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THE RANKIN–SELBERG INTEGRAL WITH A NON-UNIQUE MODEL FOR THE STANDARD $\mathcal{L}$-FUNCTION OF G_2

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In memory of Ilya Piatetski-Shapiro and Stephen Rallis

Abstract Let $\mathcal{L}^S(s, \pi, \mathfrak{st})$ be a partial $\mathcal{L}$-function of degree 7 of a cuspidal automorphic representation π of the exceptional group G_2. In this paper we construct a Rankin–Selberg integral for representations having a certain Fourier coefficient.

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1. Introduction

Until the late 1980s it was believed that a Rankin–Selberg integral must unfold to a unique model of the representation in order to be factorizable. By a unique model we mean one for which the space of functionals on the representation space with certain invariance properties is one dimensional. The commonest example is the Whittaker model, but other unique models such as the Bessel model have also been used.

In their pioneering work [14], Piatetski-Shapiro and Rallis interpreted an integral, earlier considered by Andrianov [1], as an adelic integral that unfolds to a non-unique model. Although the functional is not factorizable, the integral is, since the local integral produces the same $\mathcal{L}$-factor for **any** functional with the same invariance properties applied to a spherical vector.

There are many examples of adelic integrals that unfold with non-unique models. Only a few of them have been shown to represent $\mathcal{L}$-functions. Some more examples are detailed in [2, 3]. All of the examples rely on the knowledge of the generating function for the $\mathcal{L}$-function considered.

In this paper we consider a new Rankin–Selberg integral on the exceptional group G_2 and prove that it represents the standard $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathfrak{st})$ of degree 7 for cuspidal representations having a certain Fourier coefficient along the Heisenberg unipotent subgroup. The candidate global integral was suggested by Dihua Jiang in the course of the work on [8] and he also performed the unfolding. However, since the generating function for the $\mathcal{L}$-function was not known, the unramified computation was not completed. It is only now that we have found a way to overcome this difficulty. To the best of our

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knowledge this is the first time that the unramified computation has been performed without explicit knowledge of the generating function.

The integral introduced here binds the analytic behaviour of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ with that of a degenerate Eisenstein series of $Spin_8$ which was studied in [8]. In the last section we use information on the poles of this Eisenstein series to show that for a cuspidal representation π having a certain Fourier coefficient, the non-vanishing of the theta lift of π to the finite group scheme S_3 is equivalent to the $\mathcal{L}$-function having a double pole at $s = 2$.

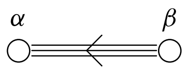
The Rankin–Selberg integral for the standard $\mathcal{L}$-function of generic representations of G_2 was constructed by Ginzburg in [9]. Recently Ginzburg and Hundley [10] have established the meromorphic continuation of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ for any cuspidal representation π using a doubling construction. Their integral representation shows that the set of poles of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ is contained in the set of poles of a degenerate Eisenstein series on the exceptional group of type E_8.

2. Preliminaries

Let k be a number field and $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ denote by k_ν the local field associated with ν. If $\nu < \infty$ denote by $\mathcal{O}_\nu$ the ring of integers of k_ν and by q_ν the cardinality of the residue field of k_ν. Let $\mathbb{A}$ denote the ring of adeles of k.

2.1. The group G_2

Let G be the split simple algebraic group of the exceptional type G_2 defined over k with maximal torus T and Borel subgroup B. Fix a root system of G and denote by α and β the short and the long simple roots respectively. The Dynkin digram of G has the form



and the set of positive roots is

$$\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

The fundamental weights are denoted by

$$\omega_1 = 2\alpha + \beta, \quad \omega_2 = 3\alpha + 2\beta.$$

For any root γ fix a one-parametric subgroup $x_\gamma : \mathbb{G}_a \to G$. For any simple root γ denote by w_γ the simple reflection with respect to it, that is an element of the Weyl group of G. Also define the coroot subgroups $h_\gamma : \mathbb{G}_m \to G$ such that for any root ϵ,

$$\epsilon\left(h_\gamma(t)\right) = t^{\langle \epsilon, \gamma^\vee \rangle}.$$

2.2. The partial $\mathcal{L}$-function

The dual Langlands group ${}^L G$ of G is isomorphic to $G_2(\mathbb{C})$. Denote the irreducible seven-dimensional complex representation of $G_2(\mathbb{C})$ by $\mathfrak{st}$. For an irreducible cuspidal representation $\pi = \otimes_v \pi_v$, unramified outside of a finite set of places S, the standard

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partial $\mathcal{L}$-function of π is defined by

$$\mathcal{L}^S(s,\pi,\mathfrak{st}) = \prod_{\nu\notin S} \frac{1}{\det\left(I - \mathfrak{st}(t_{\pi_\nu})q_\nu^{-s}\right)}.$$

Here t_{π_ν} is the Satake parameter of π_ν.

2.3. Fourier coefficients

The group G contains a Heisenberg parabolic subgroup $P = M \cdot U$. The Levi part M is isomorphic to GL_2 generated by the simple root α, while U is a five-dimensional Heisenberg group. We parametrize the elements of U by

$$u(r_1,r_2,r_3,r_4,r_5) := x_\beta(r_1)\, x_{\alpha+\beta}(r_2)\, x_{2\alpha+\beta}(r_3)\, x_{3\alpha+\beta}(r_4)\, x_{3\alpha+2\beta}(r_5).$$

The group M acts naturally on U and hence on $\mathrm{Hom}(U,\mathbb{G}_a)$. It was shown in [12] that for any field F of characteristic zero the $M(F)$-orbits of $\mathrm{Hom}(U(F),F)$ are naturally parametrized by isomorphism classes of cubic F-algebras.

Fixing an additive complex unitary character $\psi = \otimes_\nu \psi_\nu$ of $k\backslash\mathbb{A}$ this gives rise to the correspondence between $M(k)$-orbits of complex characters of $U(k)\backslash U(\mathbb{A})$ and cubic algebras over k. Let us denote by Ψ_s the character corresponding to the **split** cubic algebra $k\times k\times k$ and call it the **split character**. More explicitly,

$$\Psi_s(u(r_1,r_2,r_3,r_4,r_5)) = \psi(r_2+r_3).$$

Its stabilizer S_{Ψ_s} in $M(k)$ is isomorphic to S_3 and is generated by w_α and $h_\alpha(-1)\,x_\alpha(-1)\,x_{-\alpha}(1)$.

Denote by $\mathcal{A}(G)$ the space of automorphic forms on G. For any form φ in $\mathcal{A}(G)$ and complex character Ψ of $U(k)\backslash U(\mathbb{A})$, define the Fourier coefficient of φ with respect to (U,Ψ) by

$$L_\Psi(\varphi)(g) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\,\overline{\Psi(u)}\,du.$$

For any $g\in G$ this defines a functional $L_\Psi(\cdot)(g)$ in $\mathrm{Hom}_{U(\mathbb{A})}(\mathcal{A}(G),\mathbb{C}_\Psi)$.

For an automorphic representation π of $G(\mathbb{A})$ we say that π supports a (U,Ψ) coefficient if there exists a function φ from the underlying space of π such that $L_\Psi(\varphi)\not\equiv 0$.

It was shown in [6, Theorem 3.1] that for any cuspidal representation π there exists an étale cubic algebra such that π supports a Fourier coefficient with respect to this algebra. Conversely, in [8] it was shown that for any étale cubic algebra there exists a cuspidal representation supporting the Fourier coefficient corresponding to it. In this paper we consider only representations that support the split Fourier coefficient.

For a finite $\nu\in\mathcal{P}$ denote by K_ν the maximal compact subgroup $G(\mathcal{O}_\nu)$ of $G(k_\nu)$ and by $\mathcal{H}_\nu$ the corresponding spherical Hecke algebra. Given a complex character Ψ of $U(k_\nu)$ define

$$\mathcal{M}_\Psi = \left\{ f: G(k_\nu)\to\mathbb{C} \,\middle|\, f(ugk) = \overline{\Psi(u)}f(g)\ \ \forall u\in U(k_\nu),\, k\in K_\nu \right\}$$

$$\mathcal{M}_\Psi^0 = \left\{ f: G(k_\nu)\to\mathbb{C} \,\middle|\, f(sugk) = \overline{\Psi(u)}f(g)\ \ \forall u\in U(k_\nu),\, s\in S_\Psi, k\in K_\nu \right\}.$$

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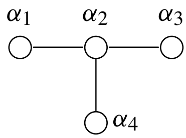
For $f \in \mathcal{H}_v$ define its Fourier transform f^{Ψ} with respect to the character Ψ by

$$f^{\Psi}(g) = \int_{U(k_v)} f(ug)\,\Psi(u)\,du.$$

Obviously f^{Ψ} belongs to $\mathcal{M}^0_{\Psi}$.

2.4. The group $Spin_8$

Let H be a simply connected split algebraic group of type D_4. We label its simple roots according to the following diagram.



The group of outer automorphisms of H is isomorphic to S_3. Fixing one-parametric subgroups, $x_\gamma : \mathbb{G}_a \to H$ defines a splitting of the sequence

$$1 \to H^{ad} \to Aut(H) \to Out(H) \to 1.$$

In particular the semidirect product $H \rtimes S_3$ can be formed. It is well known that the centralizer of S_3 in $H \rtimes S_3$ is isomorphic to the group G. We identify G with a subgroup of H in this way. The group H contains a maximal Heisenberg parabolic subgroup $P_H = M_H U_H$ such that $P = P_H \cap G$, given by

$$M_H \simeq \left\{ (g_1, g_2, g_3) \in GL_2 \times GL_2 \times GL_2 \,\middle|\, \det(g_1) = \det(g_2) = \det(g_3) \right\}.$$

The modulus character of P_H is given by $\delta_{P_H}(g_1, g_2, g_3) = |\det(g_1)|^5$.

2.5. The Eisenstein series

Consider the induced representation $I_H(s) := \operatorname{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})} \delta^s_{P_H}$. All induced representations in this paper are not normalized. For any K-finite standard section f_s define an Eisenstein series

$$E(g, f_s) = \sum_{\gamma \in P_H(k) \backslash H(k)} f_s(\gamma g).$$

It has a meromorphic continuation to the whole complex plane. The behaviour at $s = 4/5$ was studied in [8].

Proposition 2.1 ([8], Proposition 9.1). *For any standard section f_s, the Eisenstein series $E(g, f_s)$ has at most a double pole at $s = \frac{4}{5}$. The double pole is attained by the spherical section f^0_s. Also, the space*

$$Span_{\mathbb{C}} \left\{ \left(s - \frac{4}{5}\right)^2 E(g, f_s) \Big|_{s=\frac{4}{5}} \right\},$$

is isomorphic to the minimal representation Π of H.

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It is customary to define the normalized Eisenstein series

$$E^*(g, f_s) = j(s) E(g, f_s),$$

where

$$j(s) = \zeta(5s) \zeta(5s-1)^2 \zeta(10s-4).$$

3. The zeta integral

Let $\pi = \otimes \pi_\nu$ be an irreducible cuspidal representation of $G(\mathbb{A})$. For $\varphi \in \pi$ and a standard section $f_s \in I_H(s)$ we consider the following integral:

$$\mathcal{Z}(s, \varphi, f) = \int_{G(k)\backslash G(\mathbb{A})} \varphi(g) E^*(g, f_s)\, dg.$$

Since φ is cuspidal, and hence rapidly decreasing, the integral defines a meromorphic function on the complex plane. Our main result is the following.

Theorem 3.1. *Let $\pi = \otimes_\nu \pi_\nu$ be an irreducible cuspidal representation supporting the split Fourier coefficient. Let $\varphi = \otimes_\nu \varphi_\nu \in \pi$, $f_s = \otimes_\nu f_{s,\nu} \in I_H(s)$ be factorizable data. Let $S \subset \mathcal{P}$ be a finite set such that if $\nu \notin S$ then*

- $\nu \not| 2, 3, \infty$,
- $\Psi_{s,\nu}$ *is of conductor* $\mathcal{O}_\nu$,
- φ_ν *is spherical,*
- $f_{s,\nu}$ *is spherical.*

Then

$$\mathcal{Z}(s, \varphi, f) = \mathcal{L}^S(s, \pi, \mathfrak{st})\, d_S(s, \varphi_S, f_S).$$

Moreover for any s_0 there exist vectors φ_S, f_S such that $d_S(s, \varphi_S, f_S)$ is analytic in a neighbourhood of s_0 and $d_S(s_0, \varphi_S, f_S) \neq 0$.

In particular the partial $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathfrak{st})$ admits a meromorphic continuation.

Remark 3.1. If π does not support the split Fourier coefficient, the zeta integral vanishes identically. However if π supports a Fourier coefficient corresponding to an étale cubic algebra E there is a similar integral, using an Eisenstein series on the quasi-split form of $Spin_8$ corresponding to E, that is expected to represent the same $\mathcal{L}$-function. We plan to study these integrals in the near future.

The proof of the theorem will occupy the rest of the paper. In this section we will explain the main ideas, deferring the technical part to later sections and appendices.

Theorem 3.2 (unfolding)**.** *For $\mathfrak{Re}(s) \gg 0$ we have*

$$\mathcal{Z}(s, \varphi, f) = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_s}(\varphi)(g)\, F^*(g, s)\, dg, \tag{3.1}$$

where

$$F^*(g, s) = j(s) \int_{\mathbb{A}} f_s\left(w_2 w_3 x_{-\alpha_1}(1)\, x_{\alpha+\beta}(r)\right) \psi(r)\, dr.$$

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This computation was performed by Dihua Jiang, but since his proof was never published we include it in section 4.

