \operatorname{var}[(-1)^k\chi_n(t) - f_{n,k}(t)] + 2\operatorname{cov}[(-1)^k\chi_n(t) - f_{n,k}(t),\, f_{n,k}(t)].
\end{aligned}$$

Hence, it follows that

$$\left|\frac{\operatorname{var}[\chi_n(t)]}{\operatorname{var}[f_{n,k}(t)]} - 1\right| \leq \frac{\operatorname{var}[(-1)^k\chi_n(t) - f_{n,k}(t)]}{\operatorname{var}[f_{n,k}(t)]} + 2\sqrt{\frac{\operatorname{var}[(-1)^k\chi_n(t) - f_{n,k}(t)]}{\operatorname{var}[f_{n,k}(t)]}}.$$

The desired result now follows from Lemma 3.3. □

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In a similar spirit to the above two results, Lemmas 3.5 and 3.6 given below compare the limiting behaviour of $\mathrm{var}[\chi_n(t)]$ with $\mathrm{var}[\beta_{n,k}(t)]$. These results are be due to Kahle and Meckes in [18]. (The results there were established for Erdős–Rényi graphs and, hence, are applicable in our setup to $G(n, p, t)$ for any fixed t. Their notations β_k and $\tilde{\beta}_k$ correspond to $\beta_{n,k}(t)$ and $(-1)^k\chi_n(t)$ in our context.)

Lemma 3.5. (See [18].) *Fix $k \geq 1$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, for each $t \geq 0$,*

$$\lim_{n\to\infty} \frac{\mathrm{var}[\beta_{n,k}(t) - (-1)^k\chi_n(t)]}{\mathrm{var}[\chi_n(t)]} = 0.$$

Lemma 3.6. (See [18].) *Fix $k \geq 1$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, for each $t \geq 0$,*

$$\lim_{n\to\infty} \frac{\mathrm{var}[\beta_{n,k}(t)]}{\mathrm{var}[\chi_n(t)]} = 1.$$

4. Covariance

In this section we investigate the covariance functions of the processes $\bar{f}_{n,k}, \bar{\chi}_n$, and $\bar{\beta}_{n,k}$ as $n \to \infty$. We shall need these in Section 5 to show that finite-dimensional distributions of $\bar{\beta}_{n,k}$ converge to those of the stationary Ornstein–Uhlenbeck process.

Lemma 4.1. *Fix $k \geq 1$. Let $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. Then, for any $t_1, t_2 \geq 0$,*

$$\lim_{n\to\infty} \mathrm{cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = \mathrm{e}^{-\lambda|t_1-t_2|}.$$

Proof. Fix arbitrary $t_1, t_2 \geq 0$, and define $L = \mathrm{e}^{-\lambda|t_1-t_2|}$. Using (3.1), note that

$$\begin{aligned}\mathbb{E}[f_{n,k}(t_1) f_{n,k}(t_2)] &= \sum_{A_1 \in \binom{[n]}{k+1}} \sum_{A_2 \in \binom{[n]}{k+1}} \mathbb{E}[\mathbf{1}_{A_1}(t_1)\,\mathbf{1}_{A_2}(t_2)] \\ &= \binom{n}{k+1} \sum_{A_2 \in \binom{[n]}{k+1}} \mathbb{E}[\mathbf{1}_{A_1}(t_1)\,\mathbf{1}_{A_2}(t_2)],\end{aligned}$$

where in the second equality A_1 is an arbitrary, but fixed, k-face. Writing the above in terms of the number of vertices common to A_1 and A_2, applying (1.1), and (1.3), we obtain

$$\mathbb{E}[f_{n,k}(t_1) f_{n,k}(t_2)] = \binom{n}{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i}\binom{n-k-1}{k+1-i} \frac{p^{2\binom{k+1}{2}}}{p^{\binom{i}{2}}} [p + (1-p)L]^{\binom{i}{2}}.$$

Combining this with (3.2) and (3.3), it is easy to see that

$$\mathrm{cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = \frac{\sum_{i=0}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}[1 + ((1-p)/p)L]^{\binom{i}{2}} - \binom{n}{k+1}}{\sum_{i=0}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i} p^{-\binom{i}{2}} - \binom{n}{k+1}}.$$

Now using the fact that $\binom{n}{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}$, we have

$$\mathrm{cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = \frac{\sum_{i=2}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\big[(1 + ((1-p)/p)L)^{\binom{i}{2}} - 1\big]}{\sum_{i=2}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\big[(1 - p^{\binom{i}{2}})/p^{\binom{i}{2}}\big]}.$$

By expanding terms inside the square brackets and cancelling out $(1-p)/p$, we have

$$\operatorname{cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] = L\frac{\sum_{i=2}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\big[\sum_{j=1}^{\binom{i}{2}} c_{ij}(((1-p)/p)L)^{j-1}\big]}{\sum_{i=2}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\big[\sum_{j=1}^{\binom{i}{2}}(1/p)^{j-1}\big]},$$

where $c_{ij} = \binom{\binom{i}{2}}{j}$. Now observe that the term corresponding to $i = 2$ inside the summation in both the numerator as well as denominator is the same. Hence,

$$\begin{aligned}\operatorname{cov}[\bar{f}_{n,k}(t_1), \bar{f}_{n,k}(t_2)] &= L + \frac{L\sum_{i=3}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\sum_{j=1}^{\binom{i}{2}}[(c_{ij}((1-p)L)^{j-1}-1)/p^{j-1}]}{\sum_{i=2}^{k+1}\binom{k+1}{i}\binom{n-k-1}{k+1-i}\big[\sum_{j=1}^{\binom{i}{2}}(1/p)^{j-1}\big]}\\ &:= L(1+\mathfrak{Z}_{n,k}).\end{aligned}$$

To prove the desired result, it suffices to show that $\mathfrak{Z}_{n,k} \to 0$ as $n \to \infty$. If $k = 1$ then $\mathfrak{Z}_{n,k} = 0$ for each n and, hence, $\lim_{n\to\infty}\mathfrak{Z}_{n,k} = 0$ trivially. Suppose that $k \geq 2$. Observe that expansion of the term inside the inner sum of the numerator of $\mathfrak{Z}_{n,k}$ will result in a linear combination of $1, 1/p, \ldots, 1/p^{j-1}$. Hence, by multiplying the numerator and denominator of $\mathfrak{Z}_{n,k}$ by $p^{\binom{k+1}{2}-1}$, one can write $\mathfrak{Z}_{n,k}$ as

$$\mathfrak{Z}_{n,k} = \frac{\sum_{i=3}^{k+1}\sum_{j=1}^{\binom{i}{2}}\omega_{ij}\binom{n-k-1}{k+1-i}p^{\binom{k+1}{2}-j}}{\sum_{i=2}^{k+1}\sum_{j=1}^{\binom{i}{2}}\xi_{ij}\binom{n-k-1}{k+1-i}p^{\binom{k+1}{2}-j}}$$

for some real constants $\{\omega_{ij}\}$ and $\{\xi_{ij}\}$. Since $\binom{n-k-1}{k+1-i} = \Theta(n^{k+1-i})$, it follows that in order to show $\lim_{n\to\infty}\mathfrak{Z}_{n,k} = 0$ one only needs to show that $\lim_{n\to\infty}\mathfrak{Z}'_{n,k} = 0$, where

$$\mathfrak{Z}'_{n,k} := \frac{\sum_{i=3}^{k+1}\sum_{j=1}^{\binom{i}{2}}\tilde{\omega}_{ij}n^{k+1-i}p^{\binom{k+1}{2}-j}}{\sum_{i=2}^{k+1}\sum_{j=1}^{\binom{i}{2}}\tilde{\xi}_{ij}n^{k+1-i}p^{\binom{k+1}{2}-j}}$$

with $\{\tilde{\omega}_{ij}\}$ and $\{\tilde{\xi}_{ij}\}$ being additional sets of real constants. Since $p = n^{\alpha}$, the power of n in the summand of numerator as well as denominator of $\mathfrak{Z}'_{n,k}$ is of the form

$$k+1-i+\alpha\bigg[\binom{k+1}{2}-j\bigg].$$

Since $\alpha < 0$, we have

$$\operatorname*{arg\,max}_{1\leq j\leq\binom{i}{2}}\bigg(k+1-i+\alpha\bigg[\binom{k+1}{2}-j\bigg]\bigg) = \binom{i}{2}. \tag{4.1}$$

Further, the restriction that $\alpha > -1/k$ implies that, for each $i \leq k$,

$$k+1-i+\alpha\bigg[\binom{k+1}{2}-\binom{i}{2}\bigg] \geq k+1-(i+1)+\alpha\bigg[\binom{k+1}{2}-\binom{i+1}{2}\bigg]. \tag{4.2}$$

