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MAXIMAL COMPUTABILITY STRUCTURES

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Abstract. A computability structure on a metric space is a set of sequences which satisfy certain conditions. Of a particular interest are those computability structures which contain a dense sequence, so called separable computability structures. In this paper we observe maximal computability structures which are more general than separable computability structures and we examine their properties. In particular, we examine maximal computability structures on subspaces of Euclidean space, we give their characterization and we investigate conditions under which a maximal computability structure on such a space is unique. We also give a characterization of separable computability structures on a segment.

§1. Introduction. One way to impose computability notions in the context of a metric space (X, d) is to fix a dense sequence $\alpha = (\alpha_i)$ in this space with the property that the real numbers $d(\alpha_i, \alpha_j)$ can be effectively computed. This means that for each $i, j, k \in \mathbb{N}$ we can effectively compute a rational number which approximates $d(\alpha_i, \alpha_j)$ up to 2^{-k}. We say that the triple (X, d, α) is a computable metric space. A point $x \in X$ is said to be computable in this space if for each $k \in \mathbb{N}$ we can effectively compute $j \in \mathbb{N}$ such that the point α_j is 2^{-k}-close to x. Similarly, a sequence (x_i) in X is said to be computable in this space if for all $i, k \in \mathbb{N}$ we can effectively compute $j \in \mathbb{N}$ such that the point α_j is 2^{-k}-close to x_i.

Furthermore, we can define the notion of a computable subset of X. First, we fix some effective sequence (I_i) of rational open balls in (X, d, α); a rational open ball in (X, d, α) is an open ball centered in some α_j with rational radius. We say that a closed set S in (X, d) is computably enumerable in (X, d, α) if the set of all $i \in \mathbb{N}$ such that $I_i \cap S \neq \emptyset$ is recursively enumerable and we say that S is co-computably enumerable in (X, d, α) if $X \setminus S = \bigcup_{i \in A} I_i$ for some recursively enumerable set $A \subseteq \mathbb{N}$. A set S is called computable in (X, d, α) if it is computably enumerable and co-computably enumerable ([1,2]).

All these notions depend, by definition, on the sequence α. However, it turns out that if (X, d, α) and (X, d, β) are computable metric spaces, where α and β are equivalent sequences, then the notions of a computable point, a computable sequence, a (co-)computably enumerable set and a computable

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set in computable metric spaces (X, d, α) and (X, d, β) coincide. That α and β are equivalent means that α is a computable sequence in (X, d, β) and β is a computable sequence in (X, d, α).

If (X, d, α) is a computable metric space, let $\mathcal{S}_\alpha$ denote the set of all sequences which are computable in this space. If we have computable metric spaces (X, d, α) and (X, d, β), it turns out that α and β are equivalent if and only if $\mathcal{S}_\alpha = \mathcal{S}_\beta$. So the notions of a computable point, a computable sequence, a (co-)computably enumerable set and a computable set can be viewed as notions defined related to the entire set $\mathcal{S}_\alpha$ and not just to α itself.

Therefore, it makes sense to observe sets of the form $\mathcal{S}_\alpha$ and to take such sets of sequences as a basis for computability concepts on a metric space (X, d). This leads to the notion of a computability structure on a metric space (X, d).

A computability structure $\mathcal{S}$ on (X, d) is a set of sequences in X such that the following holds:

(i) if $(x_i), (y_j) \in \mathcal{S}$, then the distances $d(x_i, y_j)$ can be effectively computed;
(ii) if $(x_i) \in \mathcal{S}$ and (y_i) is a sequence in X which can be computed from (x_i), then $(y_i) \in \mathcal{S}$.

If (X, d, α) is a computable metric space, then $\mathcal{S}_\alpha$ is a computability structure on (X, d). Such computability structures on (X, d) we call separable. Computability structures have been studied by Pour-El and Richards in [7], by Yasugi, Mori, and Tsujji in [6, 11] and results related to computability structures have been studied by Melnikov in [5]. An investigation of computability structures can also be found in [4]. See also [3, 9]. Usually, to get certain results, we need the assumption that a computability structure is separable.

In this paper we focus on maximal computability structures—computability structures which are maximal with respect to inclusion. We investigate this notion and the particular relationship between maximal and separable computability structures. Although we give some observations on maximal computability structures in general metric spaces, most of the paper is devoted to the study of maximal computability structures on subspaces of Euclidean space. Using maximal computability structures, we give a description of separable computability structures on a line segment in $\mathbb{R}$ and we use this to determine the number of such computability structures obtaining a result which is a somewhat more precise form of Theorem 8.12 from [5].

We believe that the subject of this paper has a potential for further investigations. For example, one way in that direction could be a study of maximal computability structures on various well-known examples of metric spaces.

Here is how the paper is organized. In Section 2 we give basic notions and facts and study computability structures in general. In Section 3 we introduce the notion of a maximal computability structure and prove certain results related to this notion which hold in a general metric space. In Section 4 we focus on subspaces of Euclidean space. We give a characterization of maximal computability structures on such metric spaces, we examine the problem of uniqueness of a maximal computability structure and we also prove that each maximal computability structure on $\mathbb{R}^n$ is separable. In Section 5 we characterize separable computability structures among maximal computability structures on a segment and we examine the cardinality of the set of all separable computability structures on a segment.

§2. Computability structures and basic notions. A function $f : \mathbb{N}^k \to \mathbb{Q}$ is said to be *recursive* if there exist recursive functions $a, b, c : \mathbb{N}^{k+1} \to \mathbb{N}$ such that

$$f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1}$$

for each $x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be *recursive* if there exists a recursive function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x,i)| < 2^{-i}$$

for all $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$ [7, 10]. A number $x \in \mathbb{R}$ is said to be *recursive* if there exists a recursive function $f : \mathbb{N} \to \mathbb{Q}$ such that $|x - f(k)| < 2^{-k}$ for each $k \in \mathbb{N}$ [8]. A point $(x_1, \dots, x_n) \in \mathbb{R}^n$ is called *recursive* if $x_1, \dots, x_n$ are recursive numbers.

We say that a function $f : \mathbb{N}^k \to \mathbb{R}^n$ is *recursive* if the component functions of f are recursive (as functions $\mathbb{N}^k \to \mathbb{R}$).

In the following proposition we state some basic facts about recursive functions $\mathbb{N}^k \to \mathbb{R}$.

PROPOSITION 2.1. (i) *If $f, g : \mathbb{N}^k \to \mathbb{R}$ are recursive functions, then $f+g$, $f-g$, and $f \cdot g$ are recursive.*

(ii) *If $f : \mathbb{N}^k \to \mathbb{R}$ and $F : \mathbb{N}^{k+1} \to \mathbb{R}$ are functions such that F is recursive and $|f(x) - F(x,i)| < \frac{1}{2^i}$, for each $x \in \mathbb{N}^k$ and each $i \in \mathbb{N}$, then f is recursive.*

(iii) *If $f : \mathbb{N}^k \to \mathbb{R}$ is a recursive function such that $f(x) \geq 0$ for each $x \in \mathbb{N}^k$, then the function $\mathbb{N}^k \to \mathbb{R}$, $x \mapsto \sqrt{f(x)}$ is recursive.*

(iv) *If $f, g : \mathbb{N}^k \to \mathbb{R}$ are recursive functions, then the set*

$$\left\{ x \in \mathbb{N}^k \mid f(x) > g(x) \right\}$$

is recursively enumerable.

Let (X, d) be a metric space and (x_i) a sequence in X. We say that (x_i) is an *effective sequence* in (X, d) if the function $\mathbb{N}^2 \to \mathbb{R}$,

$$(i, j) \mapsto d(x_i, x_j)$$

is recursive. If (x_i) and (y_j) are sequences in X, we say that $((x_i), (y_j))$ is an *effective pair* in (X, d) and we write $(x_i) \diamond (y_j)$ if the function $\mathbb{N}^2 \to \mathbb{R}$,

$$(i, j) \mapsto d(x_i, y_j)$$

is recursive. Note that a sequence (x_i) is effective in (X, d) if and only if $(x_i) \diamond (x_i)$. Also note that $(x_i) \diamond (y_j)$ implies $(y_j) \diamond (x_i)$.

Suppose (X, d) is a metric space and (x_i) is a sequence in X. A sequence (y_i) in X is said to be *computable with respect to* (x_i) in (X, d) and we write $(y_i) \preceq (x_i)$ if there exists a recursive function $F : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d(y_i, x_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbb{N}$. A point $a \in X$ is said to be *computable with respect to* (x_i) in (X, d) if there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $d(a, x_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$.

PROPOSITION 2.2. *Let (X, d) be a metric space and let (x_i), (y_i), (z_i) be sequences in X such that $(z_i) \preceq (y_i)$ and $(y_i) \preceq (x_i)$. Then $(z_i) \preceq (x_i)$.*

PROOF. Let $F, G : \mathbb{N}^2 \to \mathbb{N}$ be recursive functions such that

$$d(z_i, y_{F(i,k)}) < 2^{-k} \text{ and } d(y_i, x_{G(i,k)}) < 2^{-k} \tag{1}$$

for all $i, k \in \mathbb{N}$. Then, for all $i, k \in \mathbb{N}$, we have $d(z_i, y_{F(i,k+1)}) < 2^{-(k+1)}$ and $d(y_{F(i,k+1)}, x_{G(F(i,k+1),k+1)}) < 2^{-(k+1)}$ and the triangle inequality implies

$$d(z_i, x_{G(F(i,k+1),k+1)}) < 2^{-k}.$$

Hence $(z_i) \preceq (x_i)$. ⊣

PROPOSITION 2.3. *Let (X, d) be a metric space and let (x_i), (y_i), (α_i), (β_i) be sequences in X such that $(x_i) \preceq (\alpha_i)$ and $(y_i) \preceq (\beta_i)$. Suppose $(\alpha_i) \diamond (\beta_i)$. Then $(x_i) \diamond (y_i)$.*

PROOF. Let $F, G : \mathbb{N}^2 \to \mathbb{N}$ be recursive functions such that

$$d(x_i, \alpha_{F(i,k)}) < 2^{-k} \text{ and } d(y_j, \beta_{G(j,k)}) < 2^{-k}$$

for all $i, j, k \in \mathbb{N}$. In general, if $a, a', b, b' \in X$, then

$$|d(a, b) - d(a', b')| \leq d(a, a') + d(b, b'),$$

which follows easily from the triangle inequality. Therefore, for all $i, j, k \in \mathbb{N}$ we have

$$|d(x_i, y_j) - d(\alpha_{F(i,k+1)}, \beta_{G(j,k+1)})| \leq$$
$$\leq d(x_i, \alpha_{F(i,k+1)}) + d(y_j, \beta_{G(j,k+1)}) < 2 \cdot 2^{-(k+1)} = 2^{-k}.$$

It follows from Proposition 2.1(ii) that the function $\mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto d(x_i, y_j)$ is recursive. ⊣

COROLLARY 2.4. *Let (X, d) be a metric space and (y_i), (x_i) sequences in X such that $(y_i) \preceq (x_i)$. Suppose (x_i) is effective. Then (y_i) is effective.*

An effective sequence (x_i) in a metric space (X, d) is said to be an *effective separating sequence* if (x_i) is dense in (X, d), i.e., if $\{x_i \mid i \in \mathbb{N}\}$ is a dense set in (X, d).

