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Classification of simple smooth modules over the Heisenberg–Virasoro algebra

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In this paper, we classify simple smooth modules over the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$, and simple smooth modules over the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$ with non-zero level. To this end we generalize Sugawara operators to smooth modules over the Heisenberg algebra, and develop new techniques. As applications, we characterize simple Whittaker modules and simple highest weight modules over $\mathfrak{D}$. A vertex-algebraic interpretation of our result is the classification of simple weak twisted and untwisted modules over the Heisenberg–Virasoro vertex algebras. We also present a few examples of simple smooth $\mathfrak{D}$-modules and $\bar{\mathfrak{D}}$-modules induced from simple modules over finite dimensional solvable Lie algebras, that are not tensor product modules of Virasoro modules and Heisenberg modules. This is very different from the case of simple highest weight modules over $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ which are always tensor products of simple Virasoro modules and simple Heisenberg modules.

Keywords: The Virasoro algebra; the mirror Heisenberg–Virasoro algebra; smooth module; Heisenberg–Virasoro vertex operator algebra; tensor product module

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1. Introduction

Throughout the paper we denote by $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}_{\leqslant 0}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^*$ the sets of integers, non-zero integers, non-negative integers, positive integers, non-positive integers, real numbers, complex numbers and non-zero complex numbers, respectively. All vector spaces and Lie algebras are assumed to be over $\mathbb{C}$. For a Lie algebra $\mathcal{G}$, the universal algebra of $\mathcal{G}$ is denoted by $\mathcal{U}(\mathcal{G})$.

The Virasoro algebra $\mathfrak{Vir}$ and the Heisenberg algebra $\mathcal{H}$ are infinite-dimensional Lie algebras with bases $\{\mathbf{c},\, d_n : n \in \mathbb{Z}\}$ and $\{\mathbf{l},\, h_n : n \in \mathbb{Z}\}$, respectively. Their Lie brackets are given by

$$[\mathfrak{Vir}, \mathbf{c}] = 0,\ [d_m, d_n] = (m-n)d_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c},\ m, n \in \mathbb{Z},$$

and

$$[\mathcal{H}, \mathbf{l}] = 0, \quad [h_m, h_n] = m\delta_{m+n,0}\mathbf{l},\ \ m, n \in \mathbb{Z},$$

respectively. The twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$ is the universal central extension of the Lie algebra

$$\left\{ f(t)\frac{d}{dt} + g(t) : f, g \in \mathbb{C}[t, t^{-1}] \right\}$$

of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t,\, t^{-t}]$. Since the Lie algebra $\bar{\mathfrak{D}}$ contains the Virasoro algebra $\mathfrak{Vir}$ and the Heisenberg algebra $\mathcal{H}$ as subalgebras (but not the semi-direct product of the two subalgebras), many properties of $\bar{\mathfrak{D}}$ are closely related to the algebras $\mathfrak{Vir}$ and $\mathcal{H}$.

The Virasoro algebra $\mathfrak{Vir}$, the Heisenberg algebra $\mathcal{H}$ and the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$ are very important infinite-dimensional Lie algebras in mathematics and mathematical physics because of their beautiful representation theory (see [**31**, **32**]), and their widespread applications to vertex operator algebras (see [**19**, **23**]), quantum physics (see [**26**]), conformal field theory (see [**18**]) and so on. Many other interesting and important algebras contain the Virasoro algebra as a subalgebra, such as the Schrödinger–Virasoro algebra (see [**29**, **30**]), the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ (see [**7**, **25**, **38**]) and so on. These Lie algebras have nice structures and perfect theory on simple Harish–Chandra modules. The mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ is the even part of the mirror $N = 2$ superconformal algebra (see [**7**]), and is the semi-direct product of the Virasoro algebra and the twisted Heisenberg algebra (see definition 2.1).