From (4.1) and (4.2), it follows that the largest power of n in the numerator of $\mathfrak{Z}'_{n,k}$ is

$$k+1-3+\alpha\bigg[\binom{k+1}{2}-\binom{3}{2}\bigg], \tag{4.3}$$

while, in the denominator, it is

$$k + 1 - 2 + \alpha\left[\binom{k+1}{2} - \binom{2}{2}\right]. \tag{4.4}$$

Since $k \geq 2$ and, hence, $\alpha > -\frac{1}{2}$, it follows that the term in (4.4) is larger than that in (4.3). This shows that $\lim_{n\to\infty} \mathfrak{Z}'_{n,k} = 0$ as desired, and so completes the proof. □

Lemma 4.2. *Fix* $k \geq 1$. *Let* $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. *Then, for any* $t_1, t_2 \geq 0$,

$$\lim_{n\to\infty} \operatorname{cov}[\bar{\chi}_n(t_1), \bar{\chi}_n(t_2)] = \mathrm{e}^{-\lambda|t_1 - t_2|}.$$

Proof. We need to show that

$$\lim_{n\to\infty} \frac{\operatorname{cov}[\chi_n(t_1), \chi_n(t_2)]}{\sqrt{\operatorname{var}[\chi_n(t_1)]\operatorname{var}[\chi_n(t_2)]}} = \mathrm{e}^{-\lambda|t_1 - t_2|}.$$

However, since Lemma 3.4 holds, it suffices to show that

$$\lim_{n\to\infty} \frac{\operatorname{cov}[\chi_n(t_1), \chi_n(t_2)]}{\sqrt{\operatorname{var}[f_{n,k}(t_1)]\operatorname{var}[f_{n,k}(t_2)]}} = \mathrm{e}^{-\lambda|t_1 - t_2|}.$$

But the term inside limit on the left-hand side is equal to

$$\begin{aligned}
&\frac{\operatorname{cov}[f_{n,k}(t_1), f_{n,k}(t_2)]}{\sqrt{\operatorname{var}[f_{n,k}(t_1)]\operatorname{var}[f_{n,k}(t_2)]}} + \frac{\operatorname{cov}[(-1)^k\chi_n(t_1) - f_{n,k}(t_1), f_{n,k}(t_2)]}{\sqrt{\operatorname{var}[f_{n,k}(t_1)]\operatorname{var}[f_{n,k}(t_2)]}}\\
&+ \frac{\operatorname{cov}[f_{n,k}(t_1), (-1)^k\chi_n(t_2) - f_{n,k}(t_2)]}{\sqrt{\operatorname{var}[f_{n,k}(t_1)]\operatorname{var}[f_{n,k}(t_2)]}}\\
&+ \frac{\operatorname{cov}[(-1)^k\chi_n(t_1) - f_{n,k}(t_1), (-1)^k\chi_n(t_2) - f_{n,k}(t_2)]}{\sqrt{\operatorname{var}[f_{n,k}(t_1)]\operatorname{var}[f_{n,k}(t_2)]}}.
\end{aligned}$$

Using Lemma 4.1 we see that the first term converges to $\mathrm{e}^{-\lambda|t_1 - t_2|}$. The remaining terms go to 0 due to Lemma 3.3 and the Cauchy–Schwarz inequality and the proof is complete. □

In the above proof, by replacing $f_{n,k}(t_i)$ with $\chi_n(t_i)$ and $\chi_n(t_i)$ with $\beta_{n,k}(t_i)$ and using Lemmas 3.6, 4.2, and 3.5, appropriately, the following result is easy to prove.

Theorem 4.1. *Fix* $k \geq 1$. *Let* $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. *Then, for any* $t_1, t_2 \geq 0$,

$$\lim_{n\to\infty} \operatorname{cov}[\bar{\beta}_{n,k}(t_1), \bar{\beta}_{n,k}(t_2)] = \mathrm{e}^{-\lambda|t_1 - t_2|}.$$

5. Convergence of finite-dimensional distributions

We now turn to the convergence of the finite-dimensional distributions of the processes $\bar{\beta}_{n,k}$, which we establish by first proving similar results for $\bar{f}_{n,k}$ and $\bar{\chi}_{n,k}$.

For random variables X and Y, write $X \overset{\mathrm{D}}{=} Y$ to indicate equivalence in distribution.

Lemma 5.1. *Fix* $k \geq 1$. *Let* $p = n^{\alpha}$, $\alpha \in (-1/k, -1/(k+1))$. *Then, for any* $m \in \mathbb{N}$ *and any* $t_1, \ldots, t_m \geq 0$, *as* $n \to \infty$,

$$(\bar{f}_{n,k}(t_1), \ldots, \bar{f}_{n,k}(t_m)) \xrightarrow{\mathrm{D}} (\mathcal{U}_{\lambda}(t_1), \ldots, \mathcal{U}_{\lambda}(t_m)).$$

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Proof. Fix $m \in \mathbb{N}$, arbitrary $t_1, \ldots, t_m \geq 0$, and arbitrary $\omega_1, \ldots, \omega_m \in \mathbb{R}$. Due to the Cramér–Wold theorem [4, Theorem 29.4], it suffices to show that, as $n \to \infty$,

$$\omega_1 \bar{f}_{n,k}(t_1) + \cdots + \omega_m \bar{f}_{n,k}(t_m) \xrightarrow{\mathrm{D}} \omega_1 \mathcal{U}_\lambda(t_1) + \cdots + \omega_m \mathcal{U}_\lambda(t_m). \tag{5.1}$$

But as $\{\mathcal{U}_\lambda(t) \colon t \geq 0\}$ is Gaussian with $\mathbb{E}[\mathcal{U}_\lambda(t)] \equiv 0$ and $\operatorname{cov}[\mathcal{U}_\lambda(t_i), \mathcal{U}_\lambda(t_j)] = \mathrm{e}^{-|t_i - t_j|}$, we have

$$\frac{\omega_1 \mathcal{U}_\lambda(t_1) + \cdots + \omega_m \mathcal{U}_\lambda(t_m)}{\sqrt{\omega_1^2 + \cdots + \omega_m^2 + 2\sum_{i<j} \omega_i \omega_j \mathrm{e}^{-|t_i - t_j|}}} \stackrel{\mathrm{D}}{=} \mathcal{N}(0, 1).$$

Further, from Lemma 4.1 we see that

$$\lim_{n\to\infty} \frac{\sqrt{\operatorname{var}\big[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)\big]}}{\sqrt{\omega_1^2 + \cdots + \omega_m^2 + 2\sum_{i<j} \omega_i \omega_j \mathrm{e}^{-|t_i - t_j|}}} = 1. \tag{5.2}$$

Hence, it follows that in order to prove (5.1) we only need show that, as $n \to \infty$,

$$\mathfrak{W}_{n,k} := \frac{\omega_1 \bar{f}_{n,k}(t_1) + \cdots + \omega_m \bar{f}_{n,k}(t_m)}{\sqrt{\operatorname{var}\big[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)\big]}} \xrightarrow{\mathrm{D}} \mathcal{N}(0, 1). \tag{5.3}$$

From (1.5) and (3.1), we have

$$\mathfrak{W}_{n,k} = \frac{\sum_{A \in \binom{[n]}{k+1}} \big[\sum_{i=1}^m \omega_i \bar{\mathbf{1}}_A(t_i)\big]}{\sqrt{\operatorname{var}\big[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)\big]}},$$

where $\bar{\mathbf{1}}_A(t_i) = ((\mathbf{1}_A(t_i) - \mathbb{E}[\mathbf{1}_A(t_i)])/\sqrt{\operatorname{var}[f_{n,k}(t_i)]}\,)$. Indexing the random variable $[\sum_{i=1}^m \omega_i \bar{\mathbf{1}}_A(t_i)]$ with the $\binom{k+1}{2}$ edges in A, it is easy to see that

$$\left\{ \frac{\big[\sum_{i=1}^m \omega_i \bar{\mathbf{1}}_A(t_i)\big]}{\sqrt{\operatorname{var}\big[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)\big]}} \colon A \in \binom{[n]}{k+1} \right\}$$

is a dissociated set of random variables. For any $A_1 \in \binom{[n]}{k+1}$, its dependency neighbourhood $\mathcal{Y}(A_1) = \{A_2 \in \binom{[n]}{k+1} \colon a_{12} \geq 2\}$. Here, a_{12} denotes the number of vertices common to A_1 and A_2. For details, see the discussion above [2, Equation (3.5)].