The function $F^*(g, s)$ is factorizable whenever the section f_s involved is. In particular,

$$F^*(g, s) = \Pi_\nu F^*_\nu(g_\nu, s),$$

where

$$F^*_\nu(g, s) = j_\nu(s) \int_{k_\nu} f_{s,\nu}\left(w_2 w_3 x_{-\alpha_1}(1) x_{\alpha+\beta}(r) g\right) \psi_\nu(r)\, dr,$$

and for almost all places $f_{s,\nu} = f^0_{s,\nu} \in \mathrm{Ind}_{P_H(k_\nu)}^{H(k_\nu)} \delta^s_{P_H}$ is a spherical vector with $f^0_{s,\nu}(1) = 1$.

Note that as the space $\mathrm{Hom}_{U(\mathbb{A})}\left(\pi, \mathbb{C}_{\Psi_s}\right)$ is usually infinite dimensional, the functional L_{Ψ_s} is not necessarily factorizable. Nevertheless it will be shown that the integral $\mathcal{Z}(\varphi, f, s)$ is factorizable. The factorizability of the integral follows from the next surprising local statement, that replaces the unramified computation.

Theorem 3.3 (unramified computation). *Let π_ν be an irreducible unramified representation of $G(k_\nu)$ and let v_0 be a fixed spherical vector in π_ν. Assume that* $\mathrm{Hom}_{U(k_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{s,\nu}}\right) \neq 0$. *There exists $s_0 \in \mathbb{R}$ such that for any $\mathfrak{Re}\, s > s_0$ and* **any** *$l \in$* $\mathrm{Hom}_{U(k_\nu)}\left(\pi_\nu, \mathbb{C}_{\Psi_{s,\nu}}\right)$ *it holds that*

$$\int_{U(k_\nu)\backslash G(k_\nu)} l\left(\pi_\nu(g) v_0\right) F^*_\nu(g, s)\, dg = \mathcal{L}(5s - 2, \pi_\nu, \mathfrak{st})\, l(v_0), \tag{3.2}$$

where $F^*_\nu(g, s)$ is the function corresponding to the normalized spherical section f^0_ν.

The identity in Theorem 3.1 follows from eq. (3.2) using standard argument as in [14]. For the sake of completeness of presentation the argument is included in section 5. This argument also defines $d_S(s, \varphi_S, f_S)$ explicitly.

The proof of Theorem 3.3 is the most non-trivial part of the paper and can be found in section 7. In fact the proof is quite amusing. Following the ideas of [14] it boils down to proving the identity between $F^*_\nu(\cdot, s)$ and a Fourier transform of the generating function Δ of $L(s, \pi_\nu, \mathfrak{st})$. We could not find the explicit formula for Δ, which must be very complicated. Instead we have proven that the two functions become equal after being convolved with a third function. Both sides are evaluated explicitly (appendices A and B). Finally we show in Proposition 7.2 that the latter convolution is in fact an invertible operation.

Theorem 3.4 (ramified computation). *For any $s_0 \in \mathbb{C}$ there exist a datum φ_S and f_S such that $d_S(s, \varphi_S, f_S)$ is holomorphic and non-vanishing in a neighbourhood of s_0.*

This theorem is proven in section 8.

4. Unfolding

The proof of Theorem 3.2 is fairly standard. First we introduce some more notation that will be used in this section and also in section 8

Denote by $Q = LV$ the maximal parabolic subgroup of G other than P. The Levi part $L \simeq GL_2$ is generated by the root β. The unipotent radical of the Borel subgroup of L

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will be denoted by N_β. The unipotent radical V of Q is a three-step nilpotent group. Denote its commutator $[V, V]$ by R. It is generated by the subgroups $x_{2\alpha+\beta}, x_{3\alpha+\beta}$ and $x_{3\alpha+2\beta}$.

The following fact will be used [15, Theorem 5]:

$$\int_{R(k)\backslash R(\mathbb{A})} \varphi(rg)\,dr = \sum_{\nu\in N_\beta(k)\backslash L(k)} W_\psi(\varphi)(\nu g), \tag{4.1}$$

where $W_\psi(\varphi)$ is the standard Whittaker coefficient of φ.

There are five $G(k)$-orbits of $P_H(k)\backslash H(k)$. The representatives of the orbits and their stabilizers are given in the next lemma [13, Lemma 2.1].

Lemma 4.1. *The following is a list of representatives of $G(k)$-orbits in $P_H(k)\backslash H(k)$ and their stabilizers:*

(1) $\mu = 1$, *and the stabilizer* $G^\mu = P$.

(2) $\mu = w_2w_1, w_2w_3, w_2w_4$, *and the stabilizer* $G^\mu = LR$.

(3) $\mu = w_2w_3x_{-\alpha_1}(1)$ *is a representative of the open orbit. The stabilizer of* $P_H(k)\mu G(k)$ *is* $G^\mu = T^\mu \cdot U^\mu$ *where*

$$T^\mu = \left\{ h_{3\alpha+2\beta}(t) \,\middle|\, t\in k^\times \right\}, \quad U^\mu = \left\{ u(r_1, r_2, r_2, r_4, r_5) \,\middle|\, r_i \in k \right\}$$

Proof of Theorem 3.2. For $\mathfrak{Re}(s) \gg 0$ it holds that

$$\begin{aligned}\int_{G(k)\backslash G(\mathbb{A})} \varphi(g) E(g, f_s)\,dg &= \int_{G(k)\backslash G(\mathbb{A})} \varphi(g) \sum_{\gamma\in P_H(k)\backslash H(k)} f_s(\gamma g)\,dg \\ &= \sum_{\mu\in P_H(k)\backslash H(k)/G(k)} I_\mu(\varphi, f_s),\end{aligned}$$

where

$$I_\mu(\varphi, f_s) = \int_{G^\mu(k)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu g)\,dg.$$

Next we show that $I_\mu(\varphi, f_s) = 0$ unless μ is a representative of the open orbit.

(1) $\mu = 1$. Then

$$\begin{aligned} I_\mu(\varphi, f_s) &= \int_{P(k)\backslash G(\mathbb{A})} \varphi(g) f_s(g)\,dg \\ &= \int_{M(k)U(\mathbb{A})\backslash G(\mathbb{A})} f_s(g) \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\,du\,dg = 0,\end{aligned}$$

since φ is cuspidal.

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(2) $\mu = w_2w_1, w_2w_3, w_2w_4$. Then

$$I_\mu(\varphi, f_s) = \int_{L(k)R(k)\backslash G(\mathbb{A})} \varphi(g) f_s(\mu g)\, dg$$
$$= \int_{L(k)R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \int_{R(k)\backslash R(\mathbb{A})} \varphi(rg)\, dr\, dg.$$

Using eq. (4.1) this equals

$$\int_{L(k)R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g) \sum_{\nu \in N_\beta(k)\backslash L(k)} W_\psi(\varphi)(\nu g)$$
$$= \int_{N_\beta(\mathbb{A})R(\mathbb{A})\backslash G(\mathbb{A})} f_s(\mu g)\, W_\psi(\varphi)(g) \left(\int_{N_\beta(k)\backslash N_\beta(\mathbb{A})} \psi(n)\, dn\right) dg = 0.$$

Now let us compute the contribution from the open orbit. For $\mu = w_2w_3x_{-\alpha_1}(1)$ it holds that

$$I_\mu(\varphi, f) = \int_{T^\mu(k)U^\mu(\mathbb{A})\backslash G(\mathbb{A})} \left(\int_{U^\mu(k)\backslash U^\mu(\mathbb{A})} \varphi(ug)\, du\right) f_s(\mu g)\, dg.$$

Expanding the function given by an inner integral along the root $\alpha + \beta$ and collapsing the sum with the outer integration, the above equals

$$\int_{U^\mu(\mathbb{A})\backslash G(\mathbb{A})} \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\, \overline{\Psi_s(u)}\, du\, f_s(\mu g)\, dg. \tag{4.2}$$

Since $U = U^\mu \cdot x_{\alpha+\beta}$ we bring the integral to its final form

$$\int_{U(\mathbb{A})\backslash G(\mathbb{A})} \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\, \overline{\Psi_s(u)}\, du \int_{\mathbb{A}} f_s\big(\mu x_{\alpha+\beta}(r)\, g\big)\, \psi(r)\, dr\, dg$$
$$= \int_{U(\mathbb{A})\backslash G(\mathbb{A})} L_{\Psi_s}(\varphi)(g)\, \frac{F^*(g,s)}{j(s)}\, dg. \quad \square \tag{4.3}$$

5. Derivation of the Main Theorem from Theorem 3.3 and 3.4

Proof of Theorem 3.1. Let $\mu = w_2w_3x_{-\alpha_1}(1)$. By definition,

$$\mathcal{Z}(s, \varphi, f) = \varinjlim_{\substack{S \subset \Omega \subset \mathcal{P} \\ |\Omega| < \infty}} \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_s}(\varphi)(g)\, F^*_\Omega(g,s)\, dg\ , \tag{5.1}$$

where $G(\mathbb{A})_\Omega = \prod_{\nu\in\Omega} G(k_\nu)$ and

$$F^*_\Omega(g,s) = j_\Omega(s) \int_{k_\Omega} f_s(\mu x_{\alpha+\beta}(r) g) \psi(r)\, dr.$$

Fix $s_0 \in \mathbb{R}$ such that the right hand side of eq. (3.1) converges for $\mathfrak{Re} s > s_0$. The integrals of the right hand side of eq. (5.1) must also converge there. Also fix $s_1 \in \mathbb{R}$ such that eq. (3.2) holds for $\mathfrak{Re} s > s_1$. For a finite set $S \subseteq \Omega$ and $\nu \notin \Omega$ we have

$$\int_{U(\mathbb{A})_{\Omega\cup\{\nu\}} \backslash G(\mathbb{A})_{\Omega\cup\{\nu\}}} L_{\Psi_s}(\varphi)(g)\, F^*_{\Omega\cup\nu}(g,s)\, dg$$

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$$= \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} \int_{U(k_\nu)\backslash G(k_\nu)} L_{\Psi_s}(\varphi)(gg_\nu) F^*_{\Omega\cup\nu}(gg_\nu, s)\, dg_\nu\, dg$$
$$= \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} F^*_\Omega(g,s) \int_{U(k_\nu)\backslash G(k_\nu)} L_{\Psi_s}(\varphi)(gg_\nu) F^*_\nu(g_\nu, s)\, dg_\nu\, dg$$
$$= \mathcal{L}(5s-2, \pi_\nu, \mathfrak{st}) \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_s}(\varphi)(g) F^*_\Omega(g,s)\, dg,$$

where the last equality is due to Theorem 3.3. *A priori* the last equality holds only for $\Re\mathfrak{e}s > \max\{s_0, s_1\}$, but since $\mathcal{L}(5s-2, \pi_\nu, \mathfrak{st})$ is a meromorphic function the equality actually holds for $\Re\mathfrak{e}s > s_0$. Plugging this into equation (5.1) we get

$$\mathcal{Z}(s,\varphi,f) = \varinjlim_{\substack{S\subset\Omega\subset\mathcal{P}\\ |\Omega|<\infty}} \int_{U(\mathbb{A})_\Omega \backslash G(\mathbb{A})_\Omega} L_{\Psi_s}(\varphi)(g) F^*_\Omega(g,s)\, dg$$
$$= \varinjlim_{\substack{S\subset\Omega\subset\mathcal{P}\\ |\Omega|<\infty}} \prod_{\nu\in\Omega\backslash S} \mathcal{L}(5s-2, \pi_\nu, \mathfrak{st}) \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_s}(\varphi)(g) F^*_S(g,s)\, dg$$
$$= \mathcal{L}^S(5s-2, \pi, \mathfrak{st}) \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_s}(\varphi)(g) F^*_S(g,s)\, dg.$$

We finish the proof by fixing our datum according to Theorem 3.4 and taking

$$d_S(s, \varphi_S, f_S) = \int_{U(\mathbb{A})_S \backslash G(\mathbb{A})_S} L_{\Psi_s}(\varphi)(g) F^*_S(g,s)\, dg. \quad \square$$

6. The generating function

Let $F = k_\nu$ with the ring of integers $\mathcal{O}$ and uniformizer ϖ for some $\nu \notin S$. By abuse of notation we drop ν from the notation and write in this section, and in section 7 and appendices A and B, π for π_ν, ψ for ψ_ν etc.

Recall that $G(F)$ contains the maximal compact subgroup $K = G(\mathcal{O})$. We fix on G the Haar measure μ such that $\mu(K) = 1$. Let $\mathcal{H} = \mathcal{H}(G,K)$ be the spherical Hecke algebra of G. In this section we construct a generating function $\Delta \in \mathcal{H}\left[\left[q^{-s}\right]\right]$ for the local $\mathcal{L}$-function $\mathcal{L}(s,\pi,\mathfrak{st})$.

Recall that the Satake isomorphism is an isomorphism of $\mathbb{C}$-algebras $\mathcal{H} \cong Rep({}^LG)$. Denote by $A_j \in \mathcal{H}$ the elements corresponding to $Sym^j(\mathfrak{st})$ under the Satake isomorphism. In particular, for any unramified representation π and a K-invariant vector $v_0 \in \pi$ it holds that

$$\int_G A_j(g)\pi(g) v_0\, dg = tr\left(Sym^j(\mathfrak{st})(t_\pi)\right) v_0, \tag{6.1}$$

where t_π is the Satake parameter of π.

For any unramified representation π the Satake isomorphism induces an algebra homomorphism that sends $f \in \mathcal{H}$ to the complex number $\hat{f}(\pi)$ such that

$$\int_G f(g)\pi(g) v_0\, dg = \hat{f}(\pi) v_0.$$

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In particular for $f_1, f_2 \in \mathcal{H}$ it holds that $\widehat{f_1 * f_2} = \hat{f}_1 \cdot \hat{f}_2$. The homomorphism $f \to \hat{f}(\pi)$ can be extended linearly to a map of formal power series algebras $\mathcal{H}[[T]] \to \mathbb{C}[[T]]$. Let $T = q^{-s}$.

