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GEOMETRIC ANOSOV FLOWS OF DIMENSION FIVE WITH SMOOTH DISTRIBUTIONS

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Abstract We classify the five-dimensional C^∞ Anosov flows which have C^∞-Anosov splitting and preserve a smooth pseudo-Riemannian metric. Up to a special time change and finite covers, such a flow is C^∞ flow equivalent either to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$, or to the geodesic flow on a three-dimensional hyperbolic manifold.

Keywords: Anosov flow; pseudo-Riemannian metric; linear connection; ergodicity

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1. Introduction

Let M be a C^∞-closed manifold. A C^∞-flow, ϕ_t, generated by the non-singular vector field X is called an Anosov flow if there exists a ϕ_t-invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-,$$

a Riemannian metric on M and two positive numbers a and b, such that

$$\forall u^\pm \in E^\pm, \quad \forall t \geqslant 0, \quad \|D\phi_{\mp t}(u^\pm)\| \leqslant a\mathrm{e}^{-bt}\|u^\pm\|,$$

where E^- and E^+ are called the strong stable and strong unstable distributions of the flow.

In general, E^- and E^+ are only continuous. If they are both C^∞ subbundles of TM, then the Anosov flow is said to have smooth distributions. This case is rather rare (see, for example, [**14**], [**11**] and [**6**]). Although the smoothness of these two distributions is dynamically so strong a condition, it is still quite weak geometrically. So to arrive at a classification result, one has to suppose in addition the existence of a smooth invariant geometric structure. For example, in [**6**], the existence of an invariant contact form is assumed.

If an Anosov flow preserves a C^∞ pseudo-Riemannian metric, then by definition, this flow is called *geometric*. In this paper, we consider the *geometric* Anosov flows with smooth distributions.

The classical examples of such flows are the suspensions of symplectic hyperbolic infranilautomorphisms and the geodesic flows on locally symmetric spaces of rank one. There exist also lots of non-classical algebraic models (see [**19**]), which makes a possible classification of such flows quite interesting. In this paper, we obtain the classification in dimension five.

In general, given an Anosov flow with C^∞ distributions ϕ_t, one gets a smooth 1-form λ, such that

$$\lambda(E^\pm) = 0, \qquad \lambda(X) = 1.$$

It is called the *canonical* 1*-form* of the flow, which is easily seen to be ϕ_t-invariant.

Definition 1.1. $\mathrm{rank}(\phi_t) := 2(\max\{k \geqslant 0 \mid \wedge^k \mathrm{d}\lambda \not\equiv 0\})$.

We call this even number the *rank* of ϕ_t. Here $\wedge^k \mathrm{d}\lambda$ denotes the exterior kth power of $\mathrm{d}\lambda$, and by convention, $\wedge^0 \mathrm{d}\lambda := 1$. Note that $\mathrm{rank}(\phi_t)$ is just the rank of the 2-form $\mathrm{d}\lambda$ (see [**16**]). If ϕ_t is topologically transitive and its rank is $2k$, then $\wedge^k \mathrm{d}\lambda$ vanishes nowhere on an open-dense subset of M.

For all $a \in \mathbb{R}$, denote by $[a]$ the biggest integer, which is smaller than a. If the dimension of M is m, then the degree of $\wedge^{[m/2]+1}\mathrm{d}\lambda$ will be bigger than m. So we have

$$\mathrm{rank}(\phi_t) \leqslant 2[m/2].$$

In § 2, we characterize the classical homogeneous models above by their ranks. More precisely, we prove the following theorem.

Theorem 1.2. *Let M be a C^∞ closed manifold of dimension m and ϕ_t be a geometric Anosov flow with C^∞ distributions on M, we have*

(i) *if* $\mathrm{rank}(\phi_t) = 0$*, then up to a constant change of time-scale, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic infranilautomorphism;*

(ii) *if* $\mathrm{rank}(\phi_t) = 2[m/2]$*, then up to finite covers, ϕ_t is C^∞ flow equivalent to a canonical perturbation of the geodesic flow on a locally symmetric Riemannian manifold of strictly negative curvature.*

A *canonical perturbation* of a smooth flow with generator X is (by definition) the flow of the field $X/(1+\alpha(X))$, where α is a C^∞ closed 1-form such that $1+\alpha(X) > 0$. It should be mentioned that Theorem 1.2 is just a more or less direct reformulation of the results of [**6**], [**4**] and [**17**].

Although there exist algebraic models of *geometric* Anosov flows with rank between 0 and $2[m/2]$, none of them is of dimension five. In fact, the principal results of this paper is the following theorem.

Theorem 1.3. *Let M be a closed manifold of dimension five and ϕ_t be a geometric Anosov flow with C^∞ distributions on M, then*

(i) *either, up to a constant change of time-scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$;*

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(ii) *or, up to finite covers, ϕ_t is C^∞ flow equivalent to a canonical perturbation of the geodesic flow on a three-dimensional Riemannian manifold of constant negative curvature.*

In the appendix, two lemmas are proved, which are used in the proof of Theorem 1.3. Lemma A.1 is about the completeness of a linear connection and Lemma A.2 is about the time change of an Anosov flow with C^∞ distributions.

If M admits a *geometric* Anosov flow, then the dimension of M must be odd (see § 2). In dimension three, an Anosov flow with C^∞ distributions is *geometric* if and only if it preserves a volume form (see [**13**]). Such flows are classified by Ghys (see [**9**]). Here Theorem 1.3 gives a classification for the case of dimension five. We should mention that such five-dimensional flows are also studied in [**11**] with the purpose to understand the contact case.

Beginning with dimension seven, we can find many algebraic models of *geometric* Anosov flows, which are neither contact nor suspensions (see [**19**]). The situation will then become much more complex and a classification is still out of reach at the moment. Indeed, our proof of Theorem 1.3 is quite specific to the case of dimension five.

2. Preliminaries

2.1. Some generalities

Let ϕ_t be an Anosov flow with C^∞ distributions on a C^∞ closed manifold M. Denote by X the generator of this flow. For each C^∞ 2-form ω on M, denote by $\operatorname{Ker}\omega$ the kernel of ω, i.e. $\operatorname{Ker}\omega := \{y \in TM \mid i_y\omega = 0\}$. Let us first prove the following lemma.

Lemma 2.1. *Under the above notation, ϕ_t is geometric, if and only if it preserves a C^∞ 2-form with $\mathbb{R}X$ as kernel.*

Proof. Suppose that ϕ_t is *geometric*. Denote by g a C^∞ ϕ_t-invariant pseudo-Riemannian metric. Then by the Anosov property of ϕ_t, we get

$$g(X, E^\pm) = 0, \qquad g(E^\pm, E^\pm) = 0.$$

Let J be the section of $T^*M \otimes TM$, such that

$$J(X) = 0, \qquad J(u^\pm) = \pm u^\pm, \quad \forall u^\pm \in E^\pm.$$

Then $g(J\cdot, \cdot)$ is easily seen to be a C^∞ ϕ_t-invariant 2-form, denoted by ω. Since g is non-degenerate, then so is $\omega|_{E^+\oplus E^-}$. Again by the Anosov property, we get $i_X\omega = 0$. So the kernel of ω is $\mathbb{R}X$.

Suppose that ϕ_t preserves a C^∞ 2-form Θ, such that $\operatorname{Ker}\Theta = \mathbb{R}X$. Then there exists a unique ϕ_t-invariant symmetric $(0, 2)$-tensor g, such that

$$g(X, X) = 1, \qquad g(X, u^\pm) = 0,$$
$$g(u^+, u^-) = g(u^-, u^+) = \Theta(u^+, u^-),$$
$$g(u^\pm, v^\pm) = 0, \quad \forall u^\pm, v^\pm \in E^\pm.$$

Since $\operatorname{Ker}\Theta = \mathbb{R}X$, then g is non-degenerate. So g is a pseudo-Riemannian metric. Thus ϕ_t is *geometric*. □

We deduce that the following Anosov flows with C^∞ distributions are *geometric*.

(i) Contact Anosov flows with C^∞ distributions.

(ii) Suspensions of symplectic hyperbolic infranilautomorphisms.

(iii) Three-dimensional volume preserving Anosov flows with C^∞ distributions (see [**13**]).

In [**19**], Tomter constructed explicitly a seven-dimensional Anosov flow, which is indeed *geometric*. By generalizing his ideas, we can then construct many non-usual algebraic models of *geometric* Anosov flows. The following lemma gives another way to construct such flows.

Lemma 2.2. *Under the above notation, if ϕ_t is geometric, then for each C^∞ 1-form β, such that $\mathcal{L}_X\,\mathrm{d}\beta = 0$ and $\beta(X) > 0$, the flow of $X/\beta(X)$ is also a geometric Anosov flow with C^∞ distributions.*

Proof. Denoted by ϕ_t^β the flow of $X/\beta(X)$. Then by Lemma A.2 proved in the appendix, ϕ_t^β is also an Anosov flow with C^∞ distributions.

Since ϕ_t is *geometric*, then by Lemma 2.1, it preserves a C^∞ 2-form ω, such that $\operatorname{Ker}\omega = \mathbb{R}X$. In particular, we have $i_X\omega = 0$. Then

$$i_X\,\mathrm{d}\omega = \mathcal{L}_X\omega - \mathrm{d}i_X\omega = 0.$$

Thus

$$\mathcal{L}_{X_\beta}\omega = i_{X_\beta}\mathrm{d}\omega + \mathrm{d}i_{X_\beta}\omega = 0.$$

So ϕ_t^β preserves also ω and $\operatorname{Ker}\omega = \mathbb{R}X_\beta$. Then by Lemma 2.1, ϕ_t^β is also *geometric*. □

Let ϕ_t be as above and *geometric*. Since ϕ_t preserves a C^∞ 2-form ω, such that $\operatorname{Ker}\omega = \mathbb{R}X$, then $\omega|_{E^+\oplus E^-}$ is non-degenerate. By the Anosov property of ϕ_t, we get $\omega(E^\pm, E^\pm) = 0$. So E^+ and E^- are both Lagrangian subspaces of $\omega|_{E^+\oplus E^-}$. We deduce that E^+ and E^- have the same dimension, denoted by n. So the dimension of M is odd.

It is easily seen that $\lambda \wedge (\wedge^n\omega)$ is a ϕ_t-invariant volume form. So ϕ_t is topologically transitive (see [**12**]). Denote by ν the probability defined by this volume form. Then by the Multiplicative Ergodic Theorem of Oseledec, there exists a ν-conull ϕ_t-invariant subset Λ of M and a decomposition of $TM|_\Lambda$ into ϕ_t-invariant measurable subbundles,

$$TM|_\Lambda = \bigoplus_{0\leqslant i\leqslant k} L_i,$$

such that for all $u_i \in L_i$,

$$\lim_{t\to\pm\infty} t^{-1}\log\|D\phi_t(u_i)\| = \chi_i,$$

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where L_i is called a Lyapunov subbundle and χ_i its Lyapunov exponent. L_i is also denoted by L_{χ_i}.

The following lemma is due to Feres and Katok (see [**11**]).

Lemma 2.3. *Under the above notation, if τ is a C^∞ ϕ_t-invariant tensor field of type $(0, r)$ and $\sum_{1\leqslant l\leqslant r} \chi_{i_l} \neq 0$, then $\tau(L_{i_1}, \dots, L_{i_r}) = 0$.*

2.2. Proof of Theorem 1.2

Let ϕ_t be a *geometric* Anosov flow with C^∞ distributions and suppose that E^+ is of dimension n. Then by the previous subsection, we have $m = 2n + 1$, where m is the dimension of M.

If $\text{rank}(\phi_t) = 2[m/2](= 2n)$, then $\wedge^n \mathrm{d}\lambda \not\equiv 0$. So the ϕ_t-invariant C^∞ m-form $\lambda \wedge (\wedge^n \mathrm{d}\lambda)$ is not identically zero. Since ϕ_t is topologically transitive, then $\exists c \neq 0$, such that $\lambda \wedge (\wedge^n \mathrm{d}\lambda) = c \cdot \lambda \wedge (\wedge^n \omega)$. We deduce that $\lambda \wedge (\wedge^n \mathrm{d}\lambda)$ vanishes nowhere, i.e. λ is a contact form. Then by the classification of contact Anosov flows with C^∞ distributions (see [**6**]), case (ii) of Theorem 1.2 is true.

If $\text{rank}(\phi_t) = 0$, then $\mathrm{d}\lambda \equiv 0$. So $E^+ \oplus E^-$ is integrable. By Theorem 3.1 of [**17**], ϕ_t admits a global section Σ (a global section is by definition a connected closed submanifold of codimension one which intersects each orbit transversally). Denote by τ the *first return time* function of Σ. Then the Poincaré map of Σ is by definition $\psi := \phi_{\tau(\cdot)}(\cdot)$. For the sake of completeness, we prove in detail the following lemma.