If $\alpha = (\alpha_i)$ is an effective separating sequence in (X, d), then the triple (X, d, α) is called a *computable metric space*. If (X, d, α) is a computable metric space, then a point $x \in X$ is said to be *computable* in (X, d, α) if x is computable with respect to α and a sequence (x_i) in X is said to be *computable* in (X, d, α) if (x_i) is computable with respect to α.

EXAMPLE 2.5. Let $n \in \mathbb{N} \setminus \{0\}$ and let d be the Euclidean metric on $\mathbb{R}^n$. Let (x_i) and (y_j) be sequences in $\mathbb{R}^n$ and let $(x_i^1), \ldots, (x_i^n)$ and $(y_j^1), \ldots, (y_j^n)$ be the component sequences of (x_i) and (y_j).

(i) Suppose (x_i) and (y_j) are recursive (as functions $\mathbb{N} \to \mathbb{R}^n$). Then $(x_i) \diamond (y_j)$ (in the metric space $(\mathbb{R}^n, d)$). Namely, for all $i, j \in \mathbb{N}$ we have

$$d(x_i, y_j) = \sqrt{(x_i^1 - y_j^1)^2 + \cdots + (x_i^n - y_j^n)^2},$$

and the claim follows from statements (i) and (iii) of Proposition 2.1. In particular, each recursive sequence in $\mathbb{R}^n$ is effective in $(\mathbb{R}^n, d)$.

(ii) Suppose (x_i) is recursive and $(y_j) \preceq (x_i)$. Then (y_j) is also recursive. Namely, if $F : \mathbb{N}^2 \to \mathbb{N}$ is a recursive function such that $d(y_j, x_{F(j,k)}) < 2^{-k}$ for all $j, k \in \mathbb{N}$, then for each $l \in \{1, \ldots, n\}$ and all $j, k \in \mathbb{N}$ we have

$$|y_j^l - x_{F(j,k)}^l| \leq d(y_j, x_{F(j,k)}) < 2^{-k}$$

and Proposition 2.1(ii) implies that $(y_j^1), \ldots, (y_j^n)$ are recursive sequences.

Now, if $\alpha : \mathbb{N} \to \mathbb{R}^n$ is a recursive sequence such that $\{\alpha_i \mid i \in \mathbb{N}\}$ is a dense set in $(\mathbb{R}^n, d)$, then α is an effective separating sequence in $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d, \alpha)$ is a computable metric space.

Let (X, d) be a metric space and let $\mathcal{S}$ be a set whose elements are sequences in X, i.e., $\mathcal{S} \subseteq X^{\mathbb{N}}$. We say that $\mathcal{S}$ is a *computability structure* on (X, d) (see [11]) if the following properties hold:

(i) if $(x_i), (y_j) \in \mathcal{S}$, then $(x_i) \diamond (y_j)$;
(ii) if $(x_i) \in \mathcal{S}$ and $(y_i) \preceq (x_i)$, then $(y_i) \in \mathcal{S}$.

Note the following: if (X, d) is a metric space and $\mathcal{S} \subseteq X^{\mathbb{N}}$ such that property (ii) above holds, then the following holds:

(iii) if $(x_i) \in \mathcal{S}$ and $f : \mathbb{N} \to \mathbb{N}$ is a recursive function, then $(x_{f(i)})_{i \in \mathbb{N}} \in \mathcal{S}$.

So if $\mathcal{S}$ is a computability structure on (X, d), then (iii) holds.

If $\mathcal{S}$ is a computability structure on (X, d), then each $(x_i) \in \mathcal{S}$ is an effective sequence in (X, d), which follows directly from (i).

Example 2.6. (i) Let (X, d) be a metric space and let $a \in X$. Let (x_i) be the sequence in X defined by $x_i = a$, $i \in \mathbb{N}$ and let $\mathcal{S} = \{(x_i)\}$. Then $\mathcal{S}$ is a computability structure on (X, d).

(ii) Let d be the Euclidean metric on $\mathbb{R}^n$. By Example 2.5 the set of all recursive sequences in $\mathbb{R}^n$ is a computability structure on $(\mathbb{R}^n, d)$.

If (X, d) is a metric space and α a sequence in (X, d), let $\mathcal{S}_\alpha$ denote the set of all sequences (x_i) in X such that $(x_i) \preceq \alpha$. Note that $\alpha \in \mathcal{S}_\alpha$.

Proposition 2.7. *Let (X, d) be a metric space and α a sequence in X. Then α is an effective sequence in (X, d) if and only if $\mathcal{S}_\alpha$ is a computability structure on (X, d).*

Proof. If $\mathcal{S}_\alpha$ is computability structure on (X, d), then α is effective in (X, d) since $\alpha \in \mathcal{S}_\alpha$.

Conversely, if α is effective in (X, d), then $\mathcal{S}_\alpha$ is computability structure on (X, d): property (i) from definition of a computability structure follows from Proposition 2.3, and property (ii) follows from Proposition 2.2. ⊣

Suppose α and β are effective sequences in a metric space (X, d). We say that α and β are *equivalent* and write $\alpha \sim \beta$ if α is computable with respect to β and β is computable with respect to α. By Proposition 2.2 for any sequences α and β in X we have

$$\alpha \preceq \beta \Longleftrightarrow \mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$$

and therefore for any effective sequences α and β in (X, d) we have

$$\alpha \sim \beta \Longleftrightarrow \mathcal{S}_\alpha = \mathcal{S}_\beta.$$

LEMMA 2.8. *Let (X, d) be a metric space and let $\alpha = (\alpha_i)$ be a sequence in X.*

(i) *The sequence α is effective if and only if for each sequence (x_i) in X the following implication holds*:

$$(x_i) \preceq \alpha \implies (x_i) \diamond \alpha. \tag{2}$$

(ii) *If α an effective separating sequence, then for each sequence (x_i) in X we have*

$$(x_i) \preceq \alpha \Longleftrightarrow (x_i) \diamond \alpha.$$

PROOF. (i) If (2) holds, then α is effective since $\alpha \preceq \alpha$.

Conversely, suppose α is effective and $(x_i) \preceq \alpha$. Then $(x_i) \in \mathcal{S}_\alpha$, which together with $\alpha \in \mathcal{S}_\alpha$ and Proposition 2.7 implies $(x_i) \diamond \alpha$.

(ii) Suppose α is an effective separating sequence and $(x_i) \diamond \alpha$.

Let $i, k \in \mathbb{N}$. Then there exists $j \in \mathbb{N}$ such that $d(x_i, \alpha_j) < 2^{-k}$. Since the set Ω of all $(i, k, j) \in \mathbb{N}^3$ such that $d(x_i, \alpha_j) < 2^{-k}$ is r.e. (by Proposition 2.1(iv)) and for all $i, k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $(i, k, j) \in \Omega$, there exists a recursive function $F : \mathbb{N}^2 \to \mathbb{N}$ such that $(i, k, F(i, k)) \in \Omega$ for all $i, k \in \mathbb{N}$. Hence

$$d(x_i, \alpha_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbb{N}$ and therefore $(x_i) \preceq \alpha$. ⊣

PROPOSITION 2.9. *Suppose (X, d) is a metric space, $\mathcal{S}$ a computability structure on (X, d), and α a dense sequence in (X, d) such that $\alpha \in \mathcal{S}$. Then α is an effective separating sequence in (X, d) and $\mathcal{S} = \mathcal{S}_\alpha$.*

PROOF. Obviously, α is an effective separating sequence in (X, d).

If $(x_i) \in \mathcal{S}_\alpha$, then $(x_i) \preceq \alpha$ and $\alpha \in \mathcal{S}$ implies $(x_i) \in \mathcal{S}$.

Conversely, let $(x_i) \in \mathcal{S}$. Then $(x_i) \diamond \alpha$ and by Lemma 2.8 $(x_i) \preceq \alpha$, i.e., $(x_i) \in \mathcal{S}_\alpha$. Hence $\mathcal{S} = \mathcal{S}_\alpha$. ⊣

Let (X, d) be a metric space. We say that $\mathcal{S}$ is a *separable computability structure* on (X, d) if $\mathcal{S}$ is a computability structure on (X, d) and there exists $\alpha \in \mathcal{S}$ such that α is a dense sequence in (X, d). Note that by Proposition 2.9 $\mathcal{S}$ is a separable computability structure on (X, d) if and only if $\mathcal{S} = \mathcal{S}_\alpha$ for some effective separating sequence α in (X, d).

EXAMPLE 2.10. Let d be the Euclidean metric on $\mathbb{R}^n$ and let α be as in Example 2.5. Then $\mathcal{S}_\alpha$ is the set of all recursive sequences in $\mathbb{R}^n$.

Indeed, if $(x_i) \in \mathcal{S}_\alpha$, then (x_i) is a recursive sequence in $\mathbb{R}^n$ by claim (ii) of Example 2.5. On the other hand, if (x_i) is a recursive sequence in $\mathbb{R}^n$, then by claim (i) of the same example we have $(x_i) \diamond \alpha$ and Lemma 2.8 implies $(x_i) \preceq \alpha$, i.e., $(x_i) \in \mathcal{S}_\alpha$.

So the set of all recursive sequences in $\mathbb{R}^n$ is a separable computability structure on $(\mathbb{R}^n, d)$.

Suppose (X, d) is a metric space and α an effective sequence in (X, d). Then

$$\alpha \text{ is a dense sequence} \Longleftrightarrow \mathcal{S}_\alpha \text{ is a separable computability structure.} \quad (3)$$

Indeed, if (x_i) is a dense sequence computable with respect to α, then α is also dense.

Let (X, d) be a metric space and $\mathcal{S}$ a computability structure on (X, d). Let $a \in X$. We say that a is a *computable point in the computability structure* $\mathcal{S}$ if there exist $(x_i) \in \mathcal{S}$ and $i \in \mathbb{N}$ such that $a = x_i$. It is easy to see that the following statements are equivalent:

(i) a is a computable point in $\mathcal{S}$;
(ii) $(a, a, a, \dots) \in \mathcal{S}$;
(iii) there exist $(x_i) \in \mathcal{S}$ and a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $d(a, x_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$.