Theorem 6.1. *There exists a generating function $\Delta(g,s) \in \mathcal{H}[[q^{-s}]]$, uniformly converging on a right half-plane, such that for any unramified representation π with a spherical vector v_0 and* **any** *functional l on π, it holds that*

$$\int_G \Delta(g,s)\, l(\pi(g)v_0)\, dg = \mathcal{L}(s,\pi,\mathfrak{st})\, l(v_0) \tag{6.2}$$

for $\mathfrak{Re}\, s \gg 0$.

Proof. We must show that there exists Δ with $\hat{\Delta}(\pi,s) = \mathcal{L}(s,\pi,\mathfrak{st})$. The construction is formal. By the well known Poincaré identity,

$$\mathcal{L}(s,\pi,\mathfrak{st}) = \frac{1}{\det\left(1 - q^{-s}\mathfrak{st}(t_\pi)\right)} = \prod_{i=1}^{7}\left(1 - q^{-s}\mathfrak{st}(t_\pi)_{ii}\right)^{-1}$$
$$= \prod_{i=1}^{7}\sum_{j=0}^{\infty}\left(q^{-s}\mathfrak{st}(t_\pi)_{ii}\right)^{j} = \sum_{j=0}^{\infty} tr\left(Sym^{j}(\mathfrak{st}(t_\pi))\right) q^{-js},$$

where t_π is the Satake parameter of π. The series converges absolutely for $\mathfrak{Re}\, s \gg 0$. Plugging eq. (6.1) into the previous equality gives

$$\mathcal{L}(s,\pi,\mathfrak{st})\, l(v_0) = l\left(\sum_{j=0}^{\infty}\left(\int_G A_j(g)\,\pi(g)v_0 q^{-js} dg\right)\right). \tag{6.3}$$

Ignoring for the moment the convergence issue and exchanging formally the sum and the integral, we obtain

$$\mathcal{L}(s,\pi,\mathfrak{st})\, l(v_0) = l\left(\int_G \left(\sum_{j=0}^{\infty} A_j(g)\, q^{-js}\right)\pi(g)v_0 dg\right)$$

for $\mathfrak{Re}(s) \gg 0$. The assertion then holds for $\Delta(\cdot,s) = \sum_{j=0}^{\infty} A_j q^{-js}$ for any unramified representation π. Uniqueness follows from the fact that the action of the spherical functions of unramified representations gives rise to a spectral decomposition of $\mathcal{H}$.

It remains to justify the exchange of the sum and the integral in eq. (6.3).

Let us introduce some standard notation. Let Λ denote the cocharacter lattice of g and for any $\gamma \in \Lambda$ denote by t_γ its representative in the maximal torus T. Let Λ^+ denote the set of dominant coweights. There is a partial order on Λ^+: $\gamma \leqslant \lambda$ if and only if $\lambda - \gamma$ can be written as a non-negative combination of the positive coroots. Let ρ be half of the sum of all the positive roots.

Making use of the Cartan decomposition $G = K\Lambda^+K$ we obtain

$$\sum_{j=0}^{\infty}\left(\int_G A_j(g)\cdot\pi(g)\, v_0 dg\right) q^{-js} = \sum_{j=0}^{\infty}\sum_{\gamma\in\Lambda^+} A_j\left(t_\gamma\right)\cdot\omega_\pi(t_\gamma)\mu\left(Kt_\gamma K\right) q^{-js} v_0,$$

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where $\omega_\pi(g) = \langle v_0^\vee, \pi(g)v_0\rangle$ is the normalized spherical function. Here $v_0^\vee$ is the K-fixed vector in $\pi^\vee$ such that $\langle v_0^\vee, v_0\rangle = 1$.

Let us show that this double series converges absolutely for any π and hence it is possible to interchange the order of the summations. For this purpose we shall produce a bound for each term.

Lemma 6.1. (1) *For any* $j \geqslant 0$,

$$\left|\left\{\gamma \in \Lambda^+ \middle| \gamma \leqslant [j, 0]\right\}\right| \leqslant (j+1)(2j+1).$$

(2) $A_j(t_\gamma) = 0$ *unless* $\gamma \leqslant [j, 0]$.

(3) *Assume that* $\gamma \leqslant [j, 0]$. *Then there exist constants* $C_1, C_2, C_3, z > 0$ *such that*

$$\begin{cases} |A_j(t_\gamma)| \leqslant C_1 j^7 \\ |\omega_\pi(t_\gamma)| \leqslant C_2 q^{jz} \\ \mu(K t_\gamma K) \leqslant C_3 q^{6j}. \end{cases}$$

Proof. For any dominant coweight λ, that can be simultaneously regarded as a dominant weight of the dual group, denote by $A_\lambda \in \mathcal{H}$ the function corresponding to the highest weight irreducible representation V_λ of ${}^L G$ via the Satake isomorphism.

Let $\gamma = n\alpha^\vee + m\beta^\vee$. Then $\gamma \leqslant [j, 0] \Rightarrow n \leqslant j, m \leqslant 2j$. Obviously, the number of such roots γ is bounded by $(j+1)(2j+1)$; this proves (1).

Recall from [15, page 836] that the j-symmetric algebra of $\mathfrak{st}$ decomposes as follows:

$$Sym^j(\mathfrak{st}) = \bigoplus_{\substack{k=0 \\ k \equiv j \pmod 2}}^{j} V_{[k,0]}.$$

Hence $A_j(g) = \sum_{\substack{k=0 \\ k \equiv j \pmod 2}}^{j} A_{[k,0]}(g)$. According to [11, § 4],

$$A_\lambda(t_\gamma) = \begin{cases} q^{-(\lambda,\rho)} P_{\lambda,\gamma}(q) & \gamma \leqslant \lambda \\ 0 & \text{otherwise}, \end{cases}$$

where $P_{\lambda,\gamma}$ is an affine Kazhdan–Lustigue polynomial of degree at most $(\lambda - \gamma, \rho)$ with non-negative integral coefficients. (2) follows immediately.

We now prove (3). In particular, for $\gamma \leqslant \lambda$ it holds that

$$0 \leqslant A_\lambda(t_\gamma) = q^{-(\lambda,\rho)} P_{\lambda,\gamma}(q) \leqslant P_{\lambda,\gamma}(1) = \dim V_\lambda(\gamma) \leqslant \dim V_\lambda.$$

By the Weyl character formula

$$\dim V_\lambda = \Pi_{\alpha>0} \frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)},$$

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there exists $C_1 > 0$ such that $\dim V_{[k,0]} \leqslant C_1 k^6$. In particular, for $\gamma \leqslant [j,0]$ one has

$$A_j\left(t_\gamma\right) = \sum_{\substack{k=0 \\ k=j (\mathrm{mod}\ 2)}}^{j} A_k\left(t_\gamma\right) \leqslant C_1 j^7.$$

This proves the first bound in (3).

Let us prove the second bound in (3). Assume that the unramified representation π is a constituent of $\mathrm{Ind}_B^G \chi$ where χ is an unramified character. There exists a constant $z > 0$ such that for any $w \in W$ one has

$$\left|w\chi\left(h_\alpha(\varpi)\right)\right|, \left|w\chi\left(h_\beta(\varpi)\right)\right| \leqslant q^z.$$

Then for $\gamma = n\alpha^\vee + m\beta^\vee \leqslant [j,0]$ one has $\left|w\chi\left(t_\gamma\right)\right| \leqslant q^{jz} \cdot q^{2jz} = q^{3jz}$.

Hence by Macdonald's formula [4, Theorem 4.2] there exists $C_2 > 0, z > 0$ such that

$$\left|\omega_\pi\left(t_\gamma\right)\right| < C_2 q^{3jz}.$$

Finally it is known that for $\gamma = n\alpha^\vee + m\beta^\vee$ one has $\mu\left(Kt_\gamma K\right) = C_3 q^{2n+2m}$ for some constant $C_3 > 0$. For $\gamma \leqslant [j,0]$ it is bounded by $C_3 q^{6j}$ and hence (3) is proven. □

Taking all the bounds into account we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{t_\gamma \in \Lambda^+} \left|A_j\left(t_\gamma\right)\right| \cdot \left|\omega_\pi(t_\gamma)\right| \mu\left(Kt_\gamma K\right) \|v_0\| \left|q^{-js}\right| \\
&\leqslant \sum_{j=0}^{\infty} \sum_{\gamma \leqslant [j,0]} \left|A_j\left(t_\gamma\right)\right| \cdot \left|\omega_\pi\left(t_\gamma\right)\right| \mu\left(Kt_\gamma K\right) \|v_0\| q^{-j\mathfrak{Re}(s)} \\
&\leqslant \sum_{j=0}^{\infty} \sum_{\gamma \leqslant [j,0]} C_1 j^7 \cdot C_2 q^{3jz} \cdot C_3 q^{6j} \cdot q^{-j\mathfrak{Re}(s)} \|v_0\| \leqslant C \cdot \sum_{j=0}^{\infty} j^9 \cdot q^{(6+3z-\mathfrak{Re}(s))j} \|v_0\|
\end{aligned}$$

which converges absolutely for $\mathfrak{Re}(s) \gg 0$.

Bounds in the lemma ensure that for $\mathfrak{Re}(s) \gg 0$ the series $\sum_{j=0}^{\infty} A_j(g) q^{-js}$ converges absolutely and uniformly (in G) and hence the function $\Delta(g,s)$ is defined for any s in some right half-plane. □

7. Unramified computation

In this section we prove Theorem 3.3. For any $l \in \mathrm{Hom}_{U(F)}\left(\pi, \mathbb{C}_{\Psi_s}\right)$ one has

$$\mathcal{L}(s,\pi,\mathfrak{st})\, l(v_0) = \int_G l\left(\pi(g)v_0\right) \Delta(g,s)\, dg = \int_{U \backslash G} l\left(\pi(g)v_0\right) \Delta^{\Psi_s}(g,s)\, dg.$$

Thus, in order to prove that for all unramified π and all $l \in \mathrm{Hom}_U\left(\pi, \mathbb{C}_{\Psi_s}\right)$ eq. (3.2) holds, it is enough to show that for $\mathfrak{Re}s \gg 0$,

$$\Delta^{\Psi_s}(g, 5s-2) = F^*(g,s). \tag{7.1}$$

Remark 7.1. Recall that the integrand of

$$\int_{U\backslash G} l\left(\pi\left(g\right)v_0\right)F\left(g,s\right)dg$$

is U-invariant and $l\in \mathrm{Hom}_U(\pi,\mathbb{C}_{\Psi_s})$; hence $F(g,s)\in\mathcal{M}_{\Psi_s}$.

While the right hand side of eq. (7.1) is given explicitly, we do not have an explicit formula for the generating function $\Delta(g,s)$.

To overcome that difficulty we introduce a new function $D\in\mathcal{H}\left[\left[q^{-s}\right]\right]$. Recall the Cartan decomposition $G=KT^{+}K$ where

$$T^{+}=\left\{t\in T\,\middle|\,\left|\gamma\left(t\right)\right|\leqslant 1\ \forall\gamma\in\Phi^{+}\right\}.$$

The function D is the bi-K-invariant defined on the torus T^{+} by

$$D(t,s)=\left|\omega_1(t)\right|^{5s+1}\qquad\forall t\in T^{+}.$$

The relation between D and Δ can be seen from the following proposition.

Proposition 7.1. *There exists $P\left(\cdot,s\right)\in\mathcal{H}\left[q^{-s}\right]$ and $s_0\in\mathbb{R}$ such that for $\mathfrak{Res}>s_0$ it holds that*

$$D\left(\cdot,s\right)=\Delta\left(\cdot,5s-2\right)*P\left(\cdot,s\right).$$

More precisely,

$$P\left(\cdot,s\right)=\frac{P_0\left(q^{2-5s}\right)A_0-P_1\left(q^{2-5s}\right)A_1}{\zeta\left(5s-1\right)\zeta\left(5s+1\right)\zeta\left(5s-2\right)},\tag{7.2}$$

where

$$P_0\left(z\right)=\frac{z^4}{q^2}+\left(\frac{1}{q^2}+\frac{1}{q}\right)z^3+\frac{z^2}{q}+\left(\frac{1}{q}+1\right)z+1,\qquad P_1\left(z\right)=\frac{z^2}{q}.$$

Proof. Let $v_0{}^{\vee}\in\pi^{\vee}$ be the unramified vector such that $\left\langle v_0{}^{\vee},v_0\right\rangle=1$ and let ω_π be the normalized spherical function associated with π given by

$$\omega_\pi\left(g\right)=\left\langle v_0{}^{\vee},\pi\left(g\right)v_0\right\rangle.$$

For any functional l of π one has

$$\int_G D\left(g,s\right)l\left(\pi(g)v_0\right)\,dg=l\left(v_0\right)\int_G D\left(g,s\right)\omega_\pi\left(g\right)\,dg.$$

Using Macdonald's formula [4, Theorem 4.2] for ω_π, this integral turns into a sum of geometric progressions that converges for $\mathfrak{Res}\gg 0$. A direct computation yields

$$\hat{D}\left(\pi,s\right)=\int_G D\left(g,s\right)\omega_\pi\left(g\right)\,dg=\mathcal{L}\left(5s-2,\pi,\mathfrak{st}\right)\cdot Q\left(\pi,s\right).\tag{7.3}$$

Here

$$Q\left(\pi,s\right)=\frac{P_0\left(q^{2-5s}\right)-P_1\left(q^{2-5s}\right)tr\left(\mathfrak{st}\right)\left(t_\pi\right)}{\zeta\left(5s-1\right)\zeta\left(5s+1\right)\zeta\left(5s-2\right)}.\tag{7.4}$$

On the other hand, $\mathcal{L}\left(5s-2,\pi,st\right)=\hat{\Delta}\left(\pi,5s-2\right)$ and obviously $Q\left(\pi,s\right)=\hat{P}\left(\pi,s\right)$. Since eq. (7.3) holds for any unramified π, the proposition follows. □

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Since the Fourier transform is a map of $\mathcal{H}$ modules, we have the following corollary:

Corollary 7.1.

$$D^{\Psi_s}(s) = \Delta^{\Psi_s}(5s-2) * P(s).$$

The basic identity Remark (7.1) will follow once we prove

$$D^{\Psi_s} = F^* * P \tag{7.5}$$

and:

Proposition 7.2. *There exists s_0 such that $f * P(\cdot, s) \equiv 0$ implies $f \equiv 0$ for any $f \in \mathcal{M}_{\Psi_s}$ whenever $\mathfrak{Res} > s_0$*

Indeed from eq. (7.5) we get

$$\left(\Delta^{\Psi_s}(\cdot, 5s-2) - F^*(\cdot, s)\right) * P(\cdot, s) = \left(D^{\Psi_s} - F^*\right) * P = 0$$

and hence by Proposition 7.2 we have $\Delta^{\Psi_s} = F^*$ for $\mathfrak{Res} \gg 0$. We now turn to proving Proposition 7.2 and eq. (7.5).