Lemma 2.4. *The previous Poincaré map ψ is a C^∞ Anosov diffeomorphism with C^∞ distributions, topologically transitive and preserving a C^∞ linear connection.*

Proof. Recall that $E^+ \oplus \mathbb{R}X$ and $E^- \oplus \mathbb{R}X$ are called the unstable and stable distributions of ϕ_t. They are both integrable (see [**12**]). Denote by $\mathcal{F}^{+,0}$ and $\mathcal{F}^{-,0}$ their corresponding foliations. Since Σ is transversal to X, then $\mathcal{F}^{+,0} \cap \Sigma$ gives a C^∞ foliation on Σ. Denote by E^+_Σ its C^∞ tangent distribution. Similarly we denote by E^-_Σ the tangent distribution of $\mathcal{F}^{-,0} \cap \Sigma$.

Since $\mathcal{F}^{+,0}$ is ϕ_t-invariant, then the foliation $\mathcal{F}^{+,0} \cap \Sigma$ is ψ-invariant. We deduce that E^+_Σ is ψ-invariant. Similarly E^-_Σ is also ψ-invariant.

Fix a Riemannian metric on M. Since $E^+|_\Sigma$ and E^+_Σ are both transversal to $\mathbb{R}X$ (along Σ), then we can project E^+_Σ onto $E^+|_\Sigma$ with respect to $\mathbb{R}X$. Denote this projection by P^+. Since Σ is compact, then we can find two positive constants M_1 and M_2, such that

$$M_1\|u\| \leqslant \|P^+u\| \leqslant M_2\|u\|, \quad \forall u \in E^+_\Sigma.$$

For all $x \in \Sigma$, take, $u \in (E^+_\Sigma)_x$. Then u splits uniquely as

$$u = P^+_x(u) + aX_x, \quad a \in \mathbb{R}.$$

We have

$$(D_x\psi)(u) = (D_x\tau(u) + a)X_{\psi(x)} + (D_x\phi_{\tau(x)})(P^+_x u).$$

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Thus

$$(D_x\psi)(u) = (P^+_{\psi(x)})^{-1}[(D_x\phi_{\tau(x)})(P^+_x u)].$$

So for all $n \in \mathbb{N}$,

$$(D_x\psi^n)(u) = (P^+_{\psi^n(x)})^{-1}(D_x\phi_{\tau(x)+\cdots+\tau(\psi^{n-1}(x))})(P^+_x u).$$

We have a similar formula for E^-_Σ. Now a simple estimation shows that ψ is an Anosov diffeomorphism with C^∞ distributions, E^+_Σ and E^-_Σ.

Since ϕ_t is *geometric*, then it preserves a C^∞ 2-form ω whose kernel is $\mathbb{R}X$. Restrict ω to a C^∞ 2-form ω_Σ on Σ. Then using the fact that $i_X\omega = 0$, ω_Σ is seen to be ψ-invariant. Since ω_Σ is non-degenerate, then ψ preserves a volume form. We deduce that ψ is topological transitive.

Now a direct calculation shows the existence of a C^∞ ψ-invariant connection ∇ on Σ, such that

$$\nabla\omega_\Sigma = 0, \qquad \nabla E^\pm_\Sigma \subseteq E^\pm_\Sigma,$$
$$\nabla_{Y^\pm}Y^\mp = P^\mp_\Sigma[Y^\pm, Y^\mp], \quad \forall Y^\pm \subseteq E^\pm_\Sigma.$$

□

By [**4**] and the previous lemma, ψ is seen to be C^∞-conjugate to a hyperbolic infranilautomorphism. Then by Corollary 3.5 of [**17**], the integral manifolds of $E^+ \oplus E^-$ are compact. So we can take a leaf of $E^+ \oplus E^-$ as Σ. With respect to this section, the *first return time* function is constant. Then Theorem 1.2 follows.

3. Homogeneity in dimension five

3.1. Remarks about rank 0 and 4

Now we begin to prove Theorem 1.3. Suppose that ϕ_t satisfies the conditions in Theorem 1.3. Denote by X the generator of ϕ_t and by ν its invariant volume form. By Lemma 2.1, ϕ_t preserves a C^∞ 2-form ω, such that $\operatorname{Ker}\omega = \mathbb{R}X$, i.e. $\omega|_{E^+\oplus E^-}$ is non-degenerate. Thus by Lemma 2.3, if a is a Lyapunov exponent of ϕ_t with respect to ν, then so is $-a$. Since M is of dimension five, then there exist only two possibilities for the Lyapunov exponents of ϕ_t:

(i) $-a < 0 < a$, and

(ii) $-a < -b < 0 < b < a$.

Lemma 3.1. *Under the above notation, we have* $\mathrm{d}\omega \equiv 0$.

Proof. Since ω is ϕ_t-invariant, then

$$\mathcal{L}_X\omega = 0, \qquad i_X\omega = 0.$$

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So

$$i_X \, \mathrm{d}\omega = \mathcal{L}_X \omega - \mathrm{d} i_X \omega = 0,$$

i.e.

$$\mathrm{d}\omega(X, \cdot, \cdot) \equiv 0.$$

If ϕ_t has only one positive Lyapunov exponent, i.e. case (i) above is true, then by Lemma 2.3, $\mathrm{d}\omega \equiv 0$.

If case (ii) above is verified, then the Lyapunov subbundles are all of dimension one. Again by Lemma 2.3, $\mathrm{d}\omega \equiv 0$. □

The rank of ϕ_t can only be 0, 2 or 4. If $\mathrm{rank}(\phi_t) = 4$, then by Theorem 1.2, ϕ_t is finitely covered by a *canonical perturbation* of the geodesic flow on a three-dimensional locally symmetric space of strictly negative curvature. But such a Riemannian space must have constant negative curvature. So Theorem 1.3 is true in this case.

If $\mathrm{rank}(\phi_t) = 0$, then by Theorem 1.2, up to a constant change of time-scale, ϕ_t is finitely covered by the suspension of a four-dimensional hyperbolic nilautomorphism. But in dimension four, such a hyperbolic nilautomorphism must be $(\mathbb{T}^4, \bar{A})$, where $\bar{A}$ is the induced application of an invertible hyperbolic matrix A in $GL(4, \mathbb{Z})$. By Lemma 3.1, $\bar{A}$ is in addition symplectic. So Theorem 1.3 is true in this case.

So to prove Theorem 1.3, we need only prove the non-existence of the case of rank 2. In the following, we suppose on the contrary that there exists a rank 2 *geometric* Anosov flow ϕ_t with C^∞ distributions on a closed five-dimensional manifold M. In § 3.2 below, this flow ϕ_t is proved to be homogeneous. Then in §§ 4–6, all the possible homogeneous models are eliminated by some dynamical and Lie theoretical arguments.

3.2. Homogeneity in rank 2

Denote by λ the *canonical* 1*-form* of ϕ_t. Since $\mathrm{rank}(\phi_t) = 2$, then

$$\mathrm{d}\lambda \not\equiv 0, \qquad \mathrm{d}\lambda \wedge \mathrm{d}\lambda \equiv 0.$$

Define $U := \{x \in M \mid (\mathrm{d}\lambda)_x \neq 0\}$. Since ϕ_t is topologically transitive and preserves $\mathrm{d}\lambda$, then U is a ϕ_t-invariant open-dense subset of M. Denote by π the projection of TM onto M. We define

$$E_1 := \{y \in E^+ \oplus E^- \mid i_y \, \mathrm{d}\lambda = 0, \ \pi(y) \in U\}$$

and

$$E_1^{\pm} := E_1 \cap E^{\pm}.$$

Since ϕ_t preserves $\mathrm{d}\lambda$, E^+ and E^-, then E_1, E_1^+ and E_1^- are all ϕ_t-invariant.

Lemma 3.2. *E_1 is a two-dimensional C^∞ subbundle of $TM|_U$. E_1^+ and E_1^- are both one-dimensional C^∞ subbundles of $TM|_U$. In addition, $E_1 = E_1^+ \oplus E_1^-$.*

Proof. Since $\mathrm{d}\lambda(X,\cdot) \equiv 0$, then we view $\mathrm{d}\lambda$ as a section of $(E^+ \oplus E^-)^*$. For all $x \in U$, we have $(\mathrm{d}\lambda)_x \neq 0$. So near x, we can find C^∞ local sections of $E^+ \oplus E^-$, V_1 and V_2, such that

$$\mathrm{d}\lambda(V_1, V_2) \equiv 1.$$

Denote by V the C^∞ local distribution spanned by V_1 and V_2 and denote by $V^\perp$ the orthogonal of V with respect to $\mathrm{d}\lambda|_{E^+\oplus E^-}$.

Since $\mathrm{d}\lambda|_V$ is non-degenerate, then

$$V \cap V^\perp = \{0\}.$$

For all $u \in E^+ \oplus E^-$, such that $\pi(u)$ near x, the following vector is contained in $V^\perp$:

$$P(u) := u - \mathrm{d}\lambda(u, V_2(\pi(u))) \cdot V_1(\pi(u)) - \mathrm{d}\lambda(V_1(\pi(u)), u) \cdot V_2(\pi(u)).$$

So we deduce that locally

$$E^+ \oplus E^- = V \oplus V^\perp.$$

In addition, we see that the projection of $E^+ \oplus E^-$ onto $V^\perp$ with respect to this direct sum decomposition is C^∞. So $V^\perp$ must be also C^∞.

Since $\mathrm{d}\lambda|_V$ is non-degenerate and $\mathrm{d}\lambda \wedge \mathrm{d}\lambda \equiv 0$, then

$$\mathrm{d}\lambda|_{V^\perp} \equiv 0.$$

Thus locally

$$E_1 = V^\perp.$$

In particular, E_1 is C^∞ and two dimensional. Since $\mathrm{d}\lambda(E^\pm, E^\pm) \equiv 0$, then for all $u \in E_1$, its projections to E^+ and E^- are also contained in E_1. Thus

$$E_1 = E_1^+ \oplus E_1^-.$$

If for some x in U, $(E_1^+)_x$ is of dimension two, then $(\mathrm{d}\lambda)_x$ will be zero, which contradicts our assumption. Thus E_1^+ and E_1^- are both of dimension one. In addition, they are evidently C^∞. □

Lemma 3.3. *Under the above notation, the Lyapunov decomposition of ϕ_t is smooth.*

Proof. By definition, the Lyapunov decomposition of ϕ_t is called smooth, if there exists a C^∞ decomposition of TM and a ϕ_t-invariant ν-conull subset $\bar{\Lambda}$ of M, such that the Lyapunov decomposition is defined on $\bar{\Lambda}$ and coincides on $\bar{\Lambda}$ with this C^∞ decomposition.

If ϕ_t has only one positive Lyapunov exponent, then its Lyapunov decomposition is just the restriction of that of Anosov onto a ν-conull subset of M. Since ϕ_t has C^∞ distributions, then the lemma is true in this case.

Suppose that ϕ_t has two positive Lyapunov exponents $b < a$. Then there exists a ν-conull subset Λ of M, such that

$$TM|_\Lambda = L_1^+ \oplus L_1^- \oplus L_2^+ \oplus L_2^+ \oplus \mathbb{R}X,$$

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where $L_1^{\pm}$ and $L_2^{\pm}$ are the Lyapunov subbundles with exponents $\pm b$ and $\pm a$ (see §§ 2.1 and 3.1).

Since U is a ϕ_t-invariant open-dense subset and the flow is ν-ergodic, then U is ν-conull. So $\nu(U \cap \Lambda) = 1$.

Take $x \in U \cap \Lambda$ and $l_i^{\pm} \in (L_i^{\pm})_x$, $i = 1, 2$. By Lemma 2.3, we have

$$\mathrm{d}\lambda(l_1^+, l_2^-) = 0, \qquad \mathrm{d}\lambda(l_1^-, l_2^+) = 0.$$

Since $(\mathrm{d}\lambda)_x \neq 0$, then we must have $\mathrm{d}\lambda(l_1^+, l_1^-) \neq 0$ or $\mathrm{d}\lambda(l_2^+, l_2^-) \neq 0$.

Suppose that $\mathrm{d}\lambda(l_2^+, l_2^-) \neq 0$. Since $\mathrm{d}\lambda \wedge \mathrm{d}\lambda \equiv 0$, then we must have $\mathrm{d}\lambda(l_1^+, l_1^-) = 0$. So $l_1^+ \in (E_1^+)_x$, i.e. $(L_1^+)_x = (E_1^+)_x$. Similarly, we get $(L_1^-)_x = (E_1^-)_x$.

