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CR-HARMONIC MAPS

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Abstract. We develop the notion of renormalized energy in Cauchy–Riemann (CR) geometry for maps from a strictly pseudoconvex pseudo-Hermitian manifold to a Riemannian manifold. This energy is a CR invariant functional whose critical points, which we call CR-harmonic maps, satisfy a CR covariant partial differential equation. The corresponding operator coincides on functions with the CR Paneitz operator.

CONTENTS

§1. Introduction

Let (M,g) and (N,h) be two Riemannian manifolds. The *Dirichlet energy* of a map $\varphi:(M,g)\to(N,h)$ is defined as

$$E(\varphi)=\frac{1}{2}\int_M \|T\varphi\|^2_{g,h}d\mathrm{vol}_g.$$

When $\dim M=2$, the energy is conformally invariant with respect to g. This is of considerable usefulness, for example, to construct conformal minimal immersions of Riemann surfaces [Mil79]. However, in higher dimension, the energy is no longer conformally invariant.

Critical points of a functional are solutions to a partial differential equation called the *Euler–Lagrange equation* of the functional; in other words, they form the kernel of a certain differential operator. In our case, the critical points of the Dirichlet energy are called *harmonic maps*, and harmonic *functions* $\varphi:(M,g)\to(\mathbb{R},\mathrm{eucl})$ coincide with the kernel of the Laplacian.

In a recent work, Bérard has shown the existence, given two Riemannian manifolds (M,g) and (N,h), with M of even dimension n, of a functional $\mathscr{E}^n_g$ on $C^\infty(M,N)$, conformally invariant with respect to g, and equal to the usual energy when $n=2$ [Bér13]. This functional is called *renormalized energy*, and its critical points are called *conformal-harmonic maps*. Conformal-harmonic maps generalize harmonic maps; moreover, when $n=4$ and $N=\mathbb{R}$, the induced operator coincides with the Paneitz operator.

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We develop here the notions of Cauchy–Riemann (CR)-harmonicity and renormalized energy in CR geometry. CR-harmonic maps also generalize CR-holomorphic maps, which are notoriously hard to come by. When $\dim M = 3$ and $N = \mathbb{R}$, the induced operator coincides with the CR Paneitz operator. This generalizes the recent work of Marugame [Mar18]. Another extension of the CR Paneitz operator to maps has been proposed by Chong, Dong, Ren and Yang [CDRY19]. The main result is the following, which summarizes Proposition 4.1 and Theorem 3.3.

THEOREM 1.1. *Let (M^{2n+1}, H, J, θ) be a compact strictly pseudoconvex pseudo-Hermitian manifold and (N, h) be a Riemannian manifold. There exists a functional F_n on $C^\infty(M, N)$ which is a CR invariant, that is, conformally invariant with respect to θ. For $\varphi \in C^\infty(M, N)$, it reads*

$$F_n(\varphi) = \frac{(-1)^{n+1}}{2n!^2} \int_M \left\langle (\delta_b^{\theta,h} \nabla^{\varphi^* h})^{n-1} \delta_b^{\theta,h} T\varphi, \delta_b^{\theta,h} T\varphi \right\rangle_h \theta \wedge d\theta^n$$
$$+ \textit{ lower order terms (in derivatives of } \varphi\textit{)},$$

where $\delta_b^{\theta,h}$ is the Webster divergence on $\Omega^1(M) \otimes \varphi^* TN$.

The Euler–Lagrange equation of F_n is a partial differential equation of order $2n+2$, which is itself CR covariant. For $\varphi \in C^\infty(M, N)$, it reads

$$0 = \frac{(-1)^n}{n!} (\delta_b^{\theta,h} \nabla^{\varphi^* h})^n \delta_b^{\theta,h} T\varphi + \textit{lower order terms (in derivatives of } \varphi\textit{)}.$$

Moreover, we provide explicit computations of P_1 and F_1 in Theorems 3.11 and 4.4, respectively.

The paper is organized as follows: in Section 2, we recall notions of asymptotically complex hyperbolic (ACH) geometry. In Section 3, we adapt the classical construction by Graham, Jenne, Mason and Sparling to obtain a CR Paneitz operator acting on maps, and we define CR-harmonicity [GJMS92]. We also provide an explicit computation of the operator in dimension 3. In Section 4, we develop the corresponding notion of renormalized energy. Section 5 presents computations in higher dimension, which do not allow for an explicit expression of the operator. Finally, Section 6 gives a correspondence between CR-harmonic maps on a pseudo-Hermitian manifold and conformal-harmonic maps on its Fefferman bundle.

We adopt the following convention: small Greek letters will denote indices in $\{1, \ldots, n\}$; capital Greek letters in $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$; small Latin letters in $\{0, 1, \ldots, n\}$; capital Latin letters in $\{0, 1, \ldots, n, \overline{0}, \overline{1}, \ldots, \overline{n}\}$. Moreover, we use the Einstein summation convention everywhere.

§2. ACHE manifolds

Asymptotically hyperbolic (AH for short) manifolds are manifolds which admit a *conformal infinity*, that is to say, a boundary equipped with a conformal structure which is, roughly speaking, a generalization of the standard conformal sphere seen as the boundary of the Poincaré disk. Reciprocally, every compact conformal manifold can be filled with an AH manifold X^{n+1} whose metric is Einstein, thus called *AH-Einstein* or *AHE*, when n is odd. When n is even, a conformally invariant obstruction to the existence of an AHE metric that is smooth up to the boundary appears [FG85, GH05]. Recently, Gursky and

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Székelyhidi have announced that an AHE metric exists locally for all $n \geqslant 3$ [GS17]. This approach provides a correspondence between a Riemannian structure on a manifold and a conformal structure on its boundary. Information on the conformal infinity can thus be read on the AHE metric.

The complex counterparts of AH manifolds, ACH manifolds, have been introduced by Epstein, Melrose and Mendoza [EMM91]. They generalize the construction by Fefferman, Cheng and Yau, of *asymptotically Bergman metrics*, which are Kähler–Einstein metrics on bounded strictly pseudoconvex domains of $\mathbb{C}^{n+1}$, which are asymptotic to the CR structure of the boundary [Fef76, CY80]. The regularity of these metrics near the boundary has been studied by Lee and Melrose [LM82]. To an ACH manifold thus corresponds a *CR infinity*. For example, the CR infinity of the complex hyperbolic space $\mathbb{C}\mathbf{H}^{n+1}$ is $\mathbb{S}^{2n+1}$ endowed with its standard CR structure.

Because of the anisotropy of their structure, pseudo-Hermitian manifolds of odd dimension N often behave, *mutatis mutandis*, like Riemannian manifolds of dimension $N+1$. They are sometimes said to have *homogeneous dimension* $N+1$ [JL89]. In particular, ACH manifolds have been known to share similarities with the "n even" real case. The asymptotic development of ACH-Einstein (ACHE) and ACH-Kähler–Einstein metrics has been extensively studied by O. Biquard, M. Herzlich and Y. Matsumoto, and obstructions to smoothness have been identified [Biq00, BH05, Mat14].

Let us consider the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ endowed with its standard contact form

$$\theta_0 = \frac{i}{4}\left(z_j d\overline{z}^j - \overline{z}_j dz^j\right)|_{\mathbb{S}^{2n+1}}.$$

Let $\gamma_0 = d\theta_0(\cdot, i\cdot)$ be the induced metric on the contact distribution $\ker \theta_0$. The *Bergman metric* on the ball $\mathbb{B}^{2n+2}$ is given in polar coordinates by

$$g_0 = dt^2 + 4\sinh^2(t)\theta_0^2 + 4\sinh^2\left(\frac{t}{2}\right)\gamma_0.$$

This metric is Kähler and has constant holomorphic sectional curvature -1. The space $(\mathbb{B}^{2n+2}, g_0)$ is known as the *complex hyperbolic space* and is denoted by $\mathbb{C}\mathbf{H}^{n+1}$.

More generally, let (M, H, J) be a $(2n+1)$-dimensional orientable compact strictly pseudoconvex CR manifold. Namely, H is an orientable hyperplane distribution in TM and J is a complex structure on H. Let θ be a compatible positive contact form and $\gamma = d\theta(\cdot, J\cdot)$ be the induced metric. Let R be the Reeb field. Let ∇^θ be the Tanaka–Webster connection of (M, H, J, θ) and τ be the pseudo-Hermitian torsion. Let $\overline{X} = [0, \varepsilon) \times M$, $\pi : \overline{X} \to M$ be the natural projection and r be the coordinate on $[0, \varepsilon)$. Let X be the interior of $\overline{X}$. Let g_0 be the metric on X

$$g_0 = \frac{dr^2}{r^2} + \frac{\theta^2}{r^2} + \frac{\gamma}{r}.$$

A function $s \in C^\infty(\overline{X}, \mathbb{R}_+)$ is called *boundary defining* if $s > 0$ on X, $s = 0$ and $ds \neq 0$ on $\{0\} \times M$. Equivalently, $s = e^f r$ for some f in $C^\infty(\overline{X}, \mathbb{R})$. A conformal change of the boundary defining function corresponds to a conformal change of the contact form. Indeed, let us consider g_0 as $g_0(r, \theta)$, then, for f in $C^\infty(\overline{X}, \mathbb{R})$,

$$g_0(e^f r, \theta) = g_0(r, e^{-f|_M}\theta).$$

We define an order O_e adapted to g_0. A normal basis with respect to g_0 is $e = (r\partial_r, rR, r^{1/2}T_A)$, where (T_A) is an orthonormal basis for γ, considered as a Hermitian metric. Its dual basis is $e^* = (r^{-1}dr, r^{-1}\theta, r^{-1/2}\theta^\alpha, r^{-1/2}\theta^{\overline{\alpha}})$. The order O_e takes e and e^* for reference. Thus, we have, for example,

$$\gamma = \theta^\alpha \circ \theta^{\overline{\alpha}} = r\left(r^{-1/2}\theta^\alpha\right) \circ \left(r^{-1/2}\theta^{\overline{\alpha}}\right) = O_e(r),$$

where $\lambda \circ \mu := \lambda \otimes \mu + \mu \otimes \lambda$.