The following observation is useful for the proof of Proposition 7.2.

Remark 7.2. We note that $\mathcal{H}$ can be completed into a C*-algebra $\hat{\mathcal{H}}$ as a closed subspace of the reduced group C*-algebra of G. One way to do this is to use the action of $\mathcal{H}$ on $L^2(K \setminus G/K)$ by convolution. This is a separable Hilbert space and $\mathcal{H}$ admits an embedding into $\mathcal{B}\left(L^2(K \setminus G/K)\right)$ in which we complete it with respect to the operator norm. In fact, for our needs we only need to know that a C*-norm and such a completion exist.

Proof of Proposition 7.2. We will show a stronger statement: there exists s_0 such that for any $\mathfrak{Re}(s) > s_0$ the element $P(\cdot, s)$ is invertible in $\hat{\mathcal{H}}$. For $\mathfrak{Re}(s) \gg 0$ this is equivalent to showing that

$$A_0 - \frac{P_1\left(q^{2-5s}\right)}{P_0\left(q^{2-5s}\right)} A_1$$

is invertible. Since $\hat{\mathcal{H}}$ is a C*-algebra it will suffice to show that $\left\| \frac{P_1(q^{2-5s})}{P_0(q^{2-5s})} A_1 \right\| < 1$. We have

$$\left\| \frac{P_1\left(q^{2-5s}\right)}{P_0\left(q^{2-5s}\right)} A_1 \right\| = \left| \frac{P_1\left(q^{2-5s}\right)}{P_0\left(q^{2-5s}\right)} \right| \|A_1\|$$

and since

$$\lim_{\mathfrak{Re}(s)\to\infty} P_0(q^{2-5s}) = 1 \quad \text{and} \quad \lim_{\mathfrak{Re}(s)\to\infty} P_1(q^{2-5s}) = 0,$$

there exists s_0 such that for $\mathfrak{Re}(s) > s_0$ we have

$$\left| \frac{P_1\left(q^{2-5s}\right)}{P_0\left(q^{2-5s}\right)} \right| < \frac{1}{\|A_1\|}. \quad \square$$

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It remains to verify eq. (7.5). We shall evaluate both functions explicitly and miraculously get the same answer. Recall that $S_{\Psi_s} = Stab_M(\Psi_s) \simeq S_3$ is generated by w_α and $h_\alpha(-1)\, x_\alpha(-1)\, x_\alpha(1)$.

Theorem 7.1. *For all $g \in G(F)$ and $\mathfrak{Re}(s) \gg 0$,*

$$D^{\Psi_s}(g,s) = (F^* * P)(g,s) \tag{7.6}$$

More precisely, both functions vanish outside of $S_{\Psi_s}UTK$. For $t = h_\alpha(t_1)h_\beta(t_2) \in T$ the values of both functions at t equal

$$\begin{cases} \frac{1+q^{1-5s}}{\zeta(5s+1)} \left|\frac{t_2}{t_1}\right| |t_1|^{5s}, & \left|\frac{t_1^2}{t_2}\right| < 1 \\ \frac{1+q^{1-5s}}{\zeta(5s+1)} \left|\frac{t_2}{t_1}\right|^{5s} |t_1|, & \left|\frac{t_1^2}{t_2}\right| > 1 \\ \frac{1+2q^{1-5s}}{\zeta(5s+1)} |t_1|^{5s+1}, & \left|\frac{t_1^2}{t_2}\right| = 1. \end{cases}$$

if $t_1, \frac{t_2}{t_1} \in \mathcal{O}$ and zero otherwise.

For the right hand side of equation (7.6) we first compute explicitly the function $F_s = \frac{F_s^*}{j(s)}$ and then perform the convolution. This tedious, but quite straightforward, computation is performed in appendix A.

Now let us explain how to evaluate the left hand side. Let SO_7 be the special orthogonal group viewed as a subgroup of GL_7, preserving the split symmetric form $(\delta_{i,7-i})$. Fix an embedding $\iota : G(F) \to SO_7(F)$ as in [13]. In appendix B we give a realization of this map. Define a function $\Gamma : G(F) \to \mathbb{R}$ by

$$\Gamma(g) = \max_{1\leqslant i,j\leqslant 7} \left|\iota(g)_{i,j}\right|.$$

The following result is easily checked.

Lemma 7.1. *Γ is a bi-K-invariant function and for $t \in T^+$,*

$$\Gamma(t) = |\omega_1(t)|^{-1}.$$

Thus $D(g,s) = \sum_{k=0}^{\infty} D_k(g)\, q^{-(5s+1)k}$, where

$$D_k(g) = \begin{cases} 1, & \Gamma(g) = q^k \\ 0, & \text{otherwise.} \end{cases}$$

For any $g \in G$ define $U_k(g) = \left\{u \in U : \Gamma(ug) \leqslant q^k\right\}$ and let

$$E_k(g) = \begin{cases} 1, & \Gamma(g) \leqslant q^k \\ 0, & \text{otherwise.} \end{cases}$$

Obviously,

$$D_k(g) = E_k(g) - E_{k-1}(g)$$

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and, in particular,

$$D^{\Psi_s}(g,s)=\sum_{k=0}^{\infty}(E_k^{\Psi_s}(g)-E_{k-1}^{\Psi_s}(g))q^{-(5s+1)k},$$

where

$$E_k^{\Psi_s}(g)=\int_U \mathbb{1}_{U_k(g)}(ug)\,\Psi_s(u)\,du=\int_{U_k(g)}\Psi_s(u)\,du.$$

The computation of $E_k^{\Psi_s}(g)$ can be further reduced to a calculation of volumes of certain sets. For a given g there are at most two values of k for which $E_k^{\Psi_s}(g)\neq 0$. The detailed computation is performed in appendix B.

8. Ramified computation

In this section we prove Theorem 3.4.

Recall from Theorem 3.2 that for the representative of the open orbit μ,

$$d_S(s,\varphi,f_s)=\int_{U_S\backslash G_S}L_{\Psi_s}(\varphi)(g)\,F^*(g,s)\,dg=j(s)\int_{U_S^{\mu}\backslash G_S}L_{\Psi_s}(\varphi)(g)\,f_s(\mu g)\,dg.$$

For $p\in G_\mu(k_\nu)$ define

$$\chi_s(p)=\delta_{P_H}^s(\mu p\mu^{-1}).$$

Since μ generates the open double coset in H, there is an inclusion

$$i:\mathrm{ind}_{G^{\mu}(k_S)}^{G(k_S)}\chi_s\hookrightarrow \mathrm{Ind}_{P_H(k_S)}^{H(k_S)}\delta_{P_H}^s$$

defined by $i(f_s)(p\mu g)=\delta_{P_H}^s(p)\,f_s(g)$, and $i(f)$ vanishes on all other double cosets $P(k_S)\,\mu' G(k_S)$.

For any $\phi_S\in\mathcal{S}(k_S)$ define an action of ϕ_S on π by

$$\phi_S*\varphi=\int_{k_S}\phi_S(r)\,\pi_S\left(x_{2\alpha+\beta}(r)\right)\varphi\,dr.$$

It is easy to see that

$$L_{\Psi_s}\left(\pi_S\left(h_{3\alpha+2\beta}(t)\right)(\phi_S*\varphi)\right)=\hat{\phi}_S(t)\,L_{\Psi_s}\left(\pi_S\left(h_{3\alpha+2\beta}(t)\right)\varphi\right),$$

where $\hat{\phi}_S$ is a Fourier transform of ϕ_S.

Let us write

$$J_S(s,\varphi)=\int_{k_S^{\times}}L_{\Psi_s}(\varphi)(t)\,|t|^{5s}\,d^{\times}t.$$

Lemma 8.1. *For any $s_0\in\mathbb{C}$ and any $\varphi\in\pi$ such that $L_{\Psi_s}(\varphi)\neq 0$, there exists $\phi_S\in\mathcal{S}(k_S)$ such that $J_S(s,\phi_S*\varphi)\neq 0$ around s_0.*

Proof. One has

$$\int_{k_S^{\times}}L_{\Psi_s}(\phi_S*\varphi)(t)\,|t|^{5s}\,d^{\times}t=\int_{k_S^{\times}}\hat{\phi}_S(t)\;L_{\Psi_s}(\varphi)(t)\,|t|^{5s}\,d^{\times}t.$$

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Since the image of $k_S^\times$ inside k_S is locally closed we may choose ϕ_S such that $\hat{\phi}_S$ is supported on a relatively compact neighbourhood of $1 \in k_S^\times$. Choose ϕ_S such that the support of $\hat{\phi}_S$ is sufficiently small to ensure the non-vanishing of $J_S(s, \phi_S * \varphi)$ around s_0. □

Consider the decomposition

$$G_S = G_S^\mu \cdot T_S^c \cdot U_S^c \cdot K_S, \quad T_S^c = \left\{ x_\beta(t) \middle| t \in k_S^\times \right\}, \quad U_S^c = \left\{ x_\alpha(r_1) x_{\alpha+\beta}(r_2) \middle| r_i \in k_S \right\}.$$

For any Schwarz function Φ_S on $(T^c \cdot U^c \cdot K)_S$ define $f_s(\Phi_S) \in \mathrm{ind}_{G_S^\mu}^{G_S} \chi_s$ by

$$f_s(\Phi_S)(g_\mu g) = \chi_s(g_\mu) \Phi_S(g).$$

Then for any $\varphi \in \pi$ it holds that

$$d_S(s, \varphi, f(\Phi)_s) = J_S(s, \Phi_S * \varphi) \tag{8.1}$$

By the Dixmier–Malliavin theorem [5] there exists a Schwarz function Φ_S on $(T^c \cdot U^c \cdot K)_S$ and $\varphi \in \pi$ such that $L_{\Psi_s}(\Phi_S * \varphi) \neq 0$. Then for any $\phi_S \in \mathcal{S}(k_S)$,

$$d_S(s, \phi_S * \varphi, f(\Phi_S)) = J_S(s, \phi_S * (\Phi * \varphi)).$$

By Lemma 8.1 there exists a Schwarz function $\phi_S \in \mathcal{S}(k_S)$ such that $d_S(s, \phi_S * \varphi, f_s(\Phi_S))$ is an entire function and does not vanish in a neighbourhood of s_0.

9. Application — the Θ-lift for the dual pair (S_3, G_2)

The theta correspondence Θ_H for the dual pair (S_3, G_2) in the group $H \rtimes S_3$ has been studied in [8]. The minimal representation Π of H can be extended to the group $H \rtimes S_3$. A cuspidal representation π belongs to the image of Θ_H if

$$\int_{G(k)\backslash G(\mathbb{A})} \varphi(g) F(g)\, dg \neq 0$$

for φ in the space π and F in the space of the minimal representation Π. It was proven in [8] that any such representation π supports the split Fourier coefficient. Besides, π is a non-tempered representation and $\mathcal{L}^S(\pi, s, \mathfrak{st})$ has a double pole at $s = 2$. Taking the residue (of depth 2) at $s = 2$ for the main equality, we obtain the converse, i.e. the double pole of the standard $\mathcal{L}$-function at $s = 2$ characterizes the image of Θ_H. In other words:

Theorem 9.1. *For a cuspidal representation π of $G(\mathbb{A})$ that supports the split Fourier coefficient, the following statements are equivalent:*

(1) $\mathcal{L}^S(s, \pi, \mathfrak{st})$ *has a double pole at* $s = 2$.

(2) $\Theta_H(\pi) \neq 0$.

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A. Computing $F(\cdot, s) * P(\cdot, s)$

Recall that

$$P(s) = \frac{P_0 A_0 + P_1 A_1}{\zeta(5s-1)\,\zeta(5s+1)\,\zeta(5s-2)}$$

and by [11, p. 231]

$$A_0 = \mathbb{1}_K, \quad A_1 = q^{-3}\left(\mathbb{1}_K + \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right);$$

hence

$$F^*(\cdot, s) * P(\cdot, s)$$
$$= j(s)\,\frac{\left(P_0(s) - q^{-3}P_1\left(q^{-s}\right)\right)F(\cdot, s) - q^{-3}P_1\left(q^{2-5s}\right)F(\cdot, s) * 1_{K\omega_2^\vee(\varpi)K}(\cdot)}{\zeta(5s-1)\,\zeta(5s+1)\,\zeta(5s-2)}. \quad \text{(A 1)}$$

We shall compute each summand separately. In this section we prove the following result.

Proposition A.1. *The following hold for* $(F(\cdot, s) * P(\cdot, s))$.

