Ergod. Th. & Dynam. Sys., (2021), **41**, 2782–2804 © The Author(s), 2020. Published by Cambridge University Press.
doi:10.1017/etds.2020.87

Limits of geodesic push-forwards of horocycle invariant measures

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(*Received* 27 *November* 2019 *and accepted in revised form* 22 *June* 2020)

Abstract. We prove several general conditional convergence results on ergodic averages for horocycle and geodesic subgroups of any continuous $\mathrm{SL}(2, \mathbb{R})$-action on a locally compact space. These results are motivated by theorems of Eskin, Mirzakhani and Mohammadi on the $\mathrm{SL}(2, \mathbb{R})$-action on the moduli space of Abelian differentials. By our argument we can derive from these theorems an improved version of the 'weak convergence' of push-forwards of horocycle measures under the geodesic flow and a short proof of weaker versions of theorems of Chaika and Eskin on Birkhoff genericity and Oseledets regularity in almost all directions for the Teichmüller geodesic flow.

Key words: Teichmüller horocycle flow, pointwise equidistribution, Birkhoff ergodic theorem, Oseledets multiplicative ergodic theorem
2020 Mathematics Subject Classification: 37A30, 30F60 (Primary); 22F30, 57M60 (Secondary)

1. *Introduction*

It has been conjectured that push-forwards, under the forward Teichmüller geodesic flow, of ergodic probability measures for the unstable Teichmüller horocycle flow, and similarly of measures uniformly distributed on unstable horocycle arcs or on arcs of the circle action, converge (to an $\mathrm{SL}(2, \mathbb{R})$-invariant measure). To the best of our knowledge, Veech was the first to ask this question, after his work [**V98**] on Siegel measures (now called Siegel–Veech measures).

Bainbridge, Smillie and Weiss have proved this conjecture for certain invariant orbifolds in the stratum $H(1, 1)$ of Abelian differentials with two simple zeros on genus-two surfaces (see [**BSW**, Theorems 1.5 and 12.7]).

The main purpose of this note is to prove that *up to removing a set of times of zero upper density* the general conjecture is in fact a corollary of results of Eskin, Mirzakhani and Mohammadi (see [**EM**] and [**EMM**]).

Our argument is based on the idea of lifting a family of (probability) measures on a compact space to measures on the space of probability measures and then to derive restrictions from the well-known extremal property of ergodic probability measures with respect to the subset of all invariant measures.

The same argument applies to limits of push-forwards under the Teichmüller geodesic flow of the Lebesgue measure on Teichmüller horocycle orbit segments or on arcs of circle orbits. In particular, our conclusion that push-forwards for circle orbits converge to an $\mathrm{SL}(2, \mathbb{R})$-invariant measure after removing a set of times of zero upper density implies, by the work of Eskin and Masur [**EMa**], a correspondingly improved version of the asymptotic for the counting function derived in [**EMM**] and [**EM**] (see [**EM**, Theorem 1.7]).

The first section (§2) of this note is devoted to the proof of the above-mentioned results for limits of push-forwards of horocycle measures and horocycle and circle arcs.

A similar argument gives a short proof of a weak version of a theorem of Chaika and Eskin, according to which, for all points in the moduli space of Abelian differentials, almost all directions are *Birkhoff generic* for the Teichmüller geodesic flow with respect to the unique absolutely continuous probability affine measure on the orbifold $\overline{\mathrm{SL}(2, \mathbb{R})x}$ (see [**CE**, Theorem 1.1]). We are unable to give a proof of the theorem of Chaika and Eskin. In our version, we prove convergence of ergodic averages outside a subset of times of zero lower density. The second section (§3) of this note is devoted to the proof of our partial result on *Birkhoff genericity* in almost all directions for general actions of $\mathrm{SL}(2, \mathbb{R})$. The third section (§4) contains a similar approach to *Oseledets regularity* in almost all directions for uniformly Lipschitz irreducible cocycles over $\mathrm{SL}(2, \mathbb{R})$ actions. Finally, in the last section (§5) we derive an equidistribution result for the push-forwards of an arbitrary horospherical leaf under the Teichmüller geodesic flow.

In fact, our results are in principle not limited to the action of $\mathrm{SL}(2, \mathbb{R})$ on the moduli space of Abelian differentials and hold more generally for general continuous actions on locally compact topological spaces.

For this reason, we present below abstract results, which can then be applied to the action on moduli spaces thanks to the celebrated theorems of Eskin, Mirzakhani and Mohammadi (see [**EM**] and [**EMM**]).

Let

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad h_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

denote the diagonal subgroup (the geodesic flow), the unstable unipotent flow (the unstable horocycle flow) and the maximal torus (circle) of $\mathrm{SL}(2, \mathbb{R})$. For all $t, s \in \mathbb{R}$, we have the commutation relation

$$g_t \circ h_s = h_{se^{2t}} \circ g_t.$$

We will consider below an arbitrary continuous (left) action of $\mathrm{SL}(2, \mathbb{R})$ on a locally compact space X. Let ν be any of the following types of Borel probability measures on X:

(1) a horocycle probability invariant measure, that is, a Borel probability measure invariant under the action of the unipotent subgroup $h_{\mathbb{R}}$ on X;

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(2) a (normalized) horocycle arc, that is, a measure of the form

$$\frac{1}{S}\int_0^S (h_s)_*(\delta_x)\,ds \quad \text{for some } (x, S) \in X \times \mathbb{R}^+;$$

(3) a (normalized) circle arc, that is, a measure of the form

$$\frac{1}{\Theta}\int_0^\Theta (r_\theta)_*(\delta_x)\,d\theta \quad \text{for some } (x, \Theta) \in X \times \mathbb{R}^+.$$

Our first result, on push-forwards of horocycle invariant measures, and of horocycle and circle arcs, can be stated as follows.

THEOREM 1.1. *Let ν be any Borel probability measure in the above list. If the weak* limit*

$$\mu := \lim_{T\to+\infty} \frac{1}{T}\int_0^T (g_t)_*(\nu)\,dt$$

exists and is $h_{\mathbb{R}}$-ergodic, then there exists a set $Z \subset \mathbb{R}$ of zero upper density such that in the weak topology*

$$\lim_{t\notin Z} (g_t)_*(\nu) = \mu.$$

Our second result, on Birkhoff genericity, is as follows.

Let $I \subset \mathbb{T}$ be a compact subinterval. Let δ^I_∞ denote the Dirac mass at the point at infinity of the one-point compactification S_I of the semi-infinite cylinder $[1, +\infty) \times I$.

THEOREM 1.2. *For any sequence (π_n) of probability measures converging to the Dirac measure δ^I_∞, the following holds. Let us assume that in the weak* sense*

$$\mu := \lim_{n\to+\infty} \int_{S_I} \frac{1}{T}\int_0^T (g_t \circ r_\theta)_*(\delta_x)\,dt\,d\pi_n(T, \theta);$$

then there exists a set $\mathcal{Z} \subset [1, +\infty) \times I$ with $\lim_{n\to+\infty} \pi_n(\mathcal{Z}) = 0$ such that in the weak topology we have*

$$\lim_{(T,\theta)\notin\mathcal{Z}} \frac{1}{T}\int_0^T (g_t \circ r_\theta)_*(\delta_x)\,dt = \mu.$$

As a consequence of Theorem 1.1 we can derive the following.

COROLLARY 1.3. *Let ν be a horocycle invariant measure, or the probability measure uniformly distributed on a horocycle or circle arc on the moduli space. Let μ be the unique affine probability $\mathrm{SL}(2, \mathbb{R})$-invariant measure supported on the affine suborbifold $\overline{\mathrm{SL}(2, \mathbb{R})(\mathrm{supp}(\nu))}$. Then there exists a set $Z \subset \mathbb{R}$ of zero upper density such that*

$$\lim_{t\notin Z} (g_t)_*(\nu) = \mu.$$

CONJECTURE 1.4. *Corollary 1.3 holds with exceptional set $Z = \emptyset$.*

As mentioned above, this conjecture has been proven by Bainbridge, Smillie and Weiss for certain invariant orbifolds in the stratum $H(1, 1)$ of Abelian differentials with two simple zeros on genus-two surfaces (see [**BSW**, Theorems 1.5 and 12.7]). Chaika, Smillie and Weiss have recently announced that the Teichmüller horocycle flow (in genus two) has orbits which are not contained in the support of their limit measures and orbits which have no (unique) limit measure. These results however do not contradict our conjecture.

By the argument of [**EMa**], we can also derive the following improved version of the 'weak asymptotic formula' for the counting function of cylinders in translation surfaces and rational billiards (compare [**EM**, Theorem 1.7] or [**EMM**, Theorem 2.12]). Let Q be a rational polygon and let $N(Q, T)$ denote the number of cylinders of periodic trajectories of length at most $T > 0$ for the billiard flow on Q.