If $\mathcal{S}$ is a computability structure on a metric space (X, d), then we will denote by $\mathcal{S}^0$ the set of all computable points in $\mathcal{S}$.

§3. Maximal computability structures in general. Let (X, d) be a metric space and $\mathcal{S}$ a computability structure on (X, d). We say that $\mathcal{S}$ is a *maximal computability structure* on (X, d) if there exists no computability structure $\mathcal{T}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$.

First we notice that each separable computability structure is maximal. Indeed, if α is an effective separating sequence in a metric space (X, d) and $\mathcal{T}$ a computability structure on (X, d) such that $\mathcal{S}_\alpha \subseteq \mathcal{T}$, then $\alpha \in \mathcal{T}$ and by Proposition 2.9 we have $\mathcal{T} = \mathcal{S}_\alpha$.

Separable computability structures on a metric space (X, d) in general need not exist. Certainly, such computability structures do not exist if (X, d) is not a separable metric space, but even if (X, d) is separable a separable computability structure on (X, d) need not exist. For example, if $X = \{0, a\}$, where a is a nonrecursive number and d is the Euclidean metric on X, then (X, d) clearly does not have a separable computability structure.

On the other hand, each metric space has maximal computability structures, moreover we have the following proposition.

Proposition 3.1. *Let $\mathcal{S}$ be a computability structure on a metric space (X, d). Then there exists a maximal computability structure $\mathcal{M}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{M}$.*

Proof. Let Λ be the family of all computability structures $\mathcal{T}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{T}$. If we partially order Λ by inclusion, it is straightforward to check that every chain in Λ has an upper bound and therefore, by Zorn's lemma, there exists a maximal element $\mathcal{M}$ in Λ. Hence $\mathcal{M}$ is a maximal computability structure on (X, d) which contains $\mathcal{S}$. ⊣

Even when a metric space has separable computability structures, a maximal computability structure need not be separable. Moreover, a maximal computability structure which is *dense* need not be separable (Example 3.2).

A computability structure $\mathcal{S}$ on a metric space (X, d) is said to be *dense* if $\mathcal{S}^0$ is a dense set in (X, d). Clearly, each separable computability structure is dense. On the other hand, a dense computability structure need not be separable. For example, if d is the Euclidean metric on $\mathbb{R}$ and $\mathcal{S}$ a set of all constant sequences $(q, q, q, \dots)$, where $q \in \mathbb{Q}$, then $\mathcal{S}$ is a dense computability structure on $(\mathbb{R}, d)$ which is not separable. Note that $\mathcal{S}$ is not a maximal computability structure.

EXAMPLE 3.2. Let d be the Euclidean metric on $[0, 1]$. Then there exists a unique separable computability structure on $([0, 1], d)$ (see [4], Example 10 or Theorem 31).

Let $\alpha : \mathbb{N} \to \mathbb{Q}$ be a recursive function whose range is $[0, 1] \cap \mathbb{Q}$ (such a function certainly exists). Then α is an effective separating sequence in $([0, 1], d)$. If $x \in [0, 1]$ is a point computable with respect to α, then x is clearly a recursive number. Hence $\mathcal{S}_\alpha$ is the only separable computability structure on $([0, 1], d)$ and every element of $\mathcal{S}_\alpha^0$ is a recursive number.

Let $c \in [0, 1]$ be a nonrecursive number. Then $\{(c, c, c, \dots)\}$ is a computability structure on $([0, 1], d)$ and by Proposition 3.1 there exists a maximal computability structure $\mathcal{M}$ on $([0, 1], d)$ such that $\{(c, c, c, \dots)\} \subseteq \mathcal{M}$. It follows $c \in \mathcal{M}^0$. Since $c \notin \mathcal{S}_\alpha^0$, we have $\mathcal{M} \neq \mathcal{S}_\alpha$, hence $\mathcal{M}$ is not a separable computability structure on $([0, 1], d)$.

Moreover, let $\mathcal{T}$ be the set of all constant sequences $(x, x, x, \dots)$, where $x \in [0, 1]$ is such that $x - c \in \mathbb{Q}$. Then $\mathcal{T}$ is a computability structure on $([0, 1], d)$ and $c \in \mathcal{T}^0$. Let $\mathcal{M}_1$ be a maximal computability structure on $([0, 1], d)$ which contains $\mathcal{T}$. We have $c \in \mathcal{M}_1^0$ and we conclude that $\mathcal{M}_1$ is not separable. If $x \in [0, 1]$ is such that $x - c \in \mathbb{Q}$, then x is a computable point in $\mathcal{M}_1$ and therefore $\mathcal{M}_1$ is a dense computability structure on $([0, 1], d)$ (which is maximal and not separable).

Actually, as we will see, $\mathcal{M} = \mathcal{M}_1$ and both of these two computability structures are equal to the set of all sequences (x_i) in $[0, 1]$ such that $(x_i - c)$ is a recursive sequence.

Let α be an effective sequence in a metric space (X, d). In contrast to equivalence (3), the equivalence

$$\alpha \text{ is a dense sequence} \iff \mathcal{S}_\alpha \text{ is a maximal computability structure}$$

does not hold in general (although the implication $\Longrightarrow$ always holds). For example, if c is a nonrecursive number, $X = \{0, c\}$, d the Euclidean metric on X and $\alpha = (0, 0, 0, \dots)$, then $\mathcal{S}_\alpha = \{\alpha\}$ and $\{\alpha\}$ is a maximal computability structure. Hence $\mathcal{S}_\alpha$ is a maximal computability structure, but α is not dense in (X, d).

EXAMPLE 3.3. Let X be a nonempty set and let d be the discrete metric on X. Suppose $\mathcal{M}$ is a maximal computability structure on (X, d). Then $\mathcal{M}^0 = X$. Indeed, if there exists $x \in X$ such that $x \notin \mathcal{M}^0$, then $\{(x, x, x, \dots)\} \cup \mathcal{M}$

is a computability structure on (X, d) which is different from $\mathcal{M}$ and which contains $\mathcal{M}$. This is impossible. Hence $\mathcal{M}^0 = X$.

Suppose additionally that the set X is uncountable. We have the following conclusion: $\mathcal{M}^0$ is uncountable, hence $\mathcal{M}$ is uncountable. Furthermore, $\mathcal{M}$ is a dense computability structure on (X, d) and (X, d) is not separable. In contrast to this, if $\mathcal{S}$ is a separable computability structure on some metric space, then that metric space is separable and $\mathcal{S}$ is countable, namely since there are only countably many recursive functions $\mathbb{N}^2 \to \mathbb{N}$, there are only countably many sequences which are computable with respect to a given effective (separating) sequence α.

Let (X, d) be a metric space and let $\mathcal{S}$ be a set whose elements are sequences in X. We say that $\mathcal{S}$ is an *effective structure* on (X, d) if for all $(x_i), (y_j) \in \mathcal{S}$ we have $(x_i) \diamond (y_i)$.

Note that each computability structure on (X, d) is an effective structure on (X, d). On the other hand, an effective structure need not be a computability structure. For example, if $\alpha : \mathbb{N} \to \mathbb{Q}$ is a recursive surjection and d the Euclidean metric on $\mathbb{R}$, then $\{\alpha\}$ is an effective structure on $(\mathbb{R}, d)$, but it is not a computability structure since $(0, 0, 0, \dots) \preceq \alpha$ and $(0, 0, 0, \dots) \notin \{\alpha\}$.

Let (X, d) be a metric space. We say that $\mathcal{S}$ is a *maximal effective structure* on (X, d) if $\mathcal{S}$ is an effective structure on (X, d) and there exists no effective structure $\mathcal{T}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$.

If $\mathcal{S}$ is a computability structure, then any subset of $\mathcal{S}$ is obviously an effective structure. Conversely, each effective structure is a subset of some computability structure. This is the contents of the following proposition.

PROPOSITION 3.4. *Let (X, d) be a metric space and $\mathcal{S} \subseteq X^{\mathbb{N}}$. Then*

(i) *$\mathcal{S}$ is an effective structure on (X, d) if and only if there exists a computability structure $\mathcal{T}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{T}$;*
(ii) *$\mathcal{S}$ is a maximal effective structure on (X, d) if and only if $\mathcal{S}$ is a maximal computability structure on (X, d).*

PROOF. If $\mathcal{S}$ is an effective structure on (X, d), then $\bigcup_{\alpha \in \mathcal{S}} \mathcal{S}_\alpha$ is a computability structure on (X, d). This follows from Proposition 2.3. Clearly $\mathcal{S} \subseteq \bigcup_{\alpha \in \mathcal{S}} \mathcal{S}_\alpha$. This proves (i).

Now, if $\mathcal{S}$ is a maximal effective structure on (X, d), then $\mathcal{S} = \bigcup_{\alpha \in \mathcal{S}} \mathcal{S}_\alpha$ and this means that $\mathcal{S}$ is a computability structure. So $\mathcal{S}$ is a maximal computability structure on (X, d).

Conversely, suppose $\mathcal{S}$ is a maximal computability structure and $\mathcal{T}$ is an effective structure such that $\mathcal{S} \subseteq \mathcal{T}$. We have $\mathcal{S} \subseteq \mathcal{T} \subseteq \bigcup_{\alpha \in \mathcal{T}} \mathcal{S}_\alpha$ and it follows $\mathcal{S} = \bigcup_{\alpha \in \mathcal{T}} \mathcal{S}_\alpha$. So $\mathcal{S} = \mathcal{T}$. Hence $\mathcal{S}$ is a maximal effective structure. ⊣

The following proposition follows from Propositions 3.1 and 3.4(ii) (or it can be proved directly as Proposition 3.1).

PROPOSITION 3.5. *Let $\mathcal{S}$ be an effective structure on a metric space (X, d). Then there exists a maximal effective structure $\mathcal{M}$ on (X, d) such that $\mathcal{S} \subseteq \mathcal{M}$.*

If $f : X \to Y$ and $\mathcal{S} \subseteq X^{\mathbb{N}}$, let $f(\mathcal{S}) = \{(f(x_i)) \mid (x_i) \in \mathcal{S}\}$. The proof of the next proposition is straightforward.