Let $S_{n,k,m}$ be the cartesian product $\binom{[n]}{k+1} \times [m]$. For $(A_1, i) \in S_{n,k,m}$, let $\aleph(A_1, i) = \mathcal{Y}(A_1) \times [m]$. Since $\mathbb{E}[\omega_i \bar{\mathbf{1}}_A(t_i)] = 0$ and $\mathbb{E}[\mathfrak{W}_{n,k}^2] = 1$, Theorem 2.1 yields

$$\begin{aligned} d_1(\mathfrak{W}_{n,k}, \mathcal{N}(0,1)) \leq{}& \frac{\rho\omega^3}{\big(\operatorname{var}\big[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)\big]\big)^{3/2}} \\ & \times \sum_{(A_1, i) \in S_{n,k,m}} \sum_{(A_2, j), (A_3, \ell) \in \aleph(A_1, i)} [\mathbb{E}[|\bar{\mathbf{1}}_{A_1}(t_i) \bar{\mathbf{1}}_{A_2}(t_j) \bar{\mathbf{1}}_{A_3}(t_\ell)|] \\ & \qquad + \mathbb{E}[|\bar{\mathbf{1}}_{A_1}(t_i) \bar{\mathbf{1}}_{A_2}(t_j)|] \mathbb{E}[|\bar{\mathbf{1}}_{A_3}(t_\ell)|]], \end{aligned}$$

where $\omega = \max_{i \in [m]} |\omega_i|$. Since

$$\begin{aligned} &\mathbb{E}[|\bar{\mathbf{1}}_{A_1}(t_i) \bar{\mathbf{1}}_{A_2}(t_j) \bar{\mathbf{1}}_{A_3}(t_\ell)|] + \mathbb{E}[|\bar{\mathbf{1}}_{A_1}(t_i) \bar{\mathbf{1}}_{A_2}(t_j)|] \mathbb{E}[|\bar{\mathbf{1}}_{A_3}(t_\ell)|] \\ &\quad \leq \frac{16\mathbb{E}[\mathbf{1}_{A_1}(t_i)\, \mathbf{1}_{A_2}(t_j)\, \mathbf{1}_{A_3}(t_\ell)]}{\sqrt{\operatorname{var}[f_{n,k}(t_i)] \operatorname{var}[f_{n,k}(t_j)] \operatorname{var}[f_{n,k}(t_\ell)]}}, \end{aligned}$$

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then

$$d_1(\mathfrak{W}_{n,k}, \mathcal{N}(0,1)) \\ \leq 16\rho\omega^3 \frac{\sum_{(A_1,i)\in S_{n,k,m}} \sum_{(A_2,j),\,(A_3,\ell)\in\aleph(A_1,i)} \mathbb{E}[\mathbf{1}_{A_1}(t_i)\,\mathbf{1}_{A_2}(t_j)\,\mathbf{1}_{A_3}(t_\ell)]}{\big(\mathrm{var}[\sum_{i=1}^m \omega_i \bar{f}_{n,k}(t_i)]\big)^{3/2}\sqrt{\mathrm{var}[f_{n,k}(t_i)]\,\mathrm{var}[f_{n,k}(t_j)]\,\mathrm{var}[f_{n,k}(t_\ell)]}}.$$

Combining this with (5.2), and defining

$$\mathfrak{R}_{n,k} := \frac{\sum_{(A_1,i)\in S_{n,k,m}} \sum_{(A_2,j),\,(A_3,\ell)\in\aleph(A_1,i)} \mathbb{E}[\mathbf{1}_{A_1}(t_i)\,\mathbf{1}_{A_2}(t_j)\,\mathbf{1}_{A_3}(t_\ell)]}{\sqrt{\mathrm{var}[f_{n,k}(t_i)]\,\mathrm{var}[f_{n,k}(t_j)]\,\mathrm{var}[f_{n,k}(t_\ell)]}},$$

it follows that in order to establish (5.3) we need only show that $\lim_{n\to\infty}\mathfrak{R}_{n,k} = 0$. Fix arbitrary $t \geq 0$ and let $\aleph(A_1) \equiv \aleph_{n,k}(A_1) := \{A_2 \in \binom{[n]}{k+1} \colon a_{12} \geq 2\}$ and

$$\mathfrak{R}'_{n,k} := \frac{\sum_{A_1 \,\in\, \binom{[n]}{k+1}} \sum_{A_2,\ A_3\,\in\,\aleph(A_1)} \mathbb{E}[\mathbf{1}_{A_1}(t)\,\mathbf{1}_{A_2}(t)\,\mathbf{1}_{A_3}(t)]}{(\mathrm{var}[f_{n,k}(t)])^{3/2}}.$$

In [18], as part of proof of Claim 2.5(ii), it was shown that $\lim_{n\to\infty}\mathfrak{R}'_{n,k} = 0$. In the remaining part of this proof, we shall show that

$$\mathfrak{R}_{n,k} \leq m^3\mathfrak{R}'_{n,k}. \tag{5.4}$$

This is clearly sufficient enough in order to establish that $\lim_{n\to\infty}\mathfrak{R}_{n,k} = 0$.

From (3.3), recall that $\mathrm{var}[f_{n,k}(t)]$ is independent of t. Hence, it follows that the denominators in $\mathfrak{R}_{n,k}$ and $\mathfrak{R}'_{n,k}$ are identical. Now, using (1.1) and (1.3) and the fact that $p + (1-p)\,\mathrm{e}^{-\tau} \leq 1$ for any $\tau \geq 0$, observe that

$$\mathbb{E}[\mathbf{1}_{A_1}(t_i)\,\mathbf{1}_{A_2}(t_j)\,\mathbf{1}_{A_3}(t_\ell)] \leq p^{3\binom{k+1}{2}-\binom{a_{12}}{2}-\binom{a_{13}}{2}-\binom{a_{23}}{2}+\binom{a_{123}}{2}} = \mathbb{E}[\mathbf{1}_{A_1}(t)\,\mathbf{1}_{A_2}(t)\,\mathbf{1}_{A_3}(t)].$$

From this and the definition of $\mathfrak{R}_{n,k}$, (5.4) easily follows. The desired result thus follows. □

Lemma 5.2. *Fix* $k \geq 1$. *Let* $p = n^\alpha$, $\alpha \in (-1/k, -1/(k+1))$. *Then, for any* $m \in \mathbb{N}$ *and any* $t_1, \ldots, t_m \geq 0$, *as* $n \to \infty$,

$$(\bar{\chi}_n(t_1), \ldots, \bar{\chi}_n(t_m)) \xrightarrow{\mathrm{D}} (\mathcal{U}_\lambda(t_1), \ldots, \mathcal{U}_\lambda(t_m)).$$

Proof. As in the proof of Lemma 5.1, it suffices to show that, as $n \to \infty$,

$$\frac{\omega_1\bar{\chi}_n(t_1) + \cdots + \omega_m\bar{\chi}_n(t_m)}{\sqrt{\mathrm{var}\big[\sum_{i=1}^m \omega_i\bar{\chi}_n(t_i)\big]}} \xrightarrow{\mathrm{D}} \mathcal{N}(0,1) \quad \text{for any } \omega_1, \ldots, \omega_m \in \mathbb{R}.$$

Since Lemmas 4.1 and 4.2 hold, it in fact suffices to show that

$$\frac{\omega_1\bar{\chi}_n(t_1) + \cdots + \omega_m\bar{\chi}_n(t_m)}{\sqrt{\mathrm{var}\big[\sum_{i=1}^m \omega_i\bar{f}_{n,k}(t_i)\big]}} \xrightarrow{\mathrm{D}} \mathcal{N}(0,1). \tag{5.5}$$

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From Lemmas 3.3 and 3.4, and the Cauchy–Schwarz inequality, note that, for all i,

$$\begin{aligned}\operatorname{var}[(-1)^k\bar{\chi}_n(t_i)-\bar{f}_{n,k}(t_i)] &= \operatorname{var}\biggl[(-1)^k\bar{\chi}_n(t_i)-\frac{(-1)^k\chi_n(t_i)-\mathbb{E}[(-1)^k\chi_n(t_i)]}{\sqrt{\operatorname{var}[f_{n,k}(t_i)]}} \\ &\qquad+\frac{(-1)^k\chi_n(t_i)-\mathbb{E}[(-1)^k\chi_n(t_i)]}{\sqrt{\operatorname{var}[f_{n,k}(t_i)]}}-\bar{f}_{n,k}(t_i)\biggr] \\ &\le \biggl(\sqrt{\operatorname{var}[\chi_n(t_i)]}\biggl|\frac{1}{\sqrt{\operatorname{var}[\chi_n(t_i)]}}-\frac{1}{\sqrt{\operatorname{var}[f_{n,k}(t_i)]}}\biggr| \\ &\qquad+\sqrt{\frac{\operatorname{var}[(-1)^k\chi_n(t_i)-f_{n,k}(t_i)]}{\operatorname{var}[f_{n,k}(t_i)]}}\biggr)^2 \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Using the above estimate and the Cauchy–Schwarz inequality, we find that

$$\operatorname{var}\biggl[\frac{\sum_{i=1}^m\omega_i[(-1)^k\bar{\chi}_n(t_i)-\bar{f}_{n,k}(t_i)]}{\sqrt{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]}}\biggr]\le\biggl(\sum_{i=1}^m|\omega_i|\sqrt{\frac{\operatorname{var}[[(-1)^k\bar{\chi}_n(t_i)-\bar{f}_{n,k}(t_i)]]}{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]}}\biggr)^2$$
$$\to 0 \quad \text{as } n\to\infty.$$