COROLLARY 1.5. *There exist a constant C_Q (a Siegel–Veech constant) and a set $Z_Q \subset \mathbb{R}$ of zero upper density such that*

$$\lim_{t \notin Z_Q} \frac{N(Q, e^t)}{e^{2t}} = C_Q.$$

For *horospherical measures*, we can prove a stronger result. By a horospherical measure we mean any measure supported on a leaf of the strong stable foliation of the Teichmüller geodesic flow, absolutely continuous with continuous density with respect to the canonical affine measure on the leaf, and with conditional measures along horocycle orbits equal to one-dimensional Lebesgue measures (see §5).

THEOREM 1.6. *Let ν be any horospherical probability measure supported on the stable leaf at $x \in \mathcal{H}_g$ and let μ denote the unique* $\mathrm{SL}(2, \mathbb{R})$*-invariant affine ergodic probability measure supported on* $\overline{\mathrm{SL}(2, \mathbb{R})x}$*. In the weak* topology, we have*

$$\lim_{t \to +\infty} (g_t)_*(\nu) = \mu.$$

Theorem 1.2 has the corollaries stated below.

COROLLARY 1.7. *Let $x \in X$ and let $I \subset \mathbb{T}$. If the weak* limit*

$$\mu := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I (g_t \circ r_\theta)_*(\delta_x)\, d\theta\, dt$$

exists and is $g_{\mathbb{R}}$-ergodic, then for Lebesgue almost all $\theta \in I$ there exists a set $Z_\theta \subset \mathbb{R}$ of zero lower density such that we have the weak limit*

$$\mu := \lim_{T \notin Z_\theta} \frac{1}{T} \int_0^T (g_t \circ r_\theta)_*(\delta_x)\, dt.$$

Corollary 1.7 has been recently proven by Khalil (see [**Kha**, Theorem 1.1]) in greater generality by a different, direct argument. Khalil's argument is based on an 'adaptation of the weak-type maximal inequality and follows similar lines to the proof of the classical Birkhoff ergodic theorem'. We believe that our indirect argument can be generalized to yield a result identical to that of Khalil.

In the motivating case when $X = \mathcal{H}_g$ is the moduli space of Abelian differentials on Riemann surfaces of genus $g \geq 2$, the results of Eskin, Mirzakhani and Mohammadi (see [EM] and [EMM]) prove that for every $x \in \mathcal{H}_g$ there exists a unique probability $SL(2,\mathbb{R})$-measure, absolutely continuous on the affine orbifold $\overline{SL(2,\mathbb{R})x}$, such that the hypotheses of Theorem 1.1 and Corollary 1.7 hold (see [EMM, Theorems 2.6 and 2.10]). In this case the Birkhoff genericity in almost all directions, which corresponds to the statement of Corollary 1.7 with exceptional sets $Z_\theta = \emptyset$, for almost all $\theta \in \mathbb{T}$, was proved by Chaika and Eskin (see [CE, Theorem 1.1]) by a different method based on ideas and results from [EM] and [EMM].

By our method, we can establish Birkhoff genericity in almost all directions in an abstract setting only under a stronger hypothesis.

COROLLARY 1.8. *Let $x \in X$ and let $I \subset \mathbb{T}$. If the weak* limit*

$$\mu := \lim_{\pi \to \delta^I_\infty} \int_{S_I} \frac{1}{T} \int_0^T (g_t \circ r_\theta)_*(\delta_x)\, dt\, d\pi(T,\theta)$$

exists as π varies over all compactly supported, absolutely continuous probability measures on S_I with smooth density, and if μ is $g_{\mathbb{R}}$-ergodic, then for almost all $\theta \in \mathbb{T}$ we have the weak limit*

$$\mu := \lim_{T \to +\infty} \frac{1}{T} \int_0^T (g_t \circ r_\theta)_*(\delta_x)\, dt.$$

Our results on the *Oseledets theorem* in the generic direction are formulated below. We consider an action of the group $SL(2,\mathbb{R})$ on a continuous vector bundle in the setting of the paper by Bonatti, Eskin and Wilkinson [BEW]. We recall briefly the main hypothesis of their work. Let $H \to X$ be a continuous vector bundle over a separable metric space X with fiber a finite-dimensional vector space. Suppose that $SL(2,\mathbb{R})$ acts on H by linear automorphisms on the fibers and a given action on the base which preserves a probability measure μ on X. Assume that H is equipped with a Finsler structure (that is, a continuous choice of norm $|\cdot|_x$ on the fibers of H). For any $g \in SL(2,\mathbb{R})$, let

$$\|g\|_x = \sup_{v \in H_x \setminus \{0\}} \frac{|g(v)|_{g(x)}}{|v|_x}.$$

A cocycle is called *uniformly Lipschitz* with respect to the Finsler structure if there exists a constant $K > 0$ such that for $(x,t) \in X \times \mathbb{R}$,

$$\log \|g_t\|_x \leq Kt.$$

We remark that all uniformly Lipschitz cocycles trivially satisfy the integrability condition of [BEW]:

$$\int_X \sup_{t \in [-1,1]} \log \|g_t\|_x \, d\mu(x) \ < \ +\infty.$$

The cocycle is called *irreducible* with respect to the $SL(2,\mathbb{R})$-invariant measure μ on X if it does not admit non-trivial μ-measurable $SL(2,\mathbb{R})$-invariant sub-bundles.

Let $I \subset \mathbb{T}$ be a compact subinterval. Let δ_∞^I denote the Dirac mass at the point at infinity of the one-point compactification S_I of the semi-infinite cylinder $[1, +\infty) \times I$.

THEOREM 1.9. *Assume that the* $\mathrm{SL}(2, \mathbb{R})$ *cocycle on* H *is uniformly Lipschitz and irreducible with respect to the* $\mathrm{SL}(2, \mathbb{R})$*-invariant probability ergodic measure* μ *on* X. *Let* λ_μ *denote the top Lyapunov exponent of the diagonal cocycle* $g_{\mathbb{R}}^H$ *on* H *with respect to the measure* μ *on* X. *For any sequence* (π_n) *of probability measures converging to the Dirac measure* δ_∞^I, *the following holds. Let* $x \in X$ *be any point such that (in the weak* topology) we have*

$$\mu := \lim_{n\to+\infty} \int_{S_I} \frac{1}{T} \int_0^T (g_t \circ r_\theta)_*(\delta_x)\, dt\, d\pi_n(T, \theta).$$

It follows that there exists a set $\mathcal{Z} \subset [1, +\infty) \times I$ *with* $\lim_{n\to+\infty} \pi_n(\mathcal{Z}) = 0$ *such that, for all* $\mathrm{v} \in H_x \setminus \{0\}$, *we have*

$$\lim_{(t,\theta)\notin\mathcal{Z}} \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} = \lambda_\mu.$$

Theorem 1.9 has the corollaries stated below.

COROLLARY 1.10. *Assume that the* $\mathrm{SL}(2, \mathbb{R})$ *cocycle on* H *is uniformly Lipschitz and irreducible with respect to the* $\mathrm{SL}(2, \mathbb{R})$*-invariant probability ergodic measure* μ *on* X. *Let* λ_μ *denote the top Lyapunov exponent of the diagonal cocycle* $g_{\mathbb{R}}^H$ *on* H *with respect to the measure* μ *on* X. *Let* $x \in X$ *be any point such that (in the weak* topology) we have*

$$\mu := \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I (g_t \circ r_\theta)_*(\delta_x)\, d\theta\, dt.$$

It follows that, for Lebesgue almost all $\theta \in I$, *there exists a set* $Z_\theta \subset \mathbb{R}$ *of zero lower density such that, for all* $\mathrm{v} \in H_x \setminus \{0\}$, *we have*

$$\lim_{t\notin Z_\theta} \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} = \lambda_\mu.$$

In the motivating case when $X = \mathcal{H}_g$ is the moduli space of Abelian differentials on Riemann surfaces of genus $g \geq 2$, Oseldets genericity in almost all directions, which corresponds for the top exponent to the statement of Corollary 1.10 with exceptional sets $Z_\theta = \emptyset$, for almost all $\theta \in \mathbb{T}$, was also proved by Chaika and Eskin (see [**CE**, Theorems 1.2 and 1.5]) by an argument based on [**EM**] and [**EMM**].

By our method, we can establish Oseledets regularity in almost all directions in an abstract setting only under a stronger hypothesis.