Since $\omega|_{E^+ \oplus E^-}$ is non-degenerate and $\omega(l_1^+, l_2^-) = 0$, then $\omega(l_1^+, l_1^-) \neq 0$. We deduce that $(\mathrm{d}\lambda \wedge \omega)_x \neq 0$. So $\lambda \wedge \mathrm{d}\lambda \wedge \omega$ is not identically zero. Then by the topological transitivity of ϕ_t, $\exists c \neq 0$, such that

$$\lambda \wedge \mathrm{d}\lambda \wedge \omega = c \cdot \lambda \wedge \omega \wedge \omega.$$

So $\lambda \wedge \mathrm{d}\lambda \wedge \omega$ is nowhere zero. We deduce that $\mathrm{d}\lambda$ vanishes nowhere and $U = M$. In particular, E_1 and $E_1^{\pm}$ are all C^{∞} subbundles of TM.

So, by the arguments above, for all $x \in \Lambda$, $(E_1^{\pm})_x = (L_1^{\pm})_x$ or $(L_2^{\pm})_x$. Define

$$\Lambda_i := \{y \in \Lambda \mid E_1^{\pm}(y) = L_i^{\pm}(y)\}, \quad i = 1, 2.$$

Then Λ_1 and Λ_2 are both measurable and ϕ_t-invariant. So one of them is ν-conull. Suppose that $\nu(\Lambda_1) = 1$. Then we have $E_1^{\pm}|_{\Lambda_1} = L_1^{\pm}|_{\Lambda_1}$.

By Lemma 2.3, we have on Λ_1,

$$L_2^{\pm} = [\mathrm{Ker}(v \mapsto \omega(L_1^{\mp}, v))] \cap E^{\pm}.$$

Define two ϕ_t-invariant C^{∞} subbundles of TM as follows,

$$E_2^{\pm} := [\mathrm{Ker}(v \mapsto \omega(E_1^{\mp}, v))] \cap E^{\pm}.$$

Then we have $E_2^{\pm}|_{\Lambda_1} = L_2^{\pm}|_{\Lambda_1}$. So the Lyapunov decomposition coincides on a conull set with a C^{∞} decomposition of TM.

If $\nu(\Lambda_2) = 1$, then a similar argument works. □

Remark 3.4. If ϕ_t has two positive Lyapunov exponents, then by the proof of Lemma 3.3, we have four C^{∞} line bundles on M, $E_1^{\pm}$ and $E_2^{\pm}$. We shall call

$$TM = \mathbb{R}X \oplus E_1^+ \oplus E_1^- \oplus E_2^+ \oplus E_2^-$$

the C^{∞} Lyapunov decomposition of ϕ_t. The Lyapunov of the corresponding Lyapunov subbundles of $E_{1,2}^{\pm}$ are called, respectively, the Lyapunov exponents of $E_{1,2}^{\pm}$. $E_i^{\pm}$ are also denoted by $E_{a_i^{\pm}}$, where $a_i^{\pm}$ are the Lyapunov exponents of $E_i^{\pm}$. If a is not a Lyapunov exponent of ϕ_t, then by convention, $E_a := \{0\}$.

If ϕ_t has only one positive Lyapunov exponent, then the C^{∞} Lyapunov decomposition of ϕ_t means $TM = \mathbb{R}X \oplus E^+ \oplus E^-$.

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Now we can construct a C^∞ connection ∇, adapted to our situation.

If the flow has two positive Lyapunov exponents, then there exists a unique C^∞ connection ∇ on M, such that

$$\begin{gathered}\nabla X = 0, \qquad \nabla\omega = 0, \qquad \nabla E_i^\pm \subseteq E_i^\pm,\\ \nabla_{Y_j^\pm} Y_i^\mp = P_i^\mp [Y_j^\pm, Y_i^\mp], \qquad \forall i, j \in \{1,2\},\\ \nabla_X Y_i^\pm := [X, Y_i^\pm] \pm a_i Y_i^\pm, \qquad \forall Y_i^\pm \subseteq E_i^\pm,\end{gathered}$$

where a_i denotes the Lyapunov exponent of E_i^+ and $P_i^\pm$ represent the projections of TM onto $E_i^\pm$.

If ϕ_t has only one positive Lyapunov exponent a, then we get a similar C^∞ connection ∇, such that

$$\begin{gathered}\nabla X = 0, \qquad \nabla\omega = 0, \qquad \nabla E^\pm \subseteq E^\pm,\\ \nabla_{Y^\pm} Y^\mp = P^\mp [Y^\pm, Y^\mp],\\ \nabla_X Y^\pm = [X, Y^\pm] \pm a Y^\pm, \quad \forall Y^\pm \subseteq E^\pm,\end{gathered}$$

where $P^\pm$ represent the projections of TM onto $E^\pm$.

If a transformation of M preserves X, ω, and the C^∞ Lyapunov decomposition, then it preserves also ∇. In particular, ∇ is ϕ_t-invariant.

Lemma 3.5. *Under the above notation, if K be a C^∞ ϕ_t-invariant tensor field of type $(1, l)$ on M, then $K(E_{a_1}, \dots, E_{a_l}) \subseteq E_{a_1+\dots+a_l}$, where $a_1, \dots, a_l$ are arbitrary Lyapunov exponents of ϕ_t. In addition, we have $\nabla K = 0$.*

Proof. By the same arguments as in Lemma 2.5 of [**5**], we get for arbitrary Lyapunov exponents, $a_1, \dots, a_l$,

$$K(E_{a_1}, \dots, E_{a_l}) \subseteq E_{a_1+\dots+a_l}.$$

Now let $Z_1, \dots, Z_l$ be the sections of the smooth subbundles, $E_{a_1}, \dots, E_{a_l}$. We have

$$\begin{aligned}(\nabla_X K)&(Z_1, \dots, Z_l)\\ &= \nabla_X (K(Z_1, \dots, Z_l)) - \sum_{1 \leqslant i \leqslant l} K(Z_1, \dots, \nabla_X Z_i, \dots, Z_l)\\ &= [X, K(Z_1, \dots, Z_l)] + \left(\sum_{1 \leqslant i \leqslant l} a_i\right) K(Z_1, \dots, Z_l) - K([X, Z_1] + a_1 Z_1, \dots) + \cdots\\ &= [X, K(Z_1, \dots, Z_l)] - \sum_{1 \leqslant i \leqslant l} K(Z_1, \dots, [X, Z_i], \dots, Z_l)\\ &= (\mathcal{L}_X K)(Z_1, \dots, Z_l) = 0.\end{aligned}$$

So $\nabla_X K = 0$. Since ∇K is a ϕ_t-invariant tensor of type $(1, l+1)$, then we have

$$(\nabla_{E_{a_0}} K)(E_{a_1}, \dots, E_{a_l}) \subseteq E_{a_0+\dots+a_l}.$$

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Since for all $a \in \mathbb{R}$, $\nabla E_a \subseteq E_a$, then

$$(\nabla_{E_{a_0}} K)(E_{a_1}, \ldots, E_{a_l}) \subseteq E_{a_1+\cdots+a_l}.$$

So if $a_0 \neq 0$, we have $\nabla_{E_{a_0}} K = 0$. We deduce that $\nabla K = 0$. □

Denote by T the torsion of ∇ and by R its curvature tensor. Then by the previous lemma, we have

$$\nabla T = 0, \qquad \nabla R = 0, \qquad T(E_{a_1}, E_{a_2}) \subseteq E_{a_1+a_2}.$$

If $a_1 + a_2 \neq 0$, then

$$R(E_{a_1}, E_{a_2}) = 0.$$

Denote by $\tilde{M}$ the universal cover of M and by $\tilde{\nabla}$ the lifted connection of ∇. Then we have the following lemma.

Lemma 3.6. *Under the above notation, the group of $\tilde{\nabla}$-affine transformations of $\tilde{M}$, which preserve $\tilde{X}$, $\tilde{\omega}$, and the lifted C^∞ Lyapunov decomposition, is a Lie group acting transitively on $\tilde{M}$.*

Proof. By Proposition 2.7 of [**5**], the ∇-geodesics, tangent to E^+ or E^-, are complete, i.e. defined on $\mathbb{R}$. Since $\nabla T = 0$ and $\nabla R = 0$, then by Lemma A.1 proved in the appendix, ∇ is complete. So $\tilde{\nabla}$ is also complete.

Recall that $E_a := \{0\}$, if a is not a Lyapunov exponent of ϕ_t. For all $a \in \mathbb{R}$, denote by $\tilde{P}_a$ the projection of $T\tilde{M}$ onto $\tilde{E}_a$. Since $\nabla E_a \subseteq E_a$, then $\tilde{P}_a$ is $\tilde{\nabla}$-parallel. Thus $\{\tilde{X}, \tilde{\omega}, \tilde{P}_a\}_{a\in\mathbb{R}}$ is a family of $\tilde{\nabla}$-parallel tensor fields. In addition, an application preserves $\{\tilde{P}_a\}_{a\in\mathbb{R}}$, if and only if it preserves the lifted C^∞ Lyapunov decomposition. So the lemma follows from the following classical result (see [**15**]).

Let N be a simply connected manifold, ∇_1 be a complete connection on N and $\mathcal{S}$ be a family of parallel tensor fields. If $\nabla_1 R^{\nabla_1} = 0$ and $\nabla_1 T^{\nabla_1} = 0$, then the group of ∇_1-affine transformations which preserve $\mathcal{S}$ is a Lie group and acts transitively on N. □

In the sense of the previous lemma, ϕ_t is called homogeneous. In particular, we deduce that $\mathrm{d}\lambda$ vanishes nowhere. So on M, we have always two C^∞ ϕ_t-invariant line bundles E_1^+ and E_1^-, which are quite essential for the following discussions.

4. The case of two positive Lyapunov exponents

4.1. Preparations

Now we begin to eliminate the possible homogeneous models. In this section, we suppose that ϕ_t has two positive Lyapunov exponents. Then by Remark 3.4, we have

$$TM = \mathbb{R}X \oplus E_1^+ \oplus E_2^+ \oplus E_1^- \oplus E_2^-.$$

Up to a constant change of time-scale, we suppose that the Lyapunov exponents of E_1^+ and E_2^+ are, respectively, 1 and a.

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In this case, the underlying geometric structure of our system is

$$g_1 := (X, E_1^+, E_2^+, E_1^-, E_2^-, \omega).$$

Let G' be the isometry group of $\tilde{g}_1$ and Γ be the fundamental group of M. By Lemma 3.6, G' acts transitively on $\tilde{M}$. The group Γ is contained as a discrete subgroup in G'. Fix $x \in \tilde{M}$ and denote by H' the isotropy subgroup of x. Let H'_e be the identity component of H'. Then we have the linear isotropy representation

$$\begin{aligned} H'_e &\overset{i}{\hookrightarrow} GL(T_x\tilde{M}) \\ h &\mapsto D_x h. \end{aligned}$$

Since each element of H' preserves ∇, then i is injective. For all $h \in H'_e$,

$$D_x h(\tilde{X}_x) = \tilde{X}_x, \qquad D_x h(\tilde{E}^{\pm}_x) \subseteq \tilde{E}^{\pm}_x.$$

So in the following, we identify $i(h)$ with its restriction to $(\tilde{E}^+ \oplus \tilde{E}^-)_x$.

Take a basis $(l_2^+, l_1^+, l_2^-, l_1^-)$ of $(\tilde{E}^+ \oplus \tilde{E}^-)_x$, such that $l_{1,2}^{\pm} \in (\tilde{E}^{\pm}_{1,2})_x$. Since each element h of H'_e preserves $\tilde{g}_1$, then we have

$$D_x h = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1/\lambda_1 & 0 \\ 0 & 0 & 0 & 1/\lambda_2 \end{pmatrix}.$$

So $i(H'_e)$ is contained in a closed subgroup of $GL(T_x\tilde{M})$, which is isomorphic to $\mathbb{R}^2$. So we can identify H'_e with $i(H'_e)$ and we deduce that H'_e is isomorphic to 0, $\mathbb{R}$ or $\mathbb{R}^2$. In any case, we have $\pi_1(H'_e) = 0$.

Let G'_e be the connected component of the identity of G'. Then it acts also transitively on $\tilde{M}$. Using the long exact sequence of homotopy, we get easily

$$H'_e = H' \cap G'_e, \qquad \pi_1(G'_e) = 0.$$

Since $\tilde{M} \cong G'/H'$, then $\tilde{M}$ admits naturally a real analytic structure. Since the geometric structure $\tilde{g}_1$ is G'-invariant, then $\tilde{g}_1$ is real analytic. Thus by [**1**] (see also [**7**]), the local Killing fields of $\tilde{g}_1$ can be extended to global ones. Since ∇ is in addition complete, then H' is easily seen to have finitely many connected components. We deduce that G' has also finitely many connected components. So up to finite covers, we can suppose that $\Gamma \subseteq G'_e$.

