Can. J. Math., Vol. XXVII, No. 1, 1975, pp. 44–49

ON THE MAPPING PROPERTIES OF CERTAIN EXCEPTIONAL SETS IN $\mathbf{R}^2$

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It is known that minimally thin and semithin sets in $\mathbf{R}^2$ are preserved by conformal mappings (see [**3**]) but it is not known whether or not analogous results hold true for ordinary thin and semithin sets respectively. In an unpublished work, Brelot has shown that ordinary thin sets at the origin are preserved by mappings of the form $f(z) = z^\alpha$ $(\alpha > 0)$ where one always considers the principal branch of the mapping. We shall prove this result along with an analogous one for ordinary semithin sets and will see that the implications established by Jackson (see [**4**, Theorem 4]), for ordinary and minimally thin sets and by Brelot (see [**2**, p. 152]) for semithin sets in a half plane hold true for any wedge shaped region with vertex at the origin.

Our main purpose, however, will be to show that ordinary thin (respectively semithin) sets in $\mathbf{R}^2$ are not preserved by conformal maps and that ordinary thinness (respectively semithinness) of a set at a Euclidean boundary point of a region does not generally imply minimal thinness (respectively minimal semithinness) at the associated minimal Martin boundary point even when the association is unique. In fact we shall demonstrate that ordinary thinness does not even imply minimal semithinness in general. For brevity, we shall in future refer to an ordinary thin set as a thin set, and employ a similar abbreviation for ordinary semithin sets. We shall generally follow the notation in Brelot [**2**], but shall specifically employ the following notation when dealing with a set E contained in some neighbourhood of 0.

Notation.

(i) J_n is the annulus $\{s^{n+1} < |z| \leqq s^n\}$ where $0 < s < 1$.

(ii) $E_n = E \cap J_n$.

(iii) $\lambda(E)$ is the logarithmic capacity of E and $\lambda_n = \lambda(E_n)$.

(iv) $\Lambda(r) = (\log 1/r)^{-1}$, if $0 < r < 1$
$= 0$, if $r = 0$.

(v) $\gamma(E) = \Lambda \circ \lambda(E)$ is the ordinary (Wiener) capacity of E and $\gamma_n = \gamma(E_n)$ (see [**1**, p. 321]). We assume that E is contained in a disk of diameter less than one.

We now begin with some preliminary lemmas.

LEMMA 1. *Let $H = \{z\colon \operatorname{Re} z > 0\}$ be the right half plane and let $E \subset H$ be thin (respectively semithin) at* 0. *If $f(z)$ is the principal branch of $z^{1/m}$ where*

Received June 22, 1973. This research was partially supported by NRC Grant 2141563.

$m > 1$ is any natural number and such that H is the domain of f, then $f(E) = E'$ is also thin (respectively semithin) at 0.

Proof. Let $D = \{z: |z| < \epsilon\}$ for some suitable $\epsilon > 0$ and suppose that E is thin at 0. Let $u = \hat{R}_1^E$ be the regularised reduced function of 1 on E with respect to D, and observe that u is a non-negative superharmonic function on D such that $u(0) < 1$ (see Brelot [**2**, p. 55]) if ϵ is chosen sufficiently small. Now let $D' = \{w : |w| < \epsilon^{1/m}\}$ and let $g(w) = w^m$ on D'. Then $v = u \circ g$ is a non-negative superharmonic function on D' such that $v(0) = u(0)$. Furthermore v dominates the regularised reduced function $\hat{R}_1^{E'}$ on D' and therefore $\hat{R}_1^{E'}(0) < 1$ which implies that E' is thin at 0.

For the semithinness part of the proof we note that E is semithin at 0 if and only if $\lim_{n\to\infty} \hat{R}_1^{E_n}(0) = 0$ (see Brelot [**2**, p. 82]). By repeating our earlier reasoning we can say that $\hat{R}_1^{E_n}(0) > \hat{R}_1^{E_n'}(0)$ where $E_n' = E' \cap J_n$ and $E_n = g(E_n')$. The lemma follows.

LEMMA 2. *Let $f(z)$ be the principal branch of the mapping $z^\alpha (\alpha \geqq 1)$ where the wedge shaped region*

$$W = \{re^{i\theta} : r > 0, 0 \leqq |\theta| < \theta_0 \leqq \pi\}$$

is the domain of f. If E is thin (respectively semithin) at 0 then $E' = f(E)$ is also thin (respectively semithin) at 0.