DEFINITION 2.1. [Biq00] A metric g on X is said to be *ACH* if $g - g_0 = o_e(1)$. The CR manifold (M, H, J) is then called the *CR infinity* of (X, g).

EXAMPLE 2.2. For $\lambda > 0$,

$$g = \frac{dr^2}{r^2} + \frac{(1-\lambda^2 r^2)^2}{r^2}\theta^2 + \frac{(1-\lambda r)^2}{r}\gamma$$

is an ACH metric on X. Moreover, if (M, H, J, θ) is *Einstein*, that is, pseudo-Einstein with vanishing pseudo-Hermitian torsion, with $\mathrm{Ric}_W(J,\theta) = 2(n+1)\lambda\gamma$, then g is an Einstein metric, satisfying

$$\mathrm{Ric}(g) = -\frac{n+2}{2}g.$$

Indeed, a complex structure $\tilde{J}$ compatible with g on X is given by $\tilde{J}|_{H\times\{r\}} = J$ and $\tilde{J}\partial_r = -R/(1-\lambda^2r^2)$, that is, $dr \circ \tilde{J} = (1-\lambda^2r^2)\theta$. Let $\theta^0 := (1/\sqrt{2})(1/(1-\lambda^2r^2)dr - i\theta)$ and let $\sigma := \theta^0 \wedge \theta^1 \wedge \cdots \wedge \theta^n$ be a section of the canonical bundle.

Then

$$d\sigma = \frac{i}{\sqrt{2}}d\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n - \theta^0 \wedge d\theta^1 \wedge \cdots \wedge \theta^n + \cdots + (-1)^n\theta^0 \wedge \theta^1 \wedge \cdots \wedge d\theta^n,$$

where the first term vanishes, and since $\tau = 0$, $d\theta^\alpha = \theta^\beta \wedge \omega_\beta^\alpha$; hence,

$$d\sigma = -\omega_\alpha^\alpha \wedge \sigma.$$

The curvature form of σ, in the sense of [BH05], is hence given by $-d\omega_\alpha^\alpha = -\mathscr{R}^{\theta}{}_{\alpha}{}^{\rho}{}_{\rho\overline{\alpha}}\theta^\alpha \wedge \theta^{\overline{\alpha}} = 2i(n+1)\lambda d\theta$. Moreover,

$$\sigma \wedge \overline{\sigma} = \frac{(-1)^{n+1} i r^{n+2}}{(1-\lambda^2r^2)^2(1-\lambda r)^{2n}}(r^{-1}dr) \wedge ((1-\lambda^2r^2)r^{-1}\theta) \wedge \left((1-\lambda r)r^{-1/2}\theta^1\right) \wedge \cdots \wedge \left((1-\lambda r)r^{-1/2}\theta^{\overline{n}}\right).$$

Consequently,

$$|\sigma|_g^2 = \frac{r^{n+2}}{(1-\lambda^2r^2)^2(1-\lambda r)^{2n}};$$

hence, $\ln|\sigma|_g^2 = (n+2)\ln r - 2\ln(1+\lambda r) - (2n+2)\ln(1-\lambda r)$. We have

$$\partial r = \tfrac{1}{2}(dr - i(1-\lambda^2r^2)\theta);$$

hence,

$$i\overline{\partial}\partial r = -\lambda^2 r dr \wedge \theta + \frac{1-\lambda^2r^2}{2}d\theta \qquad \text{and} \qquad i\overline{\partial}r \wedge \partial r = \frac{1-\lambda^2r^2}{2}dr \wedge \theta.$$

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The Ricci form of g is then given by

$$\begin{aligned}\rho_g &= -i\partial\overline{\partial}\ln|\sigma|_g^2 + id\omega_\alpha^\alpha \\ &= -i\partial\overline{\partial}\ln|\sigma|_g^2 + 2(n+1)\lambda d\theta \\ &= \frac{n+2}{2}\left(\frac{1-\lambda^2r^2}{r^2}dr\wedge\theta - \frac{(1-\lambda r)^2}{r}d\theta\right).\end{aligned}$$

With this example in mind, one may ask if there is, in general, an ACHE metric on X. Contrarily to the theorem of Cheng–Yau for domains of $\mathbb{C}^{n+1}$, such a metric may not exist in general [CY80]. Nevertheless, there are *formally determined* almost ACHE metrics in the following sense.

DEFINITION 2.3. In any asymptotic development $\sum_k a_k(p)r^k$, the term a_k, seen as a function on M, is called *formally determined* if it is a universal polynomial on a finite jet of the CR structure at $p\in M$ only.

THEOREM 2.4. [Mat14] *There is an ACH metric g_E on X, which is Einstein up to order $n+1$, that is,*

$$\mathrm{Ric}(g_E) = -\frac{n+2}{2}g_E + O_e(r^{n+1}),$$

where O_e denotes the order with respect to any basis e orthonormal for g_0. The metric g_E is formally determined modulo $O_e(r^{n+1})$. Moreover, we have the asymptotic development

$$g_E = g_0 + \Phi + O_e(r^{3/2}),$$

where

$$\Phi = -2\mathrm{Sch}_W(J,\theta) + 2\gamma(J\tau\cdot,\cdot),$$

where

$$\mathrm{Sch}_W(J,\theta) = \frac{1}{n+2}\left(\mathrm{Ric}_W(J,\theta) - \frac{\mathrm{Scal}_W(J,\theta)}{2(n+1)}\gamma\right)$$

is the CR Schouten tensor.

REMARK 2.5. Note that $\Phi = O_e(r)$.

We thus have a formally determined *almost* ACHE metric on X. A more convenient metric for our study would be an almost ACH-*Kähler*–Einstein metric on X. We have at hand the following results.

PROPOSITION 2.6. [BH05] *One can construct on X a formal complex structure J_X, entirely formally determined by the CR infinity, starting from the almost complex structure $\tilde{J}$, which is the extension of J to X with $\tilde{J}\partial_r = R$. Moreover, an extension $\tilde{\nabla}^\theta$ of ∇^θ to X is given by*

$$\tilde{\nabla}^\theta r\partial_r = \tilde{\nabla}^\theta rR = \tilde{\nabla}^\theta_{r\partial_r} r^{1/2}T_A = 0.$$

Let $\tilde{T}^\theta$ be the torsion of $\tilde{\nabla}^\theta$ and $\tilde{\tau} := \iota_R\tilde{T}^\theta$. An asymptotic development of J_X is then given by

$$J_X = \tilde{J} - 2r\tilde{\tau} + O_e(r^{5/2}).$$

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Theorem 2.7. [Fef76, BH05, Her07] *There is a formally determined ACH-Kähler metric g_{KE} on (X, J_X), which is Einstein up to order $n+\frac{3}{2}$, that is,*

$$\mathrm{Ric}(g_{KE}) = -\frac{n+2}{2} g_{KE} + O_e(r^{n+\frac{3}{2}}).$$

Moreover, g_E and g_{KE} coincide up to order $n+\frac{1}{2}$.

In dimension $2n+1=3$, the asymptotic development of g_{KE}, and therefore of g_E, is known at order $\frac{3}{2}$, which will be essential in Sections 3.4 and 4.2.

Theorem 2.8. [BH05, Her07] *When $n=1$, we have the asymptotic development*

$$g_{KE} = g_0 + \Phi_{AB}\theta^A \circ \theta^B + \Psi_{0\bar{1}}\theta^0 \circ \theta^{\bar{1}} + \Psi_{\bar{0}1}\theta^{\bar{0}} \circ \theta^1 + O_e(r^2),$$

where

$$\Psi_{0\bar{1}} = -\sqrt{2}\left(\frac{1}{6}\mathrm{Scal}_{W,\bar{1}} - \frac{2i}{3}\tau^1_{\bar{1},1}\right),$$

and Φ is given by Theorem 2.4:

$$\Phi_{1\bar{1}} = -\frac{\mathrm{Scal}_W}{4} \qquad \textit{and} \qquad \Phi_{11} = -i\tau^{\bar{1}}_1.$$

§3. CR-harmonic maps

3.1 Definitions

Let (M, H, J) be a $(2n+1)$-dimensional orientable, compact, strictly pseudoconvex CR manifold and (X, g) be an ACH manifold with CR infinity (M, H, J), where g is the approximately ACH-Kähler–Einstein metric given by Theorem 2.7. Let $\pi : X \to M$ be the standard projection. Let (N, h) be a Riemannian manifold. Let $\varphi \in C^\infty(M, N)$, and let $\tilde{\varphi} \in C^\infty(\overline{X}, N)$ be any *extension* of φ, that is, $\tilde{\varphi}|_M = \varphi$.

Let $T\tilde{\varphi}$ be the tangent map of $\tilde{\varphi}$. It is a section of the bundle $\Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$, and its norm is defined by

$$\|T\tilde{\varphi}\|^2_{g,h} := \mathrm{tr}_g(\tilde{\varphi}^*h).$$

The bundle $\Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$ is canonically equipped with the connection

$$\nabla^{g,h} := \nabla^g \otimes 1_{\tilde{\varphi}^*TN} + 1_{\Omega^1(\overline{X})} \otimes \nabla^{\tilde{\varphi}^*h},$$

where ∇^g and ∇^h are the respective Levi-Civita connections of g and h, and $\nabla^{\tilde{\varphi}^*h} := \tilde{\varphi}^*\nabla^h$.