Denote by $\mathfrak{g}'$ and $\mathfrak{h}'$ the Lie algebras of G' and H'. For all $u \in \mathfrak{g}'$, we have an induced C^∞ Killing field on $\tilde{M}$,

$$\begin{aligned} Y^u &: \tilde{M} \to T\tilde{M}, \\ a &\to \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{e}^{tu} a. \end{aligned}$$

Since ∇ is complete, $\nabla R = 0$ and $\nabla T = 0$, then we have the following classical identification of vector spaces (see Theorem 28 of [**15**, Chapter X])

$$j : \mathfrak{g}' \overset{\sim}{\mapsto} T_x\tilde{M} \oplus \mathfrak{h}'$$
$$u \mapsto (Y^u(x), (\tilde{\nabla}_{Y^u} - \mathcal{L}_{Y^u})|_x),$$

where $\mathfrak{h}'$ has been identified with $\mathrm{D}i(\mathfrak{h}')$ under $\mathrm{D}i$.

Pushing forward by j the Lie algebra structure of $\mathfrak{g}'$ onto $T_x\tilde{M} \oplus \mathfrak{h}'$, we have for all $u, v \in T_x\tilde{M}$ and $A, B \in \mathfrak{h}'$,

$$\begin{aligned}[u, v] &= -T^{\tilde{\nabla}}(u, v) - R^{\tilde{\nabla}}(u, v),\\ [A, u] &= Au,\\ [A, B] &= A \circ B - B \circ A.\end{aligned}$$

Denote by u the generating vector of the 1-parameter subgroup $\{\tilde{\phi}_t\}_{t\in\mathbb{R}}$ of G'. Then $Y^u = \tilde{X}$. Under the identification j, we have

$$u = \tilde{X}_x + (P_1^+ - P_1^- + aP_2^+ - aP_2^-) \in T_x\tilde{M} \oplus \mathfrak{h}'.$$

If $L_0 := u - \tilde{X}_x$, then $L_0 \in \mathfrak{h}'$. We deduce that $\mathfrak{h}' \cong \mathbb{R}$ or $\mathbb{R}^2$.

Lemma 4.1. *Under the above notation, $E_1^+ \oplus E_1^-$ and $E_2^+ \oplus E_2^- \oplus \mathbb{R}X$ are both integrable.*

Proof. Let Y, Z be two C^∞ sections of $E_1^+ \oplus E_1^-$, then

$$0 = \mathrm{d}\lambda(Y, Z) = -\lambda([Y, Z]).$$

So $[Y, Z]$ is a section of $E^+ \oplus E^-$.

$$\begin{aligned} i_{[Y,Z]}\mathrm{d}\lambda &= (\mathcal{L}_Y i_Z - i_Z\mathcal{L}_Y)\,\mathrm{d}\lambda\\ &= -i_Z(\mathrm{d}i_Y + i_Y\,\mathrm{d})\,\mathrm{d}\lambda\\ &= 0.\end{aligned}$$

So $[Y, Z]$ is also a section of $E_1^+ \oplus E_1^-$. Thus $E_1^+ \oplus E_1^-$ is integrable.

Since E_2^+ and E_2^- are both ϕ_t-invariant, then $[X, E_2^\pm] \subseteq E_2^\pm$. Define two tensor fields $K^\pm$ of type $(1, 2)$ on M, such that

$$K^\pm(Y, Z) = P_1^\pm[P_2^+(Y), P_2^-(Z)], \quad \forall Y, Z \subseteq TM.$$

Then $K^\pm$ are both ϕ_t-invariant. By Lemma 3.5, $K^\pm(E_2^+, E_2^-) \subseteq \mathbb{R}X$. So we have

$$[E_2^+, E_2^-] \subseteq E_2^+ \oplus E_2^- \oplus \mathbb{R}X.$$

Thus $E_2^+ \oplus E_2^- \oplus \mathbb{R}X$ is integrable. □

Up to finite covers, we suppose that E^+ and E^- are both orientable. The connection ∇ induces a connection ∇^+ on $\wedge^2 E^+$. Denote by Ω^+ its curvature form and by β^+ its connection form. Then we have

$$\Omega^+(\cdot,\cdot) = \mathrm{Tr}(R(\cdot,\cdot)|_{E^+}), \qquad \mathrm{d}\beta^+ = \Omega^+.$$

Lemma 4.2. $\mathrm{d}\lambda \wedge \Omega^+ = 0$, $\Omega^+ \wedge \Omega^+ = 0$, $\Omega^+ \wedge \omega = 0$.

Proof. Since Ω^+ is ϕ_t-invariant and the flow is topologically transitive, then there exists a constant c, such that

$$\lambda \wedge \mathrm{d}\lambda \wedge \Omega^+ = c \cdot \lambda \wedge \omega \wedge \omega.$$

So

$$\begin{aligned} c \int_M \lambda \wedge \omega \wedge \omega &= \int_M \lambda \wedge \mathrm{d}\lambda \wedge \Omega^+ \\ &= -\int_M \mathrm{d}(\lambda \wedge \mathrm{d}\lambda \wedge \beta^+) \\ &= \int_{\partial M} \lambda \wedge \mathrm{d}\lambda \wedge \beta^+ \\ &= 0. \end{aligned}$$

So $c = 0$. We deduce that

$$\mathrm{d}\lambda \wedge \Omega^+ = i_X(\lambda \wedge \mathrm{d}\lambda \wedge \Omega^+) = 0.$$

In the same way, we get $\Omega^+ \wedge \Omega^+ = 0$.

If $\lambda \wedge \Omega^+ \wedge \omega = s \cdot \lambda \wedge \omega \wedge \omega$, then

$$s \int_M \lambda \wedge \omega \wedge \omega = \int_M \beta^+ \wedge \mathrm{d}\lambda \wedge \omega.$$

If $\lambda \wedge \mathrm{d}\lambda \wedge \omega = \delta \cdot \lambda \wedge \omega \wedge \omega$, then

$$\begin{aligned} \beta^+ \wedge \mathrm{d}\lambda \wedge \omega &= \delta \cdot \beta^+ \wedge \omega \wedge \omega \\ &= \delta \cdot \beta^+(X) \lambda \wedge \omega \wedge \omega. \end{aligned}$$

By the same argument as in Lemma 2.3.3 of [**6**], we get

$$\int_M \beta^+(X) \lambda \wedge \omega \wedge \omega = 0.$$

So $s = 0$, i.e. $\Omega^+ \wedge \omega = 0$. □

Lemma 4.3. *Under the above notation, we have* $\Omega^+ = 0$.

Proof. In the direction of X, the situation is always clear. So in the following, we consider only the restrictions onto $E^+ \oplus E^-$ of the forms and endomorphisms.

Since $\omega|_{E^+ \oplus E^-}$ is non-degenerate, then we can find a section ψ of $\mathrm{End}(E^+ \oplus E^-)$, such that

$$\Omega^+(\cdot, \cdot) = \omega(\psi(\cdot), \cdot).$$

For all $y \in M$, take $l^{\pm}_{1,2} \in (E^{\pm}_{1,2})_y$ such that $(l_2^+, l_1^+, l_2^-, l_1^-)$ forms a dual basis of ω_y, i.e.

$$\omega(l_2^+, l_2^-) = \omega(l_1^+, l_1^-) = 1, \qquad \omega(l_2^+, l_1^-) = \omega(l_1^+, l_2^-) = 0.$$

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If $\psi_y(l_1^+) = 0$, then in this basis, we get

$$\psi_y = \begin{pmatrix} A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\Omega^+ \wedge \omega = 0$, then $\operatorname{Tr} \psi = 2A = 0$. By Lemma 2.3,

$$\begin{aligned} 0 &= \Omega_y^+(l_2^+, l_1^-) \\ &= \omega(\psi l_2^+, l_1^-) \\ &= B \cdot \omega(l_1^+, l_1^-). \end{aligned}$$

So $B = 0$. Thus $\psi_y = 0$.

Now suppose that $\psi_y(l_1^+) \neq 0$. Since $\Omega^+ \wedge \Omega^+ = 0$, then $\det(\psi_y) = 0$. So

$$\exists y_1^+ = \alpha l_2^+ + \delta l_1^+, \quad \alpha \neq 0,$$

such that $\psi_y(y_1^+) = 0$. Then in a dual basis with respect to ω_y, $(y_1^+, l_1^+, y_1^-, z^-)$, we have

$$\psi_y = \begin{pmatrix} 0 & A & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \end{pmatrix}.$$

As above, we have $\operatorname{Tr}(\psi_y) = 2B = 0$. By Lemma 2.3,

$$\begin{aligned} 0 &= \Omega_y^+(l_1^+, l_2^-) \\ &= \omega(A y_1^+, l_2^-) \\ &= A \cdot \alpha \cdot \omega(l_2^+, l_2^-) \\ &= A \cdot \alpha. \end{aligned}$$

So $A = 0$. We deduce that $\psi \equiv 0$, i.e. $\Omega^+ \equiv 0$. □

Define the following map

$$\begin{aligned} \mathfrak{g}' &\overset{\chi}{\mapsto} \mathbb{R}, \\ u + A &\mapsto \operatorname{Tr}(A|_{\tilde{E}_x^+}). \end{aligned}$$

Since $\Omega^+ \equiv 0$, then χ is a character of $\mathfrak{g}'$. So the kernel of χ is an ideal of $\mathfrak{g}'$, denoted by $\mathfrak{g}$,

We have seen that $\mathfrak{h}'$ is isomorphic to $\mathbb{R}$ or $\mathbb{R}^2$. In the following, these two cases are considered separately.

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4.2. $\dim \mathfrak{h}' = 1$

In this subsection, we suppose that $\dim \mathfrak{h}' = 1$. To prove the non-existence of such a flow, we shall at first calculate explicitly $\mathfrak{g}'$ using the lemmas established in the previous subsection. Then we shall get a contradiction via the non-existence of co-compact lattice in $\mathbb{R}^2 \rtimes SL(2, \mathbb{R})$.

Since $L_0 \in \mathfrak{h}'$ (see § 4.1), then $\mathfrak{h}' = \mathbb{R}L_0$. To simplify the notation, we identify $T_x\tilde{M}$ with T_xM. Thus we have

$$\mathfrak{g}' = T_xM \oplus \mathfrak{h}'.$$

Denote by $\mathfrak{g}$ the kernel of χ (see § 4.1). Then $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$. Since $\chi(L_0) = 1 + a > 0$, then we have $\mathfrak{g} = T_xM$. Recall that the Lyapunov exponents of E_1^+ and E_2^+ are 1 and a. Now we can find explicitly $\mathfrak{g}$ as follows.

Since $\mathfrak{g}$ $(= T_xM)$ is an ideal of $\mathfrak{g}'$, then for all $u, v \in T_xM$,

$$[u, v] = -T(u, v) - R(u, v) \in T_xM.$$

Thus $R(u, v) = 0$ and $[u, v] = -T(u, v)$.

Take a basis of T_xM, $(X_x, l_2^+, l_1^+, l_2^-, l_1^-)$, such that $l_{1,2}^\pm \in (E_{1,2}^\pm)_x$ and $\mathrm{d}\lambda(l_2^+, l_2^-) = -1$. Extend $l_{1,2}^\pm$ to local sections of $E_{1,2}^\pm$, denoted by $\bar{l}_{1,2}^\pm$. By the definition of ∇, we get

$$[X_x, l_1^\pm] = -T(X_x, l_1^\pm) = \mp l_1^\pm.$$

Similarly,

$$[X_x, l_2^\pm] = \mp a l_2^\pm.$$

Since $E_1^+ \oplus E_1^-$ is integrable by Lemma 4.1, then we get

$$\begin{aligned}[l_1^+, l_1^-] &= -T(l_1^+, l_1^-)\\ &= -(P_1^-[\bar{l}_1^+, \bar{l}_1^-] + P_1^+[\bar{l}_1^+, \bar{l}_1^-] - [\bar{l}_1^+, \bar{l}_1^-])\\ &= 0.\end{aligned}$$

Similarly, we get

$$[l_2^+, l_2^-] = X_x.$$

Lemma 4.4. *Under the above notation, we have* $1 < a$.

Proof. Suppose that $1 > a$. Then by Lemma 3.5, we have

$$T(E_1^+, E_2^-) \subseteq E_{1-a}.$$

If $1 - a \neq a$, then $[l_1^+, l_2^-] = -T(l_1^+, l_2^-) = 0$. If $1 - a = a$, then $\exists b \in \mathbb{R}$, such that

$$[l_1^+, l_2^-] = b \cdot l_2^+.$$

So in any case, $\exists c \in \mathbb{R}$, such that $[l_1^+, l_2^-] = c \cdot l_2^+$.