Proof. The case $\alpha = 1$ is obvious so we focus attention on the case where $\alpha > 1$. Since $|f'(z)| \to 0$ as $|z| \to 0$ we can choose $\epsilon > 0$ such that if $D = \{z: |z| < \epsilon\}$ then f is a contraction mapping on $W \cap D$. Furthermore, a subset of the annulus J_n corresponding to some s, $0 < s < 1$, is mapped by f onto a subset of J_n' corresponding to s^α. Hence $\gamma_n' = \gamma(E_n') \leqq \gamma(E_n) = \gamma_n$ (see Brelot [**1**, p. 330]) and therefore $\sum_n n\gamma_n' < \infty$ if E is thin at 0 (see Brelot [**2**, p. 81]). Similarly $n\gamma_n' \to 0$ as $n \to \infty$ if E is semithin at 0 (see Brelot [**2**, p. 82]). The lemma follows.

THEOREM 1. *Let $f(z)$ be the principal branch of the mapping $z^\alpha (\alpha > 0)$ such that H is the domain of f. If $E \subset H$ is thin (respectively semithin) at 0 then $E' = f(E)$ is also thin (respectively semithin) at 0.*

Proof. The case where $\alpha \geqq 1$ is already covered by Lemma 2. Now consider the case where $0 < \alpha < 1$. We choose a natural number m such that $m\alpha = \beta > 1$. If $g(z)$ is the principal branch of $z^{1/m}$ on H such that $W = g(H)$, and if $h(w)$ is the principal branch of w^β on W then $f = h \circ g$. By Lemma 1, g preserves thin (respectively semithin) sets at 0 and by Lemma 2, h also has the same properties. The theorem follows, and we note that similar reasoning will show that the principal branch of f^{-1} will also preserve thin (respectively semithin) sets.

THEOREM 2. Let $W = \{re^{i\theta}: r > 0, 0 \leqq |\theta| < \theta_0 \leqq \pi\}$ *be a wedge shaped region in $\mathbf{R}^2$ with vertex at 0. If $E \subset W$ is thin (respectively semithin) at 0 in $\mathbf{R}^2$ then E is minimally thin (respectively semithin) at 0 with respect to W.*

https://doi.org/10.4153/CJM-1975-007-1 Published online by Cambridge University Press

Proof. If $\alpha = 2\theta_0/\pi$ and if $f(z)$ is the principal branch of z^α then f constitutes a conformal mapping of H onto W. The function

$$h(w) = \frac{\cos \theta/\alpha}{r^{1/\alpha}}$$

is a minimal harmonic function on W with pole at 0 and we recall (see [**2**, p. 122]) that E is minimally thin at 0 with respect to W if and only if $(R_h^E)_W \not\equiv h$ on W. Let f^{-1} denote the principal branch of the inverse mapping from W onto H. If $E \subset W$ is thin (respectively semithin) at 0 in $\mathbf{R}^2$ then $E' = f^{-1}(E)$ is thin (respectively semithin) at 0 in $\mathbf{R}^2$ by Theorem 1, and therefore E' is minimally thin (respectively semithin) at 0 with respect to H (see Jackson [**4**, Theorem 4] for thinness and Brelot [**2**, p. 152] for semithinness). Since minimal thinness (respectively semithinness) is a conformal invariant (see [**3**]) therefore $E = f(E')$ is minimally thin (respectively semithin) at 0 with respect to W. This proves our theorem.

We shall now develop some criteria for thinness and semithinness at 0 which are not commonly known; but first we introduce some further notation.

Notation.

(i) If $E_n \subset J_n = \{s^{n+1} < |z| \leqq s^n: 0 < s < 1\}$ we write $E_n^* = s^{-n}E_n$.

(ii) $\lambda_n^* = \lambda(E_n^*)$ and $\gamma_n^* = \gamma(E_n^*)$.

(iii) If E is a subset of the real axis we write $m(E)$ for the linear measure of E, $m_n = m(E_n)$ and $m_n^* = m(E_n^*)$.

LEMMA 3. *E is semithin (respectively thin) at 0 if and only if*

$$\lim_{n\to\infty} n\gamma_n^* = 0 \left(\textit{respectively, } \sum_n n\gamma_n^* < +\infty\right).$$

Proof. Since $\gamma_n^* = \gamma(E_n^*) \geqq \gamma(E_n) = \gamma_n$ the "if" part of the lemma follows immediately. For the converse we observe that $n\gamma_n^* = n[\log (s^n/\lambda_n)]^{-1}$ and that an elementary calculation gives $n\gamma_n^* = n\gamma_n[1 + n\gamma_n \log s]^{-1}$. If $\lim_{n\to\infty} n\gamma_n = 0$ then $n\gamma_n^* \leqq 2n\gamma_n$ for all n sufficiently large and the lemma follows.