COROLLARY 1.11. *Let* $x \in X$ *and let* $I \subset \mathbb{T}$. *If the weak* limit*

$$\mu := \lim_{\pi\to\delta_\infty^I} \int_{S_I} \frac{1}{T} \int_0^T (g_t \circ r_\theta)_*(\delta_x)\, dt\, d\pi(T, \theta)$$

exists over all compactly supported, absolutely continuous probability measures π *on* S_I *with smooth density, and is* $g_{\mathbb{R}}$*-ergodic, then for almost all* $\theta \in \mathbb{T}$ *and for all* $\mathrm{v} \in H_x \setminus \{0\}$, *we have*

$$\lim_{t\to+\infty} \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} = \lambda_\mu.$$

Whenever Theorem 1.9 and its corollaries can be applied to all exterior products of the cocycle, it implies results for all Lyapunov exponents. In the special case of the action of $SL(2, \mathbb{R})$ on the moduli space of Abelian differentials, the theorem can indeed be applied (as in the paper by Chaika and Eskin [**CE**]) to all exterior powers of the Kontsevich–Zorich cocycle by reduction to the irreducible component thanks to the semi-simplicity theorem of Filip [**Fi**]. The uniform Lipschitz property of the Kontsevich–Zorich cocycle with respect to the Hodge norm was proved by the author in [**Fo**] (see also [**FMZ**]).

2. *Limits of geodesic push-forwards of horocycle measures*

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. We begin by explaining the argument in the case of push-forwards of horocycle invariant probability measures. The other cases can be treated similarly and require some additional considerations.

Let B_1 be the set of all Borel measures of total mass at most one on the one-point compactification $\hat{X}$ of the locally compact space X and let $\mathcal{N} : \mathbb{R} \to B_1$ be the map defined as $\mathcal{N}(t) = (g_t)_*(\nu)$ for all $t \in \mathbb{R}$. The range of the map $\mathcal{N}$ is contained in the closed subspace of probability measures, invariant under the (unstable) horocycle flow, since we are assuming that ν is invariant under the (unstable) horocycle flow.

The space B_1, endowed with the topology of the weak* convergence, is a metrizable (separable) compact space. The map $\mathcal{N} : \mathbb{R}^+ \to B_1$ is clearly continuous. In fact, every continuous function on $\hat{X}$ is bounded and all measures $(g_t)_*(\nu)$ are probability measures.

Let u_T denote the uniform measure on $[0, T]$. The push-forward $\mathcal{N}_*(u_T)$ defines a Borel probability measure on the compact metric space B_1, that is, for any continuous function F on the space B_1 (with respect to the weak* topology on B_1), we have

$$\mathcal{N}_*(u_T)(F) := \frac{1}{T} \int_0^T F((g_t)_*(\nu))\, dt.$$

Let U denote any weak* limit of $\mathcal{N}_*(u_T)$ as $T \to +\infty$ in the space of probability measures on the compact metric space B_1. We claim that U is a delta mass δ_μ at the probability measure $\mu \in B_1$. We remark that, as ν is horocycle invariant, since the subset of horocycle invariant probability measures is closed with respect to the weak* topology, the measure U is supported there.

For any continuous (compactly supported) function f on the locally compact space X, the function F_f defined as

$$F_f(\nu) = \nu(f) \quad \text{for all } \nu \in B_1$$

is continuous with respect to the weak* topology on B_1 (by definition of the weak* topology). By our assumptions, for all functions F_f, we have that

$$U(F_f) = \mu(f).$$

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In fact, for every continuous function f with compact support on the space X, we have that

$$\mathcal{N}_*(u_T)(F_f) := \frac{1}{T}\int_0^T [(g_t)_*(\nu)](f)\, dt \to \mu(f).$$

Since μ is ergodic with respect to the horocycle flow and U is supported on the subset of horocycle invariant measures, from the identity

$$U(F_f) = \int F_f(\nu)\, dU(\nu) = \int \nu(f)\, dU(\nu) = \mu(f) \tag{1}$$

we derive that $\nu = \mu$ for U-almost all $\nu \in B_1$. This in turn implies that the probability measure U, as a probability measure essentially supported on the singleton $\{\mu\} \subset B_1$, is equal to a Dirac mass δ_μ.

It follows that, for every open neighborhood $\mathcal{V}$ of μ in the (metric) space B_1, we have

$$\limsup_{T\to+\infty} \frac{1}{T} \operatorname{Leb}\{t \in [0, T] : (g_t)_*(\nu) \notin \mathcal{V}\} = 0. \tag{2}$$

In fact, let us assume that the above statement does not hold. It follows that there exist an open neighborhood $\mathcal{V}$ of μ in B_1 and a diverging sequence $\{T_n\}$ such that

$$\lim_{n\to+\infty} \frac{1}{T_n} \operatorname{Leb}\{t \in [0, T_n] : (g_t)_*(\nu) \notin \mathcal{V}\} = c > 0.$$

There exists a continuous non-negative function $F_\mathcal{V}$ on the compact metrizable space B_1 such that $F_\mathcal{V} \equiv 1$ on the complement of $\mathcal{V}$ (a closed set) and $F_\mathcal{V}(\mu) = 0$. It follows that

$$\frac{1}{T_n}\int_0^{T_n} F_\mathcal{V}((g_t)_*(\nu))\, dt \geq c > 0;$$

hence, for every weak limit U of the sequence of measures $\mathcal{N}_*(u_{T_n})$, we have $U(F_\mathcal{V}) \geq c > 0$, while $\delta_\mu(F_\mathcal{V}) = 0$, which is a contradiction.

We conclude that there exists $Z \subset \mathbb{R}^+$ of zero upper density such that

$$\lim_{t\notin Z} (g_t)_*(\nu) = \mu \quad \text{in the weak* topology.} \tag{3}$$

Let $\{\mathcal{V}_n\}$ be a basis of neighborhoods of the measure μ in B_1. By the formula (2), for every sequence $\{\epsilon_n\}$ of positive numbers, converging to zero, there exists a diverging, increasing sequence $\{T_n\}$ such that

$$\sup_{T\geq T_n} \frac{1}{T} \operatorname{Leb}\{t \in [0, T] : (g_t)_*(\nu) \notin \mathcal{V}_n\} \leq \epsilon_n.$$

Let Z be the set defined as follows:

$$Z := \bigcup_{n\in\mathbb{N}} \{t \in [T_n, T_{n+1}] : (g_t)_*(\nu) \notin \mathcal{V}_n\}.$$

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Let us find under what conditions Z has zero upper density. For any $T > 0$ sufficiently large, there exists $n \in \mathbb{N}$ such that $T \in [T_n, T_{n+1}]$. We have

$$\begin{aligned}\mathrm{Leb}(Z \cap [0, T]) &\leq \sum_{k=1}^{n-1} \mathrm{Leb}(Z \cap [T_k, T_{k+1}]) + \mathrm{Leb}(Z \cap [T_n, T]) \\ &\leq \sum_{k=1}^{n-1} \epsilon_k T_{k+1} + \epsilon_n T.\end{aligned}$$

It is therefore enough to choose the sequences recursively so that

$$\frac{1}{T_n} \sum_{k=1}^{n-1} \epsilon_k T_{k+1} + \epsilon_n \to 0.$$

It is clear by the definition of the set $Z \subset \mathbb{R}$ that the formula (3) holds.

For the cases when ν is a probability measure supported on a horocycle arc or an arc of circle, the argument is similar but we have to prove that any weak* limit U of $\mathcal{N}_*(u_T)$ as $T \to +\infty$ is supported on the subspace of horocycle invariant measures.

Let U be a weak* limit of the measures $\mathcal{N}_*(u_T)$ with support not contained in the subspace of the horocycle invariant measures. There exists a measure $\nu_0 \in \mathrm{supp}(U)$ which is not invariant under the (unstable) horocycle flow and hence there exist a function $f_0 \in C_0^0(X)$ and a real number $\tau \neq 0$ such that

$$\int f_0 \circ h_\tau \, d\nu_0 \neq \int f_0 \, d\nu_0.$$

Since $\nu_0 \in \mathrm{supp}(U)$ and since B_1 is a locally convex metrizable space, there exists a closed convex neighborhood $\mathcal{C} \subset B_1$ such that $U(\mathcal{C}) > 0$ and $U(\partial\mathcal{C}) = 0$ and, by continuity,

$$\int f_0 \circ h_\tau \, d\nu \neq \int f_0 \, d\nu_0 \quad \text{for all } \nu \in \mathcal{C}.$$

Let $\nu_{\mathcal{C}}$ denote the measure

$$\nu_{\mathcal{C}} = \frac{1}{U(\mathcal{C})} \int_{\mathcal{C}} \nu \, dU(\nu) \ \in \ \mathcal{C}.$$

We claim that $\nu_{\mathcal{C}}$ is invariant under the horocycle flow, which is a contradiction. In fact, since U is a weak* limit of the family $\{\mathcal{N}_*(u_T)\}$, there exists a diverging sequence (T_n) such that $\mathcal{N}_*(u_{T_n}) \to U$ and since, by assumption, $U(\partial C) = 0$, we have

$$U(\mathcal{C})\nu_{\mathcal{C}} = \lim_{n \to +\infty} \int_{\mathcal{C}} \nu \, d\mathcal{N}_*(u_{T_n})(\nu).$$

Now, by construction, for all $f \in C_0^0(X)$, we have

$$\left[\int_{\mathcal{C}} \nu \, d\mathcal{N}_*(u_{T_n})(\nu)\right] (f \circ h_\tau - f) = \frac{1}{T_n} \int_0^{T_n} \chi_{\mathcal{C}}((g_t)_*\nu)\nu(f \circ h_\tau - f) \, dt.$$

It is therefore enough to prove that the right-hand side in the above formula converges to zero. When ν is the uniformly distributed measure on a horocycle arc $\{h_s(x) | s \in [0, S]\}$,

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we have, uniformly with respect to $t \in \mathbb{R}$,

$$\begin{aligned}\frac{1}{S}\int_0^S (f \circ h_\tau - f)(g_t h_s x)\, ds &= \frac{1}{S}\int_0^S (f \circ h_\tau - f)(h_{e^{2t}s} \circ g_t x)\, ds \\ &= \frac{1}{S}\bigg[\int_S^{S+e^{-2t}\tau} f(h_{e^{2t}s} \circ g_t x)\, ds - \int_0^{e^{-2t}\tau} f(h_{e^{2t}s} \circ g_t x)\, ds\bigg] \to 0.\end{aligned}$$

When ν is the uniformly distributed measure on a circle arc $\{r_\theta(x) | \theta \in [0, \Theta]\}$, it is a standard argument that the push-forward of a circle arc can be well approximated by a union of horocycle arcs. We include the argument for completeness.