From this, it follows that $[\sum_{i=1}^m\omega_i[(-1)^k\bar{\chi}_n(t_i)-\bar{f}_{n,k}(t_i)]/\{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]\}^{-1/2}]$ converges to 0 in probability. Since Lemma 5.1 holds and

$$\frac{(-1)^k\sum_{i=1}^m\omega_i\bar{\chi}_n(t_i)}{\sqrt{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]}}=\frac{\sum_{i=1}^m\omega_i[(-1)^k\bar{\chi}_n(t_i)-\bar{f}_{n,k}(t_i)]}{\sqrt{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]}}+\frac{\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)}{\sqrt{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{f}_{n,k}(t_i)]}},$$

(5.5) follows via Slutsky's theorem [11, Chapter 6, Theorem 6.5], completing the proof. □

Theorem 5.1. *Fix $k\ge 1$. Let $p=n^\alpha$, $\alpha\in(-1/k,-1/(k+1))$. Then, for any $m\in\mathbb{N}$ and any $t_1,\dots,t_m$, as $n\to\infty$,*

$$(\bar{\beta}_{n,k}(t_1),\dots,\bar{\beta}_{n,k}(t_m))\xrightarrow{\text{D}}(\mathcal{U}_\lambda(t_1),\dots,\mathcal{U}_\lambda(t_m)).$$

Proof. The arguments are similar to those used in the proof of Lemma 5.2. Firstly, using Lemma 3.5, it follows that, for any $\omega_1,\dots,\omega_m\in\mathbb{R}$,

$$\lim_{n\to\infty}\operatorname{var}\biggl[\frac{\sum_{i=1}^m\omega_i[(-1)^k\bar{\chi}_n(t_i)-\bar{\beta}_{n,k}(t_i)]}{\sqrt{\operatorname{var}[\sum_{i=1}^m\omega_i\bar{\chi}_{n,k}(t_i)]}}\biggr]=0.$$

Then, using Lemma 4.2 and Theorem 4.1, the desired result follows. □

6. Tightness

In this section we show that, for each k, the sequences $\{\hat{\beta}_{n,k}\colon n\ge 1\}$ are tight. By Theorem 2.2, it suffices to establish the two conditions (C1) and (C2) for these sequences.

Lemma 6.1. *Fix $k\ge 1$. Let $p=n^\alpha$, $\alpha\in(-1/k,-1/(k+1))$. Then, for the sequence $\{\hat{\beta}_{n,k}\colon n\ge 1\}$, condition (C1) holds with $\Upsilon=2$, i.e.*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{E}[\bar{\beta}_{n,k}(\delta)-\bar{\beta}_{n,k}(0)]^2=0.$$

Proof. From Theorem 4.1 and the fact that

$$\mathbb{E}[\bar{\beta}_{n,k}(\delta) - \bar{\beta}_{n,k}(0)]^2 = 2 - 2\operatorname{cov}[\bar{\beta}_{n,k}(\delta), \bar{\beta}_{n,k}(0)],$$

we have

$$\lim_{n\to\infty} \mathbb{E}[\bar{\beta}_{n,k}(\delta) - \bar{\beta}_{n,k}(0)]^2 = 2 - 2\mathrm{e}^{-\delta},$$

and the result follows easily. □

Arguing as above, it follows that (C1) is also satisfied for the sequence of $\{\hat{f}_{n,k} : n \geq 1\}$ and $\{\hat{\chi}_{n,k} : n \geq 1\}$. We now aim to show that (C2) holds for $\{\bar{\beta}_{n,k} : n \geq 1\}$. Our approach is to first establish this result for $\bar{f}_{n,k}$, then for $\bar{\chi}_n$, and finally for $\bar{\beta}_{n,k}$.

Lemma 6.2. *Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in (-1/k, -1/(k+1))$. Then, for the sequence $\{\bar{f}_{n,k} : n \geq 1\}$, condition (C2) holds with $\Upsilon_1 = \Upsilon_2 = 2$. That is, for any $T > 0$, there exists $K_f > 0$ such that, for all $n \geq 1$, $0 \leq t \leq T+1$, and $0 \leq h \leq t$,*

$$\mathbb{E}[\bar{f}_{n,k}(t+h) - \bar{f}_{n,k}(t)]^2[\bar{f}_{n,k}(t) - \bar{f}_{n,k}(t-h)]^2 \leq K_f h^2.$$

This follows from the next result and, hence, we prove only that.

Lemma 6.3. *Fix $k \geq 1$. Let $p = n^\alpha$, $\alpha \in (-1/k, -1/(k+1))$. Then, for the sequence $\{\bar{\chi}_n : n \geq 1\}$, condition (C2) holds with $\Upsilon_1 = \Upsilon_2 = 2$. That is, for any $T > 0$, there exists $K_\chi > 0$ such that, for all $n \geq 1$, $0 \leq t \leq T+1$, and $0 \leq h \leq t$,*

$$\mathbb{E}[\bar{\chi}_{n,k}(t+h) - \bar{\chi}_{n,k}(t)]^2[\bar{\chi}_{n,k}(t) - \bar{\chi}_{n,k}(t-h)]^2 \leq K_\chi h^2.$$

Before turning to the proof of Lemma 6.3, we need some additional notation and preliminary lemmas. Fix arbitrary $n, k \geq 1$ and let p be as in Lemma 6.3. Also fix i and j such that $0 \leq i, j \leq n-1$ and let

$$\xi_{ij}(h) := \mathbb{E}[f_{n,i}(2h) - f_{n,i}(h)]^2[f_{n,j}(h) - f_{n,j}(0)]^2. \tag{6.1}$$

For $\bar{A} \equiv (A_1, A_2, A_3, A_4) \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$, let a_q be the number of vertices in A_q, a_{qr} be the number of vertices common to A_q and A_r, and so on. Note that inequalities such as $a_{1234} \leq a_{qrs} \leq a_{qr} \leq a_q$ for any $q, r, s \in \{1, \ldots, 4\}$ hold trivially. Let

$$\tau(\bar{A}) = (a_1, \ldots, a_4, a_{12}, \ldots, a_{34}, a_{123}, \ldots, a_{234}, a_{1234}),$$

$$\operatorname{ver}(\bar{A}) = \sum_{q=1}^{4} a_q - \sum_{1\leq q<r\leq 4} a_{qr} + \sum_{1\leq q<r<s\leq 4} a_{qrs} - a_{1234}, \tag{6.2}$$

$$\operatorname{pair}(\bar{A}) = \sum_{q=1}^{4}\binom{a_q}{2} - \sum_{1\leq q<r\leq 4}\binom{a_{qr}}{2} + \sum_{1\leq q<r<s\leq 4}\binom{a_{qrs}}{2} - \binom{a_{1234}}{2}, \tag{6.3}$$

and

$$g(h; \bar{A}) := [\mathbf{1}_{A_1}(2h) - \mathbf{1}_{A_1}(h)][\mathbf{1}_{A_2}(2h) - \mathbf{1}_{A_2}(h)][\mathbf{1}_{A_3}(h) - \mathbf{1}_{A_3}(0)][\mathbf{1}_{A_4}(h) - \mathbf{1}_{A_4}(0)]. \tag{6.4}$$

Here, $\tau(\bar{A})$ denotes the intersection type of $\bar{A}$, while $\operatorname{ver}(\bar{A})$ and $\operatorname{pair}(\bar{A})$ denote respectively the number of vertices and maximum possible edges in $A_1, \ldots, A_4$ with common vertices and

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edges counted only once. Terms of the form $g(h; \bar{A})$ appear in the expansion of $\xi_{ij}(h)$ and, hence, will be useful later.