Remark 1. It is proved by Jackson (see [**5**, Theorem 1′]) that E restricted to a Stolz domain in H is minimally thin at 0 with respect to H if and only if $\sum_n \gamma_n^* < \infty$. It is now obvious that for a subset of a Stolz domain, thinness strictly implies minimal thinness. On the other hand it is clear from Lemma 3 that semithinness and minimal thinness are non-comparable concepts.

LEMMA 4. *If E is semithin at 0 and if E' is the circular projection of E onto the positive real axis then E' is of finite logarithmic length. This is a strict implication.*

Proof of Lemma 4. Now E is semithin at 0 if and only if $n\gamma_n \to 0$ as $n \to \infty$ which implies that $n\gamma(E_n') \to 0$ as $n \to \infty$. We recall that $\lambda_n' \geqq m(E_n')/4$

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(see [4, p. 210]) and therefore the semithinness of E implies that

$$n\,\Lambda(m_n') = n \log \left(\frac{1}{m_n'}\right)^{-1} \to 0 \quad \text{as } n \to \infty.$$

An elementary calculation shows that $m_n' < e^{-2n}$ for all n sufficiently large, and hence $\sum_n e^n m_n' < \infty$. Since $m_n' < e^{-2n}$ for n sufficiently large it is clear that the implication is strict.

Remark 2. We recall from Jackson (see [4, Theorem 6]) that a subset E of a Stolz domain in H is minimally semithin at 0 if and only if $\lim_{n\to\infty} \gamma_n^* = 0$ or equivalently $\lim_{n\to\infty} \lambda_n^* = 0$. If each E_n is an interval on the positive real axis then E is minimally semithin at 0 if and only if $\lim_{n\to\infty} (s^{-n}m_n) = \lim_{n\to\infty} m_n^* = 0$.

For computational purposes we shall now find it convenient to consider sets which are thin (respectively semithin) at ∞ rather than at 0. Both types of thinness (respectively semithinness) are preserved by the inversion mapping $\phi(z) = z^{-1}$ along with the finite logarithmic length property. The criteria for thinness (respectively semithinness) as developed in Lemma 3 work equally well at ∞ except that we now consider circles of radius s^n where $s > 1$ and define $J_n = \{z \colon s^n \leqq |z| < s^{n+1}\}$.

LEMMA 5. *A thin (respectively semithin) set at ∞ is not necessarily preserved by the exponential mapping $g(z) = e^z$.*

Proof. We proceed by an example on the real axis. Let E_n be the interval $[2^n, 2^n + \Delta x_n)$ and $E = \bigcup_{n=1} E_n$. If we choose $\Delta x_n = e^{-n^3}$ then $\lambda_n^* \leqq m_n^* \leqq e^{-n^3}$ and hence $\gamma_n^* \leqq 1/n^3$. It follows that E is both thin and semithin at ∞ in $\mathbf{R}^2$. If $E_n' = g(E_n)$ then E_n' is an interval with left end point at e^{2^n} such that

$$e^{2^n}\Delta x_n \leqq m(E_n') \leqq e^{(2^n+\Delta x_n)}\Delta x_n.$$

Now $E_n' \subset J_{2^n}$ with respect to $s = e$ and we can say that

$$\Delta x_n/4 \leqq \lambda_{2^n}^* \leqq (e^{\Delta x_n}/4)\Delta x_n.$$

Hence $\gamma_{2^n}^* = \Lambda(\lambda_{2^n}^*)$ is asymptotic to $1/n^3$ and if $k \neq 2^n$ then $\gamma_k^* = 0$. It follows that

$$\limsup_{k\to\infty} k\gamma_k^* = \lim_{n\to\infty} (2^n n^{-3}) > 0,$$

and that $\sum_k \gamma_k^* < +\infty$. We can conclude that $g(E)$ is not semithin at ∞ in $\mathbf{R}^2$ but is minimally thin at ∞ with respect to H. The lemma follows.

LEMMA 6. *Let E be thin at ∞ and defined as in Lemma 5 where $\Delta x_n = e^{-n^3}$. If $f(z) = g(g(z))$ where $g(z) = e^z$ then $f(E)$ fails to have finite logarithmic length.*

Proof. Now

$$\int_{f(E)} \frac{du}{u} = \sum_n \int_{E_n} \frac{f'(x)}{f(x)}\,dx \quad \text{where } f(x) = e^{e^x}.$$

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Hence

$$\int_{f(E)} \frac{du}{u} = \sum_n \left(\int_{E_n} e^x dx \right) \geqq \sum_n e^{2^n} (\Delta x_n)$$

which diverges if $\Delta x_n = e^{-n^3}$. Hence $f(E)$ fails to have finite logarithmic length even though E is thin at ∞ in $\mathbf{R}^2$.