1.1. Connection with representation theory of Lie algebras

Representation theory of Lie algebras has attracted a lot of attention of mathematicians and physicists. For a Lie algebra $\mathcal{G}$ with a triangular decomposition $\mathcal{G} = \mathcal{G}_+ \oplus \mathfrak{h} \oplus \mathcal{G}_-$ in the sense of [**49**], one can study its weight and non-weight representation theory. For weight representation approach, to some extent, Harish–Chandra modules are well understood for many infinite-dimensional Lie algebras, for example, the affine Kac–Moody algebras in [**12**, **49**], the Virasoro algebra in [**20**, **32**, **44**], the twisted Heisenberg–Virasoro algebra in [**4**, **41**], the Schrödinger–Virasoro algebra (partial results) in [**29**, **30**, **37**] and the mirror Heisenberg–Virasoro algebra in [**38**]. There are also some researches about weight modules with infinite-dimensional weight spaces (see [**9**, **16**, **43**]).

Recently, non-weight module theory over Lie algebras $\mathcal{G}$ attracts more attentions from mathematicians. In particular, $\mathcal{U}(\mathfrak{h})$-free $\mathcal{G}$-modules, Whittaker modules

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and smooth modules have been widely studied for many Lie algebras. The notation of $\mathcal{U}(\mathfrak{h})$-free modules was first introduced by Nilsson [50] for the simple Lie algebra $\mathfrak{sl}_{n+1}$. At the same time these modules were introduced in a very different approach in the paper [53]. Later, $\mathcal{U}(\mathfrak{h})$-free modules for many infinite-dimensional Lie algebras are determined, for example, the Kac–Moody algebras in [11, 17, 28], the Virasoro algebra in [39, 42, 46], the Witt algebra in [53], the twisted Heisenberg–Virasoro algebra and $W(2, 2)$ algebra in [13, 15, 43], and so on.

Whittaker modules for $\mathfrak{sl}_2(\mathbb{C})$ were first constructed by Arnal and Pinczon (see [5]). Whittaker modules for arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{L}$ were introduced and systematically studied by Kostant in [34], where he proved that these modules with a fixed regular Whittaker function (Lie homomorphism) on a nilpotent radical are (up to isomorphism) in bijective correspondence with central characters of $\mathcal{U}(\mathfrak{L})$. In recent years, Whittaker modules for many other Lie algebras have been investigated (see [1, 2, 8, 10, 17, 47, 48]).

1.2. Smooth modules

The smooth modules for a $\mathbb{Z}$-graded Lie algebra are the modules in which any vector can be annihilated by sufficiently large positive part of the Lie algebra. Whittaker modules and highest weight modules are smooth modules, and, in some sense, smooth modules can be seen as generalization of Whittaker modules and highest weight modules. Understanding smooth modules for an infinite-dimensional Lie algebra with a $\mathbb{Z}$-gradation is one of the core topics in Lie theory, for this class of modules are closely connected with the modules for corresponding vertex operator algebras. The first step of studying smooth modules is to classify all simple smooth modules for a Lie algebra. But this is a difficult challenge. Up to now all simple smooth modules for the Virasoro algebra are classified in [46]. There are some partial results of simple smooth modules for other Lie algebras. Some simple smooth modules for twisted Heisenberg–Virasoro algebra and mirror Heisenberg–Virasoro algebra with level 0 were constructed in [14, 24, 38]. Different from the case of level 0, the situation of non-zero level is much more challenging, we develop new techniques to deal with the classification of simple smooth modules over the mirror Heisenberg–Virasoro algebra and the twisted Heisenberg–Virasoro algebra with non-zero level in this paper. Rudakov investigated a class of simple modules over Lie algebras of Cartan type W, S, H in [51, 52], and these modules are smooth modules over the Cartan-type Lie algebras of the formal power series.

1.3. Vertex algebraic approach

For many infinite-dimensional $\mathbb{Z}$-graded Lie algebras and superalgebras $\mathcal{G}$, one can construct the associated (universal) vertex algebra $\mathcal{V}_{\mathcal{G}}$ with the property:

- Any smooth $\mathcal{G}$-module is a weak $\mathcal{V}_{\mathcal{G}}$-module;
- Any weak module for the vertex algebra $\mathcal{V}_{\mathcal{G}}$ has the structure of a smooth $\mathcal{G}$-module.