For $\bar{A}, \bar{B} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$, we write $\bar{A} \sim \bar{B}$ if there exists a permutation π of the sets in $\bar{B}$ such that $\tau(\bar{A}) = \tau(\pi(\bar{B}))$. A priori, it may appear that the intersection type of all 24 permutations of the sets in $\bar{B}$ need to be compared with $\tau(\bar{A})$ before concluding $\bar{A} \sim \bar{B}$ or not. But this holds only when $i = j$. When $i \neq j$, many of the permutations need not be checked. For example, the permutation that interchanges the first and third set can be ignored. Clearly, '$\sim$' is an equivalence relation. Let $\Gamma_{ij} := \{[\bar{A}]\}$ denote the quotient of $\binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ under equivalence, where $[\bar{A}]$ denotes the equivalence class of $\bar{A}$. Since each a_{qr}, a_{qrs}, and a_{1234} (11 variables in total) is a number between 0 and $\max\{i+1, j+1\} \leq (i+j+1)$, the cardinality of Γ_{ij} satisfies

$$|\Gamma_{ij}| \leq (i + j + 1)^{11}. \tag{6.5}$$

We shall say that $\bar{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ has an *independent set* if there exists $q \in \{1, 2, 3, 4\}$ such that $a_{qr} \leq 1$ for all $r \neq q$. That is, there exists a special set among $A_1, \ldots, A_4$ which shares at most one vertex with the remaining three sets. Clearly, the indicator associated with this special set is independent of the indicator associated with the other three sets. Based on this description, let

$$\mathfrak{S}_{ij} := \big\{[\bar{A}] \in \Gamma_{ij} \colon \text{there exists } q \in \{1, 2, 3, 4\} \text{ such that for all } r \neq q, a_{qr} \leq 1\big\}. \tag{6.6}$$

Lemma 6.4. *Fix arbitrary $n, k \geq 1$, and let p be as in Lemma 6.3. Also fix i and j such that $0 \leq i, j \leq n-1$. Fix $\bar{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$.*

(i) *If $[\bar{A}] \in \mathfrak{S}_{ij}$ then $\mathbb{E}[g(h; \bar{A})] \equiv 0$.*

(ii) *If $[\bar{A}] \in \Gamma_{ij} \backslash \mathfrak{S}_{ij}$ then there exists some universal constant $\gamma \geq 0$ (independent of $\bar{A}$, i, j, k, and n) such that, for all $0 \leq h \leq 1$,*

$$|\mathbb{E}[g(h; \bar{A})]| \leq \gamma (i + j + 1)^4 p^{\mathrm{pair}(\bar{A})} h^2.$$

Proof. The first claim is straightforward and follows from the stationarity of the dynamic Erdős–Rényi graph. So we discuss only the second one.

Fix $\bar{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$ with $[\bar{A}] \in \Gamma_{ij} \backslash \mathfrak{S}_{ij}$. It is tedious but not difficult to see that $g(h; \bar{A})$ satisfies (B.1); see Appendix B. Hence, using (1.1) and (1.3), we have

$$\mathbb{E}[g(h; \bar{A})] = p^{\mathrm{pair}(\bar{A})} \Phi(h; \bar{A}), \tag{6.7}$$

where $\Phi(h; \bar{A})$ is as in (B.2). Note that $\Phi(h; \bar{A})$ has the form

$$\Phi(h; \bar{A}) = \sum_{\ell=1}^{16} \phi_\ell(h; \bar{A}), \tag{6.8}$$

where, for each ℓ,

$$\phi_\ell(h; \bar{A}) = \pm((1-p)\,\mathrm{e}^{-\lambda h} + p)^{c_1(\ell)}((1-p)\,\mathrm{e}^{-2\lambda h} + p)^{c_2(\ell)} \tag{6.9}$$

with

$$0 \leq c_1(\ell), \qquad c_2(\ell) \leq \sum_{1 \leq q < r \leq 4} \binom{a_{qr}}{2} + \binom{a_{1234}}{2} \leq 7(i + j + 1)^2. \tag{6.10}$$

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By analysing (B.2), it is not difficult to see that

$$\Phi(h;\bar{A})|_{h=0} = 0 \quad \text{and} \quad \frac{\partial \Phi(h;\bar{A})}{\partial h}\bigg|_{h=0} = 0.$$

Because of the above two facts, expanding $\Phi(h;\bar{A})$ using the Lagrangian form of a Taylor series shows that, for each $0 \le h \le 1$, there exists $c \in [0, h]$ such that

$$\Phi(h;\bar{A}) = \frac{1}{2}h^2 \frac{\partial^2 \Phi(h;\bar{A})}{\partial h^2}\bigg|_{h=c}. \tag{6.11}$$

Now using (6.8), (6.9), (6.10), and the fact that both $((1-p)\mathrm{e}^{-h}+p)$ and $((1-p)\mathrm{e}^{-2h}+p)$ are bounded from above by 1 for $h \ge 0$, it is not difficult to see that there exists some universal constant $\gamma_1 \ge 0$ (independent of $\bar{A}$, i, j, k, and n) such that

$$\max_{1\le \ell \le 16} \sup_{h \ge 0} \bigg| \frac{\partial^2 \phi_\ell(h;\bar{A})}{\partial h^2} \bigg| \le \gamma_1 (i+j+1)^4.$$

Combining this with (6.8) and (6.11), it follows that $|\Phi(h;\bar{A})| \le 8\gamma_1(i+j+1)^4 h^2$. Using this inequality in (6.7), the result follows. □

Lemma 6.5. *Fix arbitrary $n, k \ge 1$, and let p be as in Lemma 6.3. Also fix i and j such that $0 \le i, j \le n-1$. Fix $\bar{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2$.*

(i) *If $[\bar{A}] \in \Gamma_{ij} \backslash \mathfrak{S}_{ij}$ then*

$$\frac{n^{\mathrm{ver}(\bar{A})} p^{\mathrm{pair}(\bar{A})}}{n^{4k} p^{4\binom{k+1}{2}-2}} \le 1.$$

(ii) *If $[\bar{A}] \in \Gamma_{ij} \backslash \mathfrak{S}_{ij}$ and $(i+j) \ge 16k+15$, then*

$$\frac{n^{\mathrm{ver}(\bar{A})} p^{\mathrm{pair}(\bar{A})}}{n^{4k} p^{4\binom{k+1}{2}-2}} \le \frac{1}{n^{2k+2(i+j-16k-15)}}.$$

Proof. From (6.6), as $\bar{A} \in \Gamma_{ij} \backslash \mathfrak{S}_{ij}$, one of the following two cases must hold.

Case A. Either $a_{12}, a_{34} \ge 2$, or $a_{13}, a_{24} \ge 2$, or $a_{14}, a_{23} \ge 2$.

Case B. There exists $q \in \{1, \dots, 4\}$ such that $a_{qr} \ge 2$ for all $r \ne q$.

In both cases, using essentially the same arguments as those used to obtain [18, Erratum, Equation (8)], with the differences noted below, we have

$$n^{\mathrm{ver}(\bar{A})} p^{\mathrm{pair}(\bar{A})} \le n^{4k} p^{4\binom{k+1}{2}-2}.$$

This proves the first claim of the lemma modulo clearing up the two main differences between the arguments needed here and those used in [18]. The first relates to the fact that [18] dealt with the intersection of three sets while here we need to deal with four sets. In both cases, however, independent sets are absent, i.e. each set has at least two vertices in common with one of the remaining sets.

Secondly, in [18], an upper bound for $n^{\mathrm{ver}(\bar{A})} p^{\mathrm{pair}(\bar{A})}$, with $\mathrm{ver}(\bar{A})$ and $\mathrm{pair}(\bar{A})$ appropriately defined, was obtained by sequentially dealing with the number of vertices in the third set, then

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the second set, and so on. Here, we have to repeat the same idea by first dealing with the number of vertices in the fourth set, then third, and so on.

Now consider the second claim of the lemma. Again the conditions of case A or case B defined above must hold. Hence, from (6.2), (6.3), and (6.6), we have $\mathrm{ver}(\bar{A}) \le 2i + 2j$ and $\mathrm{pair}(\bar{A}) \ge \max\{\binom{i+1}{2}, \binom{j+1}{2}\}$.

Using these and fact that $\alpha \in (-1/k, -1/(k+1))$, we have

$$\mathrm{ver}(\bar{A}) + \alpha\,\mathrm{pair}(\bar{A}) \le 2(i+j) + \alpha \max\left\{\binom{i+1}{2}, \binom{j+1}{2}\right\}.$$

Since $\max\{\binom{i+1}{2}, \binom{j+1}{2}\} \ge \frac{1}{4}\binom{i+j+1}{2}$, it follows that

$$\mathrm{ver}(\bar{A}) + \alpha\,\mathrm{pair}(\bar{A}) \le 2(i+j) + \frac{\alpha}{4}\binom{i+j+1}{2}.$$

Consequently, in order to prove the desired result, it suffices to show that, for $i + j \ge 16k + 15$,

$$\frac{n^{2(i+j)+\alpha\binom{i+j+1}{2}/4}}{n^{4k} p^{4\binom{k+1}{2}-2}} \le \frac{1}{n^{2k+2(i+j-16k-15)}}. \tag{6.12}$$

Now, observe that if $i + j = 16k + 15$ then

$$\frac{n^{2(i+j)+\alpha\binom{i+j+1}{2}/4}}{n^{4k+\alpha(4\binom{k+1}{2}-2)}} \le \frac{1}{n^{2k}}.$$

