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IS THE SIBUYA DISTRIBUTION A PROGENY?

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Abstract

For $0 < a < 1$, the Sibuya distribution s_a is concentrated on the set $\mathbb{N}^+$ of positive integers and is defined by the generating function $\sum_{n=1}^{\infty} s_a(n)z^n = 1 - (1-z)^a$. A distribution q on $\mathbb{N}^+$ is called a progeny if there exists a branching process $(Z_n)_{n\geq 0}$ such that $Z_0 = 1$, such that $\mathbb{E}(Z_1) \leq 1$, and such that q is the distribution of $\sum_{n=0}^{\infty} Z_n$. In this paper we prove that s_a is a progeny if and only if $\frac{1}{2} \leq a < 1$. The main point is to find the values of $b = 1/a$ such that the power series expansion of $u(1-(1-u)^b)^{-1}$ has nonnegative coefficients.

Keywords: Branching process; progeny; Sibuya law

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1. Introduction

1.1. Progeny of a branching process

If p is a distribution on the set $\mathbb{N}$ of nonnegative integers with generating function

$$f_p(z) = \sum_{n=0}^{\infty} p_n z^n,$$

consider the branching process $(Z_n)_{n=0}^{\infty}$ governed by p. It is easily seen that

$$\mathbb{E}(z^{Z_n}) = f_p^{(n)}(z) = f_p(\cdots(f_p(z))\cdots) \quad n \text{ times}.$$

It is well known that Pr (there exists n: $Z_n = 0) = 1$ if and only if $m = \sum_{n=0}^{\infty} np_n \leq 1$. Under these circumstances, the random variable

$$S = \sum_{n=0}^{\infty} Z_n$$

is finite. Its distribution q is called the progeny of p and we have the following link between the generating functions of p and q. For all z such that $|z| \leq 1$ the following holds:

$$f_q(z) = zf_p(f_q(z)). \tag{1}$$

Since $Z_0 = 1$, the sum S is concentrated on $\mathbb{N}^+$. Given p, the calculation of q, or of f_q, is not easy in general. It can be done using the Lagrange–Bürmann formula (see [7, p. 129]).

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To compute q, the only simple case is when f_p is a fractional linear function (also called the Möbius or homographic function); see [6, p. 266]. Surprisingly enough, given f_q, computing f_p is easier. In fact, the function $z = g(u) = f_q^{(-1)}(u)$ valued in [0, 1] is well defined for $u \in [0, 1]$ by $u = f_q(z)$ and (1) leads to

$$f_p(u) = \frac{u}{g(u)}. \tag{2}$$

Although the correspondence $p \mapsto q$ is one-to-one, clearly not all distributions q on $\mathbb{N}^+$ can be the progeny of some p. For instance, q cannot have a bounded support, except in the degenerate case where $p_0 = 1$. Another necessary condition for q to be a progeny is $q_1 = f_q'(0) > 0$. If that were not true, the reciprocal function g of f_q would not be analytic at 0. In general, given a probability q on $\mathbb{N}^+$, no necessary and sufficient condition for q to be a progeny is known. Two classical treatises on these subjects are [3] and [1].

1.2. Sibuya distribution

In the remainder of this paper, we use the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$. The Sibuya distribution s_a with parameter $a \in (0, 1)$ is a probability on positive integers $\mathbb{N}^+ = \{1, 2, \dots\}$ such that if $S \sim s_a$ then

$$f_{s_a}(z) = \mathbb{E}(z^S) = 1 - (1-z)^a. \tag{3}$$

It is easily seen that, for $n \geq 1$,

$$\Pr(S = n) = s_a(n) = \frac{1}{n!} a(1-a)(2-a)\cdots(n-1-a) = \frac{-(-a)_n}{n!}. \tag{4}$$

Although this law s_a has been considered before, traditionally one refers to [5] for its study. In the latter, s_a appears as a particular case of what Sibuya calls a digamma law: Equations (16) and (28) of that paper for $\gamma = -\alpha = a$ yield (4) above. Surprisingly, (3) does not appear in the paper. If $S \sim s_{1/2}$, the distribution of $2S$ is well known as the law of first return to 0 of a simple random walk. Another probabilistic interpretation of $S \sim s_a$ is $S(w) = \min\{n \colon w \in A_n\}$, where $(A_n)_{n \geq 1}$ is a sequence of independent events such that $\Pr(A_n) = a/n$. Kozubowski and Podgórski [4] provided numerous observations and a rich set of references about s_a.