This approach is very prominent for the following cases:

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- Affine Kac–Moody algebra of type $X_n^{(1)}$, when the associated vertex algebra is the universal affine vertex algebra $V^k(\mathfrak{g})$ for certain simple Lie algebra $\mathfrak{g}$. This approach was used in [**2**] for studying Whittaker modules.

- Virasoro Lie algebra, when the associated vertex algebra is the universal Virasoro vertex algebra V_{Vir}^c (cf. [**35**])

- Heisenberg vertex algebra, when the associated vertex algebra is $M(1)$ (cf. [**35**]).

- Heisenberg–Virasoro algebra; super conformal algebras, etc.

From the vertex-algebraic point of view, the twisted Heisenberg–Virasoro algebra and its untwisted modules were investigated in [**3**, **27**].

The smooth representations of non-zero level for the twisted Heisenberg–Virasoro algebra corresponds to representations of the Heisenberg–Virasoro vertex algebra $\mathcal{V}^c = V_{Vir}^c \otimes M(1)$, where V_{Vir}^c is the universal Virasoro vertex algebra of central charge $c = \ell_1 - 1$, and $M(1)$ is the Heisenberg vertex algebra of level 1 (see definition 2.6). Since $M(\ell_2) \cong M(1)$ for $\ell_2 \neq 0$ (cf. [**35**]), we usually assume that the level $\ell_2 = 1$.

Moreover, the smooth representations of the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ can be treated as twisted modules for the Heisenberg–Virasoro vertex algebra $\mathcal{V}^c = V_{Vir}^c \otimes M(1)$.

We summarize the preceding discussion as follows.

- The category of smooth $\bar{\mathfrak{D}}$-modules of level 1 is equivalent to the category of weak (untwisted) modules for the vertex algebra $\mathcal{V}^c$;

- The category of smooth $\mathfrak{D}$-modules of level 1 is equivalent to the category of weak twisted modules for the vertex algebra $\mathcal{V}^c$.

1.4. Main results

In this paper, our main goal is to classify simple smooth modules for mirror Heisenberg–Virasoro algebra $\mathfrak{D}$, and classify simple smooth modules with non-zero level for the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$. As applications, we describe the simple untwisted and twisted modules for Heisenberg–Virasoro vertex algebras $\mathcal{V}^c$. The main results are the following theorems:

Main theorem A (theorem 4.13) Let S be a simple smooth module over the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ with level $\ell \neq 0$. Then

(i) $S \cong H^{\mathfrak{D}}$ where H is a simple smooth module over the Heisenberg algebra $\mathcal{H}$, or

(ii) S is an induced $\mathfrak{D}$-module from a simple smooth $\mathfrak{D}^{(0,-n)}$-module, or

(iii) $S \cong U^{\mathfrak{D}} \otimes H^{\mathfrak{D}}$ where U is a simple smooth $\mathfrak{Vir}$-module, and H is a simple smooth module over the Heisenberg algebra $\mathcal{H}$.

Main theorem B (theorem 5.8) Let M be a simple smooth module over the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$ with level $\ell \neq 0$. Then

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(i) $M \cong K(z)^{\bar{\mathfrak{D}}}$ where K is a simple smooth $\bar{\mathcal{H}}$-module and $z \in \mathbb{C}$, or

(ii) M is an induced $\bar{\mathfrak{D}}$-module from a simple smooth $\bar{\mathfrak{D}}^{(0,-n)}$-module for some $n \in \mathbb{Z}_+$, or

(iii) $M \cong K(z)^{\bar{\mathfrak{D}}} \otimes U^{\bar{\mathfrak{D}}}$ where $z \in \mathbb{C}$, K is a simple smooth $\bar{\mathcal{H}}$-module and U is a simple smooth $\mathfrak{Vir}$-module.