Since $T(E_1^+, E_2^+) \subseteq E_{1+a} = \{0\}$, then $[l_1^+, l_2^+] = 0$. By the Jacobi identity of l_1^+, l_2^+ and l_2^-, we get

$$\begin{aligned} 0 &= [l_2^+, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, l_2^+]] + [l_2^-, [l_2^+, l_1^+]] \\ &= [l_2^+, c \cdot l_2^+] + [l_1^+, -X_x] \\ &= -l_1^+, \end{aligned}$$

which is absurd. □

Since $a > 1$, then we can suppose that $[l_1^+, l_2^-] = c \cdot l_1^-$ and $[l_1^-, l_2^+] = d \cdot l_1^+$. Again by the Jacobi identity of l_1^+, l_2^+ and l_2^-, we have

$$0 = [l_2^+, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, l_2^+]] = -(1 + c \cdot d) l_1^+.$$

So $c \cdot d = -1$. Now replacing l_2^- by $(1/c) l_2^-$ and l_2^+ by $c \cdot l_2^+$, we get the following bracket relations of $\mathfrak{g}$:

$$\begin{aligned} [X_x, l_1^\pm] &= \mp l_1^\pm, & [X_x, l_2^\pm] &= \mp a l_2^\pm, \\ [l_1^+, l_1^-] &= 0, & [l_1^+, l_2^-] &= l_1^-, \\ [l_1^-, l_2^+] &= -l_1^+, & [l_2^+, l_2^-] &= X_x. \end{aligned}$$

The brackets, which have not appeared in these bracket relations, vanish by Lemma 3.5.

Since $[l_1^+, l_2^-] = l_1^-$, then $E_{1-a} \neq \{0\}$. We deduce that $a = 2$. Thus by the bracket relations above, we get clearly

$$\mathfrak{g} \cong \mathbb{R}^2 \rtimes \mathfrak{sl}(2, \mathbb{R}),$$

where the semi-direct product is given by matrix multiplication.

It is easily seen that the centre of $\mathfrak{g}'$ is $\mathbb{R}(X_x + L_0)$. Thus we have the following direct product decomposition

$$\mathfrak{g}' \cong \mathfrak{g} \oplus \mathbb{R}(X_x + L_0).$$

Let G be the connected subgroup of G'_e integrating $\mathfrak{g}$. Since G'_e is simply connected (see § 4.1), then G is also simply connected and $G'_e = G \times \mathbb{R}$, where $\mathbb{R}$ integrates $\mathbb{R}(X_x + L_0)$ in G'_e. Thus we get

$$G \cong \mathbb{R}^2 \rtimes \widetilde{SL(2, \mathbb{R})}.$$

It is easily seen that G acts transitively on $\tilde{M}$. Then by the long exact sequence of homotopy, $G \cap H'_e$ is seen to be connected. So $G \cap H'_e = \{e\}$, i.e. G acts freely on $\tilde{M}$. Thus G is identified to $\tilde{M}$.

Up to finite covers, we have $\Gamma \subseteq G'_e$ (see § 4.1). Let Γ_1 be the projection of Γ into G, with respect to the direct product $G'_e = G \times \mathbb{R}$. Since $\Omega^+ = 0$, then by the general arguments of § 5 of [**5**], Γ_1 is seen to be a co-compact lattice of G. Now we eliminate this case by proving the following lemma.

Lemma 4.5. $\mathbb{R}^2 \rtimes \widetilde{SL(2, \mathbb{R})}$ *has no co-compact lattice.*

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Proof. Suppose that there exists a co-compact lattice, denoted by Δ. Define $\Delta_1 := \Delta \cap \mathbb{R}^2$ and denote by Δ_2 the projection of Δ to $SL(2,\mathbb{R})$. Then by Corollary 8.28 of [**18**], Δ_1 is a co-compact lattice of $\mathbb{R}^2$ and Δ_2 is a lattice of $SL(2,\mathbb{R})$.

Denote by π the natural projection of $SL(2,\mathbb{R})$ onto $SL(2,\mathbb{R})$. Then $\pi(\Delta_2)$ preserves the lattice Δ_1 for the linear action. So $\pi(\Delta_2)$ is conjugate to a subgroup of $SL(2,\mathbb{Z})$.

Since Δ is co-compact, then $\pi(\Delta_2)$ is also co-compact. We deduce that $SL(2,\mathbb{Z})$ is co-compact in $SL(2,\mathbb{R})$, which is absurd. □

4.3. $\dim \mathfrak{h}' = 2$

In this subsection, we suppose that $\dim \mathfrak{h}' = 2$. To prove the non-existence of such a flow, we shall at first find $\mathfrak{g}'$. Then we shall study the action of the fundamental group of M on the space of lifted weak unstable leaves to deduce a dynamical contradiction.

Define $S := P_2^+ - P_1^+ - P_2^- + P_1^-$. Then $\mathfrak{h}'$ is generated by S and L_0 (see § 4.1). Since $\chi(S) = 0$, then we have $\mathfrak{g} = \mathbb{R}S \oplus T_xM$.

As in the previous subsection, we suppose that the Lyapunov exponents of E_1^+ and E_2^+ are 1 and a. Take a basis of T_xM, $(X_x, l_2^+, l_1^+, l_2^-, l_1^-)$, such that $l_{1,2}^{\pm} \in E_{1,2}^{\pm}$ and $\mathrm{d}\lambda(l_2^+, l_2^-) = -1$. Suppose at first that $a > 1$. Then by the same argument as in Lemma 4.4, we can find c and d, such that

$$[l_1^+, l_2^-] = c \cdot l_1^-, \qquad [l_1^-, l_2^+] = d \cdot l_1^+.$$

By the Jacobi identity of S, l_1^+ and l_2^-, we get

$$\begin{aligned} 0 &= [S, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, S]] + [l_2^-, [S, l_1^+]] \\ &= c \cdot l_1^- + [l_1^+, l_2^-] + [l_2^-, -l_1^+] \\ &= 3c \cdot l_1^-. \end{aligned}$$

Thus $c = 0$. Similarly we get $d = 0$. If $a < 1$, then we can find c' and d', such that

$$[l_1^+, l_2^-] = c' \cdot l_2^+, \qquad [l_1^-, l_2^+] = d' \cdot l_2^-.$$

Thus by the Jacobi identities, we get as above $c' = d' = 0$. We deduce that

$$[l_1^+, l_2^-] = 0, \qquad [l_1^-, l_2^+] = 0.$$

Now by similar arguments as in the previous subsection, we get the following bracket relations,

$$\begin{gathered} [S, l_1^{\pm}] = \mp l_1^{\pm}, \qquad [S, l_2^{\pm}] = \pm l_2^{\pm}, \\ [L_0, l_1^{\pm}] = \pm l_1^{\pm}, \qquad [L_0, l_2^{\pm}] = \pm a l_2^{\pm}, \\ [X_x, l_1^{\pm}] = \mp l_1^{\pm}, \qquad [X_x, l_2^{\pm}] = \mp a l_2^{\pm}, \\ [l_2^+, l_2^-] = X_x - S. \end{gathered}$$

The brackets, which have not appeared in these bracket relations, vanish. Define three elements:

$$\alpha := \frac{L_0 + S}{a+1}, \qquad \beta := \frac{L_0 - aS}{a+1}, \qquad \delta := \frac{X_x - S}{a+1}.$$

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Then $\mathfrak{g}'$ is decomposed as a direct product of three ideals,

$$\mathfrak{g}' \cong (\mathbb{R}l_1^+ \oplus \mathbb{R}l_1^- \oplus \mathbb{R}\beta) \oplus (\mathbb{R}l_2^+ \oplus \mathbb{R}l_2^- \oplus \mathbb{R}\delta) \oplus \mathbb{R}(\delta + \alpha).$$

Then by the bracket relations above, we get

$$\mathfrak{g}' \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R},$$

where the semi-direct product, $\mathbb{R}^2 \rtimes \mathbb{R}$, is given by the linear action on $\mathbb{R}^2$ of the order-two diagonal matrices of trace zero. Since G'_e is simply connected, then we have

$$G'_e \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \times \widetilde{SL(2, \mathbb{R})} \times \mathbb{R}.$$

Now we begin to study the action of Γ on the space of lifted weak unstable leaves. Let us recall at first some notation.

Let ψ_t be a C^∞ Anosov flow on a closed manifold N. Denote by $\tilde{\psi}_t$ its lifted flow on the universal covering space $\tilde{N}$. Denote by $\tilde{\mathcal{F}}^{+,0}$ the lifted foliation of $\mathcal{F}^{+,0}$ and by $\tilde{N}/\tilde{\mathcal{F}}^{+,0}$ the space of lifted weak unstable leaves with the quotient topology. Thus the fundamental group $\pi_1(N)$ acts naturally on $\tilde{N}/\tilde{\mathcal{F}}^{+,0}$. The following lemma has appeared in some special contexts (see, for example, [**6**] and [**3**]). For the sake of completeness, we prove it in detail.

Lemma 4.6. *Under the above notation, if $\gamma \in \pi_1(N)$ and $\gamma \neq e$, then each γ-fixed point of $\tilde{N}/\tilde{\mathcal{F}}^{+,0}$ is either contractive or repulsive.*

Proof. Suppose that $\tilde{W}_x^{+,0}$ is fixed by γ. Then $\exists t \in \mathbb{R}$, such that

$$\gamma \tilde{W}_x^+ = \tilde{\phi}_t \tilde{W}_x^+.$$

If $t = 0$, then we can take a curve l in $\tilde{W}_x^+$, such that $l(0) = x$ and $l(1) = \gamma x$. If $s \ll 0$, then $\phi_s(\pi(l))$ will be tiny, where π denotes the projection of $\tilde{N}$ onto N. Thus $\phi_s(\pi(l))$ is homotopically trivial. We deduce that $\pi(l)$ is also homotopically trivial, i.e. $\gamma = e$, which is a contradiction. So $t \neq 0$.

By replacing γ by γ^{-1} if necessary, we suppose that $t < 0$. We can see as follows that $\tilde{W}_x^{+,0}$ is γ-contractive.

Fix a C^∞ Riemannian metric g on N. Denote by $\tilde{g}$ the lifted metric on $\tilde{N}$. By [**2**], the induced metrics on the leaves of $\mathcal{F}^{+,0}$ are all complete. Thus with its induced metric, $\tilde{W}_x^+$ is a complete metric space. Since γ acts isometrically, then $\gamma^{-n} \circ \tilde{\phi}_{nt}$ is a contraction of $\tilde{W}_x^+$, if $n \gg 1$. Thus it admits a unique fixed point in $\tilde{W}_x^+$, denoted again by x. So we get

$$\gamma x = \tilde{\phi}_t x,$$

i.e. the orbit of x is fixed by γ.

Denote by $\bar{U}$ the saturated set of $\tilde{W}_x^-$ with respect to $\tilde{\mathcal{F}}^{+,0}$. Then by the local product structure of $\tilde{\phi}_t$, $\bar{U}$ is open. Thus the projection of $\tilde{W}_x^-$ into $\tilde{N}/\mathcal{F}^{+,0}$ is an open neighbourhood of $\tilde{W}_x^{+,0}$, denoted by U. For all $y \in \tilde{W}_x^-$, we have

$$\gamma^n \tilde{W}_y^{+,0} = \tilde{W}^{+,0}_{(\tilde{\phi}_{-nt} \circ \gamma^n)(y)}.$$

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Since $\gamma x = \tilde{\phi}_t x$, then $(\tilde{\phi}_{-nt} \circ \gamma^n)(x) = x$. So

$$(\tilde{\phi}_{-nt} \circ \gamma^n)(y) \xrightarrow[n \to +\infty]{} x.$$

We deduce that

$$\gamma^n \tilde{W}_y^{+,0} \xrightarrow[n \to +\infty]{} \tilde{W}_x^{+,0}.$$

So γ contracts on U. □

Now return to our *geometric* Anosov flow ϕ_t. Denote by P'_e the stabilizer of $\tilde{W}_x^+$ in G'_e. Then $H'_e \subseteq P'_e$ and P'_e is easily seen to be connected. So G'_e/P'_e is identified to $\tilde{M}/\tilde{\mathcal{F}}^{+,0}$. Define

$$\mathfrak{p}^+ := \mathbb{R}X_x \oplus \mathfrak{h}' \oplus \mathbb{R}l_1^+ \oplus \mathbb{R}l_2^+.$$

Then $\mathfrak{p}^+$ is the Lie algebra of P'_e and P'_e is seen to be closed in G'_e. Since G'_e is simply connected (see § 4.1), then by the long exact sequence of homotopy, we get $\pi_1(G'_e/P'_e) = 0$.