The most interesting feature of the Sibuya law from the point of view of branching processes is the semigroup property $f_{s_a} \circ f_{s_{a'}} = f_{s_{aa'}}$. This implies that if the branching process $(Z_n)_{n=0}^{\infty}$ is governed by s_a then the law of Z_n is quite explicit and is equal to s_{a^n}. This distribution is an example of an embeddable law in a continuous semigroup for composition. See [2] for instance. There are other variations of this distribution, sometimes informally called Sibuya distributions. One is the mixture of s_a with the Dirac measure at 0, $(1-\lambda)\delta_0 + \lambda s_a$, where $0 < \lambda \leq 1$, which has generating function

$$f_{(1-\lambda)\delta_0 + \lambda\, s_a}(u) = 1 - \lambda(1-u)^a. \tag{5}$$

Another one is the natural exponential family extension of the Sibuya distribution, say $s_a^{(\rho)}$, defined for $0 < \rho \leq 1$ by its generating function

$$f_{s_a^{(\rho)}}(u) = c(1 - (1-\rho u)^a), \qquad c = \frac{1}{1 - (1-\rho)^a}. \tag{6}$$

However, in the present paper 'Sibuya law' will mean s_a for some $0 < a < 1$ only.

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1.3. When is the Sibuya distribution a progeny?

Since the Sibuya distribution is concentrated on $\mathbb{N}^+$, has an unbounded support, and satisfies $q_1 = a > 0$, the natural question is the following: given $a \in (0, 1)$, does there exist p such that $q = s_a$? The following proposition gives the answer and constitutes the aim of the present paper.

Proposition 1. *The Sibuya distribution s_a is a progeny if and only if $\frac{1}{2} \le a < 1$.*

Note that the question is a natural question, since we can compute g explicitly. One should not wonder whether (5) is a progeny, since it has an atom at 0. Case (6) is more interesting.

Corollary 1. *For $\rho \in (0, 1)$, the distribution $s_a^{(\rho)}$ is a progeny if and only if $\frac{1}{2} \le a < 1$.*

2. Proofs

Proof of Proposition 1. For simplicity, we let $b = 1/a > 1$. Calculating the function g which appears in (2) is easy since, if $u \in [0, 1]$, the only solution $z \in [0, 1]$ of the equation $u = 1 - (1 - z)^{1/b}$ is $g(u) = 1 - (1 - u)^b$. We therefore have to prove that the function

$$u \mapsto h_b(u) = \frac{u}{1 - (1 - u)^b}$$

has a power series expansion with nonnegative coefficients if and only if $1 \le b \le 2$.

We let B denote the set of $b > 1$ such that $s_{1/b}$ is a progeny. The case $b = 1$ is degenerate.

Step (i): 2 is in B, 3 is not in B. We have

$$h_2(u) = \frac{1}{2}\frac{1}{1 - u/2} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u^n$$

and we obtain the well-known fact that $s_{1/2}$ is the progeny of a geometric distribution starting at 0. For $b = 3$, we have

$$h_3(u) = \frac{1}{3} \times \frac{1}{1 - u + u^2/3} = \frac{1}{3} \sum_{n=0}^{\infty} r^n \frac{\sin\,(n+1)\theta}{\sin\theta} u^n,$$

where $re^{\pm i\theta} = \frac{1}{2}(3 \pm i\sqrt{3})$ are the complex roots of the polynomial $1 - u + u^2/3$. In fact, $r = \sqrt{3}$ and $\theta = \pm\pi/6$. Clearly, $\sin\,(n+1)\theta / \sin\theta \ge 0$ for all n is impossible and therefore 3 is not in B.

Step (ii): b is in B if $1 < b < 2$. Using the Pochhammer symbol, we can write

$$(1 - u)^b = \sum_{n=0}^{\infty} \frac{(-b)_n}{n!} u^n.$$

Let

$$H(u) = \frac{1}{bu^2}(-1 + bu + (1 - u)^b) = \frac{b - 1}{2} + \sum_{n=1}^{\infty} (b - 1)(2 - b)(3 - b) \cdots (n + 1 - b) \frac{u^n}{(n + 2)!}.$$

Since $1 < b < 2$, all the coefficients of H are positive. As a consequence of the definition of H,

$$h_b(u) = \frac{1}{b}\frac{1}{1 - uH(u)} = \frac{1}{b} \sum_{n=0}^{\infty} u^n H(u)^n,$$

and, since the coefficients of H are positive, this implies that $b \in B$.