PROPOSITION 3.6. *Let (X, d) and (Y, d') be metric spaces, let $f : X \to Y$ be a surjective isometry and let $\mathcal{S} \subseteq X^{\mathbb{N}}$. Then $\mathcal{S}$ is a (maximal) computability structure on (X, d) if and only if $f(\mathcal{S})$ is a (maximal) computability structure on (Y, d'). Moreover, $\mathcal{S}$ is separable if and only if $f(\mathcal{S})$ is separable.*

§4. Maximal computability structures on Euclidean space. If (X, d) is a metric space, $n \in \mathbb{N}$ and $a_0, \dots, a_n \in X$, then we will say that $a_0, \dots, a_n$ is an *effective finite sequence* in (X, d) if $d(a_i, a_j)$ is a recursive number for all $i, j \in \{0, \dots, n\}$.

Suppose $a_0, \dots, a_n$ is an effective finite sequence in a metric space (X, d). Then $\{(a_0, a_0, \dots), \dots, (a_n, a_n, \dots)\}$ is a computability structure on (X, d) and by Proposition 3.1 this computability structure is contained in some maximal computability structure $\mathcal{M}$ on (X, d). Hence there exists a maximal computability structure $\mathcal{M}$ on (X, d) such that $a_0, \dots, a_n \in \mathcal{M}^0$. However, such a maximal computability structure $\mathcal{M}$ need not be unique. In this section we will focus on the case when (X, d) is a subspace of Euclidean space and we will examine conditions under which such a maximal computability structure is unique.

If (X, d) is a metric space and $a \in X$, let $\mathcal{R}_a^{(X,d)}$ denote the set of all sequences (x_i) in X such that the function $\mathbb{N} \to \mathbb{R}$, $i \mapsto d(x_i, a)$ is recursive. For simplicity of notation, for $a_0, \dots, a_n \in X$ we will write $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$ instead of $\mathcal{R}_{a_0}^{(X,d)} \cap \dots \cap \mathcal{R}_{a_n}^{(X,d)}$.

Note that

$$(x_i) \in \mathcal{R}_a^{(X,d)} \iff (x_i) \diamond (a, a, a, \dots).$$

From previous equivalence and Proposition 2.3 we conclude the following: if $(x_i) \in \mathcal{R}_a^{(X,d)}$ and $(y_i) \preceq (x_i)$, then $(y_i) \in \mathcal{R}_a^{(X,d)}$. Therefore

$$\mathcal{R}_{a_0,\dots,a_n}^{(X,d)} \text{ is an effective structure} \iff \mathcal{R}_{a_0,\dots,a_n}^{(X,d)} \text{ is a computability structure.}$$

Suppose $a_0, \dots, a_n$ is an effective finite sequence in a metric space (X, d). Then each of the constant sequences $(a_0, a_0, \dots)$, $\dots$, $(a_n, a_n, \dots)$ is an element of $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$. Furthermore, if $\mathcal{S}$ is an effective structure on (X, d), then

$$a_0, \dots, a_n \in \mathcal{S}^0 \implies \mathcal{S} \subseteq \mathcal{R}_{a_0,\dots,a_n}^{(X,d)}. \tag{4}$$

Therefore, if $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$ is an effective structure, then $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$ is a maximal effective structure. So, by Proposition 3.4, we have the implication

$$\mathcal{R}_{a_0,\dots,a_n}^{(X,d)} \text{ is an effective structure} \implies \mathcal{R}_{a_0,\dots,a_n}^{(X,d)} \text{ is a maximal computability structure.}$$

This, together with (4), gives the following claim.

PROPOSITION 4.1. *Let $a_0, \dots, a_n$ be an effective finite sequence in a metric space (X, d). Suppose that $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$ is an effective structure. Then $\mathcal{R}_{a_0,\dots,a_n}^{(X,d)}$ is a unique maximal computability structure $\mathcal{M}$ on (X, d) such that $a_0, \dots, a_n \in \mathcal{M}^0$.*

The converse of the previous proposition does not hold in general, i.e., it is possible that there exists a unique maximal computability structure on

(X, d) in which $a_0, \dots, a_n$ are computable points even if $\mathcal{R}^{(X,d)}_{a_0,\dots,a_n}$ is not an effective structure. Let us observe the following example.

EXAMPLE 4.2. Let d be the Euclidean metric on $\mathbb{R}$ and let $a_0 = 0$. Then any sequence in $\{-1, 1\}$ is an element of $\mathcal{R}^{(\mathbb{R},d)}_{a_0}$. But not any sequence in $\{-1, 1\}$ is effective.

Namely, if (x_i) is a sequence in $\{-1, 1\}$ which is effective in $(\mathbb{R}, d)$ and if $A = \{i \in \mathbb{N} \mid x_i = 1\}$, then A is r.e. Indeed, if $i_0 \in A$, then

$$A = \{i \in \mathbb{N} \mid d(x_i, x_{i_0}) < 1\}$$

and A is r.e. by Proposition 2.1(iv) (in fact A is recursive since $\mathbb{N} \setminus A$ is also r.e.). So if we take a subset A of $\mathbb{N}$ which is not r.e. and we define (x_i) by $x_i = 1$ for $i \in A$ and $x_i = -1$ for $i \in \mathbb{N} \setminus A$, then (x_i) is a noneffective sequence in $(\mathbb{R}, d)$ which belongs to $\mathcal{R}^{(\mathbb{R},d)}_{a_0}$.

Thus $\mathcal{R}^{(\mathbb{R},d)}_{a_0}$ is not an effective structure. On the other hand, we will see later (Theorem 4.16) that there exists a unique maximal computability structure on $(\mathbb{R}, d)$ in which a_0 is a computable point.

In what follows we concentrate on maximal computability structures on subspaces of $(\mathbb{R}^n, d)$, where d is the Euclidean metric on $\mathbb{R}^n$. From now on, if $X \subseteq \mathbb{R}^n$, we will write briefly *metric space* X instead of *metric space* $(X, d \restriction_{X \times X})$.

We first examine conditions under which $\mathcal{R}^X_{a_0,\dots,a_n}$ is an effective (and therefore a maximal computability) structure.

4.1. Characterization of maximal computability structures. If $a_0, \dots, a_k \in \mathbb{R}^n$, let

$$\mathcal{P} = \{a_0 + \sum_{i=1}^{k} t_i(a_i - a_0) \mid t_1, \dots, t_k \in \mathbb{R}\}.$$

We say that $\mathcal{P}$ is the *plane* in $\mathbb{R}^n$ spanned by $a_0, \dots, a_k$ (we take $\mathcal{P} = \{a_0\}$ if $k = 0$.)

PROPOSITION 4.3. *Let $X \subseteq \mathbb{R}^n$ be such that $d(x, y)$ is a recursive number for all $x, y \in X$. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that each element of $f(X)$ is a recursive point.*

PROOF. We may assume that X has at least two elements (otherwise the claim is obvious).

Choose $a_0 \in X$ and let V be the vector subspace of $\mathbb{R}^n$ generated by the set $\{x - a_0 \mid x \in X\}$. Then there exist $k \geq 1$ and $a_1, \dots, a_k \in X$ such that $a_1 - a_0, \dots, a_k - a_0$ is a basis for V. In particular, the vectors $a_1 - a_0, \dots, a_k - a_0$ are linearly independent, i.e., the points $a_0, \dots, a_k$ are geometrically independent.

Let $\mathcal{P}$ be the plane in $\mathbb{R}^n$ spanned by $a_0, \dots, a_k$. Then we have $X \subseteq \mathcal{P}$.

By the proof of Lemma 10 in [4] there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_0), \dots, f(a_k)$ are geometrically independent recursive points in $\mathbb{R}^n$ and such that $f(\mathcal{P}) \subseteq T$, where $T = \{(t_1, \dots, t_k, 0, \dots, 0) \in \mathbb{R}^n \mid t_1, \dots, t_k \in \mathbb{R}\}$.

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Let $x \in X$. The finite sequence $a_0, \ldots, a_k, x$ is effective and therefore $f(a_0), \ldots, f(a_k), f(x)$ is an effective sequence in T. Let $g : T \to \mathbb{R}^k$ be the isometry defined by $g(t_1, \ldots, t_k, 0, \ldots, 0) = (t_1, \ldots, t_k)$. We have that $g(f(a_0)), \ldots, g(f(a_k))$ are geometrically independent recursive points in $\mathbb{R}^k$ and $g(f(a_0)), \ldots, g(f(a_k)), g(f(x))$ is an effective sequence in $\mathbb{R}^k$. By Proposition 8 in [4] the point $g(f(x))$ is recursive in $\mathbb{R}^k$ and, by the definition of g, the point $f(x)$ is recursive in $\mathbb{R}^n$. ⊣

COROLLARY 4.4. *Let $a_0, \ldots, a_k$ be an effective sequence in $\mathbb{R}^n$. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_0), \ldots, f(a_k)$ are recursive points.*

LEMMA 4.5. *Let (x_i) be a recursive sequence in $\mathbb{R}^n$.*

(i) *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $a_1, \ldots, a_n$ be linearly independent and recursive vectors in $\mathbb{R}^n$ such that $L(a_1), \ldots, L(a_n)$ are recursive in $\mathbb{R}^m$. Then $(L(x_i))$ is also a recursive sequence in $\mathbb{R}^m$.*

(ii) *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map and let $a_0, \ldots, a_n \in \mathbb{R}^n$ be geometrically independent recursive points such that $f(a_0), \ldots, f(a_n)$ are recursive in $\mathbb{R}^m$. Then $(f(x_i))$ is a recursive sequence in $\mathbb{R}^m$.*

PROOF. (i) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. For $i \in \{1, \ldots, n\}$ let $\beta_1^i, \ldots, \beta_n^i \in \mathbb{R}$ be such that

$$e_i = \beta_1^i a_1 + \cdots + \beta_n^i a_n.$$

Then the tuple $(\beta_1^i, \ldots, \beta_n^i)$ is a unique solution to $n \times n$ system with recursive coefficients (since $e_i, a_1, \ldots, a_n$ are recursive). By applying Cramer's rule, it is easily seen that $\beta_1^i, \ldots, \beta_n^i$ are recursive. We have

$$L(e_i) = \beta_1^i L(a_1) + \cdots + \beta_n^i L(a_n)$$

implying that $L(e_i)$ is recursive. Hence $L(e_1), \ldots, L(e_n)$ are recursive elements of $\mathbb{R}^m$.

Let (x_i) be a recursive sequence in $\mathbb{R}^n$. Let $(x_i^1), \ldots, (x_i^n)$ be the component sequences of (x_i). For each $i \in \mathbb{N}$ we have

$$x_i = x_i^1 e_1 + \cdots + x_i^n e_n$$

and therefore

$$L(x_i) = x_i^1 L(e_1) + \cdots + x_i^n L(e_n).$$

So $(L(x_i))$ is a recursive sequence in $\mathbb{R}^m$.