Define $G_e^1 := (\mathbb{R}^2 \rtimes \mathbb{R}) \times SL(2, \mathbb{R}) \times \mathbb{R}$ and denote by P_e^1 the connected Lie subgroup of G_e^1 with Lie algebra $\mathfrak{p}^+$. Then G_e^1/P_e^1 is naturally identified to $\mathbb{R}^1 \times \mathbb{S}^1$. Denote by π the projection of G'_e onto G_e^1 and by P''_e the group $\pi^{-1}(P_e^1)$. Then we get

$$G'_e/P''_e \cong G_e^1/P_e^1 \cong \mathbb{R}^1 \times \mathbb{S}^1.$$

We deduce that $G'_e/P'_e \cong \mathbb{R}^1 \times \mathbb{R}^1$.

Since ϕ_t preserves a volume form, then the periodic points of ϕ_t is dense in M. Take $gH'_e \in G'_e/H'_e$ $(\cong \tilde{M})$, such that its projection in M is of period T. If $\tilde{\phi}_T(gH'_e) = gH'_e$, then each orbit of $\tilde{\phi}_t$ is periodic by the homogeneity of $\tilde{\phi}_t$. We deduce that each ϕ_t-orbit is periodic, which contradicts the topological transitivity of ϕ_t. So $\tilde{\phi}_T(gH'_e) \neq gH'_e$.

Now take $\gamma \in \Gamma$ $(\subseteq G'_e)$, such that $\gamma(gH'_e) = \tilde{\phi}_T(gH'_e)$. Then we have $\gamma \neq e$ and $\exists h \in H'_e$, such that

$$\gamma = g(h \cdot \exp(T(X_x + L_0)))g^{-1}.$$

Since γ fixes the orbit of gH'_e, then it fixes gP'_e and gP''_e. So by Lemma 4.6, the γ-action on G'_e/P''_e admits at least an isolated fixed point. Then by some direct calculations, the corresponding γ-action on $\mathbb{R}^1 \times \mathbb{S}^1$ $(\cong G'_e/P''_e)$ must be as follows:

$$\left.\begin{aligned} \mathbb{R}^1 \times \mathbb{S}^1 &\xrightarrow{\gamma} \mathbb{R}^1 \times \mathbb{S}^1, \\ (y, [u]) &\to (\mathrm{e}^{-c}y + d, [Au]), \end{aligned}\right\} \qquad (*)$$

where $c \neq 0$ and A is a matrix with two different positive eigenvalues. Here $\mathbb{S}^1$ is viewed as the set of directions, i.e.

$$\mathbb{S}^1 \cong \{[u] \mid u \in \mathbb{C}^*,\ u \sim v \iff u = tv,\ t > 0\}.$$

Then $GL(2, \mathbb{R})$ acts on $\mathbb{S}^1$ by matrix multiplication.

Up to an isomorphism of covering spaces, the projection of G'_e/P'_e onto G'_e/P''_e is as follows:

$$\left.\begin{aligned} \mathbb{R}^1 \times \mathbb{R}^1 &\mapsto \mathbb{R}^1 \times \mathbb{S}^1, \\ (x, \theta) &\mapsto (x, [\mathrm{e}^{\mathrm{i}\theta}]). \end{aligned}\right\} \qquad (**)$$

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Since the γ-action on G'_e/P'_e is just a lift of the γ-action on G'_e/P''_e, then by $(*)$ and $(**)$, we clearly see that on G'_e/P'_e, γ admits either a saddle or no fixed point. We deduce that γ admits a saddle on G'_e/P'_e, which contradicts Lemma 4.6.

5. The case of one positive exponent and $\mathbf{d}\boldsymbol{\lambda} \wedge \boldsymbol{\omega} \not\equiv \mathbf{0}$

5.1. Preparations

In this section, we suppose that ϕ_t has only one positive Lyapunov exponent and $d\lambda \wedge \omega \not\equiv 0$. Up to a constant change of time-scale, we suppose that this positive exponent is 1. By Lemma 3.6, $d\lambda \wedge \omega$ vanishes nowhere. So $\omega|_{E_1^+ \oplus E_1^-}$ is non-degenerate. As in Lemma 3.3, we define

$$E_2^{\pm} := [\mathrm{Ker}(v \mapsto \omega(E_1^{\mp}, v))] \cap E^{\pm}.$$

Then E_2^+ and E_2^- are both ϕ_t-invariant C^∞ line subbundles of TM.

In this case, the underlying geometric structure is

$$g_2 := (X, E^+, E^-, \omega).$$

Denote by G' the isometry group of $\tilde{g}_2$. Then by Lemma 3.6, G' acts transitively on $\tilde{M}$. Fix $x \in \tilde{M}$ and denote by H' the isotropy subgroup of x. Because of the existence of $E_2^{\pm}$, some arguments of § 4.1 pass through without change. In particular, we get that H'_e is isomorphic to 0, $\mathbb{R}$ or $\mathbb{R}^2$ and G'_e is simply connected.

Denote by $\mathfrak{g}'$ and $\mathfrak{h}'$ the Lie algebras of G' and H'. By using the connection corresponding to the case of one positive Lyapunov exponent, we have a similar identification of $\mathfrak{g}'$ and $T_x\tilde{M} \oplus \mathfrak{h}'$ as in § 4.1. To simplify the notation, we identify $T_x\tilde{M}$ with T_xM. If $L_0 := P^+ - P^-$, then we get $L_0 \in \mathfrak{h}'$. So $\mathfrak{h}' \cong \mathbb{R}$ or $\mathbb{R}^2$.

Lemmas 4.1 and 4.2 are also valid here. But the proof of Lemma 4.3 does not pass through in the current case.

5.2. $\dim \mathfrak{h}' = 2$

In this subsection, we suppose that $\mathfrak{h}'$ is of dimension two. So if we define

$$S := P_2^+ - P_1^+ - P_2^- + P_1^-,$$

then $\mathfrak{h}'$ is generated by S and L_0.

Lemma 5.1. *Under the above notation, we have $\Omega^+ \equiv 0$.*

Proof. As before, we consider only the restrictions onto $E^+ \oplus E^-$ of the forms and endomorphisms. Take a dual basis with respect to $\omega_x|_{E_x^+ \oplus E_x^+}$, $(l_2^+, l_1^+, l_2^-, l_1^-)$, such that $l_{1,2}^{\pm} \in E_{1,2}^{\pm}$ and $d\lambda(l_2^+, l_2^-) = -1$. Extend locally these vectors to the sections of $E_{1,2}^{\pm}$, denoted by $\bar{l}_{1,2}^{\pm}$. Then we have

$$\begin{aligned} T(l_1^+, l_2^-) &= P^-[\bar{l}_1^+, \bar{l}_2^-] - P^+[\bar{l}_2^-, \bar{l}_1^+] - [\bar{l}_1^+, \bar{l}_2^-] \\ &= d\lambda(l_1^+, l_2^-) \cdot X_x \\ &= 0. \end{aligned}$$

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Similarly, we have $T(l_1^+, l_1^-) = 0$ and $T(l_2^+, l_2^-) = -X_x$. Thus we get the constants, such that

$$\begin{aligned}[l_1^+, l_1^-] &= aS + bL_0,\\ [l_1^+, l_2^-] &= a'S + b'L_0,\\ [l_2^+, l_2^-] &= X_x + a''S + b''L_0.\end{aligned}$$

As before, we have

$$[l_1^\pm, l_2^\pm] = 0, \qquad [X_x, l_{1,2}^\pm] = \mp l_{1,2}^\pm.$$

By the Jacobi identity of l_1^+, l_2^+ and l_2^-, we get

$$\begin{aligned}0 &= [l_1^+, [l_2^+, l_2^-]] + [l_2^+, [l_2^-, l_1^+]] + [l_2^-, [l_1^+, l_2^+]]\\ &= [l_1^+, X_x + a''S + b''L_0] + [l_2^+, -a'S - b'L_0]\\ &= (1 + a'' - b'')l_1^+ + (a' + b')l_2^+.\end{aligned}$$

So $a' + b' = 0$. By the Jacobi identity of l_2^-, l_1^+ and l_1^-, we get $a' - b' = 0$. So $[l_1^+, l_2^-] = 0$. We deduce that $\Omega^+(l_1^+, l_2^-) = 0$.

Define ψ as in Lemma 4.3. View ψ_x as a matrix in the basis above, then we get $(\psi_x)_{1,2} = \Omega^+(l_1^+, l_2^-) = 0$. Since $\Omega^+ \wedge \Omega^+ = \Omega^+ \wedge \omega = 0$, then $\det \psi = \operatorname{Tr} \psi = 0$. So we get

$$\psi_x = \begin{pmatrix} 0 & 0 & 0 & 0\\ B & 0 & 0 & 0\\ 0 & 0 & 0 & B\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For all $h \in H_e'$, h preserves Ω^+. If $\psi_x \neq 0$, then the matrix of $D_x h$ must have the following form:

$$D_x h = \begin{pmatrix} c & 0 & 0 & 0\\ d & c & 0 & 0\\ 0 & 0 & 1/c & -d/c^2\\ 0 & 0 & 0 & 1/c \end{pmatrix}.$$

But h preserves also the subbundles, $E_{1,2}^\pm$. So $d = 0$. We deduce that $\dim \mathfrak{h}' = 1$, which is a contradiction. So we get $\psi_x = 0$, i.e. $\Omega_x^+ = 0$. Then by homogeneity, $\Omega^+ \equiv 0$. □

With the help of the previous lemma, we can define as in § 4.1 a character χ of $\mathfrak{g}'$. Then by similar calculations as in § 4.2, $\mathfrak{g}'$ is seen to be the same as that of § 4.3, except that $a = 1$ here. But in § 4.3, we have found three elements, α, β and δ, which have eliminated the effect of a on the structure of $\mathfrak{g}'$. So we get here the same G_e' and H_e' as in § 4.3. Thus the same arguments prove the non-existence of this case.

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5.3. $\dim \mathfrak{h}' = 1$

In this subsection, we suppose that $\dim \mathfrak{h}' = 1$. So $\mathfrak{g}' = \mathbb{R}L_0 \oplus T_xM$. Take a basis $(X_x, l_2^+, l_1^+, l_2^-, l_1^-)$ of T_xM, such that $l_{1,2}^{\pm} \in (E_{1,2}^{\pm})_x$ and $\mathrm{d}\lambda(l_2^+, l_2^-) = 1$. Since E_2^+ and E_2^- are both of dimension one, then there exists a well-defined smooth function f, such that

$$\mathrm{d}\lambda|_{E_2^+ \oplus E_2^-} = f \cdot \omega|_{E_2^+ \oplus E_2^-}.$$

Since $\mathrm{d}\lambda$, ω and $E_2^{\pm}$ are all ϕ_t-invariant, then f is also ϕ_t-invariant. We deduce that f is constant. So if we multiply ω by a constant, $(l_2^+, l_1^+, l_2^-, l_1^-)$ can be supposed to be dual with respect to $\omega_x|_{E_x^+ \oplus E_x^-}$. Using the Jacobi identities, we get directly (see § 4.2)

$$[L_0, l_{1,2}^{\pm}] = \pm l_{1,2}^{\pm},$$
$$[X_x, l_{1,2}^{\pm}] = \mp l_{1,2}^{\pm},$$
$$[l_2^+, l_2^-] = -X_x - L_0.$$

The brackets, which have not appeared in these bracket relations, vanish.

Define $\alpha := X_x + L_0$ and

$$\mathfrak{g} \cong \mathbb{R}\alpha \oplus \mathbb{R}l_2^+ \oplus \mathbb{R}l_1^+ \oplus \mathbb{R}l_2^- \oplus \mathbb{R}l_1^-.$$

Then we get

$$\mathfrak{g}' \cong \mathfrak{g} \rtimes \mathbb{R}L_0.$$

Denote by G_e the connected Lie subgroup of G_e' with Lie algebra $\mathfrak{g}$. Since G_e' is simply connected, then so is G_e. Thus by the bracket relations above, we get

$$G_e \cong \mathbb{R}^2 \times \text{Heis},$$

where 'Heis' represents the three-dimensional Heisenberg group. In addition, we have $G_e' \cong G_e \rtimes H_e'$. So G_e is naturally identified to $\tilde{M}$ as follows:

$$\psi : G_e \xrightarrow{\sim} \tilde{M},$$
$$g \mapsto gx.$$

Define $\omega_1 := \psi^* \omega$. Then ω_1 is a left-invariant 2-form on G_e. View $l_{1,2}^{\pm}$ as left-invariant vector fields on G_e. Then $\mathbb{R}l_2^+ \oplus \mathbb{R}l_1^+$ and $\mathbb{R}l_2^- \oplus \mathbb{R}l_1^-$ are identified to $\tilde{E}^+$ and $\tilde{E}^-$. The corresponding flow on G_e is given by the left-invariant field α. So the corresponding geometric structure on G_e is given by

$$g_3 := (\alpha, \mathbb{R}l_1^+ \oplus \mathbb{R}l_2^+, \mathbb{R}l_1^- \oplus \mathbb{R}l_2^-, \omega_1).$$

In addition, by the identification of $\mathfrak{g}'$ with $\mathfrak{h}' \oplus T_xM$, $(l_2^+, l_1^+, l_2^-, l_1^-)$ is dual with respect to ω_1.