These simple smooth modules over the (mirror) Heisenberg–Virasoro algebra are actually all simple weak (twisted) modules over Heisenberg–Virasoro vertex algebras $\mathcal{V}^c$. As a consequence, we obtain the classification of twisted and untwisted simple modules for the Heisenberg–Virasoro vertex algebra $\mathcal{V}^c$, i.e. we obtain all weak simple $\mathcal{V}^c$-modules and all weak simple twisted $\mathcal{V}^c$-modules.

It is important to notice that certain weak modules induced from simple smooth $\mathfrak{D}^{(0,-n)}$-modules do not have the form $M_1 \otimes M_2$ as (twisted) modules for $V^c_{Vir} \otimes M(\ell_2)$ (see § 7). This is interesting, since in the category of ordinary (twisted) modules for the vertex algebras, such modules don't exist (see [**21**, Theorem 4.7.4] and its twisted analogues).

1.5. Organization of the paper

The present paper is organized as follows. In § 2, we recall notations related to the algebras $\mathfrak{D}$ and $\bar{\mathfrak{D}}$, collect some known results and generalize Sugawara operators to smooth $\mathcal{H}$-modules. Moreover, we establish a general result for a simple module to be a tensor product module over a class of Lie algebras (theorem 2.12). In § 3, we construct a class of induced simple $\mathfrak{D}$-modules (theorem 3.1). In § 4, by taking difference of Sugawara operators and the Virasoro operators we construct a new associative algebra on the smooth module. Then the universal enveloping algebra of $\mathfrak{D}$ can be considered as a tensor product of the new associative algebra and the enveloping algebra of the Heisenberg algebra. Using this tensor product, we are able to determine all simple smooth modules over the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ (theorems 4.13 and 2.10). In § 5, we use a similar method as in § 4 to classify the simple smooth modules of level non-zero over the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$ (see theorem 5.8). In § 6, we apply theorem 4.13 to generalize the result in [**45**] to the algebra $\mathfrak{D}$, i.e. we give a new characterization of simple highest weight modules over $\mathfrak{D}$ (theorem 6.1). We also characterize simple Whittaker modules over $\mathfrak{D}$ (theorem 6.3). In § 7, we present a few examples of simple smooth $\mathfrak{D}$-modules and $\bar{\mathfrak{D}}$-modules induced from simple modules over finite dimensional solvable Lie algebras, that are not tensor product modules of Virasoro modules and Heisenberg modules. This is very different from the case of simple highest weight modules over $\mathfrak{D}$ and $\bar{\mathfrak{D}}$ which are always tensor products of simple Virasoro modules and simple Heisenberg modules.

2. Notations and preliminaries

In this section, we recall some notations and known results related to the algebras $\mathfrak{D}$ and $\bar{\mathfrak{D}}$.

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Definition 2.1. *The* **twisted Heisenberg–Virasoro algebra** $\bar{\mathfrak{D}}$ *is a Lie algebra with a basis*

$$\{d_m, h_r, \bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \bar{\mathbf{c}}_3 : m, r \in \mathbb{Z}\}$$

and subject to the commutation relations

$$\begin{aligned}
&[d_m, d_n] = (m-n)d_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}\bar{\mathbf{c}}_1,\\
&[d_m, h_r] = -rh_{m+r} + \delta_{m+r,0}(m^2+m)\bar{\mathbf{c}}_2,\\
&[h_r, h_s] = r\delta_{r+s,0}\bar{\mathbf{c}}_3,\\
&[\bar{\mathbf{c}}_1, \bar{\mathfrak{D}}] = [\bar{\mathbf{c}}_2, \bar{\mathfrak{D}}] = [\bar{\mathbf{c}}_3, \bar{\mathfrak{D}}] = 0,
\end{aligned} \tag{2.1}$$

for m, n, r, $s \in \mathbb{Z}$.