The *divergence* $\delta^{g,h}$ is then defined for $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$ by

$$\delta^{g,h}\omega := -(\nabla^{g,h}_{e_I}\omega)(e_{\bar{I}}),$$

where (e_i) is an orthonormal basis of $T^{1,0}\overline{X}$ for g, considered as a Hermitian metric. We thus have

$$\delta^{g,h}\omega = -\nabla^{\tilde{\varphi}^*h}_{e_I}(\omega(e_{\bar{I}})) + \omega(\nabla^g_{e_I} e_{\bar{I}}).$$

For $\rho \in (0, \varepsilon)$, the *energy* of $\tilde{\varphi}$ in $(\rho, \varepsilon) \times M$ is the functional

$$E(\tilde{\varphi}, \rho) = \frac{1}{2}\int_{(\rho,\varepsilon)\times M} \|T\tilde{\varphi}\|^2_{g,h} \mathrm{dvol}_g.$$

An extension $\tilde\varphi$ is said to be *harmonic* if it is a critical point of the energy for all ρ. Equivalently, $\tilde\varphi$ is harmonic if and only if $\delta^{g,h}T\tilde\varphi = 0$.

Following the ideas of Graham, Jenne, Mason and Sparling, we want to find the obstructions to the existence of a smooth harmonic extension [GJMS92]. More precisely, assuming that $\tilde\varphi$ is smooth, we want to know if the first terms of the asymptotic development of $\tilde\varphi$ are determined by the data at infinity. By similarity with the real case and based on the known asymptotic developments of the approximately ACHE metrics, we expect to find an obstruction at order $n+1$, taking the form of a CR covariant differential operator of order $2n+2$.

Here, the *asymptotic development* of $\tilde\varphi$ will denote, by identification, the asymptotic development in r of $U := \exp_\varphi^{-1} \circ \tilde\varphi \in C^\infty(\overline{X}, (\varphi\circ\pi)^*TN)$, that is,

$$\forall p \in M,\ \forall r \in (0,\varepsilon), \qquad \tilde\varphi(p,r) := \exp_{\varphi(p)}\left(U(p,r)\right),$$

where, for $p \in M$, the exponential map $\exp_{\varphi(p)}$ is a diffeomorphism between a small ball $B(0,\varepsilon) \subset T_{\varphi(p)}N$ and its image, which is a neighborhood in N of $\varphi(p)$. Note that $U(\cdot,0) = 0$. We denote $v\tilde\varphi := T\tilde\varphi(v)$ for $v \in TX$, and similarly for φ on TM, and

$$\forall k \geqslant 1, \quad \varphi_k := (\nabla^{\tilde\varphi^* h}_{\partial_r})^{k-1}\partial_r\tilde\varphi|_{r=0}.$$

Note that φ_k is a section of φ^*TN; hence, $\nabla^{\varphi^* h}\varphi_k$ is a section of $\Omega^1(M)\otimes\varphi^*TN$.

3.2 Computation of the divergence

We use the notations of Section 2. Let (T_α) be a local basis of $T^{1,0}M$ and $T_{\overline\alpha} := \overline{T_\alpha}$ such that (T_A) is orthonormal for γ, considered as a Hermitian metric. Let (θ^A) be the basis dual to (T_A). Let $T_0 := (\partial_r - iR)/\sqrt2$ and $\theta^0 := (dr + i\theta)/\sqrt2$ be its dual.

Lemma 3.1. *For $\omega \in \Omega^1(\overline X)\otimes\tilde\varphi^*TN$, we have*

$$\begin{aligned}\delta^{g_0,h}\omega &= nr\omega(\partial_r) - r^2\left(\nabla^{\tilde\varphi^*h}_{T_0}\omega(T_{\overline0}) + \nabla^{\tilde\varphi^*h}_{T_{\overline0}}\omega(T_0)\right) - r\nabla^{\tilde\varphi^*h}_{T_A}\omega(T_{\overline A})\\ &= nr\omega(\partial_r) - r^2\nabla^{\tilde\varphi^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde\varphi^*h}_{R}\omega(R) - r\nabla^{\tilde\varphi^*h}_{T_A}\omega(T_{\overline A}).\end{aligned}$$

Proof. We have

$$g_0 = r^{-2}\theta^0\circ\theta^{\overline0} + r^{-1}\theta^\alpha\circ\theta^{\overline\alpha}.$$

An orthonormal basis of $T^{1,0}\overline X$ with respect to g_0 is hence given by

$$(e_0^{(0)}, e_\alpha^{(0)}) := (rT_0, r^{1/2}T_\alpha).$$

The trace of the Levi-Civita connection of g_0 is given in this basis by the Koszul formula:

$$\nabla^{g_0}_{e_I^{(0)}} e_{\overline I}^{(0)} = g_0\big([e_{\overline J}^{(0)}, e_I^{(0)}], e_{\overline I}^{(0)}\big)e_J^{(0)}.$$

Let $\tilde\nabla^\theta$ be the extension of ∇^θ given by Proposition 2.6. We have

$$\begin{aligned}[][e_0^{(0)}, e_{\overline0}^{(0)}] &= \frac{1}{\sqrt2}\big(e_{\overline0}^{(0)} - e_0^{(0)}\big),\\ [e_0^{(0)}, e_A^{(0)}] &= \frac{1}{\sqrt2}\left(\frac12 e_A^{(0)} - i\left(\tilde\nabla^\theta_{e_0{(0)}}e_A^{(0)} - \tau(e_A^{(0)})\right)\right),\end{aligned}$$

$$[e_A^{(0)}, e_B^{(0)}] = r d\theta(T_A, T_B) R.$$

Then, since $\mathrm{tr}(\tau) = 0$,

$$\nabla^{g_0}_{e_I^{(0)}} e_{\overline{I}}^{(0)} = (n+1) r \partial_r,$$

and also,

$$\nabla^{\tilde{\varphi}^* h}_{e_0^{(0)}} \omega(e_{\overline{0}}^{(0)}) + \nabla^{\tilde{\varphi}^* h}_{e_{\overline{0}}^{(0)}} \omega(e_0^{(0)}) = r\omega(\partial_r) + r^2 \nabla^{\tilde{\varphi}^* h}_{\partial_r} \omega(\partial_r) + r^2 \nabla^{\tilde{\varphi}^* h}_{R} \omega(R),$$
$$\nabla^{\tilde{\varphi}^* h}_{e_\alpha^{(0)}} \omega(e_{\overline{\alpha}}^{(0)}) = r \nabla^{\tilde{\varphi}^* h}_{T_\alpha} \omega(T_{\overline{\alpha}}).$$

Hence, we have the announced expression for $\delta^{g_0,h}\omega$. ∎

Let us denote by $(\delta^{g,h}\omega)^{(1)}$ the remainder of $\delta^{g,h}\omega$, that is,

$$(\delta^{g,h}\omega)^{(1)} := \delta^{g,h}\omega - \delta^{g_0,h}\omega.$$

We prove the following technical lemma, which is crucial for the proof of Theorem 3.3.

LEMMA 3.2. *For $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN$, denoting by O_T the order with respect to the basis $(\tilde{\varphi}_* T_I)$ in powers of r, we have*

$$(\delta^{g,h}\omega)^{(1)} = O_T(r^2),$$

and there is no term of order 2 in the remainder of the form $r^2 \nabla^{\tilde{\varphi}^* h}_{\partial_r} \omega(\partial_r)$.

Proof. By Theorem 2.4, we have

$$g - g_0 = \Phi + O_e(r^{3/2}) = \Phi_{AB}\theta^A \circ \theta^B + O_e(r^{3/2}),$$

where we recall that $\Phi = -2\mathrm{Sch}_W(J,\theta) + 2\gamma(J\tau\cdot,\cdot)$ and that O_e denotes the order with respect to $(e_I^{(0)})$. Note that $\Phi_{AB} = \Phi_{BA}$. Since Φ is real, we have also $\Phi_{\overline{\alpha}\beta} = \overline{\Phi_{\alpha\overline{\beta}}}$ and $\Phi_{\overline{\alpha}\overline{\beta}} = \overline{\Phi_{\alpha\beta}}$.

An orthonormal basis of $T^{1,0}\overline{X}$ with respect to g induced from $e^{(0)}$ is formally given by

$$(e_0, e_\alpha) := \left(e_0^{(0)} + e_0^{(1)}, e_\alpha^{(0)} + e_\alpha^{(1)}\right),$$

where, by the Gram–Schmidt process, and since Φ is horizontal,

$$e_0^{(1)} = O_e(r^{3/2}) \qquad \text{and} \qquad e_\alpha^{(1)} = O_e(r).$$

This leads to

$$\begin{aligned}(\delta^{g,h}\omega)^{(1)} = &-\nabla^{\tilde{\varphi}^* h}_{e_I^{(0)}} \omega(e_{\overline{I}}^{(1)}) - \nabla^{\tilde{\varphi}^* h}_{e_I^{(1)}} \omega(e_{\overline{I}}^{(0)}) - \nabla^{\tilde{\varphi}^* h}_{e_I^{(1)}} \omega(e_{\overline{I}}^{(1)}) \\ &+ \omega\left(\nabla^g_{e_I^{(0)}} e_{\overline{I}}^{(0)} - \nabla^{g_0}_{e_I^{(0)}} e_{\overline{I}}^{(0)}\right) + \omega\left(\nabla^g_{e_I^{(0)}} e_{\overline{I}}^{(1)}\right) + \omega\left(\nabla^g_{e_I^{(1)}} e_{\overline{I}}^{(0)}\right) + \omega\left(\nabla^g_{e_I^{(1)}} e_{\overline{I}}^{(1)}\right),\end{aligned}$$

all terms of which are in $O_T(r^2)$ and are not of the form $r^2 \nabla^{\tilde{\varphi}^* h}_{\partial_r} \omega(\partial_r)$. ∎

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3.3 An obstruction to regularity

THEOREM 3.3. *Let (M, H, J) be a $(2n+1)$-dimensional orientable, compact, strictly pseudoconvex CR manifold and (X, g) be an ACH manifold with CR infinity (M, H, J), where g is the approximately ACH-Kähler–Einstein metric given by Theorem 2.7. Let $\pi : \overline{X} \to M$ be the standard projection. Let (N, h) be a Riemannian manifold, and let $\varphi \in C^\infty(M, N)$.*

There exists a section U of $(\varphi \circ \pi)^* TN$, unique modulo $O_T(r^{n+1})$, such that $\tilde{\varphi} = \exp_\varphi \circ U$ satisfies

$$\begin{cases} \tilde{\varphi}|_M &= \varphi, \\ \delta^{g,h} T\tilde{\varphi} &= O_T(r^{n+2}). \end{cases}$$

The asymptotic development in r of U is

$$U = U_1 r + \cdots + U_n \frac{r^n}{n!} + P_n(\varphi) \frac{r^{n+1}}{(n+1)!} \log r + O_T(r^{n+1}),$$

where $U_1, \ldots, U_n, P_n$ are formally determined by φ, g and h.