For all $c, d \in \mathbb{R}$, define an endomorphism of $\mathfrak{g}$, ρ_d^c, such that

$$\rho_d^c(l_1^{\pm}) = \mathrm{e}^{\pm c} l_1^{\pm},$$
$$\rho_d^c(l_2^{\pm}) = \mathrm{e}^{\pm d} l_2^{\pm},$$
$$\rho_d^c(\alpha) = \alpha.$$

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Then $\{\rho_d^c\}_{c,d\in\mathbb{R}}$ gives a two-parameter family of Lie algebra automorphisms of $\mathfrak{g}$. The corresponding isomorphisms of G_e form a Lie group isomorphic to $\mathbb{R}^2$. Then we observe easily that its action on G_e preserves g_3 and fixes e. We deduce that $\dim(H'_e) \geqslant 2$, which is contradictory to the assumption that $\dim \mathfrak{h}' = 1$.

6. The case of one positive exponent and $\mathbf{d}\boldsymbol{\lambda} \wedge \boldsymbol{\omega} \equiv \mathbf{0}$

6.1. Preparations

In this section, we suppose that ϕ_t has only one positive Lyapunov exponent and $\mathrm{d}\lambda \wedge \omega \equiv 0$. As before, we suppose that this positive exponent is 1. Since $\mathrm{d}\lambda \wedge \omega \equiv 0$, then $\omega|_{E_1^+ \oplus E_1^-} \equiv 0$. So in this case, we have no more the canonically defined subbundles E_2^+ and E_2^- as before (see § 5.1). Here the underlying geometric structure is

$$g_4 := (X, E^+, E^-, \omega).$$

Denote by G' the isometry group of g_4. Fix $x \in \tilde{M}$ and denote by H' the isotropy subgroup of x. Then we have $\tilde{M} \cong G'/H'$.

To simplify the notation, we identify T_xM with $T_x\tilde{M}$. Take a dual basis of $E_x^+ \oplus E_x^-$ with respect to $\omega_x|_{E_x^+ \oplus E_x^-}$, (y^+, l_1^+, l_1^-, y^-), such that $l_1^\pm \in E_1^\pm$ and $\mathrm{d}\lambda(y^+, y^-) = 1$. Denote by φ the section of $\mathrm{End}(E^+ \oplus E^-)$, such that

$$\mathrm{d}\lambda(\cdot,\cdot) = \omega(\varphi\cdot,\cdot).$$

Since $\mathrm{d}\lambda \wedge \omega = 0$, then $\mathrm{Tr}\,\varphi = 0$. So $\exists B \neq 0$, such that

$$\varphi_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For all $h \in H'_e$, D_xh preserves $\mathrm{d}\lambda_x$. So in the basis above, the matrix of D_xh must be of the following form:

$$D_xh = \begin{pmatrix} c & 0 & 0 & 0 \\ d & c & 0 & 0 \\ 0 & 0 & 1/c & -d/c^2 \\ 0 & 0 & 0 & 1/c \end{pmatrix}.$$

So H'_e is abelian and isomorphic to 0, $\mathbb{R}$ or $\mathbb{R}^2$. Then as in § 4.1, G'_e is seen to be simply connected.

Denote by $\mathfrak{g}'$ and $\mathfrak{h}'$ the Lie algebras of G' and H'. Then we get a similar identification of $\mathfrak{g}'$ and $T_xM \oplus \mathfrak{h}'$ as in § 5.1. In particular, if $L_0 := P^+ - P^-$, then $L_0 \in \mathfrak{h}'$. We deduce that $\mathfrak{h}' \cong \mathbb{R}$ or $\mathbb{R}^2$.

Lemma 4.2 is still valid here. But the proofs of Lemmas 4.3 and 5.1 do not pass through in the current case.

6.2. $\dim \mathfrak{h}' = 1$

In this subsection, we suppose that $\dim \mathfrak{h}' = 1$. Then $\mathfrak{g}' = \mathbb{R}L_0 \oplus T_xM$. By the Jacobi identities of $\mathfrak{g}'$, we get the following relations with respect to the dual basis in the previous subsection:

$$[L_0, y^{\pm}] = \pm y^{\pm}, \qquad [L_0, l_1^{\pm}] = \pm l_1^{\pm},$$
$$[X_x, y^{\pm}] = \mp y^{\pm}, \qquad [X_x, l_1^{\pm}] = \mp l_1^{\pm},$$
$$[y^+, y^-] = -X_x - L_0.$$

The brackets, which have not appeared in these bracket relations, vanish.

As in § 5.3, we define $\alpha := X_x + L_0$ and

$$\mathfrak{g} := \mathbb{R}\alpha \oplus \mathbb{R}y^+ \oplus \mathbb{R}y^- \oplus \mathbb{R}l_1^+ \oplus \mathbb{R}l_1^-.$$

Thus $\mathfrak{g}' \cong \mathfrak{g} \rtimes \mathbb{R}L_0$. Denote by G_e the connected Lie subgroup of G'_e with Lie algebra $\mathfrak{g}$. Then G_e is naturally identified to $\tilde{M}$ under ψ (see § 5.3) and the corresponding geometric structure on G_e is given by

$$g_5 := (\alpha, \mathbb{R}l_1^+ \oplus \mathbb{R}y^+, \mathbb{R}l_1^- \oplus \mathbb{R}y^-, \psi^*\omega).$$

In addition, (y^+, l_1^+, l_1^-, y^-) is dual with respect to $\psi^*\omega$.

For all $c, d \in \mathbb{R}$, there is a unique Lie algebra automorphism of $\mathfrak{g}$, ρ_d^c, such that

$$\rho_d^c(y^{\pm}) = \mathrm{e}^{\pm c}(y^{\pm} \pm d \cdot l_1^{\pm}),$$
$$\rho_1^c(l_1^{\pm}) = \mathrm{e}^{\pm c} \cdot l_1^{\pm},$$
$$\rho_d^c(\alpha) = \alpha.$$

Their corresponding isomorphisms of G_e forms a Lie group isomorphic to $\mathbb{R}^2$. Then we observe easily that its action on G_e preserves g_5 and fixes e. So $\dim(H'_e) \geqslant 2$, which is a contradiction.

6.3. $\dim \mathfrak{h}' = 2$

In this subsection, we suppose that $\dim \mathfrak{h}' = 2$.

Lemma 6.1. *$\exists c < 2$, such that $\Omega^+ = c \cdot \mathrm{d}\lambda$.*

Proof. Let φ and ψ be the same endomorphisms as in §§ 4.1 and 6.1. Take $l_1^+ \in (E_1^+)_x$. Thus $\varphi_x(l_1^+) = 0$.

Since $\Omega^+ \wedge \Omega^+ = 0$, then $\det \psi_x = 0$. So if $\psi_x(l_1^+) \neq 0$, then there exists $y^+ \neq 0$, such that $\psi_x(y^+) = 0$. Extend y^+ and l_1^+ to a dual basis, (y^+, l_1^+, z^-, y^-). Then in this basis, we get

$$\varphi_x = \begin{pmatrix} a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi_x = \begin{pmatrix} 0 & A & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \end{pmatrix}.$$

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Since $\Omega^+ \wedge \omega = 0$ and $\mathrm{d}\lambda \wedge \omega = 0$, then $\operatorname{Tr}\varphi = \operatorname{Tr}\psi = 0$, i.e. $a = B = 0$. Since $\mathrm{d}\lambda \wedge \Omega^+ = 0$, then $b \cdot A = 0$. So we get $A = 0$, i.e. $\psi_x(l_1^+) = 0$, which is a contradiction.

We deduce that $\psi_x(l_1^+) = 0$. Extend l_1^+ to a dual basis $(y_1^+, l_1^+, y_1^-, z_1^-)$. Thus in this basis, φ_x and ψ_x are proportional. Then by homogeneity, we deduce the existence of c, such that $\Omega^+ = c \cdot \mathrm{d}\lambda$.

Denote by λ the *canonical* 1-*form* of ϕ_t and by J the section of $\operatorname{End}(TM)$, such that

$$J(X) = 0, \qquad J(u^\pm) = \pm u^\pm.$$

We introduce another ϕ_t-invariant connection

$$\bar{\nabla} := \nabla - \tfrac{1}{2} c \lambda \otimes J.$$

Thus

$$\bar{\nabla}_X Y^\pm = [X, Y^\pm] \pm (1 - \tfrac{1}{2}c) Y^\pm.$$

Denote by $\bar{\Omega}^+$ the curvature form of the induced connection $\bar{\nabla}^+$ of $\bar{\nabla}$ on $\wedge^2 E^+$. Then from the definition of $\bar{\nabla}$, we easily get

$$\bar{\Omega}^+ \equiv 0.$$

Fix a nowhere-vanishing section ω^+ of $\wedge^2 E^+$. Then with respect to ω^+, the connection form of $\bar{\nabla}^+$ is given by

$$\bar{\nabla}\omega^+ = \bar{\beta}^+(\cdot)\omega^+.$$

So we have $\mathrm{d}\bar{\beta}^+ = \bar{\Omega}^+ = 0$.

Suppose that $c \geqslant 2$. Then $1 - \frac{1}{2}c \leqslant 0$. Define

$$\alpha_t := \frac{1}{t} \int_t^0 \phi_s^* \bar{\beta}^+ \, \mathrm{d}s.$$

By the arguments in § 4.4.2 of [**6**], if $t \ll 0$, then we have

$$\alpha_t(X) > 0.$$

Thus fix $t \ll 0$ and denote this α_t by α. Since $\bar{\beta}^+$ is closed, then so is α. Define $Y := X/\alpha(X)$. By Lemma 2.2, the flow of Y, ϕ_t^Y, is also a *geometric* Anosov flow with smooth distributions.

Since λ is ϕ_t-invariant, then

$$0 = \mathcal{L}_X \lambda = i_X \, \mathrm{d}\lambda = \alpha(X)(i_Y \, \mathrm{d}\lambda).$$

So

$$\mathcal{L}_Y \, \mathrm{d}\lambda = i_Y \, \mathrm{d}(\mathrm{d}\lambda) + \mathrm{d} i_Y \, \mathrm{d}\lambda = 0,$$

i.e. $\mathrm{d}\lambda$ is ϕ_t^Y-invariant. Since α is easily seen to be the *canonical* 1-*form* of ϕ_t^Y and $\mathrm{d}\alpha = 0$, then $\operatorname{rank}(\phi_t^Y) = 0$. Thus by § 3.1, ϕ_t^Y is finitely covered by the suspension of a hyperbolic automorphism of $\mathbb{T}^4$, which is given by a hyperbolic matrix in $GL(4, \mathbb{Z})$. Then by a direct calculation, using the Jordan form of this matrix, λ is seen to be closed (see [**10**] for the details). We deduce that $\operatorname{rank}(\phi_t) = 0$, which is a contradiction. □

We can see that Lemma 3.5 is also true for $\bar{\nabla}$ defined in the previous lemma. So, in particular, we get

$$\bar{\nabla}\bar{R} = 0, \qquad \bar{\nabla}\bar{T} = 0, \qquad \bar{T}(E_a, E_b) \subseteq E_{a+b},$$

and if $a + b \neq 0$, $\bar{R}(E_a, E_b) = 0$, where $\bar{T}$ and $\bar{R}$ are the torsion and curvature tensors of $\bar{\nabla}$. Since the $\bar{\nabla}$-geodesics tangent to E^+ or E^- are also complete, then by Lemma A.1 in the appendix, $\bar{\nabla}$ is complete. Thus as in § 4.1, we get the following identification via $\bar{\nabla}$:

$$\begin{aligned} &\mathfrak{g}' \overset{\sim}{\mapsto} T_x\tilde{M} \oplus \mathfrak{h}', \\ &u \to (Y^u(x), (\tilde{\bar{\nabla}}_{Y^u} - \mathcal{L}_{Y^u})|_x). \end{aligned}$$

Since $\bar{\Omega}^+ \equiv 0$, then we can define a character $\bar{\chi}$ of $\mathfrak{g}'$ as in § 4.1. Thus $\bar{\chi}^{-1}(0)$ is an ideal of $\mathfrak{g}'$, denoted again by $\mathfrak{g}$. By the same type of arguments as before, we easily get

$$\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathfrak{sl}(2, \mathbb{R}).$$

Denote by G_e the connected Lie subgroup of G'_e with Lie algebra $\mathfrak{g}$ and define $H_e := H'_e \cap G_e$. Since G'_e is simply connected, then so is G_e (see § 4.1 of [**5**]). Thus by some direct calculations, G_e and H_e can be realized as follows,

$$G_e \cong \mathbb{R}^3 \rtimes \widetilde{SO_0(1,2)},$$

where $SO_0(1, 2)$ is the identity component of the isometry group of the quadratic form: $-\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. The semi-direct product is given by the composition of the projection of $\widetilde{SO_0(1,2)}$ onto $SO_0(1, 2)$ and the linear action of $SO_0(1, 2)$ on $\mathbb{R}^3$. Let $((0, 0, 1), 0) \in \mathbb{R}^3 \rtimes \mathfrak{so}(1, 2)$. Then H_e is just the 1-parameter subgroup generated by this vector, denoted also by $\mathbb{R}$.