There exists a constant $C > 0$ such that, for all $\theta \in [-\pi/4, \pi/4]$ and for all $t \in \mathbb{R}$, we have

$$\operatorname{dist}(g_t \circ r_\theta, g_t \circ h_\theta) \leq C(e^t + e^{-t})\theta^2.$$

As a consequence, for any $\alpha \in (1/2, 2)$, we can approximate the push-forward of a circle arc $\{r_\theta x \mid \theta \in [0, \Theta]\}$ by a union of at most $\Theta e^{\alpha t}$ push-forwards of horocycle arcs of length $\ell_t \in [e^{-\alpha t}, 2e^{-\alpha t}]$. By the above estimate, the error in computing the integral of a continuous function *f*, of unit Lipschitz constant with respect to the $\mathrm{SL}(2, \mathbb{R})$ action, will be of size

$$4\Theta C(e^t + e^{-t})e^{-2\alpha t} \leq 8C\Theta e^{(1-2\alpha)t} \to 0.$$

The claim is thus reduced to prove that for any family of intervals $\{[a_t, b_t]\}$, we have

$$\lim_{t\to\infty} \frac{1}{\ell_t}\int_{a_t}^{b_t} (f \circ h_\tau - f)(g_t h_s x)\, ds = 0 \quad \text{uniformly on } x \in X.$$

The proof of the above limit is straightforward since

$$\begin{aligned}&\bigg|\frac{1}{\ell_t}\int_{I_t} (f \circ h_\tau - f)(g_t h_s x)\, ds\bigg| \\ &\quad = \frac{1}{\ell_t}\bigg|\int_{b_t}^{b_t+e^{-2t}\tau} f(h_{e^{2t}s} \circ g_t x)\, ds - \int_{a_t}^{a_t+e^{-2t}\tau} f(h_{e^{2t}s} \circ g_t x)\, ds\bigg|;\end{aligned}$$

hence, as $\ell_t \geq e^{-\alpha t}$ with $\alpha < 2$, it follows that for any $f \in C_0^0(X)$, the above averages converge to zero, uniformly with respect to $x \in X$.

We have thus completed the proof of the claim that $\nu_{\mathcal{C}}$ is in all cases invariant under the horocycle flow and, since $\nu_{\mathcal{C}} \in \mathcal{C}$, we have reached a contradiction. □

3. *Birkhoff genericity in almost all directions*

In this section we prove Theorem 1.2 and Corollaries 1.7 and 1.8.

Proof of Theorem 1.2. Given $x \in X$, let us consider the map $G : [1, +\infty) \times I \to B_1$ to the space of measures on $\hat{X}$ of total mass at most one, given for every $T \geq 1$, for every

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$\theta \in I \subset \mathbb{T}$ and for every $f \in C^0(\hat{X})$ by the formula

$$G(T,\theta)(f) := \frac{1}{T}\int_0^T f(g_t r_\theta x)\,dt.$$

For any compact interval $I \subset \mathbb{T}$, let us consider the family of push-forwards

$$\{G_*(\pi) \text{ for all probability measure } \pi \text{ on } [1,+\infty)\times I\}.$$

Let Π be any weak* limit of the above family in the following sense. There exists a sequence (π_n) which converges in the weak* topology to the delta mass at the one-point compactification S_I of the cylinder $[1,+\infty)\times I$ such that

$$G_*(\pi_n) \to \Pi$$

in the weak* topology on the space of measures on B_1.

We claim that Π is a Dirac mass supported at μ. By our hypotheses, for all functions F_f we have

$$\Pi(F_f) = \mu(f).$$

In fact, by hypothesis we have

$$\begin{aligned} G_*(\pi_n)(F_f) &= \int_1^{+\infty}\int_I F_f(G(T,\theta))\,d\pi_n(T,\theta) \\ &= \int_1^{+\infty}\int_I \frac{1}{T}\int_0^T f(g_t r_\theta x)\,dt\,d\pi_n(T,\theta) \to \mu(f). \end{aligned}$$

It the follows that, for all $f \in C^0(X)$,

$$\mu(f) = \Pi(F_f) = \int \nu(f)\,d\Pi(\nu). \tag{4}$$

We claim that the support of Π is contained in the closed subspace of B_1 of probability measures invariant under the geodesic flow. Let us assume that it is not the case. It follows that there exists a measure $\nu_0 \in \operatorname{supp}(\Pi)$ which is not invariant under the geodesic flow. There exist a function $f_0 \in C_0^0(X)$ and a real number $\tau \neq 0$ such that

$$\int f_0 \circ g_\tau\,d\nu_0 \neq \int f_0\,d\nu_0.$$

Since $\nu_0 \in \operatorname{supp}(\Pi)$ and since B_1 is a locally convex metrizable space, there exists a closed convex neighborhood $\mathcal{C} \subset B_1$ such that $\Pi(\mathcal{C}) > 0$ and $\Pi(\partial\mathcal{C}) = 0$ and, by continuity,

$$\int f_0 \circ g_\tau\,d\nu \neq \int f_0\,d\nu_0 \quad \text{for all } \nu \in \mathcal{C}.$$

Let $\nu_{\mathcal{C}}$ denote the measure

$$\nu_{\mathcal{C}} = \frac{1}{\Pi(\mathcal{C})}\int_{\mathcal{C}} \nu\,d\Pi(\nu) \ \in \ \mathcal{C}.$$

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We claim that $\nu_{\mathcal{C}}$ is invariant under the geodesic flow, which is a contradiction. In fact, since Π is a weak* limit of the family $\{G_*(\pi_n)\}$ and $\Pi(\partial\mathcal{C}) = 0$, we have

$$\Pi(\mathcal{C})\nu_{\mathcal{C}} = \lim_{n\to+\infty} \int_{\mathcal{C}} \nu \, dG_*(\pi_n)(\nu).$$

Now, by construction, for all $f \in C_0^0(X)$, we have

$$\begin{aligned}
&\left[\int_{\mathcal{C}} \nu \, dG_*(\pi_n)(\nu)\right](f \circ g_\tau - f) \\
&\quad = \frac{1}{|I|} \int_{G^{-1}(\mathcal{C})} \frac{1}{T} \left(\int_0^T (f \circ g_\tau - f)(g_t r_\theta x)\, dt\right) d\pi_n(T,\theta) \\
&\quad = \frac{1}{|I|} \int_{G^{-1}(\mathcal{C})} \frac{1}{T} \left(\int_T^{T+\tau} f(g_t r_\theta x)\, dt - \int_0^\tau f(g_t r_\theta x)\, dt\right) d\pi_n(T,\theta) \to 0.
\end{aligned}$$

The claim that $\nu_{\mathcal{C}}$ is invariant under the geodesic flow follows and, since $\nu_{\mathcal{C}} \in \mathcal{C}$, we reached a contradiction. It follows that Π is supported on the subspace of $g_{\mathbb{R}}$-invariant measures.