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Step (ii): b is not in B if $b > 3$. Consider the numbers $(P_n)_{n\geq 2}$ defined by

$$\frac{bu}{1-(1-u)^b} = 1 + \frac{b-1}{2}u + \sum_{n=2}^{\infty} P_n u^n. \tag{7}$$

A simple calculation shows that

$$P_2 = \frac{b^2-1}{6}, \quad P_3 = \frac{b^2-1}{4}, \quad P_4 = \frac{(19-b^2)(b^2-1)}{30}, \quad P_5 = \frac{(9-b^2)(b^2-1)}{4}.$$

Therefore, $P_5 < 0$ if $b > 3$.

Step (iv): b is not in B if $2 < b < 3$. This point is more difficult. Let us introduce the numbers $(p_n)_{n\geq 2}$ defined by

$$\begin{aligned}\frac{1-(1-u)^b}{bu} &= \int_0^1 (1-ux)^{b-1}\,\mathrm{d}x \\ &= 1 - \frac{b-1}{2}u + \sum_{n=2}^{\infty} \frac{(1-b)_n}{(n+1)!}u^n \\ &= 1 - \frac{b-1}{2}u + \sum_{n=2}^{\infty} p_n u^n, \end{aligned} \tag{8}$$

where

$$p_n = \frac{1}{(n+1)!}(b-1)(b-2)(3-b)\cdots(n-b). \tag{9}$$

The radius of convergence of power series (8) is 1. Observe that, since $n \geq 2$, from (9), we have $p_n > 0$ if $2 < b < 3$. To simplify, we now let

$$v = \frac{b-1}{2}u, \qquad u = \frac{2}{b-1}v, \qquad A_n = P_n\frac{2^n}{(b-1)^n}, \qquad a_n = p_n\frac{2^n}{(b-1)^n},$$

where P_n was defined in (7). With this notation, equalities (7) and (8) become

$$\frac{2b}{b-1}\,\frac{v}{1-(1-2v/(b-1))^b} = 1 + v + \sum_{n=2}^{\infty} A_n v^n, \tag{10}$$

$$\frac{b-1}{2b}\,\frac{1-(1-2v/(b-1))^b}{v} = 1 - v + \sum_{n=2}^{\infty} a_n v^n. \tag{11}$$

Since $2 < b < 3$, the radius of convergence of the power series on the right-hand side of (11) is $(b-1)/2 < 1$. Because $a_n > 0$, this remark implies that

$$\sum_{n=2}^{\infty} a_n = \infty. \tag{12}$$

Now we multiply the right-hand sides of (11) and (10). Their product is 1. We obtain

$$\sum_{n=2}^{\infty}(a_n + A_n)v^n + \sum_{n=2}^{\infty}(a_n - A_n)v^{n+1} + \bigg(\sum_{n=2}^{\infty} a_n v^n\bigg)\bigg(\sum_{n=2}^{\infty} A_n v^n\bigg) = v^2,$$

$$\sum_{n=2}^{\infty}(a_n + A_n)v^n + \sum_{n=3}^{\infty}(a_{n-1} - A_{n-1})v^n + \sum_{n=4}^{\infty}\bigg(\sum_{k=2}^{n-2} A_{n-k}a_k\bigg)v^n = v^2.$$

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From this last equality, watching the coefficient on v^n for $n \geq 4$, we obtain

$$a_n + a_{n-1} + \sum_{k=2}^{n-2} A_{n-k} a_k = A_{n-1} - A_n. \tag{13}$$

Now assume that $A_n \geq 0$ for all $n \geq 2$, which is equivalent to assuming that $b \in B$. Then (13) implies that $a_n \leq A_{n-1} - A_n$. Summing from $n = 4$ to N we obtain, for all N,

$$\sum_{n=4}^{N} a_n \leq A_3 - A_N \leq A_3,$$

which contradicts (12). Therefore, there exists at least one n such that $A_n < 0$ and the proposition is proved. □

Proof of Corollary 1. If $f_q(z) = c(1 - (1 - \rho z)^{1/b})$ then

$$g(u) = \frac{1}{\rho}\left(1 - \left(1 - \frac{u}{c}\right)^b\right).$$

From the proposition, the corresponding fecundity distribution p does exist if and only if $1 < b \leq 2$, since

$$\frac{f_p(cu)}{c\rho} = \frac{u}{1 - (1 - u)^b}.$$ □

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