$P_n(\varphi)$ is an obstruction to the regularity of U and is given by

$$\begin{aligned} P_n(\varphi) &= \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^n \tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} - n \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{n-1} \nabla^{\tilde{\varphi}^* h}_R R\tilde{\varphi}\Big|_{r=0} \\ &\quad + \frac{1}{n+1} \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{n+1} \left(\delta^{g,h} T\tilde{\varphi}\right)^{(1)}\Big|_{r=0} \\ &= \frac{(-1)^n}{n!} (\delta^{\theta,h}_b \nabla^{\varphi^* h})^n \delta^{\theta,h}_b T\varphi + \textit{lower order terms (in derivatives of } \varphi\textit{)}. \end{aligned}$$

Proof. For $m \in \mathbb{N}$, we have

$$\delta^{g,h} T\tilde{\varphi} = O_T(r^{m+1}) \quad \iff \quad \forall k \leqslant m, \quad \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^k \delta^{g,h} T\tilde{\varphi}\Big|_{r=0} = 0.$$

We recall the notation

$$\varphi_k := \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{k-1} \partial_r \tilde{\varphi}\Big|_{r=0}.$$

Now, by Lemma 3.1, we have, for $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN$,

$$\nabla^{\tilde{\varphi}^* h}_{\partial_r} \delta^{g,h} \omega\Big|_{r=0} = n\, \omega(\partial_r)|_{r=0} + \delta^{\theta,h}_b (\omega|_{r=0}),$$

and, for all $2 \leqslant k \leqslant n$,

$$\begin{aligned} \frac{1}{k} \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^k \delta^{g,h} \omega\Big|_{r=0} &= (n-k+1) \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{k-1} \omega(\partial_r)\Big|_{r=0} + \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{k-1} \tilde{\delta}^{\theta,h}_b \omega\Big|_{r=0} \\ &\quad - (k-1) \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^{k-2} \nabla^{\tilde{\varphi}^* h}_R \omega(R)\Big|_{r=0} + \frac{1}{k} \left(\nabla^{\tilde{\varphi}^* h}_{\partial_r}\right)^k \left(\delta^{g,h} \omega\right)^{(1)}\Big|_{r=0}, \end{aligned}$$

where

$$\forall \omega_0 \in \Omega^1(M) \otimes \varphi^* TN, \qquad \delta^{\theta,h}_b \omega_0 := -\nabla^{\varphi^* h}_{T_A} \omega_0(T_{\overline{A}})$$

and

$$\forall \omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN, \qquad \tilde{\delta}^{\theta,h}_b \omega := -\nabla^{\tilde{\varphi}^* h}_{T_A} \omega(T_{\overline{A}}).$$

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Consequently, $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ is equivalent to

$$\begin{cases} n\varphi_1 &= -\delta_b^{\theta,h}T\varphi, \\ (n-k+1)\varphi_k &= -\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{k-1} \tilde{\delta}_b^{\theta,h}T\tilde{\varphi}\Big|_{r=0} - D_{k-1}(\varphi) \quad \forall 2 \leqslant k \leqslant n, \end{cases}$$

where

$$D_{k-1}(\varphi) := -(k-1)\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{k-2} \nabla_R^{\tilde{\varphi}^*h} R\tilde{\varphi}\Big|_{r=0} + \frac{1}{k}\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{k}\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\Big|_{r=0}.$$

By Lemma 3.2, $D_{k-1}(\varphi)$ only depends on $\varphi, \varphi_1, \ldots, \varphi_{k-1}$. This observation comes from the fact that, although

$$(\nabla_{\partial_r}^{\tilde{\varphi}^*h})^k \left(r^2 \nabla_{\partial_r}^{\tilde{\varphi}^*h}\partial_r\tilde{\varphi}\right)\Big|_{r=0} = 2\varphi_k,$$

$$\forall X, Y \in \{\partial_r, R, T_A\}, \quad (X, Y) \neq (\partial_r, \partial_r), \quad (\nabla_{\partial_r}^{\tilde{\varphi}^*h})^k \left(r^2 \nabla_X^{\tilde{\varphi}^*h} Y\tilde{\varphi}\right)\Big|_{r=0} \text{does not depend on } \varphi_k.$$

By induction, $D_{k-1}(\varphi)$ is thus well defined.

In conclusion, requiring $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ gives an asymptotic development for $\tilde{\varphi}$ in powers of r, and this development is unique up to order n with respect to T.

Assume now that $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ and that $\tilde{\varphi}$ admits a Taylor development up to order $n+1$. Then

$$\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+2}) \Longleftrightarrow \left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^n \tilde{\delta}_b^{\theta,h}T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi) = 0.$$

This equality cannot be true in general. Consequently, we introduce a term in $r^{n+1}\log r$ in the development of $\tilde{\varphi}$:

$$U = U_1 r + \cdots + U_n \frac{r^n}{n!} + P_n(\varphi)\frac{r^{n+1}}{(n+1)!}\log r + O_T(r^{n+1}).$$

The coefficient $P_n(\varphi)$ verifies

$$\frac{1}{n+1}\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{n+1} \delta^{g,h}T\tilde{\varphi}\Big|_{r=0} = -P_n(\varphi) + \left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^n \tilde{\delta}_b^{\theta,h}T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi);$$

hence,

$$\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+2}) \Longleftrightarrow P_n(\varphi) = \left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^n \tilde{\delta}_b^{\theta,h}T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi).$$

This yields the announced obstruction, which only depends on φ. Since

$$\varphi_k = -\frac{1}{n-k+1}\delta_b^{\theta,h}\nabla^{\varphi^*h}\varphi_{k-1} + \text{lower order terms (in derivatives of } \varphi),$$

we have the announced leading term. ∎

Proposition 3.4. *P_n does not depend on whether we take $g = g_E$ or g_{KE} on X.*

Proof. To compute P_n, it is sufficient to be able to compute

$$\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{n+1}\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\Big|_{r=0};$$

that is, by the proof of Lemma 3.2, to know the $e_I^{(1)}$ at order $n+1/2$ with respect to $e^{(0)}$. By the Gram–Schmidt process, it is thus sufficient to know g at order $n+1/2$ with respect to $e^{(0)}$. Hence, by Theorems 2.4 and 2.7, we can equivalently consider g_E or g_{KE}. ∎

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PROPOSITION 3.5. *Let $f \in C^\infty(\overline{X}, \mathbb{R})$ and $f_0 := f|_M$, and let $\hat{r} = e^f r$ be a conformal change of boundary defining function. Then*

$$\hat{P}_n(\varphi) = e^{-(n+1)f_0} P_n(\varphi).$$

The obstruction $P_n(\varphi)$ to the regularity of $\tilde{\varphi}$ is therefore CR covariant.

Proof. We have

$$U = U_1 r + \cdots + U_n \frac{r^n}{n!} + P_n(\varphi) \frac{r^{n+1}}{(n+1)!} \log r + O_T(r^{n+1}).$$

Now, since $\exp_\varphi : (\varphi \circ \pi)^* TN \to N$ does not depend on r, neither does U. Moreover, since M is compact, $\forall k,\ O_T(\hat{r}^k) = O_T(r^k)$. We thus have

$$\begin{aligned} U &= \hat{U}_1 \hat{r} + \cdots + \hat{U}_n \frac{\hat{r}^n}{n!} + \hat{P}_n(\varphi) \frac{\hat{r}^{n+1}}{(n+1)!} \log \hat{r} + O_T(\hat{r}^{n+1}) \\ &= \hat{U}_1 e^f r + \cdots + \hat{U}_n e^{nf} \frac{r^n}{n!} + \hat{P}_n(\varphi) e^{(n+1)f} \frac{r^{n+1}}{(n+1)!} \log r + O_T(r^{n+1}). \end{aligned}$$

Since the function f itself has a Taylor expansion in r, all polynomial terms are mixed. However, there is only one term with order $r^{n+1} \log r$. By identification, this yields the result. ∎

We then introduce *CR-harmonic maps* as maps for which the obstruction vanishes.

DEFINITION 3.6. If $P_n(\varphi) = 0$, φ is said to be *CR-harmonic*.

EXAMPLE 3.7. Let us assume that (M, H, J, θ) is Einstein with $\mathrm{Ric}_W = 2\lambda(n+1)\gamma$. We know from Example 2.2 that

$$g = \frac{dr^2}{r^2} + \frac{(1 - \lambda^2 r^2)^2}{r^2} \theta^2 + \frac{(1 - \lambda r)^2}{r} \gamma$$

satisfies $\mathrm{Ric}(g) = -((n+2)/2) g$. In this case, we can explicitly compute the divergence $\delta^{g,h} \omega$, for $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN$.