(1) $(F^*(\cdot, s) * P(\cdot, s)) \in \mathcal{M}^0_{\Psi_s}$.

(2) $(F^*(\cdot, s) * P(\cdot, s))(g) = 0$ *unless* $g \in S_{\Psi_s}UTK$.

(3) *Let* $t = h_\alpha(t_1)\,h_\beta(t_2) \in T$. *If* $t_1, \frac{t_2}{t_1} \in \mathcal{O}$ *it holds that*

$$\left(F^*(\cdot, s) * P(\cdot, s)\right)(t) = \begin{cases} \frac{1+q^{1-5s}}{\zeta(5s+1)}\left|\frac{t_2}{t_1}\right| |t_1|^{5s}, & \left|\frac{t_1^2}{t_2}\right| < 1 \\ \frac{1+q^{1-5s}}{\zeta(5s+1)}\left|\frac{t_2}{t_1}\right|^{5s} |t_1|, & \left|\frac{t_1^2}{t_2}\right| > 1\,; \\ \frac{1+2q^{1-5s}}{\zeta(5s+1)}|t_1|^{5s+1}, & \left|\frac{t_1^2}{t_2}\right| = 1 \end{cases} \quad \text{(A 2)}$$

otherwise $(F^*(\cdot, s) * P(\cdot, s))(t) = 0$.

A.1. The spaces $\mathcal{M}_{\Psi_s}, \mathcal{M}^0_{\Psi_s}$

In this subsection we list some properties of the spaces $\mathcal{M}_{\Psi_s}$ and $\mathcal{M}^0_{\Psi_s}$, defined in section 2, which will be used in this section and in appendix B. By Iwasawa decomposition any function $H \in \mathcal{M}_{\Psi_s}$ is determined by the values that it attains on $B_M/(B_M \cap K)$,

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i.e. on the elements

$$g = h_\alpha(t_1)\, h_\beta(t_2)\, x_\alpha(d),$$

where $d \in F/\mathcal{O}$. In this appendix if $d \in \mathcal{O}$ we choose $d = 1$ as a representative. In appendix B if $d \in \mathcal{O}$ we choose $d = 0$ as a representative.

Note that for any positive root γ and $|d| \geqslant 1$ one has

$$x_{-\gamma}(d) = x_\gamma(d^{-1})h_\gamma(d^{-1})k$$
$$x_\gamma(d) = h_\gamma(d)\, x_{-\gamma}(d)\, k', \tag{A 3}$$

for some $k, k' \in K$.

Using the invariance properties one easily checks the following lemma.

Lemma A.1. *Let $g = h_\alpha(t_1)\, h_\beta(t_2)\, x_\alpha(d)$.*

(1) *Let $H \in \mathcal{M}_{\Psi_s}$. Then $H(g) = 0$ unless*

$$t_1,\ dt_1 + \frac{t_2}{t_1},\ 2d\frac{t_2}{t_1} + d^2 t_1 \in \mathcal{O}. \tag{A 4}$$

(2) *Let $H \in \mathcal{M}^0_{\Psi_s}$. Then $H(g) = 0$ unless*

$$t_1, \frac{t_2}{t_1}, d^2 t_1, d\frac{t_2}{t_1} \in \mathcal{O}. \tag{A 5}$$

Proof.

(1) Note that for any $u \in U \cap K$ it holds that

$$H(g) = H(gu) = H\left(u^g g\right) = \Psi_s\left(u^g\right) H(g).$$

So if $H(g) \neq 0$ then

$$\Psi_s\left(u^g\right) = 1 \quad \forall u \in U \cap K.$$

Note that for $u(r_1, r_2, r_3, r_4, r_5) \in U \cap K$ it holds that

$$u^g = u\left(r_1', \frac{dt_2}{2t_1} r_1 + \frac{t_2}{t_1} r_2, \frac{d^2 t_1}{4} r_1 + dt_1 r_2 + t_1 r_3, r_4', r_5'\right),$$

for some r_1', r_4', r_5'. Applying Ψ_s to u^g yields

$$\Psi_s\left(u^g\right) = \Psi_s\left(\left(\frac{dt_2}{2t_1} + \frac{d^2 t_1}{4}\right) r_1 + \left(\frac{t_2}{t_1} + dt_1 r_2\right) r_2 + t_1 r_3\right).$$

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In order that $\Psi_s(u^g) \neq 1$, the coefficients of r_1, r_2 and r_3 must belong to $\mathcal{O}$. It then follows that

$$\Psi_s\left(u^g\right) = 1 \forall r_1, r_2, r_3 \in \mathcal{O} \ \Rightarrow \ \frac{dt_2}{2t_1} + \frac{d^2 t_1}{4}, \frac{t_2}{t_1} + dt_1, t_1 \in \mathcal{O}.$$

(2) This follows from the previous item by applying w_α-invariance. □

The following lemma will be useful in the computation of the second summand of eq. (A 1).

Lemma A.2. *Let $H \in \mathcal{M}_{\Psi_s}$, $t \in T$ and $\mathbb{1}_{KtK}$ be a characteristic function of the double coset KtK. Then*

$$H * \mathbb{1}_{KtK}(g) = \sum_i H(gb_i^{-1}),$$

where $KtK = \coprod Kb_i$. Note that the representatives b_i can be taken in the Borel subgroup B of G.

A.2. Computation of F_s

Proposition A.2. *Assume that $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d) \in M$ satisfies eq. (A 5). It holds that*

$$F(g,s) = \begin{cases} \frac{\zeta(5s-1)}{\zeta(5s)} |t_1|^{5s} \left|\frac{t_2}{t_1}\right| \left(1 - \left|\varpi \frac{t_2}{t_1}\right|^{5s-1}\right), \\ \quad \left|d^2 \frac{t_1^2}{t_2} + d\right| \leqslant 1 \\ \frac{\zeta(5s-1)}{\zeta(5s)} |t_1|^{5s} \left|\frac{t_2}{t_1}\right| \left|d^2 \frac{t_1^2}{t_2} + d\right|^{1-5s} \left(1 - \left|\varpi \left(d^2 \frac{t_1^2}{t_2} + d\right) \frac{t_2}{t_1}\right|^{5s-1}\right), \\ \quad \left|d^2 \frac{t_1^2}{t_2} + d\right| \geqslant 1. \end{cases}$$

For $g \in M$ violating eq. (A 5), we have $F(g,s) = 0$.

Proof. We recall that

$$F(g,s) = \int_F f_s\left(\mu x_{\alpha+\beta}(r) g\right) \psi(r)\, dr,$$

where f_s here is the spherical section such that $f_s(1) = 1$. For g as above we have

$$\begin{aligned} F(g,s) &= \int_F f_s\left(w_2 w_3 x_{-\alpha_1}(1) x_{\alpha+\beta}(r) h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)\right) \psi(r)\, dr \\ &= |t_1|^{5s} \int_F f_s\left(w_2 w_3 x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right) x_{\alpha_2+\alpha_3}\left(\frac{t_1}{t_2} r\right) x_{\alpha_1}(d) x_{\alpha_3}(d)\right) \psi(r)\, dr. \end{aligned}$$

Making a change of variables $r' = \frac{t_1}{t_2}$ and conjugating w_3 to the right we get

$$F(g,s) = |t_1|^{5s} \left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right) x_{\alpha_2}(r') x_{\alpha_1}(d) x_{-\alpha_3}(d)\right) \psi\left(\frac{t_2}{t_1} r'\right) dr'.$$

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Due to eq. (A 3) we have

$$F(m,s) = |t_1|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right) x_{\alpha_2}(r') x_{\alpha_1}(d)\, x_{\alpha_3}(d^{-1}) h_{\alpha_3}(d^{-1})\right) \psi\left(\frac{t_2}{t_1}r'\right) dr'.$$

Conjugating the elements associated with α_3 to the left and using a similar equality for α_1 yields

$$\begin{aligned}
F(g,s) &= \left|\frac{t_1}{d}\right|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right) x_{\alpha_2}\left(\frac{r'}{d}\right) x_{\alpha_1}(d)\right) \psi\left(\frac{t_2}{t_1}r'\right) dr' \\
&= \left|\frac{t_1}{d}\right|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{-\alpha_1}\left(\frac{t_1^2}{t_2}\right) x_{\alpha_2}\left(\frac{r'}{d}\right) h_{\alpha_1}(d)\, x_{-\alpha_1}(d)\right) \psi\left(\frac{t_2}{t_1}r'\right) dr' \\
&= |t_1|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{\alpha_2}(r') x_{-\alpha_1}\left(d^2\frac{t_1^2}{t_2}+d\right)\right) \psi\left(\frac{t_2}{t_1}r'\right) dr'.
\end{aligned}$$

If $\left|d^2\frac{t_1^2}{t_2}+d\right| \leqslant 1$ we have

$$F(g,s) = |t_1|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s\left(w_2 x_{\alpha_2}(r')\right) \psi\left(\frac{t_2}{t_1}r'\right) dr'.$$

The integral is evaluated by separation to $\mathcal{O}$ and $F\setminus\mathcal{O}$ and once again using eq. (A 3). It holds that

$$\begin{aligned}
&\int_F f_s\left(w_2 x_{\alpha_2}(r')\right) \psi\left(\frac{t_2}{t_1}r'\right) dr' \\
&= \int_{\mathcal{O}} f_s\left(w_2 x_{\alpha_2}(r')\right) \psi\left(\frac{t_2}{t_1}r'\right) dr' + \int_{F\setminus\mathcal{O}} f_s\left(w_2 x_{\alpha_2}(r')\right) \psi\left(\frac{t_2}{t_1}r'\right) dr' \\
&= 1 + \int_{F\setminus\mathcal{O}} |r'|^{-5s} \psi\left(\frac{t_2}{t_1}r'\right) dr' \\
&= 1 + \sum_{1<q^k<\left|\frac{t_2}{t_1}\right|} q^{-5ks}\int_{|r|=q^k} \psi(r)\, dr - \left|\frac{t_2}{t_1}\right|^{-5s} \int_{|r|=\left|\frac{t_2}{t_1}\right|} \psi(r)\, dr \\
&= \frac{\zeta(5s-1)}{\zeta(5s)}\left(1-\left|\varpi\frac{t_2}{t_1}\right|^{5s-1}\right).
\end{aligned}$$

And hence

$$F(g,s) = \frac{\zeta(5s-1)}{\zeta(5s)} |t_1|^{5s}\left|\frac{t_2}{t_1}\right|\left(1-\left|\varpi\frac{t_2}{t_1}\right|^{5s-1}\right).$$

Assume now that $\left|d^2\frac{t_1^2}{t_2}+d\right| > 1$ and define $p = d^2\frac{t_1^2}{t_2}+d$. It holds that

$$F(g,s) = |t_1|^{5s}\left|\frac{t_2}{t_1}\right| \int_F f_s(w_2 x_{\alpha_2}(r') x_{\alpha_1}(p^{-1})\alpha_1^\vee(p^{-1}))\psi\left(\frac{t_2}{t_1}r'\right) dr'$$

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$$= \left|\frac{t_1}{p}\right|^{5s} \left|\frac{t_2}{t_1}\right| \int_F f_s \left(w_2 x_{\alpha_2}\left(\frac{r'}{p}\right)\right) \psi\left(\frac{t_2}{t_1} r'\right) dr' \quad \left(r'' = \frac{r'}{p}\right)$$
$$= \left|\frac{t_1}{p}\right|^{5s} \left|p\frac{t_2}{t_1}\right| \int_F f_s \left(w_2 x_{\alpha_2}(r'')\right) \psi\left(\frac{t_2}{t_1} p r''\right) dr''.$$

If $|p\frac{t_2}{t_1}| \leqslant 1$ then, as in the previous case,

$$F(g,s) = \frac{\zeta(5s-1)}{\zeta(5s)} |t_1|^{5s} \left|\frac{t_2}{t_1}\right| \left|d^2\frac{t_1^2}{t_2} + d\right|^{1-5s} \left(1 - \left|\varpi\left(d^2\frac{t_1^2}{t_2} + d\right)\frac{t_2}{t_1}\right|^{5s-1}\right).$$

If on the other hand $|p\frac{t_2}{t_1}| > 1$, like for the previous cases it holds that

$$\int_F f_s\left(w_2 x_{\alpha_2}(r)\right)\psi\left(\frac{t_2}{t_1} pr\right) dr = \int_{\mathcal{O}} \psi\left(\frac{t_2}{t_1} pr\right) dr + \int_{F\setminus\mathcal{O}} |r|^{-5s} \psi\left(\frac{t_2}{t_1} pr\right) dr = 0 + 0,$$

since ψ is of conductor $\mathcal{O}$. Note that when $\left|p\frac{t_2}{t_1}\right| > 1$, eq. (A 5) is violated. □

Corollary A.1. (1) $F(\cdot, s) \in \mathcal{M}^0_{\Psi_s}$.

(2) $F(\cdot, s) * P(\cdot, s) \in \mathcal{M}^0_{\Psi_s}$.

Proof. In order to prove (1) it is enough to prove

$$F(wg, s) = F(g, s)$$

for any $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)$ and any generator w of S_{Ψ_s}. Recall that S_{Ψ_s} is generated by w_α and $\tilde{w} = h_\alpha(-1) x_\alpha(-1) x_{-\alpha}(1)$.