(ii) As f is an affine map, there exist a linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^m$ such that $f(x) = L(x) + c$ for all $x \in \mathbb{R}^n$. We have $c = f(a_0) - L(a_0)$, so

$$f(x) = f(a_0) + L(x - a_0)$$

for all $x \in \mathbb{R}^n$. Vectors $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent and recursive and for $i \in \{1, \ldots, n\}$ we have

$$L(a_i - a_0) = f(a_i) - f(a_0),$$

so $L(a_1 - a_0), \ldots, L(a_n - a_0)$ are recursive in $\mathbb{R}^m$. Let (x_i) be a recursive sequence in $\mathbb{R}^n$. Obviously, $(x_i - a_0)$ is also a recursive sequence in $\mathbb{R}^n$.

By claim (i), $(L(x_i - a_0))$ is a recursive sequence in $\mathbb{R}^m$ and since $f(x_i) = f(a_0) + L(x_i - a_0)$, for all $i \in \mathbb{N}$, $(f(x_i))$ is a recursive sequence. ⊣

LEMMA 4.6. *Let $a_0, \dots, a_k$ be recursive points in $\mathbb{R}^n$ and let $\mathcal{P}$ be the plane spanned by $a_0, \dots, a_k$. If (x_i) is a sequence in $\mathcal{P}$ such that $(x_i) \in \mathcal{R}^{\mathbb{R}^n}_{a_0,\dots,a_k}$, then (x_i) is a recursive sequence in $\mathbb{R}^n$.*

PROOF. We may assume that $a_0, \dots, a_k$ are geometrically independent (otherwise we take $i_0, \dots, i_l \in \{0, \dots, k\}$ so that $a_{i_0}, \dots, a_{i_l}$ are geometrically independent and span $\mathcal{P}$).

As $\mathcal{P}$ is a k-plane, it is isometric to $\mathbb{R}^k$. Let $g : \mathcal{P} \to \mathbb{R}^k$ be an isometry. Obviously, $g(a_0), \dots, g(a_k)$ is a geometrically independent effective sequence in $\mathbb{R}^k$, so by Corollary 4.4, there exists an isometry $h : \mathbb{R}^k \to \mathbb{R}^k$ such that $h(g(a_0)), \dots, h(g(a_k))$ are recursive points. We define $f = h \circ g$. Obviously, f is an isometry $\mathcal{P} \to \mathbb{R}^k$ such that $f(a_0), \dots, f(a_k)$ are recursive points.

We have $(f(x_i)) \in \mathcal{R}^{\mathbb{R}^k}_{f(a_0),\dots,f(a_k)}$, so by Proposition 8 in [4], $(f(x_i))$ is a recursive sequence in $\mathbb{R}^k$. The function $f^{-1} : \mathbb{R}^k \to \mathcal{P}$ is also an isometry. Note that $\mathcal{P} - a_0$ is a vector subspace of $\mathbb{R}^n$. If we compose f^{-1} by the map $\mathcal{P} \to \mathcal{P} - a_0$, $x \mapsto x - a_0$, we get a surjective isometry $\mathbb{R}^k \to \mathcal{P} - a_0$ which is therefore an affine map (Mazur–Ulam theorem). We conclude that f^{-1}, as a function $\mathbb{R}^k \to \mathbb{R}^n$, is affine. Clearly $f^{-1}(f(a_0)), \dots, f^{-1}(f(a_k))$ are recursive points in $\mathbb{R}^n$ and, by Lemma 4.5(ii), $(x_i) = (f^{-1}(f(x_i)))$ is a recursive sequence in $\mathbb{R}^n$. ⊣

LEMMA 4.7. *Let $a_0, \dots, a_{k+1}$ be an effective sequence in $\mathbb{R}^n$ and let $\mathcal{P}$ be the plane spanned by points $a_0, \dots, a_k$. If (x_i) is a sequence in $\mathcal{P}$ such that $(x_i) \in \mathcal{R}^{\mathbb{R}^n}_{a_0,\dots,a_k}$, then $(x_i) \in \mathcal{R}^{\mathbb{R}^n}_{a_{k+1}}$.*

PROOF. Let $\mathcal{Q}$ be a plane spanned by $a_0, \dots, a_{k+1}$. If $\mathcal{Q}$ is a one-point set, then the claim is obvious. Otherwise, there exists $l \in \{1, \dots, k+1\}$ and an isometry $f : \mathcal{Q} \to \mathbb{R}^l$ with the property that $f(a_0), \dots, f(a_{k+1})$ are recursive in $\mathbb{R}^l$.

Let $g : \mathcal{Q} - a_0 \to \mathcal{Q}$, $g(x) = x + a_0$. As in the proof of Lemma 4.6 we conclude that $f \circ g$ is an (injective) affine map. It follows that $f(\mathcal{P})$ is the plane in $\mathbb{R}^l$ spanned by points $f(a_0), \dots, f(a_k)$. We have that $(f(x_i))$ a sequence in $f(\mathcal{P})$ such that $(f(x_i)) \in \mathcal{R}^{\mathbb{R}^{k+1}}_{f(a_0),\dots,f(a_k)}$. Now Lemma 4.6 implies that $(f(x_i))$ is a recursive sequence $\mathbb{R}^l$. Therefore $(f(x_i)) \in \mathcal{R}^{\mathbb{R}^l}_{f(a_{k+1})}$ (claim (i) of Example 2.5) and so $(x_i) \in \mathcal{R}^{\mathbb{R}^n}_{a_{k+1}}$. ⊣

Let $X \subseteq \mathbb{R}^n$ and let $a_0, \dots, a_k$ be a geometrically independent effective sequence in X. We say that $a_0, \dots, a_k$ is a *maximal geometrically independent effective sequence in X* if there exists no $a_{k+1} \in X$ such that $a_0, \dots, a_k, a_{k+1}$ is a geometrically independent effective sequence.

THEOREM 4.8. *Let $X \subseteq \mathbb{R}^n$ and let $a_0, \dots, a_k$ be a maximal geometrically independent effective sequence in X. Then $\mathcal{R}^X_{a_0,\dots,a_k}$ is a maximal computability structure. Moreover, this is a unique maximal computability structure on X in which $a_0, \dots, a_k$ are computable points.*

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Proof. Let $f : \mathcal{P} \to \mathbb{R}^k$ be an isometry such that $f(a_0), \dots, f(a_k)$ are recursive points in $\mathbb{R}^k$ (the existance of such an isometry can be proved as in the proof of Lemma 4.6).

Note that by maximality of $a_0, \dots, a_k$

$$(x_i) \in \mathcal{R}^X_{a_0,\dots,a_k} \Rightarrow x_i \in \mathcal{P}, \text{ for all } i \in \mathbb{N}.$$

Therefore, for $(x_i), (y_i) \in \mathcal{R}^X_{a_0,\dots,a_k}$ sequences $(f(x_i)), (f(y_i))$ are well defined and clearly $(f(x_i)), (f(y_i)) \in \mathcal{R}^{\mathbb{R}^k}_{f(a_0),\dots,f(a_k)}$. It follows from Lemma 4.6 that $(f(x_i)), (f(y_i))$ are recursive sequences in $\mathbb{R}^k$. It follows $(f(x_i)) \diamond (f(y_i))$ (Example 2.5) and consequently $(x_i) \diamond (y_j)$. Now we have that $\mathcal{R}^X_{a_0,\dots,a_k}$ is an effective structure, so the claim follows from Proposition 4.1. ⊣

Corollary 4.9. *Let $X \subseteq \mathbb{R}^n$. If $a_0, \dots, a_n \in X$ is a geometrically independent effective sequence, then $\mathcal{R}^X_{a_0,\dots,a_n}$ is a unique maximal computability structure on X in which $a_0, \dots, a_n$ are computable points.*

The following theorem gives a precise description of maximal computability structures on a subspace X of Euclidean space: each maximal computability structure on X is of the form $\mathcal{R}^X_{a_0,\dots,a_k}$.

Theorem 4.10. *Let $X \subseteq \mathbb{R}^n$. Let $\mathcal{M}$ be a maximal computability structure on X and $k \in \mathbb{N}$ the largest number with the property that there are $k+1$ geometrically independent points in $\mathcal{M}^0$. If $a_0, \dots, a_k \in \mathcal{M}^0$ are geometrically independent, then $a_0, \dots, a_k$ is a maximal geometrically independent effective sequence in X and $\mathcal{M} = \mathcal{R}^X_{a_0,\dots,a_k}$.*

Proof. Obviously we have

$$\mathcal{M} \subseteq \mathcal{R}^X_{a_0,\dots,a_k}. \tag{5}$$

We claim that there is no $a_{k+1} \in X$ such that $a_0, \dots, a_{k+1}$ is a geometrically independent effective sequence. Let us suppose that such $a_{k+1} \in X$ exists. If $\mathcal{P}$ is the plane spanned by $a_0, \dots, a_k$, then obviously $\mathcal{M}^0 \subseteq \mathcal{P}$, so $a_{k+1} \notin \mathcal{M}^0$. Also, if $(x_i) \in \mathcal{M}$, then (x_i) is a sequence in $\mathcal{P}$ and from (5) it follows that $(x_i) \in \mathcal{R}^X_{a_0,\dots,a_k}$. By Lemma 4.7, we have $(x_i) \in \mathcal{R}^X_{a_{k+1}}$, so $\mathcal{M} \cup \{(a_{k+1}, a_{k+1}, \dots)\}$ is a computability structure. This contradicts the fact that $\mathcal{M}$ is a maximal computability structure and $a_{k+1} \notin \mathcal{M}^0$. Therefore, such a_{k+1} does not exist.

Now by Theorem 4.8, $\mathcal{R}^X_{a_0,\dots,a_k}$ is a computability structure, so the claim of the theorem follows from (5). ⊣

4.2. Canonical computability structures. Suppose $X \subseteq \mathbb{R}^n$ and let $\mathcal{S}$ be the set of all sequences (x_i) in X which are recursive in $\mathbb{R}^n$. Since the set of all recursive sequences in $\mathbb{R}^n$ is a computability structure on $\mathbb{R}^n$ (even separable, Example 2.10), we have that $\mathcal{S}$ is a computability structure on X. We say that $\mathcal{S}$ is a *canonical computability structure* on X.