Indeed, an orthonormal basis of $T^{1,0}\overline{X}$ with respect to g induced from $e^{(0)}$ is given by

$$(e_0, e_\alpha) := \left(\frac{1}{\sqrt{2}} \left(r\partial_r - i \frac{r}{1 - \lambda^2 r^2} R \right), \frac{r^{1/2}}{1 - \lambda r} T_\alpha \right);$$

hence,

$$\begin{aligned} [e_0, e_{\overline{0}}] &= \frac{1}{\sqrt{2}} \frac{1 + \lambda^2 r^2}{1 - \lambda^2 r^2} (e_{\overline{0}} - e_0), \\ [e_0, e_A] &= \frac{1}{2\sqrt{2}} \frac{1 + \lambda r}{1 - \lambda r} e_A, \\ [e_A, e_B] &= \frac{r}{(1 - \lambda r)^2} d\theta(T_A, T_B) R. \end{aligned}$$

Then

$$\nabla^g_{e_I} e_{\overline{I}} = \left(n \frac{1 + \lambda^2 r^2}{1 - \lambda^2 r^2} + \frac{1 + \lambda r}{1 - \lambda r} \right) r\partial_r,$$

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and also,

$$\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\overline{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\overline{0}}}\omega(e_0) = r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + \frac{r^2}{(1-\lambda^2r^2)^2}\nabla^{\tilde{\varphi}^*h}_{R}\omega(R),$$

$$\nabla^{\tilde{\varphi}^*h}_{e_\alpha}\omega(e_{\overline{\alpha}}) = \frac{r}{(1-\lambda r)^2}\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}).$$

The divergence is hence given by

$$\delta^{g,h}\omega = \left(n\frac{1+\lambda^2r^2}{1-\lambda^2r^2} + \frac{1+\lambda r}{1-\lambda r} - 1\right) r\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - \frac{r^2}{(1-\lambda^2r^2)^2}\nabla^{\tilde{\varphi}^*h}_{R}\omega(R) + \frac{r}{(1-\lambda r)^2}\tilde{\delta}^{\theta,h}_b\omega.$$

From Example 3.7, we get the following results.

Corollary 3.8. *If (M, H, J, θ) is Einstein, then subharmonic maps which verify $\nabla^{\varphi^*h}_R R\varphi = 0$ are CR-harmonic.*

Proof. Indeed, let φ be subharmonic, that is, $\delta^{\theta,h}_b T\varphi = 0$, and such that $\nabla^{\varphi^*h}_R R\varphi = 0$. Let $\tilde{\varphi}$ be the extension of φ given by Theorem 3.3. We thus have $\varphi_1 = 0$. Moreover, by Example 3.7, we have

$$\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)} = \alpha(r)\partial_r\tilde{\varphi} + \beta(r)\nabla^{\tilde{\varphi}^*h}_R R\tilde{\varphi} + \gamma(r)\tilde{\delta}^{\theta,h}_b T\tilde{\varphi},$$

where $\alpha(r) = O(r^2)$, $\beta(r) = O(r^4)$ and $\gamma(r) = O(r^2)$. Since $\varphi_1 = \nabla^{\varphi^*h}_R R\varphi = 0$, we get that

$$(n-1)\varphi_2 = -\left.\nabla^{\tilde{\varphi}^*h}_{\partial_r}\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\right|_{r=0} - \nabla^{\varphi^*h}_R R\varphi - \frac{1}{2}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^2\left.\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\right|_{r=0} = 0.$$

By induction, we similarly have $\forall k \leqslant n,\ \varphi_k = 0$, which implies that $P_n(\varphi) = 0$. ∎

Corollary 3.9. *If (M, H, J, θ) is Einstein and (N, h) is a Kähler manifold, then CR-holomorphic maps which verify $R\varphi = 0$ are CR-harmonic.*

Proof. Indeed, assuming that $T\varphi \circ J = J_N \circ T\varphi$ and extending J by taking $J(R) = 0$, we have

$$\begin{aligned}
\nabla^{\varphi^*h}_{T_\alpha} T_{\overline{\alpha}}\varphi &= \nabla^{\varphi^*h}_{JT_\alpha} JT_{\overline{\alpha}}\varphi \\
&= J_N\nabla^{\varphi^*h}_{JT_\alpha} T_{\overline{\alpha}}\varphi \\
&= J_N\nabla^{\varphi^*h}_{T_{\overline{\alpha}}} JT_\alpha\varphi + J([JT_\alpha, T_{\overline{\alpha}}])\varphi \\
&= -\nabla^{\varphi^*h}_{T_{\overline{\alpha}}} T_\alpha\varphi + iJ([T_\alpha, T_{\overline{\alpha}}])\varphi \\
&= -\nabla^{\varphi^*h}_{T_{\overline{\alpha}}} T_\alpha\varphi - nJ(R)\varphi;
\end{aligned}$$

hence,

$$\delta^{\theta,h}_b T\varphi = nJ_N(R\varphi).$$

Consequently, φ is CR-harmonic by Corollary 3.8. ∎

Example 3.10. Let (M, H, J) be a circle bundle over a Riemann surface Σ admitting an Einstein contact form. Then the projection $\pi : M \to \Sigma$ is CR-harmonic.

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3.4 Explicit obstruction in dimension 3

When $n=1$, that is, $\dim(M)=3$, the asymptotic development of g is given at order $\frac{3}{2}$ in $e^{(0)}$ by Theorem 2.8. Hence, by Proposition 3.4, we can explicitly compute the obstruction.

THEOREM 3.11. *Still denoting $v\varphi := T\varphi(v)$ for $v\in TM$, and also $(\nabla^{\varphi^*h}v)\varphi := \nabla^{\varphi^*h}(v\varphi)$, we have*

$$P_1(\varphi) = -\delta_b^{\theta,h}\nabla^{\varphi^*h}\delta_b^{\theta,h}T\varphi - \nabla_R^{\varphi^*h}R\varphi + 4\mathrm{Im}\left(\nabla_{T_{\bar{1}}}^{\varphi^*h}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi - S_b\left(\delta_b^{\theta,h}T\varphi\right),$$

where

$$S_b(X) := \mathscr{R}^h_{X,T_1\varphi}T_{\bar{1}}\varphi + \mathscr{R}^h_{X,T_{\bar{1}}\varphi}T_1\varphi.$$

Proof. An orthonormal basis of $T^{1,0}X$ with respect to g is given by

$$(e_0,e_1) := \left(e_0^{(0)} - r^{3/2}\Psi_{0\bar{1}}e_1^{(0)}, \left(1-r\Phi_{1\bar{1}}\right)e_1^{(0)} - r\Phi_{11}e_{\bar{1}}^{(0)}\right) + O_e(r^2).$$

We have

$$\begin{aligned}
[e_0,e_{\bar{0}}] &= \frac{1}{\sqrt{2}}\left(e_{\bar{0}} - e_0 - r^{3/2}\Psi_{\bar{0}1}e_{\bar{1}} + r^{3/2}\Psi_{0\bar{1}}e_1\right) + O_e(r^2),\\
[e_0,e_1] &= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{2} - r\Phi_{1\bar{1}}\right)e_1 - r\Phi_{11}e_{\bar{1}} - i\big(\tilde{\nabla}^\theta_{e_0}e_1 - \tau(e_1)\big)\right) + O_e(r^2),\\
[e_0,e_{\bar{1}}] &= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{2} - r\Phi_{1\bar{1}}\right)e_{\bar{1}} - r\Phi_{\bar{1}\bar{1}}e_1 - i\big(\tilde{\nabla}^\theta_{e_0}e_{\bar{1}} - \tau(e_{\bar{1}})\big)\right) + O_e(r^2),\\
[e_1,e_{\bar{1}}] &= r^{3/2}\big(\Phi_{1\bar{1},\bar{1}} - \Phi_{\bar{1}\bar{1},1}\big)e_1 - r^{3/2}\big(\Phi_{1\bar{1},1} - \Phi_{11,\bar{1}}\big)e_{\bar{1}} + O_e(r^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla^g_{e_I}e_{\bar{I}} = {}& 2r\left(1-r\Phi_{1\bar{1}}\right)\partial_r\\
&- r^2\left(\sqrt{2}\Psi_{0\bar{1}} + \Phi_{1\bar{1},\bar{1}} - \Phi_{\bar{1}\bar{1},1}\right)T_1\\
&- r^2\left(\sqrt{2}\Psi_{\bar{0}1} + \Phi_{1\bar{1},1} - \Phi_{11,\bar{1}}\right)T_{\bar{1}} + O_T(r^{5/2}).
\end{aligned}$$

We also have, for $\omega\in\Omega^1(X)\otimes\tilde{\varphi}^*TN$,

$$\begin{aligned}
\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\bar{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\bar{0}}}\omega(e_0) &= r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_R\omega(R)\\
&\quad - \sqrt{2}r^2\Psi_{0\bar{1}}\omega(T_1) - \sqrt{2}r^2\Psi_{\bar{0}1}\omega(T_{\bar{1}}) + O_T(r^{5/2}),\\
\nabla^{\tilde{\varphi}^*h}_{e_1}\omega(e_{\bar{1}}) &= r\nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - r^2\nabla^{\tilde{\varphi}^*h}_{T_1}\left(\Phi_{1\bar{1}}\omega(T_{\bar{1}})\right) - r^2\nabla^{\tilde{\varphi}^*h}_{T_1}\left(\Phi_{\bar{1}\bar{1}}\omega(T_1)\right)\\
&\quad - r^2\Phi_{1\bar{1}}\nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - r^2\Phi_{11}\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\omega(T_{\bar{1}}) + O_T(r^3).
\end{aligned}$$