Remark 3. From Lemma 6 and [**4**, Theorem 5], we notice that a conformal mapping may carry a set E which is thin at ∞ in $\mathbf{R}^2$ to a set which is not minimally thin at ∞ with respect to H. Even stronger results can be obtained.

LEMMA 7. *Let $h(z) = 2^z$, $g(w) = e^w$ and $f(z) = g \circ h(z)$. If E is thin at ∞ and defined as in Lemma 5 where $\Delta x_n = e^{-n^3}$ then $f(E)$ will not be minimally semithin at ∞ with respect to H.*

Proof. We recall that $E = \bigcup_{n=1}^{\infty} E_n$ where $E_n = [2^n, 2^n + e^{-n^3}]$. Now $f(E_n)$ is an interval with left end point $e^{(2^{2^n})}$. We will concentrate on $f(E) \cap J_k$, (note $s = e$ in defining J_k) where $k = 2^{2^n}$. Now

$$m(f(E_n)) = \int_{E_n} f'(x)dx \geqq e^{(2^{2^n})} 2^{2^n} (\log 2) e^{-n^3}$$

so that if $k = 2^{2^n}$ then $m_k e^{-k} \geqq \inf\{2^{2^n} e^{-n^3} \log 2, e - 1\}$ which implies that $\limsup_{k \to \infty} (m_k e^{-k}) > 0$. By Remark 2, it follows that $f(E)$ cannot be minimally semithin at ∞ with respect to H.

Remark 4. Let D be a bounded, simply connected region in $\mathbf{R}^2$ whose Euclidean boundary, ∂D, is a piecewise smooth curve. Since the Martin boundary points of D can be identified with the prime ends of D therefore we can identify ∂D and $\Delta(D) = \Delta_1(D)$ in a natural way. We shall now show that if $E \subset D$ is thin at $x_0 \in \partial D$ in $\mathbf{R}^2$ then E is not necessarily minimally semithin at $x_0 \in \Delta_1(D)$ with respect to D.

THEOREM 3. *Let D be a bounded simply connected region in $\mathbf{R}^2$ whose boundary ∂D is a piecewise smooth curve such that $0 \in \partial D$. We can construct D so that if $E \subset D$ is thin at 0 then E is not necessarily minimally semithin at $0 \in \Delta_1(D)$.*

Proof. As before we let $\phi(z) = 1/z$, $h(z) = 2^z$, $g(z) = e^z$ and $f = g \circ h$. Now define the horizontal half strip to be $D_2 = \{(x, y): x > 2$ and $|y| < \pi/2\}$. Then $g(D_2) = D_3$ is the right half plane H minus the disk of centre 0 and radius e^2. Now let L denote the principal branch of the relation $\log_2 = h^{-1}$ and then define $D_1 = L(D_2)$ and $D = \phi(D_1)$ respectively. We note that D is a bounded simply connected region with piecewise smooth boundary. Furthermore D contains the interval $(0, 1)$ of the real axis and D_1 contains $(1, \infty)$. If $F = f \circ \phi$ then F is a conformal mapping from D onto D_3. Furthermore $E \subset D$ is minimally semithin at 0 in D if and only if $F(E)$ is minimally semithin at ∞ with respect to D_3 or H. Now define $E \subset D$ such that

$$\phi(E) = E_1 = \bigcup_{n=1}^{\infty} [2^n, 2^n + e^{-n^3}]$$

and recall that E_1 is thin at ∞. Then E is thin at 0 in $\mathbf{R}^2$ but $F(E) = f(E_1)$ is not minimally semithin at ∞ with respect to H or D_3 as a consequence of Lemma 7. Hence E is not minimally semithin at 0 with respect to D and our theorem is proved.

Remark 5. For the region D as constructed in the proof of Theorem 3 it is evident that ordinary thinness (respectively semithinness) and minimal thinness (respectively semithinness) are independent concepts at the boundary point 0. The function which defines ∂D at 0 is asymptotic to the function $x^2 e^{-1/x}$.

REFERENCES

1. M. Brelot, *Points irréguliers et transformations continues en théorie du potentiel*, J. Math. Pures Appl. *19* (1940), 319–337.

2. ——— *On topologies and boundaries in potential theory*, Lecture Notes in Math., vol. *175* (Springer).

3. M. Brelot and J. L. Doob, *Limites angulaires et limites fines*, Ann. Inst. Fourier (Grenoble) *13* (1963), 395–415.

4. H. L. Jackson, *Some results on thin sets in a half plane*, Ann. Inst. Fourier (Grenoble) *20* (1970), 201–218.

5. ——— *On minimally thin sets in a Stolz domain*, Can. Math. Bull. *15* (1972), 219–223.

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