A canonical computability structure need not be maximal. For example, if X does not contain any recursive point, then $\mathcal{S} = \emptyset$ and therefore $\mathcal{S}$ is not maximal (if $X \neq \emptyset$). Another example is the set X defined as the line segment in $\mathbb{R}^2$ with endpoints $(0,0)$ and $(1,\gamma)$, where γ is a nonrecursive

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number. The only recursive point in X is $(0,0)$ and so $\mathcal{S}$ contains only the constant sequence $((0,0),(0,0),\dots)$. On the other hand, it is clear that there exist points in $X \setminus \{(0,0)\}$ whose distances from $(0,0)$ are recursive numbers. So $\mathcal{S}$ is not a maximal computability structure on X.

Although canonical computability structures need not be maximal, each maximal computability structure is canonical "up to isometry." This is the contents of the next theorem.

THEOREM 4.11. *Let $X \subseteq \mathbb{R}^n$ and let $\mathcal{M}$ be a maximal computability structure on X. Then there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\{(f(x_i)) \mid (x_i) \in \mathcal{M}\}$ is a canonical computability structure on $f(X)$.*

PROOF. By Theorem 4.10 we have $\mathcal{M} = \mathcal{R}^X_{a_0,\dots,a_k}$ for some maximal geometrically independent effective sequence $a_0,\dots,a_k$ in X. By Corollary 4.4 there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a_0),\dots,f(a_k)$ are recursive points. It follows

$$\{(f(x_i)) \mid (x_i) \in \mathcal{M}\} = \mathcal{R}^{f(X)}_{f(a_0),\dots,f(a_k)}. \tag{6}$$

Let $\mathcal{S}$ be a canonical computability structure on $f(X)$. Then clearly $\mathcal{S} \subseteq \mathcal{R}^{f(X)}_{f(a_0),\dots,f(a_k)}$. On the other hand, each element of $\mathcal{R}^{f(X)}_{f(a_0),\dots,f(a_k)}$ is by (6) equal to $(f(x_i))$ for some $(x_i) \in \mathcal{M}$, i.e., $(x_i) \in \mathcal{R}^X_{a_0,\dots,a_k}$. By maximality of $a_0,\dots,a_k$ we have that (x_i) is a sequence in the plane spanned by $a_0,\dots,a_k$ and so $(f(x_i))$ is a sequence in the plane spanned by $f(a_0),\dots,f(a_k)$ which, together with $(f(x_i)) \in \mathcal{R}^{f(X)}_{f(a_0),\dots,f(a_k)}$ and Lemma 4.6, gives that $(f(x_i))$ is a recursive sequence in $\mathbb{R}^n$. So $(f(x_i)) \in \mathcal{S}$ and this proves that $\mathcal{S} = \mathcal{R}^{f(X)}_{f(a_0),\dots,f(a_k)}$. From this and (6) follows the claim of the theorem. ⊣

As we saw, a maximal computability structure on a metric space (X,d) need not be separable, even when (X,d) is a subspace of Euclidean space (Example 3.2). However, the situation is different when the ambient space is entire Euclidean space.

THEOREM 4.12. *Every maximal computability structure on $\mathbb{R}^n$ is separable.*

PROOF. Let $\mathcal{M}$ be a maximal computability structure on $\mathbb{R}^n$. By Theorem 4.11 there exists an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathcal{M})$ is a canonical computability structure on $f(\mathbb{R}^n)$. Since each isometry $\mathbb{R}^n \to \mathbb{R}^n$ is surjective, $f(\mathcal{M})$ is a canonical computability structure on $\mathbb{R}^n$ which is separable by Example 2.10. So $f(\mathcal{M})$ is separable and thus $\mathcal{M}$ is also separable (Proposition 3.6). ⊣

4.3. More about uniqueness of maximal computability structures.

LEMMA 4.13. *Let $a_0,\dots,a_{n-1}$ be geometrically independent recursive points in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be such that $d(a_0,x),\dots,d(a_{n-1},x)$ are recursive numbers. Then x is a recursive point in $\mathbb{R}^n$.*

PROOF. If $x \in \mathcal{P}$, the claim follows from Lemma 4.6 and the fact that x is a recursive point if and only if the constant sequence $(x,x,x,\dots)$ is recursive.

If $x \notin \mathcal{P}$, we have

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$$\begin{aligned} d(x,a_k)^2 &= \langle x - a_k, x - a_k \rangle \\ &= \|x\|^2 - 2\langle x, a_k \rangle + \|a_k\|^2, \end{aligned} \tag{7}$$

for all $k \in \{0, \ldots, n-1\}$. If we subtract the first equation in (7) from other $n-1$ equations, we get

$$\langle x, -2a_k + 2a_0 \rangle = d(x,a_k)^2 - d(x,a_0)^2 - \|a_k\|^2 + \|a_0\|^2, \tag{8}$$

for all $k \in \{1, \ldots, n-1\}$. The number on the left in (8) is recursive, so after dividing the equation (8) by -2, we have that

$$s_k = \langle x, a_k - a_0 \rangle \tag{9}$$

is a recursive number, for all $k \in \{1, \ldots, n-1\}$. Let A be the $(n-1) \times n$ matrix whose k-th row is the n-tuple $a_k - a_0$, i.e.,

$$A = \begin{pmatrix} a_1 - a_0 \\ \vdots \\ a_{n-1} - a_0 \end{pmatrix}.$$

If $x = (x_1, \ldots, x_n)$, from (9) we get

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}.$$

Since $a_1 - a_0, \ldots, a_{n-1} - a_0$ are linearly independent, the rank of matrix A is $n-1$. Therefore, there exists a column in A which is a linear combination of other columns. Let us denote by p the number of that column. Let B be the matrix that we get from A by deleting the p-th column. We have

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ x_{p+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} s_1 + t_1 x_p \\ \vdots \\ s_{n-1} + t_{n-1} x_p \end{pmatrix},$$

for some recursive numbers $t_1, \ldots, t_{n-1}$. Since B is invertible, we have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ x_{p+1} \\ \vdots \\ x_n \end{pmatrix} = B^{-1} \begin{pmatrix} s_1 + t_1 x_p \\ \vdots \\ s_{n-1} + t_{n-1} x_p \end{pmatrix}. \tag{10}$$

As the coefficients of B are recursive, the inverse matrix B^{-1} also has recursive coefficients. It follows from (10) that for all $i \in \{1, \ldots, n\}$, $i \neq p$ there are recursive numbers α_i, β_i such that

$$x_i = \alpha_i + \beta_i x_p. \tag{11}$$

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We have

$$\|x\|^2 - 2\langle x, a_0\rangle + \|a_0\|^2 = d(x, a_0)^2,$$

i.e.,

$$x_1^2 + \cdots + x_n^2 + \gamma_1 x_1 + \cdots + \gamma_n x_n + \delta = 0, \tag{12}$$

for some recursive numbers $\gamma_1, \dots, \gamma_n, \delta$. From (11) we get

$$\alpha x_p^2 + \beta x_p + \gamma = 0,$$

where α, β, γ are recursive and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Namely, if $(\alpha, \beta, \gamma) = (0, 0, 0)$, then for any $x_p \in \mathbb{R}$ we would have that the n-tuple $(x_1, \dots, x_n)$ with the property (11) satisfies the equation (12). This is impossible because the equation (12) determines a sphere in $\mathbb{R}^n$ (and (11), for $x_p \in \mathbb{R}$, determines a line in $\mathbb{R}^n$). Therefore, x_p is a solution of the quadratic (or linear) equation with recursive coefficients, so x_p is a recursive number. Now from (11) it follows that $x_1, \dots, x_n$ are recursive numbers, so x is a recursive point in $\mathbb{R}^n$. ⊣

The previous lemma does not hold uniformly, i.e., if (x_i) is a sequence in $\mathbb{R}^n$ such that $(d(a_0, x_i))_{i\in\mathbb{N}}, \dots, (d(a_{n-1}, x_i))_{i\in\mathbb{N}}$ are recursive sequences, then (x_i) need not be recursive. Namely, with notation of Example 4.2, we have that $(d(a_0, x_i))_{i\in\mathbb{N}}$ is a recursive sequence, but (x_i) is not recursive in $\mathbb{R}$ (since it is not effective). This also shows that Lemma 4.6 does not hold if we remove the assumption that (x_i) is a sequence in $\mathcal{P}$ (we have $(x_i) \in \mathcal{R}^{\mathbb{R}}_{a_0}$, but (x_i) is not recursive).

By Theorems 4.8 and 4.10, $\mathcal{M}$ is a maximal computability structure on a subspace X of $\mathbb{R}^n$ if and only if $\mathcal{M} = \mathcal{R}^X_{a_0,\dots,a_k}$, where $a_0, \dots, a_k$ is a maximal geometrically independent effective sequence in X. The question here is, can we reduce the number of points which are needed in this description of $\mathcal{M}$? More precisely, if $a_0, \dots, a_k$ is a maximal geometrically independent effective sequence in X and $k \geq 1$, does it hold $\mathcal{R}^X_{a_0,\dots,a_k} = \mathcal{R}^X_{a_0,\dots,a_{k-1}}$? The answer in negative, as the following simple example shows.

Example 4.14. Let $a_0 = 0$, $a_1 = 1$ and $X = \mathbb{R}$. Then a_0, a_1 is a maximal geometrically independent effective sequence in X, but $\mathcal{R}^X_{a_0,a_1} \neq \mathcal{R}^X_{a_0}$. To see this, choose a sequence (x_i) in $\{-1, 1\}$ which is nonrecursive as a function $\mathbb{N} \to \mathbb{R}$ (there are uncountably many sequences in $\{-1, 1\}$, but only countably many recursive functions $\mathbb{N} \to \mathbb{R}$). Then $(x_i) \in \mathcal{R}^X_{a_0}$, but $(x_i) \notin \mathcal{R}^X_{a_0,a_1}$ (otherwise the equality $x_i = 1 - d(x_i, a_1)$ would imply the recursiveness of (x_i)).

On the other hand, the reduction of number of points which determine a maximal computability structure can be viewed in the following way. If $a_0, \dots, a_k$ is a maximal geometrically independent effective sequence in X, then there exists a unique maximal computability structure $\mathcal{M}$ on X in which $a_0, \dots, a_k$ are computable points (Theorem 4.8). In this sense we can say that a maximal geometrically independent effective sequence in X determines a *unique* maximal computability structure on X. The question is can some geometrically independent effective sequence in X which is not maximal also determine a *unique* maximal computability structure on X?

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More specifically, is $\mathcal{M}$ also a *unique* maximal computability structure in which $a_0, \ldots, a_{k-1}$ are computable points?