The divergence is hence given by

$$\begin{aligned}
\delta^{g,h}\omega &= r(1-2r\Phi_{1\bar{1}})\left(\omega(\partial_r) - \nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - \nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\omega(T_1)\right) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_R\omega(R)\\
&\quad + 4r^2\mathrm{Im}\left(\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}\omega(T_{\bar{1}})\right)\right) + O_T(r^{5/2})\\
&= \delta^{g_0,h}\omega - 2r^2\Phi_{1\bar{1}}\nabla^{\tilde{\varphi}^*h}_{\partial_r}\delta^{g,h}\omega + 4r^2\mathrm{Im}\left(\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}\omega(T_{\bar{1}})\right)\right) + O_T(r^{5/2}).
\end{aligned}$$

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Then, by Theorem 3.3, we have

$$P_1(\varphi) = \nabla^{\tilde{\varphi}^*h}_{\partial_r}\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}|_{r=0} - \nabla^{\varphi^*h}_R R\varphi + 4\mathrm{Im}\left(\nabla^{\varphi^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi$$
$$= \delta^{\theta,h}_b\nabla^{\varphi^*h}\varphi_1 + \mathscr{R}^h_{\varphi_1,T_1\varphi}T_{\bar{1}}\varphi + \mathscr{R}^h_{\varphi_1,T_{\bar{1}}\varphi}T_1\varphi - \nabla^{\varphi^*h}_R R\varphi + 4\mathrm{Im}\left(\nabla^{\varphi^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi,$$

hence, since $\varphi_1 = -\delta^{\theta,h}_b T\varphi$, the announced obstruction. ∎

Note that on functions, meaning that $N = \mathbb{R}$, P_1 reduces to a multiple of the CR Paneitz operator. Since the construction follows the ideas of Graham *et al.*, this was expected. A similar phenomenon appears in the real case [Bér13].

EXAMPLE 3.12. Let us consider $\mathrm{id} : (M, H, J, \theta) \to (M, g := g_{J,\theta})$.
Since $\nabla^g_R R = 0$ by [DT06, Lemma 1.3], we have, using the Koszul formula,

$$\begin{aligned}\delta^{\theta,g}_b T\mathrm{id} &= -\nabla^g_{T_1}T_{\bar{1}} - \nabla^g_{T_{\bar{1}}}T_1\\ &= -g\left([T_1, T_{\bar{1}}], T_1\right)T_{\bar{1}} - g\left([T_{\bar{1}}, T_1], T_{\bar{1}}\right)T_1\\ &\quad - g\left([R, T_{\bar{1}}], T_1\right)R - g\left([R, T_1], T_{\bar{1}}\right)R\\ &\quad - g\left([T_1, R], R\right)T_{\bar{1}} - g\left([T_{\bar{1}}, R], R\right)T_1\\ &= 0;\end{aligned}$$

hence,

$$P_1(\mathrm{id}) = 4\mathrm{Im}\nabla^g_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right).$$

Consequently, the identity is CR-harmonic if and only if $\mathrm{Im}\nabla^g_{T_{\bar{1}}}(\tau_1^{\bar{1}}T_{\bar{1}}) = 0$. This is, in particular, verified when θ is normal, that is, when $\tau = 0$.

§4. Renormalized energy

4.1 Definition

Let $\varphi \in C^\infty(M, N)$ and $\tilde{\varphi}$ be the extension of φ constructed in Theorem 3.3. For ρ in $(0, \varepsilon)$, let

$$E(\tilde{\varphi}, \rho) = \frac{1}{2}\int_{(\rho,\varepsilon)\times M}\|T\tilde{\varphi}\|^2_{g,h}\mathrm{dvol}_g$$

be the energy of $\tilde{\varphi}$ in $(\rho, \varepsilon)\times M$. We have

$$\|T\tilde{\varphi}\|^2_{g,h} = f_0 r + f_1 r^2 + \cdots + f_n r^{n+1} + O(r^{n+2}\log r),$$

where $\forall k \leqslant n$, f_k depends only on U_j for $j \leqslant k$ and on g at order k in $e^{(0)}$, and

$$\mathrm{dvol}_g = r^{-n-2}\sqrt{\det g}\,dr\wedge\theta\wedge d\theta^n.$$

Consequently,

$$\|T\tilde{\varphi}\|^2_{g,h}\mathrm{dvol}_g = \left(a_0 r^{-n-1} + a_1 r^{-n} + \cdots + a_n r^{-1} + O(\log r)\right)dr\wedge\theta\wedge d\theta^n,$$

where $\forall k \leqslant n$, a_k depends only on U_j for $j \leqslant k$ and on g at order k. Hence, E admits the development, when $\rho \to 0$,

$$E(\tilde{\varphi}, \rho) = E_0(\varphi)\rho^{-n} + E_1(\varphi)\rho^{1-n} + \cdots + E_{n-1}(\varphi)\rho^{-1} + F_n(\varphi)\log\rho + E_n(\varphi) + o(1),$$

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where $\forall k \leqslant n-1$, E_k depends only on U_j for $j \leqslant k$ and on g at order k and F_n depends only on U_j for $j \leqslant n$ and on g at order n. The coefficient $F_n(\varphi)$ can be written as

$$F_n(\varphi) = -\frac{1}{2}\int_M a_n \theta \wedge d\theta^n = -\frac{1}{2n!}\int_M \partial_r^n \left(r^{n+1}\|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g\right)\big|_{r=0}.$$

By construction, F_n is formally determined by φ, g and h. Moreover, we have the following.

PROPOSITION 4.1. *$F_n(\varphi)$ is a CR invariant:*

$$\hat{F}_n(\varphi) = F_n(\varphi).$$

Proof. The proof is similar to the proof of Proposition 3.5. Indeed, if $\hat{r} = e^f r$, then

$$\begin{aligned}\|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g &= \left(a_0 r^{-n-1} + a_1 r^{-n} + \cdots + a_n r^{-1} + a_{n+1} + O(r)\right) dr \wedge \theta \wedge d\theta^n \\ &= \left(\hat{a}_0 \hat{r}^{-n-1} + \hat{a}_1 \hat{r}^{-n} + \cdots + \hat{a}_n \hat{r}^{-1} + \hat{a}_{n+1} + O(\hat{r})\right) d\hat{r} \wedge \theta \wedge d\theta^n;\end{aligned}$$

hence, when integrating over $(r = \rho, r = \varepsilon) \times M$,

$$\begin{aligned}E(\tilde{\varphi}, \rho) &= E_0(\varphi)\rho^{-n} + E_1(\varphi)\rho^{1-n} + \cdots + E_{n-1}(\varphi)\rho^{-1} + F_n(\varphi)\log\rho + E_n(\varphi) + o(1) \\ &= \hat{E}_0(\varphi)\rho^{-n} + \hat{E}_1(\varphi)\rho^{1-n} + \cdots + \hat{E}_{n-1}(\varphi)\rho^{-1} + \hat{F}_n(\varphi)\log\rho + \hat{E}_n(\varphi) + o(1).\end{aligned}$$

Again, since the function f itself has a Taylor expansion in r, all polynomial terms are mixed. However, the only $\log\rho$ term which appears when integrating with respect to $\hat{r}$ comes from the $\hat{r}^{-1}$ term. Hence, we have the result. □

The principal term of $F_n(\varphi)$ is the following: since

$$\begin{aligned}r^{n+1}\|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g &= \left(\langle T_A\tilde{\varphi}, T_{\overline{A}}\tilde{\varphi}\rangle_h + r\|\partial_r\tilde{\varphi}\|^2_h\right) dr \wedge \theta \wedge d\theta^n \\ &\qquad + \text{lower order (in derivations of } \varphi\text{) terms},\end{aligned}$$

we have

$$\begin{aligned}F_n(\varphi) &= -\frac{1}{2n!}\int_M \left(\sum_{k=0}^{n}\binom{n}{k}\langle \delta_b^{\theta,h}\nabla^{\varphi^*h}\varphi_k, \varphi_{n-k}\rangle_h + n\sum_{k=0}^{n-1}\binom{n-1}{k}\langle\varphi_{k+1}, \varphi_{n-k}\rangle_h\right)\theta \\ &\qquad \wedge d\theta^n + \text{l.o.t.} \\ &= \frac{(-1)^{n+1}}{2n!^2}\int_M \langle(\delta_b^{\theta,h}\nabla^{\varphi^*h})^{n-1}\delta_b^{\theta,h}T\varphi, \delta_b^{\theta,h}T\varphi\rangle_h \theta \wedge d\theta^n + \text{lower order terms}.\end{aligned}$$

DEFINITION 4.2. $F_n(\varphi)$ is called the *renormalized energy* of φ.