Suppose that for i' and j' with $(i' + j') \ge 16k + 15$, the desired result holds. Now consider i and j satisfying $(i + j) = (i' + j') + 1$. Since $(i' + j') \ge 16k + 15$,

$$2(i+j) - 2(i'+j') + \alpha\left[\frac{1}{4}\binom{i+j+1}{2} - \frac{1}{4}\binom{i'+j'+1}{2}\right] = 2 + \frac{\alpha}{4}(i'+j'+1) \le -2.$$

By induction, (6.12) follows and so does the claim. □

Lemma 6.6. *Fix arbitrary $n, k \ge 1$, and let p be as in Lemma 6.3. Also fix i and j such that $0 \le i, j \le n-1$. Let $\xi_{ij}(h)$ be as in (6.1) and γ as in Lemma 6.4.*

(i) *If $(i + j) < 16k + 15$ then*

$$\frac{\xi_{ij}(h)}{n^{4k} p^{4\binom{k+1}{2}-2}} \le \gamma (i+j+1)^{15} h^2.$$

(ii) *If $(i + j) \ge 16k + 15$ then*

$$\frac{\xi_{ij}(h)}{n^{4k} p^{4\binom{k+1}{2}-2}} \le \gamma \frac{(i+j+1)^{15}}{n^{2k+2(i+j-16k-15)}} h^2.$$

Proof. From (6.1) and (6.4), it is easy to see that

$$\xi_{ij}(h) = \sum_{\bar{A} \in \binom{[n]}{i+1}^2 \times \binom{[n]}{j+1}^2} \mathbb{E}[g(h; \bar{A})].$$

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Collecting terms based on their equivalence classes under '$\sim$', it follows that

$$\xi_{ij}(h) = \sum_{[\bar{B}]\in\Gamma_{ij}} \sum_{\left\{\bar{A}\in\binom{[n]}{i+1}^2\times\binom{[n]}{j+1}^2:\ \bar{A}\sim\bar{B}\right\}} \mathbb{E}[g(h;\bar{A})].$$

Applying Lemma 6.4, we obtain

$$\xi_{ij}(h) \le \gamma(i+j+1)^4 h^2 \sum_{[\bar{B}]\in\Gamma_{ij}\setminus\mathfrak{S}_{ij}} \sum_{\left\{\bar{A}\in\binom{[n]}{i+1}^2\times\binom{[n]}{j+1}^2:\ \bar{A}\sim\bar{B}\right\}} p^{\mathrm{pair}(\bar{A})}.$$

Now, note that, from (6.2) and (6.3), if $\bar{A}\sim\bar{B}$ then $\mathrm{ver}(\bar{A}) = \mathrm{ver}(\bar{B})$ and $\mathrm{pair}(\bar{A}) = \mathrm{pair}(\bar{B})$. Further, the cardinality of the set $\{\bar{A}\in\binom{[n]}{i+1}^2\times\binom{[n]}{j+1}^2:\ \bar{A}\sim\bar{B}\}$ is bounded above by $n^{\mathrm{ver}(\bar{B})}$. From these observations, it follows that

$$\xi_{ij}(h) \le \gamma(i+j+1)^4 h^2 \sum_{[\bar{B}]\in\Gamma_{ij}\setminus\mathfrak{S}_{ij}} n^{\mathrm{ver}(\bar{B})} p^{\mathrm{pair}(\bar{B})}.$$

Using (6.5) and Lemma 6.5, both the desired statements are now easy to see. $\square$

Proof of Lemma 6.3. Since $\{G(n,p,t)\colon t\ge 0\}$ and, hence, $\{\bar{\chi}_n(t)\colon t\ge 0\}$ are stationary, to prove the desired result, it suffices to show that there exists $K_\chi > 0$ such that

$$\mathbb{E}[\bar{\chi}_n(2h) - \bar{\chi}_n(h)]^2[\bar{\chi}_n(h) - \bar{\chi}_n(0)]^2 \le K_\chi h^2 \quad \text{for } 0\le h\le 1,\ n\ge 1. \tag{6.13}$$

From Lemmas 3.2 and 3.4, $\mathrm{var}[\chi_n(t)] = \Theta(n^{2k}p^{2\binom{k+1}{2}-1})$. Hence, to prove (6.13), it suffices to show that there exists $K_\chi > 0$ such that

$$\Omega_{n,k}(h) := \frac{\mathbb{E}[\chi_n(2h) - \chi_n(h)]^2[\chi_n(h) - \chi_n(0)]^2}{n^{4k}p^{4\binom{k+1}{2}-2}} \le K_\chi h^2.$$

Using (1.4) and the triangle inequality, we have

$$\sqrt{\Omega_{n,k}(h)} \le \sum_{0\le i,j\le n-1} \sqrt{\frac{\xi_{ij}(h)}{n^{4k}p^{4\binom{k+1}{2}-2}}},$$

where $\xi_{ij}(h)$ is as in (6.1). Collecting terms based on the sum $(i+j)$, we have

$$\sqrt{\Omega_{n,k}(h)} \le \sum_{0\le\ell\le 2(n-1)} \sum_{(i+j)=\ell} \sqrt{\frac{\xi_{ij}(h)}{n^{4k}p^{4\binom{k+1}{2}-2}}}.$$

This implies that

$$\sqrt{\Omega_{n,k}(h)} \le \sum_{0\le\ell<\infty} \sum_{(i+j)=\ell} \sqrt{\frac{\xi_{ij}(h)}{n^{4k}p^{4\binom{k+1}{2}-2}}}.$$

From this, it follows that $\sqrt{\Omega_{n,k}(h)} \le \mathrm{term}_1 + \mathrm{term}_2$, where

$$\mathrm{term}_1 := \sum_{0\le\ell<16k+15} \sum_{(i+j)=\ell} \sqrt{\frac{\xi_{ij}(h)}{n^{4k}p^{4\binom{k+1}{2}-2}}}$$

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and

$$\text{term}_2 := \sum_{16k+15\le\ell<\infty} \sum_{(i+j)=\ell} \sqrt{\frac{\xi_{ij}(h)}{n^{4k} p^{4\binom{k+1}{2}-2}}}.$$

As $\{(i,j)\colon i,j \ge 0, i+j=\ell\}$ has $\ell+1$ elements, using Lemma 6.6(i), we have $\text{term}_1 \le \sqrt{\gamma h^2} K_1$, where $K_1 := \sum_{0\le\ell<16k+15}(l+1)^{15/2+1}$. Note that K_1 is a constant independent of n and $0 \le h \le 1$. Similarly, using Lemma 6.6(ii), we obtain $\text{term}_2 \le \sqrt{\gamma h^2} K_2(n)$, where

$$K_2(n) := \sum_{16k+15\le\ell<\infty} \frac{(\ell+1)^9}{n^{k+(\ell-16k-15)}}.$$

Clearly, $K_2(n)$ is finite for each $n \ge 2$ and is monotonically decreasing. Consequently, if we let $K_\chi := \gamma(K_1 + K_2(2))^2$ then the desired result follows. □

Theorem 6.1. *Fix $k \ge 1$. Let $p = n^\alpha$, $\alpha \in (-1/k, -1/(k+1))$. Then, for the sequence $\{\bar\beta_{n,k}\colon n \ge 1\}$, condition (C2) holds with $\Upsilon_1 = \Upsilon_2 = 2$, i.e. for any $T > 0$, there exists $K_\beta > 0$ such that, for all $n \ge 1$, $0 \le t \le T+1$, and $0 \le h \le t$,*

$$\mathbb{E}[\bar\beta_{n,k}(t+h) - \bar\beta_{n,k}(t)]^2[\bar\beta_{n,k}(t) - \bar\beta_{n,k}(t-h)]^2 \le K_\beta h^2.$$

Proof. From Lemmas 3.2, 3.4, and 3.6, we have $\operatorname{var}[\beta_{n,k}(t)] = \Theta(n^{2k} p^{2\binom{k+1}{2}-1})$. Hence, as discussed in Lemma 6.3, to prove the desired result it suffices to show that there exists $K_\beta > 0$ such that

$$\Omega_{n,k}(h) := \frac{\mathbb{E}[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2}{n^{4k} p^{4\binom{k+1}{2}-2}} \le K_\beta h^2 \quad \text{for all } n \ge 1,\ 0 \le h \le 1.$$

Now fix an arbitrary $h \in [0,1]$ and consider the event

$$E = \{(-1)^k \chi_n(0) = \beta_{n,k}(0)\} \cap \{(-1)^k \chi_n(h) = \beta_{n,k}(h)\} \cap \{(-1)^k \chi_n(2h) = \beta_{n,k}(2h)\}. \tag{6.14}$$

Then, observe that

$$\mathbb{E}[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2 = \text{term}_1 + \text{term}_2, \tag{6.15}$$

where

$$\text{term}_1 = \mathbb{E}[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2\, \mathbf{1}_E$$

and

$$\text{term}_2 = \mathbb{E}[\beta_{n,k}(2h) - \beta_{n,k}(h)]^2[\beta_{n,k}(h) - \beta_{n,k}(0)]^2\, \mathbf{1}_{E^c}. \tag{6.16}$$