Note that for $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)$ it holds that

$$h_\alpha(-1) x_\alpha(-1) x_{-\alpha}(1) g w_\alpha = h_\alpha\left(-\frac{t_2}{t_1}\right) h_\beta(t_2) x_{-\alpha}\left(-\left(d + \frac{t_2}{t_1^2}\right)\right).$$

Now, note that $g \in S_{\Psi_s} UTK$ if and only if $d + \frac{t_2}{t_1^2} \in \mathcal{O}$; when $|d| > 1$ it holds that $\left|\frac{t_2}{t_1^2}\right| = |d|$. In particular, one may write

$$\left|d^2\frac{t_1^2}{t_2} + d\right| = \left|\frac{t_2}{t_1^2}\right| \left|\frac{dt_1^2}{t_2} + 1\right| = \left|d + \frac{t_2}{t_1^2}\right| \leqslant 1.$$

Case 1 – $g \in T$. For $g = h_\alpha(t_1) h_\beta(t_2)$ it holds that

$$w_\alpha h_\alpha(t_1) h_\beta(t_2) w_\alpha^{-1} = h_\alpha\left(\frac{t_2}{t_1}\right) h_\beta(t_2) = h_\alpha(t_1') h_\beta(t_2').$$

It then holds that $F(g,s) = F\left(w_\alpha g w_\alpha^{-1}, s\right) = F(w_\alpha g, s)$ since

$$\left|\frac{t_1'^2}{t_2'}\right| = \left|\frac{t_2}{t_1^2}\right|$$

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and

$$\left|t_1' \cdot \frac{t_2'}{t_1'^2}\right| = |t_1|\,, \qquad \left|\frac{t_2'}{t_1'} \cdot \frac{t_1'^2}{t_2'}\right| = \left|\frac{t_2}{t_1}\right|.$$

Also,

$$\tilde{w} h_\alpha\,(t_1)\, h_\beta\,(t_2)\, w_\alpha = h_\alpha\left(-\frac{t_2}{t_1}\right) h_\beta\,(t_2)\, x_{-\alpha}\left(-\frac{t_2}{t_1^2}\right).$$

If $\left|\frac{t_2}{t_1^2}\right| \leqslant 1$ then $x_{-\alpha}\left(-\frac{t_2}{t_1^2}\right) \in K$ and also $\left|\frac{\left(\frac{t_2}{t_1}\right)^2}{t_2}\right| = \left|\frac{t_2}{t_1^2}\right| \leqslant 1$. Write

$$\tilde{w} g w_\alpha = h_\alpha\left(-\frac{t_2}{t_1}\right) h_\beta\,(t_2) = h_\alpha(t_1')h_\beta(t_2').$$

Then $F\,(g,s) = F\,(\tilde{w} g w_\alpha, s) = F\,(w_\alpha g, s)$ follows from

$$\left|\frac{t_1'^2}{t_2'}\right| = \left|\frac{t_2}{t_1^2}\right|$$

and

$$\left|t_1' \frac{t_2'}{t_1'^2}\right| = |t_1|\,, \qquad \left|\frac{t_2'}{t_1'} \cdot \frac{t_1'^2}{t_2'}\right| = \left|\frac{t_2}{t_1}\right|.$$

If, on the other hand, $\left|\frac{t_2}{t_1^2}\right| > 1$ then $x_{-\alpha}\left(-\frac{t_2}{t_1^2}\right) \notin K$ and hence

$$\tilde{w} g w_\alpha = h_\alpha\,(t_1)\, h_\beta\,(t_2)\, x_\alpha\left(-\frac{t_2}{t_1^2}\right) = h_\alpha(t_1')h_\beta(t_2')x_\alpha(d').$$

Note that

$$\left|d'^2 \frac{t_1'^2}{t_2'} + d'\right| = \left(\frac{t_2}{t_1^2}\right)^2 \cdot \frac{t_1^2}{t_2} - \frac{t_2}{t_1^2} = 0 \in \mathcal{O},$$

and hence $F\,(g,s) = F\,(\tilde{w} g w_\alpha, s) = F\,(w_\alpha g, s)$ is automatic since $t_1' = t_1$ and $t_2' = t_2$ and $\left|\frac{t_1^2}{t_2}\right| < 1$.

Case 2 – $g \notin T$. Let $g = h_\alpha\,(t_1)\, h_\beta\,(t_2)\, x_\alpha\,(d) \notin S_{\Psi_s} UTK$. In particular, $|d|\,, |d^2 \frac{t_1^2}{t_2} + d|$, $|d + \frac{t_2}{t_1^2}| > 1$.

Note that

$$w_\alpha g w_\alpha = h_\alpha\left(\frac{t_2}{d t_1}\right) h_\beta\,(t_2)\, x_\alpha\,(d)\, k,$$

for some $k \in K$. We write

$$h_\alpha\left(\frac{t_2}{d t_1}\right) h_\beta\,(t_2)\, x_\alpha\,(d) = h_\alpha(t_1')h_\beta(t_2')x_\alpha(d').$$

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Hence $F(g,s) = F(wg,s)$ follows from the fact that

$$\left| d'^2 \frac{t_1'^2}{t_2'} + d' \right| = \left| d + \frac{t_2}{t_1^2} \right| > 1$$

and

$$t_1' \cdot \frac{1}{d'^2 \frac{t_1'^2}{t_2'} + d'} = \frac{t_1}{d^2 \frac{t_1^2}{t_2} + d}, \quad \frac{t_2'}{t_1'} \cdot \left(d'^2 \frac{t_1'^2}{t_2'} + d' \right) = \frac{t_2}{t_1} \cdot \left(d^2 \frac{t_1^2}{t_2} + d \right).$$

Also,

$$\tilde{w} g w_\alpha = h_\alpha \left(\frac{t_2}{t_1 \left(d + \frac{t_2}{t_1^2} \right)} \right) h_\beta(t_2)\, x_\alpha \left(- \left(d + \frac{t_2}{t_1^2} \right) \right) k$$

for some $k \in K$. We write

$$h_\alpha \left(\frac{t_2}{t_1 \left(d + \frac{t_2}{t_1^2} \right)} \right) h_\beta(t_2)\, x_\alpha \left(- \left(d + \frac{t_2}{t_1^2} \right) \right) = h_\alpha(t_1') h_\beta(t_2') x_\alpha(d').$$

Hence $F(g,s) = F(wg,s)$ follows from the fact that

$$\left| d'^2 \frac{t_1'^2}{t_2'} + d' \right| = |d| > 1$$

and

$$\left| t_1' \cdot \frac{1}{d'^2 \frac{t_1'^2}{t_2'} + d'} \right| = \left| \frac{t_1}{d^2 \frac{t_1^2}{t_2} + d} \right|, \quad \left| \frac{t_2'}{t_1'} \cdot \left(d'^2 \frac{t_1'^2}{t_2'} + d' \right) \right| = \left| \frac{t_2}{t_1} \cdot \left(d^2 \frac{t_1^2}{t_2} + d \right) \right|.$$

This completes the proof of (1). Item (2) follows immediately. □

A.3. Decomposition of $K\omega_2^\vee(\varpi)K$ into left K cosets

We recover the list of left K cosets in $K\omega_2^\vee(\varpi)K$ from [7, Propositions 13.3 and 14.2]. The decomposition of $K\omega_2^\vee(\varpi)K = \coprod b_i' K$ as a union of right K cosets is described there; after listing them we will make them into left cosets. Write $b_i' = u_i \tilde{b}_i$ where $u_i \in U$, $\tilde{b}_i \in B \cap M$. Fix Y to be Teichmüller representatives in $\mathcal{O}$ of $\mathcal{O}/(\varpi)$ (or any other set of representatives) and Z to be a set of representatives in $\mathcal{O}$ of $\mathcal{O}/(\varpi^2)$.

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$t \in K_M \backslash M / K_M$: $UtK \cap K\omega_2^\vee(\varpi)K \neq \emptyset$	$\#\{u_i\tilde{b}_i K \subset K\omega_2^\vee(\varpi)K$: $\tilde{b}_i \in K_M t K_M\}$	representatives $u_i\tilde{b}_i$
$h_{-\omega_2}(\varpi)$	1	$h_{-\omega_2}(\varpi)$
$h_{-\beta}(\varpi)$	q^6	$u(r_1, r_2, r_3, r_4, r_5)\, h_{\omega_2}(\varpi)$ $r_1, r_2, r_3, r_4 \in Y,\ \ r_5 \in Z$
$h_\alpha(\varpi)\, h_\beta(\varpi)$	$q(q+1)$	$u(r_1, 0, 0, 0, 0)\, h_{-\alpha}(\varpi)\, h_{-\beta}(\varpi)$ $r_1 \in Y$; $u(0, 0, 0, r_4, 0)\, x_\alpha(z)\, h_{-\beta}(\varpi)$ $r_4, z \in Y$
$h_{\omega_2}(\varpi)$	$q^4(q+1)$	$u(r_1, r_2, 0, 0, r_5)\, h_\beta(\varpi)$ $r_2, r_5 \in Y,\ \ r_1 \in Z$; $u(0, 0, r_3, r_4, r_5)\, x_\alpha(z)\, h_\alpha(\varpi)\, h_\beta(\varpi)$ $r_3, r_5, z \in Y,\ \ r_4 \in Z$
1	$q^3 - 1$	$u\left(0, 0, 0, 0, \frac{r_5}{\varpi}\right)$ $r_5 \in Y,\ \ r_5 \not\equiv 0$; $u\left(\frac{r_1}{\varpi}, 0, 0, 0, \frac{r_5}{\varpi}\right)$ $r_1, r_5 \in Y,\ \ r_1 \not\equiv 0$; $u\left(\frac{y^3 r_1}{\varpi}, \frac{y^2 r_1}{\varpi}, \frac{y r_1}{\varpi}, \frac{r_1}{\varpi}, \frac{r_5}{\varpi}\right)$ $r_1, r_5, y \in Y,\ \ r_1 \not\equiv 0$

We need now to make the right coset representatives $\{b_i'\}$ into left coset representatives. Let $w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta \in K$ be the longest element in the Weyl group of G. Recall that w_0 sends γ to $-\gamma$ for all $\gamma \in \Phi$. Also note that $\{w_0 b_i'\}$ is also a full set of representatives of right cosets.

Denote by θ the Cartan antiinvolution, fixing the torus T such that $\theta(x_\gamma(r)) = x_{-\gamma}(r)$. Let $w_0 \in K$ be a lifting to G of the longest Weyl group element such that $\theta(w_0) = w_0$. Then

$$K\omega_2^\vee(\varpi)K = \theta\left(K\omega_2^\vee(\varpi)K\right) = \theta\left(\coprod_i w_0 b_i' K\right) = \coprod_i K\theta\left(b_i'\right) w_0 = \coprod_i K t_i^{-1} n_i. \qquad \text{(A 6)}$$

Fixing $b_i = t_i^{-1} n_i$ gives a set of left coset representatives of $K\omega_1(\varpi)K$.

A.4. Convolution

Combining eq. (A 1), Proposition A.2 and eq. (A 6), the computation of the convolution $F^*(\cdot, s) * P(\cdot, s)$ is straightforward. We shall present the computation for toral elements only, thus proving eq. (A 2). The vanishing $F^*(\cdot, s) * P(\cdot, s)$ outside of $S_{\Psi_s} UTK$ is proved similarly.

By Lemma A.2 we have

$$\left(F(\cdot, s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(g) = \sum_i F(g b_i^{-1}, s).$$

Assume that $g = t = h_\alpha(t_1)\, h_\beta(t_2)$. By S_{Ψ_s}-invariance we may assume that $\left|\frac{t_1^2}{t_2}\right| \leqslant 1$. The case $\left|\frac{t_1^2}{t_2}\right| > 1$ follows by symmetry from $\left|\frac{t_1^2}{t_2}\right| < 1$, since $F(\cdot, s) \in \mathcal{M}^0_{\Psi_s}$. We can write

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$\left(F(\cdot,s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t)$ as follows:

$$\begin{aligned}\left(F(\cdot,s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t) = {} & F\left(h_{-\omega_1}(\varpi)\,t, s\right) + q^6 F\left(h_{\omega_1}(\varpi)\,t, s\right) \\ & + q F\left(h_{-\alpha}(\varpi)\,h_{-\beta}(\varpi)\,t, s\right) \\ & + q \sum_{y\in\mathcal{O}/(\varpi)} F\left(h_{-\beta}(\varpi)\,t\,x_\alpha\left(-\frac{y}{\varpi}\right), s\right) + q^4 F\left(h_\beta(\varpi)\,t, s\right) \\ & + q^4 \sum_{y\in\mathcal{O}/(\varpi)} F\left(h_\alpha(\varpi)\,h_\beta(\varpi)\,t\,x_\alpha\left(-\frac{y}{\varpi}\right), s\right) \\ & + \left(q \sum_{r,y\in\mathcal{O}/(\varpi)} \psi\left(-\frac{y}{\varpi}\left(\frac{t_2}{t_1}y + t_1\right) r\right) - 1\right) F(t,s). \qquad \text{(A 7)}\end{aligned}$$

We separate this computation into four cases depending on the absolute value of t_1 and t_2. All the following results follow by applying Proposition A.2 to the summands in eq. (A 7). Define $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$.