From the formula (4) and from the ergodicity of the measure μ with respect to the geodesic flow, it follows that $\Pi = \delta_\mu$ is a Dirac mass at μ. We have thus proved that

$$\lim_{n\to+\infty} G_*(\pi_n) = \delta_\mu.$$

From the above conclusion we immediately derive that, for every neighborhood $\mathcal{V}$ of μ in the (metric) space B_1, we have

$$\lim_{n\to+\infty} \pi_n(\{(T,\theta) \in [1,+\infty) \times I \mid G(T,\theta) \notin \mathcal{V}\}) = 0. \tag{5}$$

Since $\pi_n(\{\infty\}) = 0$, it follows that for every $\epsilon > 0$ there exists $T_\epsilon > 1$ such that

$$\pi_n(\{(T,\theta) \in [T_\epsilon,+\infty) \times I \mid G(T,\theta) \notin \mathcal{V}\}) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let $(\mathcal{V}_k)$ be a basis of neighborhoods of μ in B_1 and let (ϵ_k) be any *summable* sequence of positive real numbers. There exists an increasing diverging sequence (T_k) such that

$$\pi_n(\{(T,\theta) \in [T_k,+\infty) \times I \mid G(T,\theta) \notin \mathcal{V}_k\}) < \epsilon_k \quad \text{for all } n \in \mathbb{N}.$$

Let then $\mathcal{Z}$ be the set such that

$$\mathcal{Z} \cap [T_k, T_{k+1}) = \{(T,\theta) \in [T_k, T_{k+1}) \times I \mid G(T,\theta) \notin \mathcal{V}_k\}.$$

It is clear that by construction we have

$$\lim_{(T,\theta)\notin\mathcal{Z}} G(T,\theta) = \mu.$$

Finally, since $\pi_n \to \delta_\infty^I$, it follows that, for any $k \in \mathbb{N}$, we have

$$\lim_{n\to+\infty} \pi_n([1,T_k) \times I) = 0,$$

while, by construction, for all $n \in \mathbb{N}$, we have

$$\lim_{k\to+\infty} \pi_n\,(\mathcal{Z}\cap([T_k,+\infty)\times I)) \le \lim_{k\to+\infty}\sum_{j\ge k}\epsilon_j = 0.$$

The statement of the theorem follows. □

Proof of Corollary 1.7. Let (τ_n) be any sequence of probability measures on $[1,+\infty)$ which converges to the Dirac mass at the point at infinity. Let Θ_I denote the normalized Lebesgue measure on the interval $I \subset \mathbb{T}$ and let then (π_n) be the sequence of probability measures on $[1,+\infty)\times I$ defined as

$$\pi_n := \tau_n \times \Theta_I \qquad \text{for all } n \in \mathbb{N}.$$

By the hypothesis of the corollary that

$$\mu := \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{1}{|I|}\int_I (g_t\circ r_\theta)_*(\delta_x)\,d\theta\,dt,$$

it follows that the hypothesis of Theorem 1.2 holds for the sequence (π_n). Therefore, there exists a set $\mathcal{Z}$ such that $\lim_{n\to+\infty}\pi_n(\mathcal{Z}) = 0$ and such that in the weak* topology we have

$$\lim_{(T,\theta)\notin\mathcal{Z}} G(T,\theta) = \lim_{(T,\theta)\notin\mathcal{Z}}\frac{1}{T}\int_0^T (g_t\circ r_\theta)_*(\delta_x)\,dt = \mu.$$

From the above conclusion we derive that, for every sequence of probability measures (τ_n) weakly converging to the delta mass at $+\infty \in [1,+\infty]$ and for every neighborhood $\mathcal{V}$ of μ in the (metric) space B_1, we have

$$\lim_{n\to+\infty}(\tau_n\times\Theta_I)(\{(T,\theta)\in[1,+\infty)\times I \mid G(T,\theta)\notin\mathcal{V}\}) = 0. \qquad (6)$$

We claim that for every $\mathcal{V}$, there exists a full-measure set $\mathcal{F}_\mathcal{V}\subset I$ such that for all $\theta\in\mathcal{F}_\mathcal{V}$, we have

$$\liminf_{\mathcal{T}\to+\infty}\frac{1}{\mathcal{T}}\,\mathrm{Leb}(\{T\in[1,\mathcal{T}] \mid G(T,\theta)\notin\mathcal{V}\}) = 0.$$

Otherwise, there exists a positive-measure set $\mathcal{P}_\mathcal{V}\subset I$ such that for all $\theta\in\mathcal{P}_\mathcal{V}$,

$$\liminf_{\mathcal{T}\to+\infty}\frac{1}{\mathcal{T}}\mathrm{Leb}(\{T\in[1,\mathcal{T}] \mid G(T,\theta)\notin\mathcal{V}\}) > 0.$$

By the Egorov theorem, it follows that there exist a sequence of times $(\mathcal{T}_n)$ and a positive-measure subset $\mathcal{P}'_\mathcal{V}\subset\mathcal{P}_\mathcal{V}$ such that the sequence

$$\inf_{\mathcal{T}\ge\mathcal{T}_n}\frac{1}{\mathcal{T}}\mathrm{Leb}(\{T\in[1,\mathcal{T}] \mid G(T,\theta)\notin\mathcal{V}\})$$

converges uniformly to a continuous positive function on $\mathcal{P}'_\mathcal{V}$. It is then possible to construct a sequence (τ_n) of probability measures on $[1,+\infty)$, weakly converging to the delta mass at $+\infty\in[1,+\infty]$, which contradicts the conclusion in the formula (6). Hence, the above claim is proved.

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Let $(\mathcal{V}_n)$ be a basis of neighborhoods of μ in B_1 and let $\mathcal{F}_I$ denote the full-measure set defined as

$$\mathcal{F}_I := \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\mathcal{V}_n}.$$

From the above claim it follows that for all $\theta \in \mathcal{F}_I$ and for all $n \in \mathbb{N}$, we have

$$\liminf_{\mathcal{T}\to+\infty} \frac{1}{\mathcal{T}}\mathrm{Leb}(\{T \in [1, \mathcal{T}] \mid G(T, \theta) \notin \mathcal{V}_n\}) = 0.$$

In particular, for any sequence (ϵ_n) of positive real numbers converging to zero, we have that there exists an increasing diverging sequence $(\mathcal{T}_n) \subset [1, +\infty)$ such that

$$\frac{1}{\mathcal{T}_n} \mathrm{Leb}(\{T \in [1, \mathcal{T}_n] \mid G(T, \theta) \notin \mathcal{V}_n\}) < \epsilon_n.$$

Such sequence can be constructed recursively as follows. For any finite increasing sequence $\{\mathcal{T}_k | k \le n\}$ and for any $\mathcal{T}^*_{n+1} > 0$, there exists $\mathcal{T}_{n+1} \ge \mathcal{T}^*_{n+1}$ such that

$$\frac{1}{\mathcal{T}_{n+1}} \mathrm{Leb}(\{T \in [1, \mathcal{T}_{n+1}] \mid G(T, \theta) \notin \mathcal{V}_n\}) < \epsilon_{n+1}.$$

Let Z_θ be the set defined as follows:

$$Z_\theta := \bigcup_{n\in\mathbb{N}} \{T \in [\mathcal{T}_n, \mathcal{T}_{n+1}] : G(T, \theta) \notin \mathcal{V}_n\}.$$

Let us find under what conditions Z_θ has zero lower density. We have

$$\mathrm{Leb}(Z_\theta \cap [0, \mathcal{T}_n]) \le \sum_{k=1}^{n-1} \mathrm{Leb}(Z_\theta \cap [\mathcal{T}_k, \mathcal{T}_{k+1}]) \le \sum_{k=1}^{n-1} \epsilon_{k+1}\mathcal{T}_{k+1}.$$

It is therefore enough to choose the sequences recursively so that

$$\frac{1}{\mathcal{T}_n} \sum_{k=1}^{n-2} \epsilon_{k+1}\mathcal{T}_{k+1} + \epsilon_n \to 0.$$

It is clear by the definition of the set $Z_\theta \subset \mathbb{R}$ that, for $\theta \in \mathcal{F}_I$, we have

$$\lim_{T\notin Z_\theta} G(T, \theta) = \mu.$$

The argument is therefore complete. □

Proof of Corollary 1.8. For all $(T, \theta) \in [1, +\infty) \times \mathbb{T}$, let $d(T, \theta)$ denote the distance from the measure $G(T, \theta)$ to μ with respect to any metric which induces the weak* topology. Let us assume by contraposition that there exists a positive-measure set $\mathcal{P} \subset I$ such that, for all $\theta \in \mathcal{P}$, we have

$$\limsup_{T\to+\infty} d(T, \theta) > 0.$$

This implies that there exist $\epsilon > 0$ and a set $\mathcal{P}_\epsilon$ of positive Lebesgue measure such that for all $\theta \in \mathcal{P}_\epsilon$, there exists a sequence $(T_n(\theta))$ such that, for all $n \in \mathbb{N}$,

$$d(T_n(\theta), \theta) \ge \epsilon.$$

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By a straightforward argument, for any continuous function $f \in C^0(X)$ and for any $h \in \mathbb{R}$, we have

$$|[G(T+h,\theta) - G(T,\theta)](f)| \le 2\frac{h}{T}|f|_{L^\infty}.$$

It follows that there exists $\delta > 0$ such that, for all $T \in [(1-\delta)T_n(\theta), (1+\delta)T_n(\theta)]$, we have

$$d(T,\theta) \ge \epsilon/2$$

and hence it is possible to construct a sequence of compactly supported measures π_n on $[1,+\infty) \times I$ with smooth bounded density and conditional measure on $\mathbb{T}$ equal to the Lebesgue measure, such that

$$\lim_{n\to+\infty} \pi_n(\{(T,\theta)/d(T,\theta) \ge \epsilon/2\}) > 0.$$

This contradicts the conclusion of Theorem 1.2 and hence the corollary is proven. □

4. *Oseledets regularity in almost all directions*

In this section we prove the Oseledets regularity result stated in Theorem 1.9.