Clearly,

$$\text{term}_1 = \mathbb{E}[\chi_n(2h) - \chi_n(h)]^2[\chi_n(h) - \chi_n(0)]^2\, \mathbf{1}_E$$

and, hence, using Lemma 6.3, it follows that

$$\frac{\text{term}_1}{n^{4k} p^{4\binom{k+1}{2}-2}} \le K_\chi h^2. \tag{6.17}$$

To obtain a bound on term_2, we consider an alternate but equivalent description of the dynamic Erdős–Rényi graph. Specifically, to each edge e, independently associate two independent sequences $T^e := \{T_i^e\}_{i\ge1}$ and $I^e := \{I_i^e\}_{i\ge0}$, where the T^e are arrival times of

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a Poisson process with parameter λ and the I^e are independent and identically distributed Bernoulli random variables which take the 'on' state with probability p and 'off' state with probability $1-p$. Let $T_0^e = 0$. If we define the state of the edge e at time t as

$$e(t) := \sum_{i\geq 0} \mathbf{1}_{\{T_i^e \leq t < T_{i+1}^e\}}\, I_i^e,$$

then it follows that the behaviour of edge e is that of an edge in the dynamic Erdős–Rényi graph. Firstly, the initial configuration $e(0) = I_0^e$ almost surely and so $\mathbb{P}\{e(0) = \text{on}\} = p$, as required. Fix $t_1 < t_2$. Let $\#_T$ be the cardinality of $\{i : T_i^e \in (t_1, t_2]\}$ and, if $\#_T > 0$, let $i_{\text{last}} := \arg\max\{i : T_i^e \in (t_1, t_2]\}$. Then

$$\begin{aligned}\mathbb{P}\{e(t_2) = \text{on} \mid e(t_1) = \text{on}\} &= \mathbb{P}\{\#_T = 0\} + \sum_{\ell>0} \mathbb{P}\{\#_T = \ell, T_{i_{\text{last}}}^e = \text{on}\} \\ &= \mathrm{e}^{-\lambda(t_2-t_1)} + \sum_{\ell>0} \mathrm{e}^{-\lambda(t_2-t_1)} \frac{[\lambda(t_2-t_1)]^\ell}{\ell!} p,\end{aligned}$$

where the last equality follows due to independence of T^e and I^e. From this, it is easy to see that (1.1) holds. Similarly, one can check that (1.2) also holds. This verifies the equivalence of the two descriptions of the dynamic Erdős–Rényi graph.

Let $S_{0,h} := \sum_e \sum_{i\geq 1} \mathbf{1}_{\{T_i^e \leq h\}}$ denote the sum of arrivals that occurred across each edge in time $(0, h]$. Let $\tau_1, \tau_2, \ldots,$ with $\tau_i \leq \tau_{i+1}$, denote the sequence of arrival times in $(0, h]$ at which these $S_{0,h}$ arrivals occurred. Note that τ_i and τ_{i+1} could correspond to arrivals along different edges. Separately, let $\tau_0 = 0$. Let $\mathcal{P}_0$ denote the event that no arrival occurs at time 0, i.e. for all $i \geq 1$, $\tau_i > 0$. Then,

$$|\beta_{n,k}(h) - \beta_{n,k}(0)|\, \mathbf{1}_{\mathcal{P}_0} \leq \sum_{i=1}^{S_{0,h}} |\beta_{n,k}(\tau_i) - \beta_{n,k}(\tau_{i-1})|\, \mathbf{1}_{\mathcal{P}_0}.$$

Using [23, Lemma 2.2], it then follows that

$$\begin{aligned}&|\beta_{n,k}(h) - \beta_{n,k}(0)|\, \mathbf{1}_{\mathcal{P}_0} \\ &\quad\leq \sum_{i=1}^{S_{0,h}} |f_{n,k}(\tau_i) - f_{n,k}(\tau_{i-1})|\, \mathbf{1}_{\mathcal{P}_0} + \sum_{i=1}^{S_{0,h}} |f_{n,k+1}(\tau_i) - f_{n,k+1}(\tau_{i-1})|\, \mathbf{1}_{\mathcal{P}_0}.\end{aligned}$$

However, $|f_{n,k}(\tau_i) - f_{n,k}(\tau_{i-1})| \leq \binom{n}{k+1}$ and $|f_{n,k+1}(\tau_i) - f_{n,k+1}(\tau_{i-1})| \leq \binom{n}{k+2}$. Hence,

$$|\beta_{n,k}(h) - \beta_{n,k}(0)|\, \mathbf{1}_{\mathcal{P}_0} \leq \left[\binom{n}{k+1} + \binom{n}{k+2}\right] S_{0,h}\, \mathbf{1}_{\mathcal{P}_0} \leq 2n^{k+2} S_{0,h}.$$

Similarly, if we let $S_{h,2h}$ denote the total number of arrivals across edges in $(h, 2h]$, then

$$|\beta_{n,k}(2h) - \beta_{n,k}(h)|\, \mathbf{1}_{\mathcal{P}_h} \leq 2n^{k+2} S_{h,2h},$$

where $\mathcal{P}_h$ denotes the event that no arrivals happened at time h. Since $\mathbf{1}_{\mathcal{P}_0}$ and $\mathbf{1}_{\mathcal{P}_h}$ are almost sure events, the above inequalities, combined with (6.16), show that

$$\text{term}_2 \leq 16n^{4k+8}\mathbb{E}[S_{0,h}^2 S_{h,2h}^2\, \mathbf{1}_{E^c}].$$

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Now, using (6.14), note that

$$\mathbf{1}_{E^c} \leq \mathbf{1}_{\{(-1)^k \chi_n(0) \neq \beta_{n,k}(0)\}} + \mathbf{1}_{\{(-1)^k \chi_n(h) \neq \beta_{n,k}(h)\}} + \mathbf{1}_{\{(-1)^k \chi_n(2h) \neq \beta_{n,k}(2h)\}}.$$

Consequently, we have

$$\begin{aligned}\text{term}_2 \leq 16n^{4k+8}\{&\mathbb{E}[S_{0,h}^2 S_{h,2h}^2\, \mathbf{1}_{\{(-1)^k \chi_n(0) \neq \beta_{n,k}(0)\}}] + \mathbb{E}[S_{0,h}^2 S_{h,2h}^2\, \mathbf{1}_{\{(-1)^k \chi_n(h) \neq \beta_{n,k}(h)\}}] \\ &+ \mathbb{E}[S_{0,h}^2 S_{h,2h}^2\, \mathbf{1}_{\{(-1)^k \chi_n(2h) \neq \beta_{n,k}(2h)\}}]\}.\end{aligned}$$

However, for any $t \geq 0$, note that $\mathbf{1}_{\{(-1)^k \chi_n(t) \neq \beta_{n,k}(t)\}}$ is a function of only $G(n, p, t)$ which, in turn, is a function of only $\{I^e_{i_e(t)}\}$, where

$$i_e(t) := \min\{i : T^e_i \leq t < T^e_{i+1}\}.$$

Since for each e, the independent and identically distributed sequence $\{I^e_i\}$ and the sequence $\{T^e_i\}$ are independent, it is not difficult to see that $\bigcup_e \{I^e_{i_e(t)}\}$ is independent of $\bigcup_e \{T^e_i\}$. So, $S_{0,h}$, $S_{h,2h}$ (both of which depend only upon $\cup_e \{T^e_i\}$) and $\mathbf{1}_{\{(-1)^k \chi_n(t) \neq \beta_{n,k}(t)\}}$ (which depends only upon $\bigcup_e \{I^e_{i_e(t)}\}$) are mutually independent for any $t \geq 0$. Since $S^2_{0,h}$ and $S^2_{h,2h}$ are Poisson with parameter $\binom{n}{2}\lambda h$, we have

$$\mathbb{E}[S^2_{0,h}] = \mathbb{E}[S^2_{h,2h}] = \binom{n}{2}\lambda h + \binom{n}{2}^2 \lambda^2 h^2 \leq 2n^4\lambda^2 h,$$

where the last inequality follows since $0 \leq h \leq 1$. Consequently, we have

$$\begin{aligned}\text{term}_2 \leq 64n^{4k+16}\lambda^2 h^2\{&\mathbb{P}\{(-1)^k \chi_n(0) \neq \beta_{n,k}(0)\} \\ &+ \mathbb{P}\{(-1)^k \chi_n(h) \neq \beta_{n,k}(h)\} + \mathbb{P}\{(-1)^k \chi_n(2h) \neq \beta_{n,k}(2h)\}\}.\end{aligned}$$

However, from Theorem 1.1,

$$\mathbb{P}\{(-1)^k \chi_n(t) \neq \beta_n(t)\} = o(n^{-M}) \quad \text{for any } M > 0.$$

Using this, it is not difficult to see that there exists $K'_\beta > 0$ such that

$$\frac{\text{term}_2}{n^{4k} p^{4\binom{k+1}{2}-1}} \leq K'_\beta h^2. \tag{6.18}$$

Combining (6.15), (6.17), and (6.18), the desired result follows. □

Combining Lemma 6.1 and Theorem 6.1 shows that the sequence of processes $\{\bar{\beta}_{n,k} : n \geq 1\}$ is tight. Combining this with Theorem 5.1 completes the proof for Theorem 1.3, as desired. Note that along the way we have also proved that if $p = n^\alpha$ with $\alpha \in (-1/k, -1/(k+1))$, then the sequences of processes $\{\bar{f}_{n,k} : n \geq 1\}$ and $\{\bar{\chi}_n : n \geq 1\}$ converge in distribution to the stationary Ornstein–Uhlenbeck process.