PROPOSITION 4.3. *The gradient of $F_n(\varphi)$ is $(1/2n!)P_n(\varphi)$, that is to say, for all $\dot{\varphi} \in \Gamma(\varphi^*TN)$,*

$$d_\varphi F_n(\dot{\varphi}) = \frac{1}{2n!}\int_M \langle\dot{\varphi}, P_n(\varphi)\rangle_h \theta \wedge d\theta^n.$$

Proof. Let $\dot{\varphi} \in \Gamma(\varphi^*TN)$. Let $(\varphi_t)_{t\in[-1,1]}$ be a one-parameter family in $C^\infty(M, N)$ such that

$$\begin{cases}\varphi_0 = \varphi, \\ \partial_t\varphi_t|_{t=0} = \dot{\varphi}.\end{cases}$$

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Let us equip $X \times [-1,1]$ with the metric $\overline{g} = g + dt^2$ and let $\xi \in C^\infty(X \times [-1,1], N)$ be the map

$$\forall p \in X,\ \forall t \in [-1,1], \quad \xi(p,t) = \tilde{\varphi}_t(p).$$

We then have

$$\begin{aligned}
\partial_t \|T\tilde{\varphi}_t\|^2_{g,h} &= \partial_t \left(\|T\xi\|^2_{\overline{g},h} - \|\partial_t \xi\|^2_h\right) \\
&= \left\langle \nabla^{\overline{g},h}_{\partial_t} T\xi, T\xi \right\rangle_{\overline{g},h} - \left\langle \nabla^{\xi^* h}_{\partial_t} \partial_t \xi, \partial_t \xi \right\rangle_h \\
&= \left\langle \nabla^{\xi^* h}_{\partial_t} e_I \xi, e_{\overline{I}} \xi \right\rangle_h \\
&= \left\langle \nabla^{\xi^* h}_{e_I} \partial_t \xi, e_{\overline{I}} \xi \right\rangle_h \\
&= \left\langle \nabla^{\tilde{\varphi}_t^* h}_{e_I} \partial_t \tilde{\varphi}_t, e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h \\
&= e_I \left\langle \partial_t \tilde{\varphi}_t, e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h - \left\langle \partial_t \tilde{\varphi}_t, \nabla^{\tilde{\varphi}_t^* h}_{e_I} e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h ;
\end{aligned}$$

hence,

$$\partial_t E(\tilde{\varphi}_t, \rho)|_{t=0} = \frac{1}{2} \int_{(\rho,\varepsilon)\times M} \left(e_I \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, e_{\overline{I}} \tilde{\varphi} \right\rangle_h - \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, \nabla^{\tilde{\varphi}^* h}_{e_I} e_{\overline{I}} \tilde{\varphi} \right\rangle_h \right) d\mathrm{vol}_g.$$

There is no $\log\rho$ term in the second part, and

$$\begin{aligned}
\frac{1}{2} \int_{(\rho,\varepsilon)\times M} e_I \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, e_{\overline{I}} \tilde{\varphi} \right\rangle_h d\mathrm{vol}_g &= \frac{1}{2} \int_M \rho^{-n} \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, \partial_\rho \tilde{\varphi} \right\rangle_h \theta \wedge d\theta^n \\
&\quad + \text{lower order terms},
\end{aligned}$$

whose $\log\rho$ term is

$$\frac{1}{2n!} \int_M \langle \dot{\varphi}, P_n(\varphi) \rangle_h \, \theta \wedge d\theta^n;$$

hence, we have the result. ∎

4.2 Explicit energy in dimension 3

Here again, when $n = 1$, that is, $\dim(M) = 3$, knowing the asymptotic development of g at order $\frac{3}{2}$ in $e^{(0)}$ allows for an explicit computation of the renormalized energy.

THEOREM 4.4. *We have*

$$F_1(\varphi) = -\frac{1}{2} \int_M \left(\|\delta_b^{\theta,h} T\varphi\|^2_h + \|R\varphi\|^2_h - 4\mathrm{Im}\left(\tau_1^{\overline{1}} \|T_{\overline{1}}\varphi\|^2_h\right)\right) \theta \wedge d\theta.$$

Proof. We have

$$\begin{aligned}
\|T\tilde{\varphi}\|^2_{g,h} &= 2 \left\langle e_0 \tilde{\varphi}, e_{\overline{0}} \tilde{\varphi} \right\rangle_h + 2 \left\langle e_1 \tilde{\varphi}, e_{\overline{1}} \tilde{\varphi} \right\rangle_h \\
&= 2r \left\langle T_1 \varphi, T_{\overline{1}} \varphi \right\rangle_h \\
&\quad + r^2 \left(\|\varphi_1\|^2_h + \|R\varphi\|^2_h - 4\Phi_{1\overline{1}} \left\langle T_1 \varphi, T_{\overline{1}} \varphi \right\rangle_h - 2\Phi_{11} \|T_{\overline{1}}\varphi\|^2_h - 2\Phi_{\overline{1}\overline{1}} \|T_1 \varphi\|^2_h \right) \\
&\quad + O(r^{5/2}),
\end{aligned}$$

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and

$$d\mathrm{vol}_g = \left(1 + 2r\Phi_{1\overline{1}} + O(r^2)\right) r^{-3} dr \wedge \theta \wedge d\theta.$$

Consequently,

$$\begin{aligned} r^2 \|T\tilde{\varphi}\|_g^2 d\mathrm{vol}_g = & \big(2\langle T_1\varphi, T_{\overline{1}}\varphi\rangle_h \\ & + r\big(\|\varphi_1\|_h^2 + \|R\varphi\|_h^2 - 2\Phi_{11}\|T_{\overline{1}}\varphi\|_h^2 - 2\Phi_{\overline{1}\overline{1}}\|T_1\varphi\|_h^2\big) \\ & + O(r^{3/2})\big) dr \wedge \theta \wedge d\theta, \end{aligned}$$

and finally,

$$F_1(\varphi) = -\frac{1}{2}\int_M \left(\|\varphi_1\|_h^2 + \|R\varphi\|_h^2 - 4\mathrm{Im}\big(\tau_1^{\overline{1}}\|T_{\overline{1}}\varphi\|_h^2\big)\right)\theta \wedge d\theta. \qquad \square$$

As an example, for $\mathrm{id} : (M, H, J, \theta) \to (M, g_{J,\theta})$, we have

$$F_1(\mathrm{id}) = -\tfrac{1}{2}\mathrm{Vol}(M, \theta).$$

§5. Further computations in the general case

We give here a more precise computation for $\delta^{g,h}\omega$ and $r^{n+1}\|T\tilde{\varphi}\|_g^2 d\mathrm{vol}_g$ in the general case, using Theorem 2.4. We show that this computation does not allow for an explicit expression of the obstruction and of the renormalized energy respectively.

5.1 Computation of the divergence

By Theorem 2.4, we have

$$g = g_0 + \Phi_{AB}\theta^A \circ \theta^B + O_e(r^{3/2}),$$

where, denoting by $R_{\alpha\overline{\beta}}$ the components of the Webster Ricci tensor,

$$\Phi_{\alpha\overline{\beta}} = -\frac{1}{n+2}\left(R_{\alpha\overline{\beta}} - \frac{\mathrm{Scal}_W(J,\theta)}{2(n+1)}\delta_{\alpha\overline{\beta}}\right), \qquad \text{and} \qquad \Phi_{\alpha\beta} = -i\tau_\alpha^{\overline{\beta}}.$$

By Proposition 2.6, we can equip $\{r\} \times H$ with a complex structure $J_r = J_0 + rJ_1 + O_T(r^2)$, with

$$J_1 T_\alpha = -2\Phi_{\alpha\beta}T_{\overline{\beta}}.$$

An orthonormal basis of $T^{1,0}X$ with respect to g is given by

$$(e_0, e_\alpha) := \left(r\partial_r - irR, \left(\delta_{\alpha\overline{\beta}} - r\Phi_{\alpha\overline{\beta}}\right) r^{1/2}T_\beta - r\Phi_{\alpha\beta}r^{1/2}T_{\overline{\beta}}\right) + O_e(r^{3/2}).$$

Now, g can be rewritten as

$$g = (r^{-1}\theta^0) \circ (r^{-1}\theta^{\overline{0}}) + (r^{-1/2}\theta^\alpha) \circ (r^{-1/2}\theta^{\overline{\alpha}}) + r\Phi_{AB}(r^{-1/2}\theta^A) \circ (r^{-1/2}\theta^B) + O_e(r^{3/2}).$$

We have, modulo $O_e(r^{3/2})$,

$$\begin{aligned} [e_0, e_{\overline{0}}] &= \frac{1}{\sqrt{2}}\left(e_{\overline{0}} - e_0\right), \\ [e_0, e_\alpha] &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}e_\alpha - r\Phi_{\alpha\overline{\beta}}e_\beta - r\Phi_{\alpha\beta}e_{\overline{\beta}} - i\big(\tilde{\nabla}^\theta_{e_0}e_\alpha - \tau(e_\alpha)\big)\right), \end{aligned}$$

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$$[e_0, e_{\overline{\alpha}}] = \frac{1}{\sqrt{2}}\left(\frac{1}{2}e_{\overline{\alpha}} - r\Phi_{\overline{\alpha}\beta}e_{\overline{\beta}} - r\Phi_{\overline{\alpha}\overline{\beta}}e_\beta - i\big(\tilde{\nabla}^\theta_{e_0}e_{\overline{\alpha}} - \tau(e_{\overline{\alpha}})\big)\right),$$
$$[e_\alpha, e_\beta] = r^{3/2}\big(\Phi_{\alpha\overline{\delta},\beta} - \Phi_{\beta\overline{\delta},\alpha}\big)e_\delta + r^{3/2}\left(\Phi_{\alpha\delta,\beta} - \Phi_{\beta\delta,\alpha}\right)e_{\overline{\delta}},$$
$$[e_\alpha, e_{\overline{\beta}}] = r^{3/2}\big(\Phi_{\alpha\overline{\delta},\overline{\beta}} - \Phi_{\overline{\beta}\overline{\delta},\alpha}\big)e_\delta - r^{3/2}\big(\Phi_{\delta\overline{\beta},\alpha} - \Phi_{\alpha\delta,\overline{\beta}}\big)e_{\overline{\delta}}.$$

Hence,

$$\begin{aligned}\nabla^g_{e_i}e_{\bar{i}} + \nabla^g_{e_{\bar{i}}}e_i = {} & r\left(n+1-2r\Phi_{\alpha\overline{\alpha}}\right)\partial_r \\ & - r^2\big(2\Phi_{\beta\overline{\beta},\overline{\alpha}} - \Phi_{\overline{\alpha}\overline{\beta},\beta} - \Phi_{\overline{\alpha}\beta,\overline{\beta}}\big)T_\alpha \\ & - r^2\big(2\Phi_{\beta\overline{\beta},\alpha} - \Phi_{\alpha\beta,\overline{\beta}} - \Phi_{\alpha\overline{\beta},\beta}\big)T_{\overline{\alpha}} + O_T(r^{5/2}),\end{aligned}$$

with $\Phi_{\alpha\overline{\alpha}} = -\mathrm{tr}(\mathrm{S}_\theta) = -\mathrm{Scal}_W/2(n+1)$.