Proof. Let $\mathbb{P}^1(H)$ denote the projectivization of the irreducible bundle H over the separable metric space X. Let $x \in X$ satisfy the hypothesis with respect to the SL(2, $\mathbb{R}$)-invariant measure μ on X. Let us recall that there exists a sequence $\{\sigma_\ell\}$ of continuous functions $\sigma_\ell : \mathbb{P}^1(H) \to \mathbb{R}$ such that the following holds. For any $g_{\mathbb{R}}^H$-invariant probability measure ν on $\mathbb{P}^1(H)$ which projects to the $g_{\mathbb{R}}$-invariant probability measure μ on X, and for any $\ell \in \mathbb{N}$, we have

$$\int_{\mathbb{P}^1(H)} \sigma_\ell(\mathrm{v})\, d\nu(\mathrm{v}) \le \lambda_\mu. \tag{7}$$

If, in addition, the measure ν is supported on the Oseledets subspace of the top Lyapunov exponent λ_μ of $g_{\mathbb{R}}^H$ with respect to the measure μ on X, then we have

$$\int_{\mathbb{P}^1(H)} \sigma_\ell(\mathrm{v})\, d\nu(\mathrm{v}) = \lambda_\mu. \tag{8}$$

Finally, since the cocycle is uniformly Lipschitz, for any $x \in X$ and any $\mathrm{v} \in \mathbb{P}^1(H_x)$, uniformly with respect to $T \ge 1$, we have

$$\frac{1}{T}\log |g_T^H(\mathrm{v})|_{g_T(x)} = \lim_{\ell\to+\infty} \frac{1}{T}\int_0^T \sigma_\ell(g_t^H(\mathrm{v}))\, dt. \tag{9}$$

The functions $\sigma_\ell : \mathbb{P}^1(H) \to \mathbb{R}$ can be defined as follows: for all $\ell \in \mathbb{N}$, for all $x \in X$ and for all $\mathrm{v} \in \mathbb{P}^1(H_x)$, let

$$\sigma_\ell(\mathrm{v}) := \ell \log\left(\frac{|g_{1/\ell}^H(\mathrm{v})|_{g_{1/\ell}(x)}}{|\mathrm{v}|_x}\right).$$

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It is immediate to verify that, for all $L \in \mathbb{N}$, by telescopic summation we have

$$\log\left(\frac{|g_L^H(\mathrm{v})|_{g_L(x)}}{|\mathrm{v}|_x}\right) = \frac{1}{\ell}\sum_{j=0}^{\ell L-1} \sigma_\ell(g^H_{j/\ell}(\mathrm{v})).$$

By the uniform Lipschitz property, we also have the estimate

$$\left|\frac{1}{\ell L}\sum_{j=0}^{\ell L-1} \sigma_\ell(g^H_{j/\ell}(\mathrm{v})) - \frac{1}{L}\int_0^L \sigma_\ell(g_t^H(\mathrm{v}))\,dt\right| \le \frac{K}{\ell}.$$

The above claims (7), (8) and (9) follow from Birkhoff ergodic theorem and Oseledets' multiplicative ergodic theorem.

By a result of Bonatti, Eskin and Wilkinson (see [**BEW**, Theorem 1.3]) under the irreducibility assumption, any P-invariant probability measure ν on $\mathbb{P}^1(H)$ which projects to the $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure μ on X is supported on the Oseledets subspace of the top Lyapunov exponent and hence the identity (8) holds.

Let $x \in X$ be any point satisfying the assumption of the theorem. For any $\mathrm{v} \in \mathbb{P}^1(H_x)$, let us consider the measures ν_n on $\mathbb{P}^1(H)$ given, for all $f \in C^0_0(\mathbb{P}^1(H))$, by the formula

$$\nu_n(f) := \int_{S_I} \frac{1}{T}\int_0^T f(g_t^H(r_\theta(\mathrm{v}))\,dt\,d\pi_n(T,\theta).$$

Let ν be any weak* limit point (along a subsequence) of the above family of measures. The measure ν is P-invariant (invariant under the action of the maximal parabolic subgroup generated by the diagonal subgroup and the unstable unipotent) and, by the hypothesis on $x \in X$, it projects to μ under the canonical projection $\mathbb{P}^1(H) \to X$ and hence the identity (8) holds.

Let us now consider the map $\mathcal{L} : I \to B_1$ from the cylinder S_I to the space of measures on the compactification $\hat{\mathbb{P}}^1(H)$ of the bundle $\mathbb{P}^1(H)$, given for every $f \in C^0(\hat{\mathbb{P}}^1(H))$ by the formula

$$\mathcal{L}(T,\theta)(f) := \frac{1}{T}\int_0^T f(g_t^H(r_\theta(\mathrm{v}))\,dt.$$

Let $\mathcal{L}_\infty$ be any weak* limit of the push-forwards $\mathcal{L}_n := (\mathcal{L})_*(\pi_n)$ under the above maps. By construction, for all $\ell \in \mathbb{N}$, we have the identity

$$\begin{aligned}\int F_{\sigma_\ell}\,d\mathcal{L}_\infty &= \lim_{n\to+\infty}\int F_{\sigma_\ell}\,d\mathcal{L}_n \\ &= \lim_{n\to+\infty}\int_{S_I}\frac{1}{T}\int_0^T \sigma_\ell(g_t^H(r_\theta(\mathrm{v}))\,dt\,d\pi_n(T,\theta) = \int \sigma_\ell\,d\nu = \lambda_\mu.\end{aligned}$$

We claim that $\mathcal{L}_\infty$ is supported on the closed subset $\mathcal{C}$ of $g^H_{\mathbb{R}}$-invariant probability measures ν on the sub-bundle $\mathbb{P}^1(H)$ such that

$$\int_{\mathbb{P}^1(H)} \sigma_\ell\,d\nu = \lambda_\mu \quad \text{for all } \ell \in \mathbb{N}. \tag{10}$$

In fact, by an argument similar to that of §3, it can be proved that $\mathcal{L}_\infty$ is supported on the set of all $g^H_{\mathbb{R}}$-invariant measures on $\hat{\mathbb{P}}^1(H)$ which project to μ under the projection

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$\mathbb{P}^1(H) \to X$ and, since λ_μ is the top Lyapunov exponent, for all such measures ν on $\mathbb{P}^1(H)$, we have the inequalities in the formula (7), that is,

$$\int \sigma_\ell \, d\nu \leq \lambda_\mu \quad \text{for all } \ell \in \mathbb{N}.$$

It follows by definition that $F_{\sigma_\ell} \leq \lambda_\mu$ on $\text{supp}(\mathcal{L}_\infty)$ and since, as proved above,

$$\int F_{\sigma_\ell} \, d\mathcal{L}_\infty = \lambda_\mu \quad \text{for all } \ell \in \mathbb{N},$$

it follows that $\mathcal{L}_\infty$ is supported on the set $\mathcal{C}$, as claimed.

From the above conclusion we immediately derive that, for every neighborhood $\mathcal{V}$ of the closed subset $\mathcal{C}$ in the (metric) space B_1, we have

$$\lim_{n \to +\infty} \pi_n(\{(T, \theta) \in [1, +\infty) \times I \mid \mathcal{L}(T, \theta) \notin \mathcal{V}\}) = 0. \tag{11}$$

Since $\pi_n(\{\infty\}) = 0$, it follows that for every $\epsilon > 0$, there exists $T_\epsilon > 1$ such that

$$\pi_n(\{(T, \theta) \in [T_\epsilon, +\infty) \times I \mid \mathcal{L}(T, \theta) \notin \mathcal{V}\}) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let $(\mathcal{V}_k)$ be a basis of neighborhoods of μ in B_1 and let (ϵ_k) be any *summable* sequence of positive real numbers. There exists an increasing diverging sequence (T_k) such that

$$\pi_n(\{(T, \theta) \in [T_k, +\infty) \times I \mid \mathcal{L}(T, \theta) \notin \mathcal{V}_k\}) < \epsilon_k \quad \text{for all } n \in \mathbb{N}.$$

Let then $\mathcal{Z}$ be the set such that

$$\mathcal{Z} \cap [T_k, T_{k+1}) = \{(T, \theta) \in [T_k, T_{k+1}) \times I \mid \mathcal{L}(T, \theta) \notin \mathcal{V}_k\}.$$

It is clear that by construction all weak limits of the family $\{\mathcal{L}(T, \theta) \mid (T, \theta) \notin \mathcal{Z}\}$ belong to the closed set $\mathcal{C}$ (see its definition in the formula (10)) and hence, for all $\ell \in \mathbb{N}$,

$$\lim_{(T,\theta) \notin \mathcal{Z}} \frac{1}{T} \int_0^T \sigma_\ell(g_t^H r_\theta \text{v}) \, dt = \lambda_\mu.$$

Finally, by the uniform approximation property stated in formula (9), we have

$$\lim_{(T,\theta) \notin \mathcal{Z}} \frac{1}{T} \log |g_T^H(\text{v})|_{g_T(x)} = \lambda_\mu.$$

It remains to prove that the limit of the sequence $\{\pi_n(Z)\}$ is equal to zero. Since $\pi_n \to \delta_\infty^I$, it follows that, for any $k \in \mathbb{N}$, we have

$$\lim_{n \to +\infty} \pi_n([1, T_k) \times I) = 0,$$

while, by construction, for all $n \in \mathbb{N}$, we have

$$\lim_{k \to +\infty} \pi_n\left(\mathcal{Z} \cap ([T_k, +\infty) \times I)\right) \leq \lim_{k \to +\infty} \sum_{j \geq k} \epsilon_j = 0.$$

The statement of the theorem follows. □

We conclude the section by proving Corollaries 1.10 and 1.11.