Since $\bar{\Omega}^+ \equiv 0$, then the same argument as in § 4.2 of [**5**] works in our case, if we replace the metric entropy there by $2(1 - \frac{1}{2}c)$. Thus the general argument of § 5 of [**5**] gives a discrete subgroup of G_e, acting freely, properly and co-compactly on G_e/H_e. Now we finish the proof by proving the following lemma.

Lemma 6.2. $\mathbb{R}^3 \rtimes \widetilde{SO_0(1,2)}$ *admits no discrete subgroup, which acts properly, freely and co-compactly on* $(\mathbb{R}^3 \rtimes \widetilde{SO_0(1,2)})/\mathbb{R}$.

Proof. Recall that $\mathbb{R}$ denotes H_e and G_e denotes $\mathbb{R}^3 \rtimes \widetilde{SO_0(1,2)}$. Suppose the existence of a subgroup Γ satisfying the conditions in the lemma. Denote by $\bar{\Gamma}$ the Zariski closure of Γ in G_e. (Here the Zariski topology of G_e means the lifted topology of the Zariski topology of $\mathbb{R}^3 \rtimes SO_0(1, 2)$ by the canonical projection.)

If Γ is solvable, then $\bar{\Gamma}$ is also solvable. Then by [**18**], there exists a connected closed subgroup H of $\bar{\Gamma}$, such that $\Gamma \subseteq H$ and H/Γ is compact. Let $\mathrm{cd}(\cdot)$ denote the co-homological dimension of a group. Since Γ acts co-compactly on $G_e/\mathbb{R}$, then $\mathrm{cd}(\Gamma) = 5$. We deduce that $\mathrm{cd}(H) = 5$. So H is a closed solvable subgroup of G_e of dimension five.

Then the single possibility is $\mathbb{R}^3 \rtimes AN$ (where KAN is the Iwasawa decomposition of $SO_0(1,2)$). But $\mathbb{R}^3 \rtimes AN$ is not unimodular. So it has no co-compact lattice. We deduce that Γ is not solvable. Then $\bar{\Gamma}$ must contain $SO_0(1,2)$.

Since Γ acts co-compactly on $G_e/\mathbb{R}$, then $SO_0(1,2) \subsetneq \bar{\Gamma}$. We deduce that $\bar{\Gamma} \cap \mathbb{R}^3 \neq 0$. Since the representation of $SO_0(1,2)$ on $\mathbb{R}^3$ is irreducible, then $\bar{\Gamma}$ must be G_e, i.e. Γ is Zariski-dense in G_e. Let Δ be the projection of Γ into $SO_0(1,2)$, then by [**18**], Δ is discrete in $SO_0(1,2)$. We deduce that $\Gamma \cap \mathbb{R}^3 \neq 0$. Since the semi-direct product is given by an irreducible representation, $\Gamma \cap \mathbb{R}^3$ is in fact co-compact in $\mathbb{R}^3$.

Since Γ acts properly on $G_e/\mathbb{R}$, then $\Gamma \cap \mathbb{R}^3$ acts properly on $\mathbb{R}^3/\mathbb{R}$ which is a closed subset of $G_e/\mathbb{R}$. We deduce that $\mathbb{R}^3$ acts also properly on $\mathbb{R}^3/\mathbb{R}$. But it is absurd. □

Appendix A.

At first, we prove the following elementary lemma, which is used in the proof of Lemma 3.6.

Lemma A.1. *Let ∇ be a smooth linear connection on a connected manifold M of dimension n. Let $X_1, \dots, X_k$ be complete fields on M and $E_1, \dots, E_l$ be smooth distributions on M, such that*

(1) $\nabla X_i = 0$, $\forall 1 \leqslant i \leqslant k$, $\nabla E_j \subseteq E_j$, $\forall 1 \leqslant j \leqslant l$,

(2) $TM = \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_k \oplus E_1 \oplus \cdots \oplus E_l$,

(3) $\nabla R = 0$, $\nabla T = 0$,

(4) $\forall 1 \leqslant j \leqslant l$, *the geodesics of* ∇, *tangent to* E_j, *are defined on* $\mathbb{R}$,

then ∇ is complete.

Proof. For the terminology below, our reference is Volume I of [**15**]. For all $1 \leqslant i \leqslant k$, since X_i is complete and parallel, then any geodesic tangent to $\mathbb{R}X_i$ is defined on $\mathbb{R}$. So without any loss of generality, we suppose that $k = 0$.

Let $\mathcal{F}(M)$ be the frame bundle of M and π the projection of $\mathcal{F}(M)$ onto M. The linear connection ∇ gives a horizontal distribution $\mathcal{H}$ on $\mathcal{F}M$ and $\mathcal{F}M$ is foliated by holonomy subbundles. $\mathcal{H}$ is tangent to each holonomy subbundle, then so is any standard horizontal field. For all $u \in \mathcal{F}M$, denote by $P(u)$ the holonomy subbundle containing u. The induced fields on $P(u)$ of the standard horizontal fields of $\mathcal{F}M$ are also called standard horizontal. By [**15**], ∇ is complete, if and only if for all $x \in M$, $\exists u \in \pi^{-1}(x)$, such that the standard horizontal fields of $P(u)$ are all complete.

Take $x \in M$ and $u \in \pi^{-1}(x)$, such that

$$u = (v_1^1, \dots, v_{i_1}^1, \dots, v_1^l, \dots, v_{i_l}^l),$$

where $\{v_1^j, \dots, v_{i_j}^j\}$ is a basis of $E_j(x)$, $\forall 1 \leqslant j \leqslant l$. For all $\xi \in \mathbb{R}^n$, the standard horizontal field on $P(u)$ corresponding to ξ is denoted by $B^u(\xi)$ and the canonical basis of $\mathbb{R}^n$ is denoted by $(e_1, \ldots, e_n)$. Take $v \in P(u)$. Because of assumption (1), v has the same form as u. Then for all $1 \leqslant m \leqslant n$, the integral curve of $B^u(e_m)$, beginning at v, is just the horizontal lift, beginning at v, of the geodesic tangent to $Pr_m(v)$. By assumption (4), such a geodesic is defined on $\mathbb{R}$. We deduce that $B^u(e_m)$ is complete.

Fix a basis of the holonomy algebra of ∇ and denote the corresponding vertical fields of $P(u)$ by $\{V_1, \ldots, V_s\}$. By assumption (3), the fields

$$\{V_1, \ldots, V_s, B^u(e_1), \ldots, B^u(e_n)\}$$

generate a Lie algebra. Since these fields are all complete, then this Lie algebra must be induced by the smooth action on $P(u)$ of a simply connected Lie group. Thus for all $\xi \in \mathbb{R}^n$, the field $B^u(\xi)$ $(= \sum_{1 \leqslant i \leqslant n} \xi_i B^u(e_i))$ is complete. We deduce that ∇ is complete. □

The following lemma is used in the proof of Lemma 2.2.

Lemma A.2. *Let ϕ_t be an Anosov flow with C^∞ distributions on a closed manifold M. If f is a smooth positive function on M and the flow of fX (X is the generator of ϕ_t) has also C^∞ distributions, then there exists a C^∞ 1-form α on M, such that $\mathcal{L}_X \,\mathrm{d}\alpha = 0$ and $f = 1/\alpha(X)$. Conversely, if α is a C^∞ 1-form on M, such that $\mathcal{L}_X \,\mathrm{d}\alpha = 0$ and $\alpha(X) > 0$, then the flow of $X/\alpha(X)$ has also C^∞ distributions.*

Proof. Recall at first that a C^∞ time change of an Anosov flow is also Anosov. Let fX be a time change of ϕ_t with smooth distributions. Denote by ϕ_t^{fX} the flow of fX and by λ_1 its *canonical* 1-*form*. Then $\lambda_1(fX) = 1$, i.e. $f = 1/\lambda_1(X)$. Since λ_1 is ϕ_t^{fX}-invariant, then $i_{fX} \,\mathrm{d}\lambda_1 = 0$. So $i_X \,\mathrm{d}\lambda_1 = 0$. We deduce that $\mathcal{L}_X \,\mathrm{d}\lambda_1 = 0$.

If hX is a smooth time change of ϕ_t, then its strong stable distribution is given by (see Lemma 1.2 of [**8**])

$$E^-_{hX} = \{Y^- + \beta(Y^-)X \mid Y^- \in E^-_X\},$$

where E^-_{hX} denotes the strong stable distribution of hX and β is a C^0 section of $(E^-_X)^*$, such that

$$\mathcal{L}_X(h^{-1}\beta) = h^{-2} \,\mathrm{d}h. \qquad (*)$$

Denote by λ the *canonical* 1-*form* of ϕ_t. If α is a smooth 1-form on M, such that $\mathcal{L}_X \,\mathrm{d}\alpha = 0$ and $\alpha(X) > 0$, then by a simple calculation, $-(\alpha - \lambda)/\alpha(X)$ satisfies the previous Equation $(*)$ about β with $h := 1/\alpha(X)$. So $E^-_{X/\alpha(X)}$ is smooth. Similarly, $E^+_{X/\alpha(X)}$ is also smooth. □

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References

1. A. M. Amores, Vector fields of a finite type G-structure, *J. Diff. Geom.* **14** (1979), 1–6.
2. V. D. Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Inst. Steklov* **90** (1967), 1–235.

3. T. Barbot, Caractérisation des flots d'Anosov en dimension 3 par leurs feuilletages faibles, *Ergod. Theory Dynam. Sys.* **15** (1995), 247–270.
4. Y. Benoist and F. Labourie, Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables, *Invent. Math.* **111** (1993), 285–308.
5. Y. Benoist, P. Foulon and F. Labourie, Flots d'Anosov à distributions de Liapounov différentiables, I, *Annls Inst. H. Poincaré* **53** (1990), 395–412.
6. Y. Benoist, P. Foulon and F. Labourie, Flots d'Anosov à distributions stable et instable différentiables, *J. Am. Math. Soc.* **5** (1992), 33–74.
7. A. Candel and R. Quiroga-Barranco, Gromov's centralizer theorem, *Geom. Dedicata* **100** (2003), 123–155.
8. R. de la Llave, J. Marco and R. Moriyon, Canonical perturbation theory of Anosov systems and regularity results for Livsic cohomology equation, *Ann. Math.* **123**(3) (1986), 537–612.
9. É. Ghys, Flots d'Anosov dont les feuilletages stables sont différentiables, *Annls Scient. Éc. Norm. Sup.* **20** (1987), 251–270.
10. Y. Fang, A remark about hyperbolic infranilautomorphisms, *C. R. Acad. Sci. Paris Sér. I* **336**(9) (2003), 769–772.
11. R. Feres and A. Katok, Invariant tensor fields of dynamical systems with pinched Lyapunov exponents and rigidity of geodesic flows, *Ergod. Theory Dynam. Sys.* **9** (1989), 427–432.
12. B. Hasselblatt and A. Katok, *Introduction to the modern theory of dynamical systems* (Cambridge University Press, 1995).
13. S. Hurder and A. Katok, Differentiability, rigidity and Godbillon–Vey classes for Anosov flows, *Publ. Math. IHES* **72** (1990), 5–61.
14. M. Kanai, Geodesic flows of negatively curved manifolds with smooth stable and instable foliations, *Ergod. Theory Dynam. Sys.* **8** (1988), 215–240.
15. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vols I, II (Wiley Interscience, New York and London, 1963).
16. A. Lichnérowicz, *Théorie globale des connexions et des groupes d'holonomie* (Edizioni Cremonese, Roma, 1962).
17. J. F. Plante, Anosov flows, *Am. J. Math.* **94** (1972), 729–754.
18. M. S. Raghunathan, *Discrete subgroups of Lie groups* (Springer, 1972).
19. P. Tomter, Anosov flows on infra-homogeneous spaces, in *Proc. Symp. Pure Mathematics, Global Analysis*, Vol. XIV, pp. 299–327 (American Mathematical Society, 1970).