EXAMPLE 4.15. Let $a_0 = (0,0)$ and $a_1 = (1,0)$. Choose a point b on the unit circle $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ such that $d(b, a_1)$ is a nonrecursive number. Let $X = \{a_0, a_1, b\}$. Then a_0, a_1 is a maximal geometrically independent effective sequence in X and therefore $\mathcal{R}^X_{a_0,a_1}$ is a maximal computability structure on X, but this is not the only maximal computability structure on X in which a_0 is a computable point. Namely, a_0, b is also a maximal geometrically independent effective sequence in X and so $\mathcal{R}^X_{a_0,b}$ is a maximal computability structure on X which is clearly different from $\mathcal{R}^X_{a_0,a_1}$.

However, the statement that $a_0, \ldots, a_{k-1}$ determine a unique computability structure on X will hold if we assume that the geometrically independent sequence $a_0, \ldots, a_k$ is maximal in X, not just as an effective sequence. This is the contents of the following theorem, but first we need the following definition.

If $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, we define the number $\dim X$ (the *dimension* of X) as the largest number $k \in \mathbb{N}$ such that there exists a geometrically independent finite sequence $a_0, \ldots, a_k$ in X. So if $k = \dim X$, then X is contained in some k-plane in $\mathbb{R}^n$, but it is not contained in any $(k-1)$-plane.

THEOREM 4.16. *Let $X \subseteq \mathbb{R}^n$, $k = \dim X$, and suppose $k \geq 1$. Let $a_0, \ldots, a_{k-1}$ be a geometrically independent effective sequence in X. Then there exists a unique maximal computability structure on X in which $a_0, \ldots, a_{k-1}$ are computable points.*

PROOF. If $a_0, \ldots, a_{k-1}$ is a maximal geometrically independent effective sequence in X, then the claim follows from Theorem 4.8.

Otherwise, there exists $a_k \in X$ such that $a_0, \ldots, a_k$ is a geometrically independent effective sequence in X and since $k = \dim X$ by Theorem 4.8 we have that $\mathcal{R}^X_{a_0,\ldots,a_k}$ is a maximal computability structure on X. We have to prove that this is the only maximal computability structure on X in which $a_0, \ldots, a_{k-1}$ are computable points.

Suppose $\mathcal{M}$ is a maximal computability structure on X such that $a_0, \ldots, a_{k-1} \in \mathcal{M}^0$. Then clearly

$$\mathcal{M} \subseteq \mathcal{R}^X_{a_0,\ldots,a_{k-1}}. \tag{13}$$

Let $\mathcal{Q}$ be the plane spanned by points $a_0, \ldots, a_{k-1}$. We claim that there exists $b \in \mathcal{M}^0$ such that $b \notin \mathcal{Q}$. Suppose the opposite, i.e., $\mathcal{M}^0 \subseteq \mathcal{Q}$. Let $(x_i) \in \mathcal{M}$. Then (x_i) is a sequence in $\mathcal{Q}$ and by (13) we have $(x_i) \in \mathcal{R}^X_{a_0,\ldots,a_{k-1}}$. Now Lemma 4.7 implies that $(x_i) \in \mathcal{R}^X_{a_k}$. Hence $(x_i) \in \mathcal{R}^X_{a_0,\ldots,a_k}$ and this proves that $\mathcal{M} \subseteq \mathcal{R}^X_{a_0,\ldots,a_k}$. Since $\mathcal{M}$ is maximal, we have $\mathcal{M} = \mathcal{R}^X_{a_0,\ldots,a_k}$ implying that $a_k \in \mathcal{M}^0$ which is impossible since $a_k \notin \mathcal{Q}$.

Hence there exists $b \in \mathcal{M}^0$ such that $b \notin \mathcal{Q}$. Let $\mathcal{P}$ be the plane spanned by points $a_0, \ldots, a_k$. Since $k = \dim X$, we have $X \subseteq \mathcal{P}$.

Let $f : \mathcal{P} \to \mathbb{R}^k$ be an isometry such that $f(a_0), \ldots, f(a_k)$ are recursive points in $\mathbb{R}^k$. Since $b \in \mathcal{M}^0$, we have by (13) that $a_0, \ldots, a_{k-1}, b$ is an effective sequence in X. Moreover, this sequence is geometrically independent since $b \notin \mathcal{Q}$.

It follows that $f(a_0), \dots, f(a_{k-1}), f(b)$ is a geometrically independent effective sequence in $\mathbb{R}^k$ (its geometrical independence follows from the fact that $f^{-1} : \mathbb{R}^k \to \mathbb{R}^n$ is an affine map). By Lemma 4.13 we have that $f(b)$ is a recursive point in $\mathbb{R}^k$.

Let $(x_i) \in \mathcal{M}$. It follows from (13) and $b \in \mathcal{M}^0$ that $(x_i) \in \mathcal{R}^X_{a_0,\dots,a_{k-1},b}$. Now we have $(f(x_i)) \in \mathcal{R}^{\mathbb{R}^k}_{f(a_0),\dots,f(a_{k-1}),f(b)}$ and Lemma 4.6 implies that $(f(x_i))$ is a recursive sequence in $\mathbb{R}^k$. This and the fact that $f(a_k)$ is a recursive point give $(f(x_i)) \diamond (f(a_k), f(a_k), f(a_k), \dots)$. So we have $(x_i) \diamond (a_k, a_k, a_k, \dots)$, i.e., $(x_i) \in \mathcal{R}^X_{a_k}$, which together with (13) gives $(x_i) \in \mathcal{R}^X_{a_0,\dots,a_k}$.

We conclude that $\mathcal{M} \subseteq \mathcal{R}^X_{a_0,\dots,a_k}$. The maximality of $\mathcal{M}$ implies $\mathcal{M} = \mathcal{R}^X_{a_0,\dots,a_k}$ and this proves the claim of the theorem. ⊣

§5. Characterization of separable computability structures on a segment. If $X \subseteq \mathbb{R}^n$ is such that $\dim X \geq 1$ and $k = \dim X$, then by Theorem 4.16 for any geometrically independent effective sequence $a_1, \dots, a_k$ in X there exists a unique maximal computability structure $\mathcal{M}_{a_1,\dots,a_k}$ on X in which these points are computable. The general question here is this: under what conditions on X and the points $a_1, \dots, a_k$ the computability structure $\mathcal{M}_{a_1,\dots,a_k}$ is separable?

In particular, if $X \subseteq \mathbb{R}$ and $a \in X$, then there exists a unique maximal computability structure $\mathcal{M}_a$ on X in which a is computable. In this section, we will observe the case when X is a segment and we will give necessary and sufficient conditions that $\mathcal{M}_a$ is a separable. First, let us recall some facts about left and right recursive numbers.

A real number x is said to be *left recursive* if there exists a recursive sequence of rational numbers (q_i) such that $\sup \operatorname{Im} q = x$.

It is easily seen that if $F : \mathbb{N}^k \to \mathbb{Q}$ is a recursive function and $x = \sup \operatorname{Im} F$, then x is a left recursive number.

Proposition 5.1. *If $f : \mathbb{N}^k \to \mathbb{R}$ is a recursive function and $a = \sup \operatorname{Im} f$, then a is left recursive.*

Proof. Let $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ be a recursive function such that

$$|f(x) - F(x,i)| < 2^{-i}, \tag{14}$$

for all $x \in \mathbb{N}^k$, $i \in \mathbb{N}$. Let $G : \mathbb{N}^k \to \mathbb{Q}$ be the function defined by

$$G(x,i) = F(x,i) - 2^{-i},$$

$x \in \mathbb{N}^k$, $i \in \mathbb{N}$. Inequality (14) gives us

$$G(x,i) < f(x),$$

for all $x \in \mathbb{N}^k$, $i \in \mathbb{N}$, and therefore $\sup \operatorname{Im} G \leq \sup \operatorname{Im} f$. On the other hand, from (14) we also get

$$f(x) < F(x,i) + 2^{-i} = G(x,i) + 2 \cdot 2^{-i},$$

for all $x \in \mathbb{N}^k$, $i \in \mathbb{N}$. Hence, $\sup \operatorname{Im} f \leq \sup \operatorname{Im} G$, so $\sup \operatorname{Im} f = \sup \operatorname{Im} G$. As G is recursive $\mathbb{N}^k \to \mathbb{Q}$, $a = \operatorname{Im} G$ is left recursive. ⊣

Similarly, we say that a real number x is *right recursive* if there exists a recursive sequence of rational numbers (r_i) such that $\inf \operatorname{Im} r = x$. We also have a similar proposition.

PROPOSITION 5.2. *If $f : \mathbb{N}^k \to \mathbb{R}$ is a recursive function and $a = \inf \operatorname{Im} f$, then a is right recursive.*

PROPOSITION 5.3. *If $x \geq 0$ is left recursive, then there exists a recursive sequence of rational numbers (r_i) such that $\overline{\operatorname{Im} r} = [0, x]$ and $0 \in \operatorname{Im} r$.*

PROOF. By definition, there exists a recursive sequence of rational numbers (q_i) such that $\sup \operatorname{Im} q = x$. The set

$$S = \{i \in \mathbb{N} \mid q_i \geq 0\}$$

is recursive, so the sequence (s_i) defined by

$$s_i = q_i \cdot \chi_S(i), \quad i \in \mathbb{N}$$

is a recursive sequence of rational numbers such that $s_i \geq 0$, for all $i \in \mathbb{N}$ and $\sup \operatorname{Im} s = x$. Let (t_i) be a recursive sequence of rational numbers such that $\operatorname{Im} t = \mathbb{Q} \cap [0, 1]$ and let $\sigma_1, \sigma_2 : \mathbb{N} \to \mathbb{N}$ be recursive functions such that $\mathbb{N}^2 = \{(\sigma_1(x), \sigma_2(x)) \mid x \in \mathbb{N}\}$. We define

$$r_i = s_{\sigma_1(i)} \cdot t_{\sigma_2(i)},$$

$i \in \mathbb{N}$. Now it is easily seen that (r_i) is the desired sequence. ⊣

Before a characterization of separable computability structures on a segment, we give a characterization of maximal computability structures on $X \subseteq \mathbb{R}$.

PROPOSITION 5.4. *Let $X \subseteq \mathbb{R}$ and $a \in X$. Let $\mathcal{S}$ be the set of all sequences (x_i) in X such that $(x_i - a)$ is a recursive sequence. Then $\mathcal{S}$ is a maximal computability structure on X (and clearly $a \in \mathcal{S}^0$).*

PROOF. Let $Y = \{x - a \mid x \in X\}$. The function $f : X \to Y$, $f(x) = x - a$, is a surjective isometry. So it suffices to prove that $f(\mathcal{S})$ is a maximal computability structure on Y.