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Proof of Corollary 1.10. The argument is similar to the proof of Corollary 1.7.

Let (τ_n) be any sequence of probability measures on $[1, +\infty)$ which converges to the Dirac mass at the point at infinity. Let Θ_I denote the normalized Lebesgue measure on the interval $I \subset \mathbb{T}$ and let then (π_n) be the sequence of probability measures on $[1, +\infty) \times I$ defined as

$$\pi_n := \tau_n \times \Theta_I \qquad \text{for all } n \in \mathbb{N}.$$

By the hypothesis of the corollary that

$$\mu := \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I (g_t \circ r_\theta)_*(\delta_x)\, d\theta\, dt,$$

it follows that the hypothesis of Theorem 1.9 holds for the sequence (π_n).

Therefore, there exists a set $\mathcal{Z}$ such that $\lim_{n\to+\infty} \pi_n(\mathcal{Z}) = 0$ and such that, for all $\mathrm{v} \in H_x \setminus \{0\}$, we have

$$\lim_{(t,\theta)\notin\mathcal{Z}} \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} = \lambda_\mu.$$

From the above conclusion we derive that, for every sequence of probability measures (τ_n) weakly converging to the delta mass at $+\infty \in [1, +\infty]$ and for every $\delta > 0$, we have

$$\lim_{n\to+\infty} (\tau_n \times \Theta_I)\left(\left\{(t,\theta) \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta, \lambda_\mu + \delta)\right\}\right) = 0. \quad (12)$$

We claim that, for every $\delta > 0$, there exists a full-measure set $\mathcal{F}_\delta \subset I$ such that, for all $\theta \in \mathcal{F}_\delta$, we have

$$\liminf_{T\to+\infty} \frac{1}{T}\mathrm{Leb}\left(\left\{t \in [1, T] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta, \lambda_\mu + \delta)\right\}\right) = 0.$$

Otherwise, there exists a positive-measure set $\mathcal{P}_\delta \subset I \subset \mathbb{T}$ such that, for all $\theta \in \mathcal{P}_\delta$,

$$\liminf_{T\to+\infty} \frac{1}{T}\mathrm{Leb}\left(\left\{t \in [1, T] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta, \lambda_\mu + \delta)\right\}\right) > 0.$$

By the Egorov theorem, it follows that there exist a sequence of times (T_n) and a positive-measure subset $\mathcal{P}'_\delta \subset \mathcal{P}_\delta \subset I$ such that the sequence

$$\frac{1}{T_n}\mathrm{Leb}\left(\left\{t \in [1, T_n] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta, \lambda_\mu + \delta)\right\}\right)$$

converges uniformly to a continuous positive function on $\mathcal{P}'_\delta$.

It is then possible to construct a sequence (τ_n) of probability measures on $[1, +\infty)$, weakly converging to the delta mass at $+\infty \in [1, +\infty]$, which contradicts the conclusion in the formula (12). Hence, the claim is proved.

For any (fixed) sequence (δ_n) converging to zero, let $\mathcal{F}_I$ denote the full-measure set defined as

$$\mathcal{F}_I := \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\delta_n}.$$

From the above claim it follows that, for all $\theta \in \mathcal{F}_I$ and for all $n \in \mathbb{N}$, we have

$$\liminf_{T\to+\infty} \frac{1}{T}\mathrm{Leb}\left(\left\{t \in [1, T] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta_n, \lambda_\mu + \delta_n)\right\}\right) = 0.$$

In particular, for any sequence (ϵ_n) of positive real numbers converging to zero, we have that there exist an increasing diverging sequence $(T_n) \subset [1, +\infty)$ such that

$$\frac{1}{T_n}\,\mathrm{Leb}\left(\left\{t \in [1, T_n] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta_n, \lambda_\mu + \delta_n)\right\}\right) < \epsilon_n.$$

Such sequence can be constructed recursively as follows. For any finite increasing sequence $\{T_k \mid k \le n\}$ and for any $T^*_{n+1} > 0$, there exists $T_{n+1} \ge T^*_{n+1}$ such that

$$\frac{1}{T_{n+1}}\,\mathrm{Leb}\left(\left\{t \in [1, T_{n+1}] \mid \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta_n, \lambda_\mu + \delta_n)\right\}\right) < \epsilon_{n+1}.$$

Let Z_θ be the set defined as follows:

$$Z_\theta := \bigcup_{n\in\mathbb{N}} \left\{T \in [T_n, T_{n+1}] : \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} \notin (\lambda_\mu - \delta_n, \lambda_\mu + \delta_n)\right\}.$$

Let us find under what conditions Z_θ has zero lower density. We have

$$\mathrm{Leb}(Z_\theta \cap [0, T_n]) \le \sum_{k=1}^{n-1} \mathrm{Leb}(Z_\theta \cap [T_k, T_{k+1}]) \le \sum_{k=1}^{n-1} \epsilon_{k+1} T_{k+1}.$$

It is therefore enough to choose the sequences recursively so that

$$\frac{1}{T_n} \sum_{k=1}^{n-2} \epsilon_{k+1} T_{k+1} + \epsilon_n \to 0.$$

It is clear by the definition of the set $Z_\theta \subset \mathbb{R}$ that, for $\theta \in \mathcal{F}_I$, we have

$$\lim_{t\notin Z_\theta} \frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} = \lambda_\mu.$$

The argument is therefore complete. □

Proof of Corollary 1.11. Let us assume by contraposition that there exists a positive-measure set $\mathcal{P} \subset I$ such that, for all $\theta \in \mathcal{P}$, we have

$$\limsup_{t\to+\infty} \left|\frac{1}{t} \log |g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} - \lambda_\mu\right| > 0.$$

This implies that there exist $\epsilon > 0$ and a set $\mathcal{P}_\epsilon$ of positive Lebesgue measure such that the following holds. For all $\theta \in \mathcal{P}_\epsilon$, there exists a diverging sequence $(t_n) = (t_n(\theta))$ such that, for all $n \in \mathbb{N}$, we have

$$\left|\frac{1}{t_n} \log |g_{t_n}^H(r_\theta(\mathrm{v}))|_{g_{t_n}(r_\theta(x))} - \lambda_\mu\right| \ge \epsilon.$$

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Since the cocycle is by hypothesis uniformly Lipschitz, there exists $\delta > 0$ such that, for all $t \in [(1-\delta)t_n(\theta), (1+\delta)t_n(\theta)]$, we have

$$\left|\frac{1}{t}\log|g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} - \lambda_\mu\right| \geq \epsilon/2$$

and hence it is possible to construct a sequence of compactly supported measures π_n on $[1,+\infty) \times I$ with smooth bounded density and conditional measure on $\mathbb{T}$ equal to the Lebesgue measure, such that

$$\lim_{n\to+\infty} \pi_n\left(\left\{(T,\theta)/\left|\frac{1}{t}\log|g_t^H(r_\theta(\mathrm{v}))|_{g_t(r_\theta(x))} - \lambda_\mu\right| \geq \epsilon/2\right\}\right) > 0.$$

This contradicts the conclusion of Theorem 1.2 and hence the corollary is proven. □

5. *Limits of geodesic push-forwards of horospherical measures*

Let X be a stratum of the moduli space of Abelian differentials. Let $\mathcal{H}_X$ denote the set of all compactly supported probability measures on X supported on a leaf $\mathcal{F}^s(x)$ of the stable foliation of the Teichmüller geodesic flow such that the following properties hold:

(1) the measure is absolutely continuous with continuous density with respect to the canonical affine measure on $\mathcal{F}^s(x)$;

(2) almost all of its conditional measures along the stable Teichmüller horocycle are restrictions of Lebesgue measures along horocycle orbits.

In particular, we may consider the restriction of the canonical affine measure to a compact subset of a leaf of the stable foliation.