Appendix A. The processes $\bar{\beta}_{n,k}$ are not Markovian

Although the dynamic Erdős–Rényi graph $\{G(n, p, t) : t \geq 0\}$ is a continuous-time Markov chain, and the processes $\{\beta_{n,k}(t) : t \geq 0\}$ are pointwise functions of them, they themselves are not Markovian. To prove this, we need the following result from [5, Theorem 4].

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Theorem A.1. *Let $\{X(t) : t \geq 0\}$ be a Markov chain on the state space $M = \{1, \ldots, m\}$, with arbitrary initial distribution, and stationary transition probability function $P(t) = (p_{ij}(t))$, continuous in t. Assume that $\lim_{t \to 0} P(t) = \mathbb{I}$. Let ψ be a function M on $Y(t) = \psi(X(t))$. If the states if Y are $y_1, \ldots, y_r$, $r \leq m$, define r disjoint subsets of M by $S_j = \{i \in M : \psi(i) = y_j\}$. Then Y is Markovian if, and only if, for each $j = 1, \ldots, r$, either one of the following conditions holds.*

(i) $p_{i,S_j}(t) \equiv 0$ *for all* $i \notin S_j$.

(ii) $p_{i,S_j}(t) = C_{S_{j'}}, S_j(t)$ *for every* $i \in S_{j'}$ *for* $j' = 1, \ldots, r$, *where* $C_{S_{j'},S_j}(t)$ *is a constant that depends only on* $S_{j'}$, S_j, *and* t.

(Note that (ii) implies (i), and so (i) is irrelevant for the 'only if' part of the theorem.)

An example which shows that the process $\{\beta_{n,k}(t) : t \geq 0\}$ is not Markov for finite n is the following. Consider the dynamic Erdős–Rényi graph with $n = 4$, arbitrary $p \in (0, 1)$, and arbitrary $\lambda > 0$. At any given time t, each of its six edges, say $e_1, \ldots, e_6$, can be either in 'on' or 'off' state. Thus, $G(4, p, t)$ has $m = 64$ possible configurations. However, the process $\{\beta_{4,1}(t) : t \geq 0\}$ can only take $r = 2$ values, i.e. 0 or 1. This can be inferred from Figure 1, in which we show the different edge configurations when $\beta_{4,1}(t) = 1$, and the fact that if more than one of these configurations occur simultaneously then the resulting complex will have $\beta_{4,1}(t) = 0$. Hence, using (1.1) and (1.2), we have

$$\mathbb{P}\{\beta_{4,1}(t + s) = 1 \mid e_1(s) = \cdots = e_6(s) = \text{off}\} = 3p^4(1 - \mathrm{e}^{-\lambda t})^4((1 - p) + p\mathrm{e}^{-\lambda t})^2,$$

while

$$\mathbb{P}\{\beta_{4,1}(t + s) = 1 \mid e_1(s) = \cdots = e_6(s) = \text{on}\} = 3(p + (1 - p)\mathrm{e}^{-\lambda t})^4(1 - p)^2(1 - \mathrm{e}^{-\lambda t})^2.$$

Clearly, for a generic p and t, the above two equations are not equal. On the other hand, $\beta_{4,1}(s) = 0$ when either $e_1(s) = \cdots = e_6(s) = \text{off}$, or $e_1(s) = \cdots = e_6(s) = \text{on}$. These facts along with Theorem A.1 show that the process $\{\beta_{4,1}(t) : t \geq 0\}$ is not Markovian.

Appendix B. Exact expression for $\mathbb{E}[g(h; \bar{A})]$

Consider the notations defined below Lemma 6.3. Clearly,

$$\begin{aligned} g(h; \bar{A}) = {} & \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(h) + \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(0) \\ & + \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(0) + \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(h) \\ & + \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(0) + \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(0) \\ & + \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(h) + \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(h) \\ & - \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(0) - \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(h) \\ & - \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(h) - \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(0) \\ & - \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(h) - \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(h)\,\mathbf{1}_{A_4}(h) \\ & - \mathbf{1}_{A_1}(2h)\,\mathbf{1}_{A_2}(h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(0) - \mathbf{1}_{A_1}(h)\,\mathbf{1}_{A_2}(2h)\,\mathbf{1}_{A_3}(0)\,\mathbf{1}_{A_4}(0). \end{aligned} \tag{B.1}$$

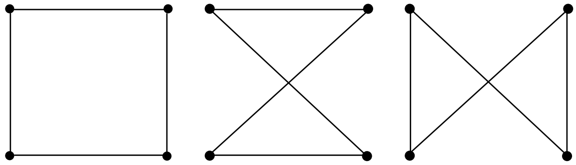

FIGURE 1: Configurations of $G(4, p, t)$ with $\beta_{4,1}(t) = 1$ (no vertices at intersections).

Using (1.1) and (1.3), it is not difficult to see that if $\tau(h) := p + (1-p)\mathrm{e}^{-\lambda h}$ then $\mathbb{E}[g(h;\, \bar{A})] = p^{\mathrm{pair}(\bar{A})}\Phi(h;\, \bar{A})$, where

$$
\begin{aligned}
\Phi(h;\, \bar{A}) = {} & [\tau(h)]^{\binom{a_{13}}{2}+\binom{a_{14}}{2}+\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& + [\tau(h)]^{\binom{a_{13}}{2}+\binom{a_{14}}{2}+\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& + [\tau(2h)]^{\binom{a_{13}}{2}+\binom{a_{14}}{2}+\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} + 1 \\
& + [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{13}}{2}+\binom{a_{24}}{2}+\binom{a_{34}}{2}-\binom{a_{234}}{2}-\binom{a_{123}}{2}}[\tau(2h)]^{\binom{a_{14}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}+\binom{a_{1234}}{2}} \\
& + [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{23}}{2}+\binom{a_{14}}{2}+\binom{a_{34}}{2}-\binom{a_{134}}{2}-\binom{a_{123}}{2}}[\tau(2h)]^{\binom{a_{24}}{2}-\binom{a_{124}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& + [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{14}}{2}+\binom{a_{23}}{2}+\binom{a_{34}}{2}-\binom{a_{234}}{2}-\binom{a_{124}}{2}}[\tau(2h)]^{\binom{a_{13}}{2}-\binom{a_{123}}{2}-\binom{a_{134}}{2}+\binom{a_{1234}}{2}} \\
& + [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{24}}{2}+\binom{a_{13}}{2}+\binom{a_{34}}{2}-\binom{a_{134}}{2}-\binom{a_{124}}{2}}[\tau(2h)]^{\binom{a_{23}}{2}-\binom{a_{123}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{13}}{2}+\binom{a_{23}}{2}+\binom{a_{34}}{2}-\binom{a_{123}}{2}}[\tau(2h)]^{\binom{a_{14}}{2}+\binom{a_{24}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{14}}{2}+\binom{a_{24}}{2}+\binom{a_{34}}{2}-\binom{a_{124}}{2}}[\tau(2h)]^{\binom{a_{13}}{2}+\binom{a_{23}}{2}-\binom{a_{123}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{14}}{2}+\binom{a_{24}}{2}+\binom{a_{34}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{13}}{2}+\binom{a_{23}}{2}+\binom{a_{34}}{2}-\binom{a_{123}}{2}-\binom{a_{134}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{13}}{2}+\binom{a_{14}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{234}}{2}}[\tau(2h)]^{\binom{a_{13}}{2}+\binom{a_{14}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{134}}{2}+\binom{a_{1234}}{2}} \\
& - [\tau(h)]^{\binom{a_{12}}{2}+\binom{a_{13}}{2}+\binom{a_{14}}{2}-\binom{a_{134}}{2}}[\tau(2h)]^{\binom{a_{23}}{2}+\binom{a_{24}}{2}-\binom{a_{123}}{2}-\binom{a_{124}}{2}-\binom{a_{234}}{2}+\binom{a_{1234}}{2}}.
\end{aligned}
\tag{B.2}
$$

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