Also,

$$\begin{aligned}\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\overline{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\overline{0}}}\omega(e_0) &= r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_R\omega(R) + O_T(r^2),\\ \nabla^{\tilde{\varphi}^*h}_{e_\alpha}\omega(e_{\overline{\alpha}}) &= r\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}) - r^2\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\big(\Phi_{\overline{\alpha}\beta}\omega(T_{\overline{\beta}})\big) - r^2\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\big(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\beta)\big)\\ &\quad - r^2\Phi_{\alpha\overline{\beta}}\nabla^{\tilde{\varphi}^*h}_{T_\beta}\omega(T_{\overline{\alpha}}) - r^2\Phi_{\alpha\beta}\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\omega(T_{\overline{\alpha}}) + O_T(r^{5/2}).\end{aligned}$$

Coming back to the divergence, we have

$$\begin{aligned}\delta^{g,h}\omega = {} & r(n-2r\Phi_{\alpha\overline{\alpha}})\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_R(\omega(R))\\ & - r(1-2r\Phi_{\alpha\overline{\alpha}})\big(\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}) + \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\omega(T_\alpha)\big)\\ & + 2r^2\big(\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\big(\Phi_{\alpha\beta}\omega(T_{\overline{\alpha}})\big) + \nabla^{\tilde{\varphi}^*h}_{T_\beta}\big(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\alpha)\big)\big)\\ & + 2r^2\big(\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\Phi_{\overline{\alpha}\beta}\omega(T_{\overline{\beta}}) + \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\Phi_{\alpha\overline{\beta}}\omega(T_\beta) - \nabla^{\tilde{\varphi}^*h}_{T_\alpha}\Phi_{\beta\overline{\beta}}\omega(T_{\overline{\alpha}}) - \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\Phi_{\beta\overline{\beta}}\omega(T_\alpha)\big)\\ & + O_T(r^2).\end{aligned}$$

The term of order 2 is consequently not known, which does not allow for an explicit computation of P_n. Note that

$$\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\left(\Phi_{\alpha\beta}\omega(T_{\overline{\alpha}})\right) + \nabla^{\tilde{\varphi}^*h}_{T_\beta}\big(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\alpha)\big) = 2\mathrm{Im}\big(\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\big(\tau^{\overline{\beta}}_\alpha\omega(T_{\overline{\alpha}})\big)\big)$$

and that the potentially hidden r^2 terms are necessarily of the form $C^\alpha r^2\omega(T_\alpha) + D^{\overline{\alpha}}r^2\omega(T_{\overline{\alpha}})$.

5.2 Computation of the integrand of the energy

We have

$$\begin{aligned}\|T\tilde{\varphi}\|^2_{g,h} &= 2\left\langle e_0\tilde{\varphi}, e_{\overline{0}}\tilde{\varphi}\right\rangle_h + 2\left\langle e_\alpha\tilde{\varphi}, e_{\overline{\alpha}}\tilde{\varphi}\right\rangle_h\\ &= 2r\left\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\right\rangle_h\\ &\quad + r^2\left(\|\varphi_1\|^2_h + \|R\varphi\|^2_h - 2\Phi_{\alpha\beta}\big\langle T_{\overline{\alpha}}\varphi, T_{\overline{\beta}}\varphi\big\rangle_h - 2\Phi_{\overline{\alpha}\overline{\beta}}\left\langle T_\alpha\varphi, T_\beta\varphi\right\rangle_h\right.\\ &\quad \left. -2\Phi_{\alpha\overline{\beta}}\left\langle T_{\overline{\alpha}}\varphi, T_\beta\varphi\right\rangle_h - 2\Phi_{\overline{\alpha}\beta}\langle T_\alpha\varphi, T_{\overline{\beta}}\varphi\rangle_h\right)\\ &\quad + O(r^2),\end{aligned}$$

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and

$$d\mathrm{vol}_g = \big(1 + 2r\Phi_{\alpha\overline{\alpha}} + O(r^{3/2})\big) r^{-n-2} dr \wedge \theta \wedge d\theta^n.$$

Consequently,

$$\begin{aligned} r^{n+1}\|T\tilde{\varphi}\|_g^2 d\mathrm{vol}_g = \big(&2\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\rangle_h \\ &+ r\big(\|\varphi_1\|_h^2 + \|R\varphi\|_h^2 - 2\Phi_{\alpha\beta}\langle T_{\overline{\alpha}}\varphi, T_{\overline{\beta}}\varphi\rangle_h - 2\Phi_{\overline{\alpha}\overline{\beta}}\langle T_\alpha\varphi, T_\beta\varphi\rangle_h \\ &- 2\Phi_{\alpha\overline{\beta}}\langle T_{\overline{\alpha}}\varphi, T_\beta\varphi\rangle_h - 2\Phi_{\overline{\alpha}\beta}\langle T_\alpha\varphi, T_{\overline{\beta}}\varphi\rangle_h + 4\Phi_{\alpha\overline{\alpha}}\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\rangle_h\big) \\ &+ O(r)\big) dr \wedge \theta \wedge d\theta^n. \end{aligned}$$

The term of order 1 is consequently not known, which does not allow for an explicit computation of F_n.

§6. Relation with the Fefferman bundle in dimension 3

We describe here the correspondence between the obstruction to CR-harmonicity on a given CR 3-manifold and the obstruction to conformal harmonicity on its Fefferman bundle. It generalizes the Appendix B. of [CY13].

Let (M, H, J) be a compact strictly pseudoconvex CR 3-manifold and let (N, h) be a Riemannian manifold. Let (F, g_F) be the Fefferman bundle of (M, H, J). For a detailed construction of the Fefferman bundle, see [Far86, Lee86, Her09]. Let $\pi : F \to M$ be the natural bundle projection. Let θ be a positive contact form on M and let ϖ be the S^1-invariant connection 1-form induced by the Weyl structure attached to θ on F. The *Fefferman metric* attached to θ on F is the Lorentzian metric

$$g_F = i\varpi \circ \pi^*\theta + \tfrac{1}{2}\pi^*\gamma.$$

By analogy with the Riemannian case [Bér13], given $\varphi \in C^\infty(F, N)$, the obstruction to the existence of a smooth harmonic extension of φ on the interior of (F, g_F) is given by

$$P_F(\varphi) = -\tfrac{1}{16}\big(\delta^{g_F,h}\nabla^{\varphi^*h}\delta^{g_F,h}T\varphi - \delta^{g_F,h}\big(2\mathrm{Ric}_{g_F} - \tfrac{2}{3}\mathrm{Scal}_{g_F}\big)T\varphi + S(\delta^{g_F,h}T\varphi)\big),$$

where Ric_{g_F} is understood as an endomorphism of TF, and $\mathrm{Ric}_{g_F}T\varphi := T\varphi(\mathrm{Ric}_{g_F}(\cdot))$, and

$$S(X) := \sum_{i=1}^{4} \mathscr{R}^h_{X, T\varphi(e_i)} T\varphi(e_i).$$

Proposition 6.1. *For all $\varphi \in C^\infty(M^3, N)$,*

$$\pi_*\left(\delta^{g_F,h}\nabla^{\varphi^*h}\delta^{g_F,h}T(\pi^*\varphi)\right) = 4\delta_b^{\theta,h}\nabla^{\varphi^*h}\delta_b^{\theta,h}T\varphi,$$

$$\pi_*\left(\delta^{g_F,h}\left(2\mathrm{Ric}_{g_F} - \tfrac{2}{3}\mathrm{Scal}_{g_F}\right)T(\pi^*\varphi)\right) = -4\nabla_R^{\varphi^*h}R\varphi + 16\mathrm{Im}\big(\nabla_{T_{\overline{1}}}^{\varphi^*h}\big(\tau_1^{\overline{1}}T_{\overline{1}}\big)\big)\varphi,$$

and for X in TN,

$$\pi_*\left(S((\pi^*\varphi)^*X)\right) = 4S_b(\varphi^*X).$$

Proof. The first and third equalities are straightforward from the expression of g_F. The second equality comes from the fact that, see [Lee86],

$$\mathrm{Sch}_{g_F} = -\varpi^2 - \mathbf{S}\theta^2 + \tfrac{1}{2}\mathrm{Sch}_W - \tfrac{1}{2}\gamma(J\tau\cdot, \cdot) + \tfrac{1}{2}\mathbf{T}J \circ \theta,$$

where

$$\mathbf{T} = \tfrac{1}{3}\left(\tfrac{1}{4}d_b\mathrm{Scal}_W + i\delta\tau\right) \qquad \text{and} \qquad \mathbf{S} = \delta\mathbf{T} - |\mathrm{Sch}_W|^2 + |\tau|^2.$$

Indeed, since $\mathrm{Scal}_{g_F} = 3\mathrm{Scal}_W$ and $\mathrm{Sch}_W = \frac{1}{4}\mathrm{Scal}_W\gamma$, we have then

$$\begin{aligned}2\mathrm{Ric}_{g_F} - \tfrac{2}{3}\mathrm{Scal}_{g_F}g_F &= 4\mathrm{Sch}_{g_F} - \tfrac{1}{3}\mathrm{Scal}_{g_F}g_F\\ &= -4\varpi^2 - 4\mathbf{S}\theta^2 - 2\gamma(J\tau\cdot,\cdot) + 2\mathbf{T}J\circ\theta - \mathrm{Scal}_W i\varpi\circ\theta,\end{aligned}$$

which gives the second equality. ∎

From the latter comes directly the following.

THEOREM 6.2. *For all* $\varphi \in C^\infty(M^3, N)$,

$$\pi_*\left(P_F(\pi^*\varphi)\right) = \tfrac{1}{4}P_1(\varphi).$$

In particular, a map $\varphi : M \to N$ is CR-harmonic if and only if $\pi^*\varphi$ is conformal-harmonic.

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