By the results of Eskin, Mirzakhani and Mohammadi (see [**EM**] and [**EMM**]) and by condition (2) above, we can deduce that for any horospherical probability measure $\nu \in \mathcal{H}_X$ there exists a unique $\mathrm{SL}(2,\mathbb{R})$-invariant affine ergodic probability measure μ on X such that, in the weak* topology, we have

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T (g_t)_*(\nu)\,dt = \mu.$$

By the argument explained in §2, we can then deduce that there exists a set $Z \subset \mathbb{R}$ of zero upper density such that, in the weak* topology, we have

$$\lim_{t\notin Z}(g_t)_*(\nu) = \mu. \tag{13}$$

Our goal in this section is to prove Theorem 1.6.

Let $\|\cdot\|_X$ denote the Hodge norm on the tangent space TX of an (affine) $\mathrm{SL}(2,R)$-invariant suborbifold X of the moduli space of Abelian differentials and let $d_X : X \times X \to \mathbb{R}$ denote the corresponding distance function. Let $\mathrm{Lip}(X)$ denote the space of Lipschitz continuous functions with respect to the metric d_X on X endowed with the norm

$$\|f\|_{\mathrm{Lip}} := |f|_{C^0(X)} + \sup_{x\neq y}\frac{|f(x)-f(y)|}{d_X(x,y)} \quad \text{for all } f \in \mathrm{Lip}(X).$$

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We recall that by the Ascoli–Arzelà theorem, for any compact set $K \subset K$ a ball $\mathrm{Lip}(X, R)$ of radius $R > 0$ in $\mathrm{Lip}(X)$ maps under the restriction map $R_K : C^0(X) \to C^0(K)$ into a compact subset.

Let $\mathcal{F}^s$ denote the strong stable foliation of the (Teichmüller) geodesic flow. For all $x \in X$, let $D^s(x, r) \subset \mathcal{F}^s(x)$ denote the stable disk

$$D^s(x, r) := \{y \in \mathcal{F}^s(x) \mid d_X(x, y) \le r\}.$$

Let $\mathcal{I}^s_r : C^0(X) \to C^0(X)$ denote the averaging operator along the stable disks with respect to the Hodge volume vol^s on stable leaves, that is,

$$\mathcal{I}^s_r(f)(x) := \frac{1}{\mathrm{vol}(D^s(x, r))} \int_{D^s(x,r)} f \, d\mathrm{vol}^s \quad \text{for all } f \in C^0(X).$$

Let $\mathcal{F}^{wu}$ denote the weak-unstable foliation of the geodesic flow. Let $\mathrm{Lip}^{wu}(X)$ denote the space of continuous functions which are Lipschitz along the weak-unstable foliation, that is,

$$\mathrm{Lip}^{wu}(X) := \left\{ f \in C^0(X) \mid \sup_{x\in X} \sup_{y\in\mathcal{F}^{wu}(x)} \frac{|f(x) - f(y)|}{d_X(x, y)} < +\infty \right\},$$

endowed with the norm

$$\|f\|_{\mathrm{Lip}^{wu}} := |f|_{C^0(X)} + \sup_{x\in X} \sup_{y\in\mathcal{F}^{wu}(x)} \frac{|f(x) - f(y)|}{d_X(x, y)} \quad \text{for all } f \in \mathrm{Lip}^{wu}(X).$$

We have the following immediate result.

LEMMA 5.1. *For every $r > 0$, the averaging operator $\mathcal{I}^s_r$ maps $\mathrm{Lip}^{wu}(X)$ continuously into $\mathrm{Lip}(X)$ and hence for any compact set $K \subset X$ the composition*

$$\mathcal{I}^s_r \circ R_K : \mathrm{Lip}^{wu}(X) \to C^0_c(X)$$

is a compact operator.

Proof. We can prove by an immediate estimate that the averaging map $\mathcal{I}^s_r$ maps the Banach space $\mathrm{Lip}^{wu}(X)$ continuously to the Banach space $\mathrm{Lip}(X)$ and, for any compact set $K \subset X$, it maps $\mathrm{Lip}^{wu}(K) := \mathrm{Lip}^{wu}(X) \cap C^0(K)$ into $\mathrm{Lip}(K) := \mathrm{Lip}(X) \cap C^0(K)$. By the Ascoli–Arzelà theorem, the embedding $\mathrm{Lip}(K)$ into $C^0(K)$ is a compact operator. Finally, the composition of a continuous (bounded) operator and a compact operator is a compact operator. □

We finally prove the convergence of push-forwards of horospherical measures.

Proof of Theorem 1.6. By Lemma 5.1, the operator $\mathcal{I}^s_r \circ R_K : \mathrm{Lip}^{wu}(X) \to C^0_c(X)$ is compact and hence the dual operator

$$(\mathcal{I}^s_r \circ R_K)^* : C^0(K)^* \to \mathrm{Lip}^{wu}(X)^*$$

from the space $C^0_c(X)^*$ of linear continuous functionals on $C^0_c(X)$ to the space $\mathrm{Lip}^{wu}(X)^*$ of linear continuous functionals on $\mathrm{Lip}^{wu}(X)$ is also compact. In particular, for any weakly converging sequence $(\nu_n) \subset \mathcal{M}(X)$ of probability measures on X, the sequence $R^*_K(\mathcal{I}^s_r)^*(\nu_n)$ is (strongly) convergent in $\mathrm{Lip}^{wu}(X)^*$. By construction, we have that for all

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$t \geq 0$, the pull-back operator $(g_{-t})^* : \mathrm{Lip}^{wu}(X) \to \mathrm{Lip}^{wu}(X)$ is a weak contraction, in the sense that

$$\|f \circ g_{-t}\|_{\mathrm{Lip}^{wu}} \leq \|f\|_{\mathrm{Lip}^{wu}} \quad \text{for all } f \in \mathrm{Lip}^{wu}(X)$$

and hence the dual operator $(g_t)_* : \mathrm{Lip}^{wu}(X)^* \to \mathrm{Lip}^{wu}(X)^*$ defined as

$$(g_t)_*(\nu)(f) := \nu(f \circ g_{-t}) \quad \text{for all } \nu \in \mathrm{Lip}^{wu}(X)^* \text{ and for all } f \in \mathrm{Lip}^{wu}(X)$$

is also a weak contraction with respect to the dual norm $\| \cdot \|^*_{\mathrm{Lip}^{wu}}$ on $\mathrm{Lip}^{wu}(X)^*$:

$$\|(g_t)_*(\nu)\|^*_{\mathrm{Lip}^{wu}} \leq \|\nu\|^*_{\mathrm{Lip}^{wu}} \quad \text{for all } \nu \in \mathrm{Lip}^{wu}(X)^*.$$

Let ν be any horospherical measure supported on the stable leaf $\mathcal{F}^s(x)$ at a point $x \in X$. Let μ denote the unique affine probability measure supported on the orbit closure $\overline{\mathrm{SL}(2, \mathbb{R})x}$. As we have remarked above, see the formula (13), there exists a sequence (t_n) such that

$$(g_{t_n})_*(\nu) \to \mu \quad \text{in the weak* topology.}$$

Since ν is a *horospherical measure*, it follows that

$$\lim_{t \to +\infty} \|(\mathcal{I}^s_r)^*(g_{t_n})_*(\nu) - (g_{t_n})_*(\nu)\|^*_{\mathrm{Lip}^{wu}} = 0$$

and hence, for any compact set $K \subset X$, we have

$$\lim_{n \to +\infty} \|R^*_K(g_{t_n})_*(\nu) - R^*_K(\mu)\|^*_{\mathrm{Lip}^{wu}} = 0.$$

Since $(g_t)_*$ is a weak contraction on $\mathrm{Lip}^{wu}(X)^*$, uniformly with respect to $t \geq 0$, we have that

$$\lim_{n \to +\infty} \|(g_t)_* R^*_K(g_{t_n})_*(\nu) - R^*_K(\mu)\|_{\mathrm{Lip}^{wu}} = 0.$$

Finally, since the set of all probability measures supported on horocycle arcs is *tight* (see [**MW**] and [**EMa**]), for any $\epsilon > 0$ there exists a compact set K_ϵ such that

$$\|(g_t)_* R^*_{K_\epsilon}(g_{t_n})_*(\nu) - (g_{t+t_n})_*(\nu)\|_{\mathcal{M}(X)} = \|R^*_{K_\epsilon}(g_{t_n})_*(\nu) - (g_{t_n})_*(\nu)\|_{\mathcal{M}(X)} \leq \epsilon$$

and hence the measure μ is the unique weak* limit of the set $\{(g_{t+t_n})_*(\nu)\}$, as claimed. □

Acknowledgements. We are very grateful to Barak Weiss and Ronggang Shi for pointing out an error in the first draft of this paper. As a consequence our results on Birkhoff genericity and Oseledets regularity have been significantly weakened. We are grateful to D. Aulicino, A. Brown, A. Eskin, C. Matheus and R. Treviño for several discussions about the topics of this paper. This research was supported by the NSF grant DMS 1600687.

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