(1) Assume that $|t_2| = |t_1| = 1$:

$$\left(F(\cdot,s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t) = \frac{q^{6-10s} + q^{5-5s} + 3q^{4-5s} + 2q^2 - 1}{\zeta(5s)}$$

and also

$$F(1,s) = \frac{1}{\zeta(5s)}.$$

Plugging this into eq. (A 1) yields

$$\left(F^*(\cdot,s) * P(\cdot,s)\right)(t) = \frac{1 + 2q^{1-5s}}{\zeta(5s+1)}.$$

(2) Assume that $|t_2| < |t_1|$ and $\left|\frac{t_1^2}{t_2}\right| = 1$:

$$\begin{aligned}\left(F(\cdot,s) * \mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t) = {} & \frac{\zeta(5s-1)}{\zeta(5s)} \left(\frac{q^{1+5s-n-5ns}}{\zeta(5ns)} + \frac{q^{5-5s-n-5ns}}{\zeta(5(n+2)s)} + \frac{q^{2-n-5ns}}{\zeta(5ns)}\right. \\ & + \frac{q^{4-n-5(n+1)s}}{\zeta(5(n+1)s)} + (q^3-1)\frac{q^{-n-5ns}}{\zeta(5(n+1)s)} \\ & + q\left(2\frac{q^{1-n-5ns}}{\zeta(5ns)} + (q-2)\frac{q^{2-n-5(n+1)s}}{\zeta(5(n-1)s)}\right) \\ & \left. + q^4\left(2\frac{q^{-n-5(n+1)s}}{\zeta(5(n+1)s)} + (q-2)\frac{q^{1-n-5(n+2)s}}{\zeta(5ns)}\right)\right)\end{aligned}$$

and also

$$F(t,s) = \frac{\zeta(5s-1)}{\zeta(5s)} \frac{q^{-n-5ns}}{\zeta(5(n+1)s)}.$$

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Plugging this into eq. (A 1) yields

$$\left(F^*(\cdot,s)*P(\cdot,s)\right)(t)=\frac{1+2q^{1-5s}}{\zeta(5s+1)}\left|t_1\right|^{5s+1}.$$

(3) Assume that $|t_2|<|t_1|$ and $\left|\frac{t_1^2}{t_2}\right|<1$:

$$\begin{aligned}\left(F(\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t)=\frac{\zeta(5s-1)}{\zeta(5s)}&\left(\frac{q^{1+5s-m+n-5ns}}{\zeta(5(m-n)s)}+\frac{q^{5-5s-m+n-5ns}}{\zeta(5(m-n+2)s)}\right.\\&+\frac{q^{1-m+n-5ns+5s}}{\zeta(5(m-n+1)s)}+\frac{q^{3-5ns-m+n}}{\zeta(5(m-n+2)s)}\\&+q\left(\frac{q^{1-5ns-m+n}}{\zeta(5(m-n)s)}+(q-1)\frac{q^{2-5s-m+n-5ns}}{\zeta(5(m-n-1)s)}\right)\\&+q^4\left(\frac{q^{n-m-5s-5ns}}{\zeta(5(m-n+1)s)}+(q-1)\frac{q^{1-10s-m+n-5ns}}{\zeta(5(m-n)s)}\right)\\&\left.+(q^3-1)\frac{q^{n-m-5ns}}{\zeta(5(m-n+1)s)}\right)\end{aligned}$$

and also

$$F(t,s)=\frac{\zeta(5s-1)}{\zeta(5s)}\frac{q^{n-m-5ns}}{\zeta(5(m-n+1)s)}.$$

Plugging this into eq. (A 1) yields

$$\left(F^*(\cdot,s)*P(\cdot,s)\right)(t)=\frac{1+q^{1-5s}}{\zeta(5s+1)}\left|\frac{t_2}{t_1}\right|\left|t_1\right|^{5s}.$$

(4) Assume that $|t_2|=|t_1|$ and $\left|\frac{t_1^2}{t_2}\right|<1$:

$$\begin{aligned}\left(F(\cdot,s)*\mathbb{1}_{K\omega_2^\vee(\varpi)K}\right)(t)=\frac{\zeta(5s-1)}{\zeta(5s)}&\left(\frac{q^{5-5s-5ns}}{\zeta(10s-2)}+\frac{q^{1+5s-5ns}}{\zeta(5s-1)}+\frac{q^{3-5ns}}{\zeta(10s-2)}\right.\\&\left.+\frac{q^{4-5s-5ns}}{\zeta(5s-1)}+(q^2-1)\frac{q^{-5ns}}{\zeta(5s-1)}\right)\end{aligned}$$

and also

$$F(t,s)=\frac{\zeta(5s-1)}{\zeta(5s)}\frac{q^{-5ns}}{\zeta(5s-1)}.$$

Plugging this into eq. (A 1) yields

$$\left(F^*(\cdot,s)*P(\cdot,s)\right)(t)=\frac{1+q^{1-5s}}{\zeta(5s+1)}\left|t_1\right|^{5s}.$$

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B. Computation of D_{Ψ_s}

Recall from §7 that our aim is to compute

$$E_k^{\Psi_s}(g) = \int_U \mathbb{1}_{U_k(g)}(ug)\,\Psi_s(u)\,du = \int_{U_k(g)} \Psi_s(u)\,du,$$

where

$$U_k(g) = U_k(g) = \left\{u \in U : \Gamma(ug) \leqslant q^k\right\}.$$

We treat first the case where $g \in S_{\Psi_s}UTK$ and then the case where $g \notin S_{\Psi_s}UTK$.

We note the following helpful fact that will be used repeatedly through out this section.

Lemma B.1. *For $a, b, c \in \mathbb{N}$ with $a + b \geqslant c$ it holds that*

$$\mu\left\{(x, y)\,\middle|\, |x| \leqslant q^a, |y| \leqslant q^b, |xy| \leqslant q^c\right\} = q^c(1 + (a + b - c)(1 - q^{-1})),$$

where μ is the Haar measure on G such that $\mu(K) = 1$.

B.1. Toral elements

For $t = h_\alpha(t_1)h_\beta(t_2)$ and $u = u(r_1, r_2, r_3, r_4, r_5)$,

$$\iota(ut) = \begin{pmatrix} 1 & 0 & r_2 & r_3 & \frac{-r_4}{2} & \frac{r_2r_3 + r_5}{2} & \frac{r_2r_4 - r_3^2}{2} \\ 0 & 1 & r_1 & r_2 & \frac{-r_3}{2} & \frac{r_1r_3 - r_2^2}{2} & \frac{r_1r_4 - 2r_2r_3 - r_5}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{r_3}{2} & \frac{r_4}{2} \\ 0 & 0 & 0 & 1 & 0 & -r_2 & -r_3 \\ 0 & 0 & 0 & 0 & 1 & -r_1 & -r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} t_1 &&&&&& \\ & \frac{t_2}{t_1} &&&&& \\ && \frac{t_1^2}{t_2} &&&& \\ &&& 1 &&& \\ &&&& \frac{t_2}{t_1^2} && \\ &&&&& \frac{t_1}{t_2} & \\ &&&&&& \frac{1}{t_1} \end{pmatrix}. \tag{B 1}$$

Consider an element $g \in T$ and denote it by $t = h_\alpha(t_1)h_\beta(t_2)$. Define $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$. By eq. (A 5), $E_k(t) = 0$ unless $|t_1|, |\frac{t_2}{t_1}| \leqslant 1$. Recall from §2 that $E_k^{\Psi_s} \in \mathcal{M}^0_{\Psi_s}$; hence

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we may assume that $|\alpha(t)| = |\frac{t_1^2}{t_2}| \leqslant 1$ or $|t_1| \leqslant |\frac{t_2}{t_1}|$. The case $|\frac{t_1^2}{t_2}| > 1$ follows from $|\frac{t_1^2}{t_2}| < 1$ by w_α-invariance. Also, $U_k(t) = \emptyset$ unless $|t_1| \geqslant q^{-k}$. To sum up we have to compute $E_k^{\Psi_s}(t)$ for t_1, t_2 and k satisfying

$$q^{-k} \leqslant |t_1| \leqslant \left|\frac{t_2}{t_1}\right| \leqslant 1.$$

We may exchange integration over $U_k(t)$ to integration over a smaller and simpler set, namely:

Lemma B.2.

$$E_k^{\Psi_s}(t) = \int_{\overline{U_k(t)}} \Psi_s(u)\, du,$$

where

$$\overline{U_k(t)} = \left\{ u(r_1, r_2, r_3, r_4, r_5) \in U_k(t) \,\middle|\, |r_2|, |r_3| \leqslant q \right\}.$$

Proof. For any $x, y \in F$ define

$$U_k^{(x,y)}(t) = \left\{ u(r_1, r_2, r_3, r_4, r_5) \in U_k(t) \,\middle|\, r_2 = x, r_3 = y \right\}$$

and note that for $s_1, s_2 \in \mathcal{O}^\times$,

$$U_k^{(s_1 x, s_2 y)}(t) = h(s_1, s_2)\, U_k^{(x,y)}(t)\, h^{-1}(s_1, s_2),$$

where $h(s_1, s_2) = h_\beta(s_1)\, h_{2\alpha+\beta}(s_2)$. Since $\delta_P(h(s_1, s_2)) = 1$ it follows that $\mu\left(U_k^{(s_1 x, s_2 y)}(t)\right) = \mu\left(U_k^{(x,y)}(t)\right)$ which means that it depends only on t, $|x|$ and $|y|$. In particular, if $|x| = q^i$, $|y| = q^j$ we denote $\mu\left(U_k^{(x,y)}(t)\right)$ by $\mu\left(U_k^{i,j}(t)\right)$.

Thus

$$\begin{aligned} E_k^{\Psi_s}(t) &= \int_{U_k(t)} \Psi_s(u)\, du = \int_{F\times F} \mu\left(U_k^{(x,y)}(t)\right) \psi(x+y)\, dx\, dy \\ &= \sum_{i,j=-\infty}^{\infty} \mu(U_k^{i,j}(t)) \int_{|x|=q^i} \psi(x)\, dx \int_{|y|=q^j} \psi(y)\, dy. \end{aligned}$$

Since $\int_{|z|=q^l} \psi(z)\, dz = 0$ for $l > 1$, the proposition follows. □

Remark B.1. We can describe $\overline{U_k(t)}$ by giving a short list of inequalities. Namely, $u \in \overline{U_k(t)}$ if and only if

$$\begin{gathered} k \geqslant n \\ |r_2|, |r_3| \leqslant q \\ |r_1|, |r_2|, |r_3|, |r_2 r_3 + r_5|, \left|r_1 r_3 - r_2^2\right| \leqslant q^{k+n-m} \\ |r_2|, |r_3|, |r_4|, \left|r_2 r_4 - r_3^2\right|, |r_1 r_4 - 2 r_2 r_3 - r_5| \leqslant q^{k-n}. \end{gathered}$$

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Corollary B.1.

$$E_k^{\Psi_s}(t) = \mu\left(U_k^{0,0}(t)\right) - \mu\left(U_k^{0,1}(t)\right) - \mu\left(U_k^{1,0}(t)\right) + \mu\left(U_k^{1,1}(t)\right).$$

Proof. We first note that for every $i, i' \leqslant 0$, $j \leqslant 1$ and k it holds that

$$\mu\left(U_k^{i,j}(t)\right) = \mu\left(U_k^{i',j}(t)\right), \quad \mu\left(U_k^{j,i}(t)\right) = \mu\left(U_k^{j,i'}(t)\right).$$

We also recall that

$$\int_{|r|\leqslant 1} \psi(r) = 1, \quad \int_{|r|=q} \psi(r) = -1.$$

The claim then follows by a simple computation:

$$\begin{aligned} E_k^{\Psi_s}(t) &= \sum_{i,j=-\infty}^{\infty} \mu(U_k^{i,j}(t)) \int_{|x|=q^i} \psi(x)\, dx \int_{|y|=q^j} \psi(y)\, dy \\ &= \mu\left(U_k^{0,0}(t)\right) \int_{|x|\leqslant 1} \psi(x)\, dx \int_{|y|\leqslant 1} \psi(y)\, dy \\ &\quad - \mu\left(U_k^{0,1}(t)\right) \int_{|x|\leqslant 1} \psi(x)\, dx \int_{|y|=q} \psi(y)\, dy \\ &\quad - \mu\left(U_k^{1,0}(t)\right) \int_{|x|=q} \psi(x)\, dx \int_{|y|\leqslant 1} \psi(y)\, dy \\ &\quad + \mu\left(U_k^{1,1}(t)\right) \int_{|x|=q} \psi(x)\, dx \int_{|y|=q} \psi(y)\, dy \\ &= \mu\left(U_k^{0,0}(t)\right) - \mu\left(U_k^{0,1}(t)\right) - \mu\left(U_k^{1,0}(t)\right) + \mu\left(U_k^{1,1}(t)\right). \quad \square \end{aligned}$$

Proposition B.1. *For t as above, with $|t_1| = q^{-n}$, it holds that:*

(1) $E_k^{\Psi_s}(t) = 0$ *for* $k \neq n, n+1$.

(2) $E_n^{\Psi_s}(t) = \begin{cases} 1 & |\alpha(t)| = 1 \\ |\alpha(t)|^{-1} & |\alpha(t)| < 1, \end{cases}$ $E_{n+1}^{\Psi_s}(t) = \begin{cases} 2q^2 & |\alpha(t)| = 1 \\ 2q^2\, |\alpha(t)|^{-1} & |\alpha(t)| < 1. \end{cases}$

Proof. We separate the proof into sections according to the absolute value of $\alpha(t)$.

- Assume that $|\alpha(t)| = 1$, i.e. $\left|\frac{t_1^2}{t_2}\right| = 1$.

  (1) Assume that $k = n$; then $u \in \overline{U_k(t)}$ if and only if

  $$|r_1|, |r_2|, |r_3|, |r_4|, |r_5| \leqslant 1.$$

  In this case $U_k^{0,1}(t), U_k^{1,0}(t), U_k^{1,1}(t) = \emptyset$ and $U_k^{0,0}(t) = \mathcal{O}^3$. Hence

  $$E_n^{\Psi_s}(g) = 1$$

  (2) Assume that $k = n+1$; then $u \in \overline{U_k(t)}$ if and only if

  $$|r_1|, |r_2|, |r_3|, |r_4| \leqslant q$$

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$$|r_2r_3 + r_5|, \left|r_1r_3 - r_2^2\right|, \left|r_2r_4 - r_3^2\right|, |r_1r_4 - 2r_2r_3 - r_5| \leqslant q.$$

We demonstrate the measurement of $U_k^{i,j}(t)$ in this case as an example to indicate the calculation carried out in all other cases. Assume that $|r_2|, |r_3| \leqslant 1$; then $u \in \overline{U_k(t)}$ if and only if

$$|r_1|, |r_4|, |r_5|, |r_1r_4| \leqslant q$$

and hence, by Lemma B.1,

$$\mu\left(U_k^{0,0}(t)\right) = q\left(q + (q-1)\right) = 2q^2 - q.$$

Assume that $|r_2| = q$ and $|r_3| \leqslant 1$. Then $|r_1r_3| \leqslant q$ but also $\left|r_1r_3 - r_2^2\right| \leqslant q$ which contradicts the fact that $\left|r_2^2\right| = q^2$. Hence $U_k^{0,1}(t) = \emptyset$, and by a similar argument $U_k^{1,0}(t) = \emptyset$.