By the definition of $\mathcal{S}$ we have that $f(\mathcal{S})$ is the set of all sequences in Y which are recursive. If 0 is the only recursive point in Y, then $f(\mathcal{S}) = \{(0, 0, 0, \dots)\}$ and $f(\mathcal{S})$ is a maximal computability structure on Y. Otherwise, there exists $b \in Y \setminus \{0\}$ such that b is a recursive number. Each recursive sequence in Y belongs to $\mathcal{R}^Y_{0,b}$. Conversely, each element of $\mathcal{R}^Y_{0,b}$ is a recursive sequence in $\mathbb{R}$ by Lemma 4.6. Hence $f(\mathcal{S}) = \mathcal{R}^Y_{0,b}$, so $f(\mathcal{S})$ is a maximal computability structure on Y (Theorem 4.8). ⊣

THEOREM 5.5. *Let $\gamma > 0$. For $a \in [0, \gamma]$ let $\mathcal{M}_a$ be the unique maximal computability structure on $[0, \gamma]$ such that $a \in \mathcal{M}^0_a$. Then $\mathcal{M}_a$ is a separable computability structure if and only if a and $\gamma - a$ are left recursive numbers.*

PROOF. If $\mathcal{M}_a$ is a separable computability structure, then there exists an effective separating sequence α in $[0, \gamma]$ such that $\mathcal{M}_a = \mathcal{S}_\alpha$. Let α' be the sequence in $[-a, \gamma - a]$ defined by $\alpha'_i = \alpha_i - a$, $i \in \mathbb{N}$. By Proposition 5.4 α' is a recursive sequence. From

$$\gamma - a = \sup \operatorname{Im} \alpha',$$
$$a = \sup \operatorname{Im}(-\alpha')$$

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and Proposition 5.1 it follows that $\gamma - a$ and a are left recursive numbers.

Let us assume now that a and $\gamma - a$ are left recursive numbers. By Proposition 5.3, there exist recursive sequences of rational numbers (r_i) and (q_i) such that

$$\overline{\operatorname{Im} q} = [0, a],\ 0 \in \operatorname{Im} q,$$
$$\overline{\operatorname{Im} r} = [0, \gamma - a].$$

Let α' be the sequence of rational numbers defined by

$$\alpha'(2i) = r_i,$$
$$\alpha'(2i+1) = -q_i,$$

$i \in \mathbb{N}$. Then α' is a recursive sequence of rational numbers and obviously $\overline{\operatorname{Im} \alpha'} = [-a, \gamma - a]$, $0 \in \operatorname{Im} \alpha'$. Let α be the sequence defined by

$$\alpha_i = \alpha'_i + a,$$

$i \in \mathbb{N}$. Then α is an effective sequence and $\overline{\operatorname{Im} \alpha} = [0, \gamma]$, so α is an effective separating sequence in $[0, \gamma]$ and $a \in \operatorname{Im} \alpha$. Hence $a \in \mathcal{S}_\alpha^0$. Since $\mathcal{S}_\alpha$ is a maximal computability structure and $a \in \mathcal{S}_\alpha^0$, it must be $\mathcal{S}_\alpha = \mathcal{M}_a$, so $\mathcal{M}_a$ is separable. ⊣

If $c, d \in \mathbb{R}$, $c < d$, and $a \in [c, d]$, then the function $[c, d] \to [0, d - c]$, $x \mapsto x - c$ is a surjective isometry and this, together with Theorem 5.5, gives the following conclusion.

COROLLARY 5.6. *Let $c, d \in \mathbb{R}$, $c < d$. For $a \in [c, d]$ let $\mathcal{M}_a$ be the unique maximal computability structure on $[c, d]$ such that $a \in \mathcal{M}_a^0$. Then $\mathcal{M}_a$ is separable if and only if $a - c$ and $d - a$ are left recursive numbers.*

Suppose $a \in \mathbb{R}$ and let (x_i) and (y_i) be recursive sequences of rational numbers such that $a = \sup \operatorname{Im} x = \inf \operatorname{Im} y$. Then for each $k \in \mathbb{N}$ there exist $i, j \in \mathbb{N}$ such that $|x_i - y_j| < 2^{-k}$ and since the latter condition is decidable (recursive), there exist recursive functions $\varphi_1, \varphi_2 : \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$ we have $|x_{\varphi_1(k)} - y_{\varphi_2(k)}| < 2^{-k}$ which implies $|a - x_{\varphi_1(k)}| < 2^{-k}$. Hence each real number which is both left and right recursive is recursive. On the other hand, it is easy to conclude that each recursive number is left and right recursive. Furthermore, it is obvious that the sum of two left (right) recursive numbers is a left (right) recursive number. Note also the following: if a is left (right) recursive and r is a nonnegative rational number, then $-a$ is right (left) recursive and $r \cdot a$ is left (right) recursive.

Let (X, d) be a metric space. There are two general questions:

(1) What is the number of separable computability structures on (X, d)?
(2) What is the number of nonisometric separable computability structures on (X, d)?

Two computability structures $\mathcal{S}$ and $\mathcal{T}$ on (X, d) are said to be *isometric* if there exists a surjective isometry $f : X \to X$ such that $\mathcal{T} = f(\mathcal{S})$.

Let γ be a positive real number. In [5] the author asks the question (2) in the case of the metric space $[0, \gamma]$ and he gives an answer in Theorem 8.12: if γ is recursive, then every two separable computable structures on $[0, \gamma]$ are isometric, and if γ is left recursive but not recursive, then there are infinitely many nonisometric separable computability structures on $[0, \gamma]$ (see also Fact 8.8. in [5]).

We will now see how these results in a somewhat more general form can be obtained from Theorem 5.5. Let us first notice that there are only two isometries $[0, \gamma] \to [0, \gamma]$. Therefore, there are infinitely many separable computability structures on $[0, \gamma]$ if and only if there are infinitely many nonisometric separable computability structures on $[0, \gamma]$. The same holds if we replace "infinitely" by "countably."

COROLLARY 5.7. *Let γ be a positive real number.*

1. *If γ is recursive, then there exists a unique separable computability structure on* $[0, \gamma]$.
2. *If γ is left recursive, but not recursive, then there exist infinitely many, but only countably, separable computability structure on* $[0, \gamma]$.
3. *If γ is not left recursive, then there exists no separable computability structure on* $[0, \gamma]$.

PROOF. For $a \in [0, \gamma]$ let $\mathcal{M}_a$ be the unique maximal computability structure on $[0, \gamma]$ in which a is a computable point. If $a, b \in [0, \gamma]$, then $\mathcal{M}_a = \mathcal{M}_b$ if and only if $b - a$ is a recursive number. Indeed, if $\mathcal{M}_a = \mathcal{M}_b$, then a and b are computable points in the same computability structure and therefore their Euclidian distance is a recursive number. Conversely, if $b - a$ is a recursive number, then b is a computable point in $\mathcal{M}_a$ (by Proposition 5.4) and so we have $\mathcal{M}_a = \mathcal{M}_b$.

1. Suppose γ is recursive. By Theorem 5.5 $\mathcal{M}_0$ is a separable computability structure on $[0, \gamma]$. Suppose $\mathcal{S}$ is some other separable computability structure on $[0, \gamma]$. Then $\mathcal{S}$ is also a maximal computability structure on $[0, \gamma]$ and we have $\mathcal{S} = \mathcal{M}_a$ for some $a \in [0, \gamma]$. By Theorem 5.5 the numbers a and $\gamma - a$ are left recursive. Since $-\gamma$ is recursive and $-a = (\gamma - a) + (-\gamma)$, we have that $-a$ is left recursive. This, together with the fact that a is left recursive, gives that a is recursive. It follows $\mathcal{M}_a = \mathcal{M}_0$, i.e., $\mathcal{S} = \mathcal{M}_0$. Hence $\mathcal{M}_0$ is a unique separable computability structure on $[0, \gamma]$.
2. Suppose γ is left recursive and not recursive. Each separable computability structure on $[0, \gamma]$ equals $\mathcal{M}_a$ for some $a \in [0, \gamma]$ which is a left recursive number (Theorem 5.5). Since there are only countably many left recursive numbers, we have that there are only countably many separable computability structures on $[0, \gamma]$. On the other hand, there exist infinitely many such computability structures. Namely, for each rational number $r \in [0, 1]$ we have that $r\gamma$ and $\gamma - r\gamma = (1 - r)\gamma$ are left recursive numbers and by Theorem 5.5 $\mathcal{M}_{r\gamma}$ is a separable computability structure on $[0, \gamma]$. The mapping $r \mapsto \mathcal{M}_{r\gamma}$ is injective: if $r, s \in \mathbb{Q} \cap [0, 1]$ are such that $\mathcal{M}_{r\gamma} = \mathcal{M}_{s\gamma}$, then the number $s\gamma - r\gamma$ is recursive and this is possible only if $r = s$ (for $r \neq s$ we have $\gamma = \frac{1}{s-r}(s\gamma - r\gamma)$ and the recursiveness of $s\gamma - r\gamma$ would imply the recursiveness of γ).

3. Suppose there exists a separable computability structure on $[0, \gamma]$. Then this computability structure must be of the form $\mathcal{M}_a$, where $a \in [0, \gamma]$ is such that a and $\gamma - a$ are left recursive. But γ is the sum of these two numbers and it follows that γ is left recursive. ⊣

Let us note (related to the claim 2 of Corollary 5.7) that there are metric spaces on which there exist uncountably many separable computability structures. For example, any Euclidean space $\mathbb{R}^n$ is such a space. First notice that the set of all computable points in some separable computability structure $\mathcal{S}$ is countable (since $\mathcal{S}$ is countable). Now each point of $\mathbb{R}^n$ is a computable point in some maximal (and thus separable) computability structure on $\mathbb{R}^n$ and therefore the set of all separable computability structures on $\mathbb{R}^n$ cannot be countable.

§6. Conclusion. In this paper we have studied maximal computability structures on a metric space. Although they can be viewed as a generalization of separable computability structures, they are much less convenient and practical to deal with than separable computability structures. We gave certain general observations regarding maximal computability structures and then we concentrated on subspaces of Euclidean space and properties of maximal computability structures on these spaces. We gave a precise description of such computability structures and we investigated conditions under which a maximal computability structure is unique.

The following question naturally arises: among all maximal computability structures on a metric space, which are separable? We proved that in the case of the entire Euclidean space, each maximal computability structure is separable. Furthermore, in the case of a segment in $\mathbb{R}$ we gave a characterization of separable computability structures and applied that result to determine the cardinality of the set of all separable computability structures on a segment.

The latter question shows a possible direction of further investigations. For a given subspace X of Euclidean space (for example a ball or a cube in Euclidean space), how can we characterize separable computability structures on X among maximal computability structures? Of course, instead of Euclidean space we can observe some other metric spaces which usually occur in analysis and topology.

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