Assume that $|r_2|, |r_3| = q$. Let us parametrize $U_k^{1,1}(t)$ in the following way:

$$r_1 = \frac{x + r_2^2}{r_3}$$
$$r_4 = \frac{y + r_3^2}{r_2}$$
$$r_5 = z - r_2r_3.$$

The domain of integration for the new variables is $|x|, |y|, |z| \leqslant q$. Also

$$dr_1 = \frac{dx}{q}, \quad dr_4 = \frac{dy}{q}, \quad dr_5 = dz.$$

Note that now

$$|r_1r_4 - 2r_2r_3 - r_5| = \left|\frac{x + r_2^2}{r_3} \cdot \frac{y + r_3^2}{r_2} - r_2r_3 - z\right| = \left|\frac{xy + xr_3^2 + yr_2^2}{r_2r_3} - z\right| \leqslant q.$$

Hence

$$\mu\left(U_k^{1,1}(t)\right) = \int_{\varpi^{-1}\mathcal{O}} \frac{dx}{q} \int_{\varpi^{-1}\mathcal{O}} \frac{dy}{q} \int_{\varpi^{-1}\mathcal{O}} dz = q.$$

Combining the computed $\mu\left(U_k^{i,j}(t)\right)$ yields

$$\begin{aligned} E_{n+1}^{\Psi_s}(t) &= \mu\left(U_k^{0,0}(t)\right) - \mu\left(U_k^{0,1}(t)\right) - \mu\left(U_k^{1,0}(t)\right) - \mu\left(U_k^{1,1}(t)\right) \\ &= \left(2q^2 - q\right) - 0 - 0 + q = 2q^2. \end{aligned}$$

(3) Assume that $k > n+1$; then $u \in \overline{U_k(t)}$ if and only if

$$|r_2|, |r_3| \leqslant q$$
$$|r_1|, |r_4|, |r_5|, |r_2r_4|, |r_1r_3|, |r_1r_4| \leqslant q^{k-n}.$$

Hence, according to Lemma B.1,

$$\mu\left(U_k^{0,0}(t)\right) = q^{2(k-n)}\left(1 + (k-n)\left(1 - q^{-1}\right)\right)$$

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$$\mu\left(U_k^{1,0}(t)\right)=\mu\left(U_k^{0,1}(t)\right)=q^{2(k-n)}\left(1+(k-n-1)\left(1-q^{-1}\right)\right)$$
$$\mu\left(U_k^{1,1}(t)\right)=q^{2(k-n)}\left(1+(k-n-2)\left(1-q^{-1}\right)\right),$$

and then

$$E_k^{\Psi_s}(g)=0.$$

Evaluating D^{Ψ_s} at t yields

$$D^{\Psi_s}(t)=q^{-n}E_n^{\Psi_s}(t)+q^{-n-1}\left(E_{n+1}^{\Psi_s}(t)-E_n^{\Psi_s}(t)\right)+q^{-n-2}E_{n+1}^{\Psi_s}(t)$$
$$=\frac{1+2q^{1-5s}}{\zeta(5s+1)}|t_1|^{5s+1}.$$

- Assume that $|\alpha(t)|<1$, i.e. $\left|\frac{t_1^2}{t_2}\right|<1$.

(1) Assume that $k=n$; then $u\in\overline{U_k(t)}$ if and only if

$$|r_2|,|r_3|,|r_4|,|r_1r_4-r_5|\leqslant 1$$
$$|r_1|,|r_5|\leqslant q^{2n-m}.$$

By making a change of variables $r_5=x+r_1r_4$, this is equivalent to

$$|r_2|,|r_3|,|r_4|,|x|\leqslant 1$$
$$|r_1|,|r_1r_4|\leqslant q^{2n-m}.$$

Hence, according to Lemma B.1,

$$\mu\left(U_k^{0,0}(t)\right)=q^{2n-m}$$
$$\mu\left(U_k^{1,0}(t)\right)=\mu\left(U_k^{0,1}(t)\right)=\mu\left(U_k^{1,1}(t)\right)=0,$$

and then

$$E_n^{\Psi_s}(t)=q^{2n-m}.$$

(2) Assume that $k=n+1$; then $u\in\overline{U_k(t)}$ if and only if

$$|r_2|,|r_3|,|r_4|,\left|r_2r_4-r_3^2\right|,|r_1r_4-2r_2r_3-r_5|\leqslant q$$
$$|r_1|,|r_5|,|r_1r_3|\leqslant q^{2n-m+1}.$$

Hence, according to Lemma B.1 and arguments similar to those for case 2 with $|\alpha(t)|=1$,

$$\mu\left(U_k^{0,0}(t)\right)=q^{2n-m+2}\left(1+\left(1-q^{-1}\right)\right),\quad \mu\left(U_k^{1,0}(t)\right)=q^{2n-m+2}$$
$$\mu\left(U_k^{0,1}(t)\right)=0,\quad \mu\left(U_k^{1,1}(t)\right)=q^{2n-m+1},$$

and then

$$E_{n+1}^{\Psi_s}(t)=q^{2n-m+2}.$$

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(3) Assume that $k > n+1$; then $u \in \overline{U_k(g)}$ if and only if

$$|r_2|, |r_3| \leqslant q$$
$$|r_1|, |r_5|, |r_1 r_3| \leqslant q^{k+n-m}$$
$$|r_4|, |r_2 r_4|, |r_1 r_4 - r_5| \leqslant q^{k-n}.$$

By making a change of variables $r_5 = x + r_1 r_4$, this is equivalent to

$$|r_2|, |r_3| \leqslant q$$
$$|r_1|, |r_1 r_4|, |r_1 r_3| \leqslant q^{k+n-m}$$
$$|r_4|, |r_2 r_4|, |x| \leqslant q^{k-n}.$$

Hence, according to Lemma B.1,

$$\mu\left(U_k^{0,0}(t)\right) = q^{k-n} q^{k+n-m}\left(1 + (k-n)\left(1-q^{-1}\right)\right)$$
$$\mu\left(U_k^{1,0}(t)\right) = q^{k-n} q^{k+n-m}\left(1 + (k-n-1)\left(1-q^{-1}\right)\right)$$
$$\mu\left(U_k^{0,1}(t)\right) = q^{k-n} q^{k+n-m}\left(1 + (k-n-1)\left(1-q^{-1}\right)\right)$$
$$\mu\left(U_k^{1,1}(t)\right) = q^{k-n} q^{k+n-m}\left(1 + (k-n-2)\left(1-q^{-1}\right)\right),$$

and then

$$E_k^{\Psi_s}(t) = 0.$$

Evaluating D^{Ψ_s} at t yields

$$D^{\Psi_s}(t) = \frac{1+q^{1-5s}}{\zeta(5s+1)}\left|\frac{t_2}{t_1}\right| |t_1|^{5s}. \quad \square$$

B.2. The non-toral case

This case is technically more involved than the case of the toral elements, but all the ideas for the toral elements can be carried over to this case as well. We will prove the following result.

Proposition B.2. $E_k^{\Psi_s}(g) = 0$ *for* $g \notin S_{\Psi_s} UTK$.

Let $g = t x_\alpha(d)$, where $t = h_\alpha(t_1) h_\beta(t_2)$ and $|d| > 1$. Since $g \notin S_{\Psi_s} UTK$ it holds that $\left|d^2 \alpha(t) + d\right| \geqslant 1$. By eq. (A 5) $E_k(t) = 0$ unless

$$t_1, \frac{t_2}{t_1}, d^2 t_1, d\frac{t_2}{t_1} \in \mathcal{O}.$$

Since $E_k^{\Psi_s} \in \mathcal{M}_{\Psi_s}^0$ it is enough to compute $E_k^{\Psi_s}(g)$ when $|d\alpha(t)| = \left|d\frac{t_1^2}{t_2}\right| \leqslant 1$; the dual case follows by w_α-invariance.

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The matrix $\iota\left(x_{\alpha}(d)\right)$ has the form

$$\begin{pmatrix} 1 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d & -\frac{d^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $|t_1| = q^{-n}$, $|t_2| = q^{-m}$ and $|d\alpha(t)| = q^l$. With this notation, $U_k(g) = \emptyset$ when $k < n$ and so we may assume that $k \geqslant n$.

We now reduce the domain of integration. The proof of this lemma is similar to the proof of Lemma B.2 and is omitted.

Lemma B.3.

$$E_k^{\Psi_s}(t) = \int_{\widehat{U_k(g)}} \Psi_s(u)\, du,$$

where

$$\widehat{U_k(g)} = \left\{ u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \,\middle|\, |r_2 + r_3| \leqslant q \right\}.$$

Remark B.2. Define $b = d\frac{t_1^2}{t_2}$. When $|d\alpha(t)| \leqslant 1$, we have $u \in \widehat{U_k(g)}$ if and only if

$$\begin{gathered} k \geqslant n \\ |r_2 + r_3| \leqslant q \\ |r_1|, |r_2|, |r_3|, |r_2 r_3 + r_5|, \left|r_1 r_3 - r_2^2\right| \leqslant q^{k+n-m} \\ |br_1 - r_2|, |br_2 - r_3|, |br_3 - r_4| \leqslant q^{k-n} \\ \left|r_2 r_4 - r_3^2 - br_2 r_3 - br_5\right|, \left|r_1 r_4 - 2r_2 r_3 + br_2^2 - br_1 r_3 - r_5\right| \leqslant q^{k-n}. \end{gathered}$$

We are now ready to prove Proposition B.2.

Proof. • Assume that $|b| < 1$, i.e. $l < 0$. Note that under this assumption,

$$\widehat{U_k(g)} = \overline{U_k(g)}$$

and thus

$$E_k^{\Psi_s}(g) = \mu\left(U_k^{0,0}(g)\right) - \mu\left(U_k^{0,1}(g)\right) - \mu\left(U_k^{1,0}(g)\right) + \mu\left(U_k^{1,1}(g)\right).$$

(1) Assume that $k = n$; then $u \in \widehat{U_k(g)}$ if and only if

$$\begin{gathered} |r_2| \leqslant q \\ |r_1|, |r_5| \leqslant q^{2n-m} \\ |r_3|, |r_4|, |br_1 - r_2|, |r_2 r_4 - br_5|, |r_1 r_4 - 2r_2 r_3 + br_2^2 - br_1 r_3 - r_5| \leqslant 1. \end{gathered}$$

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Hence, according to Lemma B.1 and arguments given for the toral case,

$$\mu\left(U_k^{0,0}(g)\right) = q^{-2l}, \quad \mu\left(U_k^{1,0}(g)\right) = q^{-2l}$$
$$\mu\left(U_k^{0,1}(g)\right) = 0, \quad \mu\left(U_k^{1,1}(g)\right) = 0,$$

and then

$$E_k^{\Psi_s}(g) = 0.$$

(2) Assume that $k = n+1$; then $u \in \widehat{U_k(g)}$ if and only if

$$|r_1| \leqslant q^{1-l}$$
$$|r_2|, |r_3|, |r_4|, |r_2 r_4 - r_3^2 - b r_5|, |r_1 r_4 - 2 r_2 r_3 - b r_1 r_3 - r_5| \leqslant q.$$

Hence, according to Lemma B.1 and arguments given for the toral case,

$$\mu\left(U_k^{0,0}(g)\right) = q^{2-l}\left(1 + \left(1 - q^{-1}\right)\right), \ \mu\left(U_k^{1,0}(g)\right) = q^{2-l}\left(1 + \left(1 - q^{-1}\right)\right)$$
$$\mu\left(U_k^{0,1}(g)\right) = (1-l)\, q^{2-l}\left(q - q^{-1}\right), \mu\left(U_k^{1,1}(g)\right) = (1-l)\, q^{2-l}\left(q - q^{-1}\right),$$

and then

$$E_k^{\Psi_s}(g) = 0.$$

(3) Assume that $k > n+1$ and define $x = r_2 + r_3$. Then $u \in \widehat{U_k(g)}$ if and only if

$$|x| \leqslant q$$
$$\left|r_5 - r_3^2\right|, \left|r_1 r_3 - r_3^2\right| \leqslant q^{k+n-m}$$
$$|r_4|, |b r_1|, \left|b r_5 + r_3 r_4 - (b+1) r_3^2\right|, \left|r_1 (r_4 - b r_3) + (b+2) r_3^2 - 2 x r_3 - r_5\right| \leqslant q^{k-n}.$$

The set $\widehat{U_k(g)}$ is invariant under the change of variables

$$(r_1, x, r_3, r_4, r_5) \mapsto \left(r_1, x + \varpi^{-1}, r_3, r_4, r_5 + 2 r_3 \varpi^{-1}\right).$$

Making this change of variables in the integral yields

$$E_k^{\Psi_s}(g) = \overline{\psi\left(\varpi^{-1}\right)} E_k^{\Psi_s}(g),$$

and hence

$$E_k^{\Psi_s}(g) = 0.$$

- When $|b| = \left|d \frac{t_1^2}{t_2}\right| = 1$ the calculation is more involved and is omitted. Nonetheless $E_k^{\Psi_s}$ vanishes on such elements and hence D^{Ψ_s} also vanishes. □

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