It is clear that $\bar{\mathfrak{D}}$ contains a copy of the Virasoro subalgebra $\mathfrak{Vir} = \mathrm{span}\{\bar{\mathbf{c}}_1, d_i : i \in \mathbb{Z}\}$ and the Heisenberg algebra $\bar{\mathcal{H}} = \bigoplus_{r\in\mathbb{Z}} \mathbb{C}h_r \oplus \mathbb{C}\bar{\mathbf{c}}_3$. So $\bar{\mathfrak{D}}$ has a quotient algebra that is isomorphic to a copy of **Heisenberg–Virasoro algebra**

$$\widetilde{\mathfrak{D}} = \mathrm{span}_{\mathbb{C}}\{d_m, h_r, \bar{\mathbf{c}}_1, \bar{\mathbf{c}}_3 : m, r \in \mathbb{Z}\}$$

whose relations are defined by (2.1) (but the second and fourth equalities are replaced by $[d_m, h_r] = -rh_{m+r}$ and $[\bar{\mathbf{c}}_1, \widetilde{\mathfrak{D}}] = [\bar{\mathbf{c}}_3, \widetilde{\mathfrak{D}}] = 0$).

Note that $\bar{\mathfrak{D}}$ is $\mathbb{Z}$-graded and equipped with a triangular decomposition: $\bar{\mathfrak{D}} = \bar{\mathfrak{D}}^+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{D}}^-$, where

$$\bar{\mathfrak{D}}^{\pm} = \bigoplus_{n,r\in\mathbb{Z}_+} (\mathbb{C}d_{\pm n} \oplus \mathbb{C}h_{\pm r}), \quad \mathfrak{h} = \mathbb{C}d_0 \oplus \mathbb{C}h_0 \oplus \mathbb{C}\bar{\mathbf{c}}_1 + \mathbb{C}\bar{\mathbf{c}}_2 + \mathbb{C}\bar{\mathbf{c}}_3.$$

Moreover, $\bar{\mathfrak{D}} = \oplus_{i\in\mathbb{Z}}\bar{\mathfrak{D}}_i$ is $\mathbb{Z}$-graded with $\bar{\mathfrak{D}}_i = \mathbb{C}d_i \oplus \mathbb{C}h_i$ for $i \in \mathbb{Z}^*$, $\bar{\mathfrak{D}}_0 = \mathfrak{h}$.

Another compatible $\mathbb{Z}$-grading on $\mathfrak{D}$ can be given by $\deg(\mathfrak{Vir}) = 0$, $\deg(h_r)=1$, $\deg(c_2) = 2$.

Definition 2.2. *The* **mirror Heisenberg–Virasoro algebra** $\mathfrak{D}$ *is a Lie algebra with a basis*

$$\left\{d_m, h_r, \mathbf{c}_1, \mathbf{c}_2 \mid m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\right\}$$

and subject to the commutation relations

$$\begin{aligned}
&[d_m, d_n] = (m-n)d_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}\mathbf{c}_1,\\
&[d_m, h_r] = -rh_{m+r},\\
&[h_r, h_s] = r\delta_{r+s,0}\mathbf{c}_2,\\
&[\mathbf{c}_1, \mathfrak{D}] = [\mathbf{c}_2, \mathfrak{D}] = 0,
\end{aligned}$$

for m, $n \in \mathbb{Z}$, r, $s \in \frac{1}{2} + \mathbb{Z}$.

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It is clear that $\mathfrak{D}$ is the semi-direct product of the Virasoro subalgebra $\mathfrak{Vir} = \mathrm{span}\{\mathbf{c}_1, d_i \mid i \in \mathbb{Z}\}$ and the twisted Heisenberg algebra $\mathcal{H} = \bigoplus_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{C}h_r \oplus \mathbb{C}\mathbf{c}_2$. Note that $\mathfrak{D}$ is $\frac{1}{2}\mathbb{Z}$-graded and equipped with triangular decomposition: $\mathfrak{D} = \mathfrak{D}^+ \oplus \mathfrak{D}^0 \oplus \mathfrak{D}^-$, where

$$\mathfrak{D}^{\pm} = \bigoplus_{n\in\mathbb{Z}_+} \mathbb{C}d_{\pm n} \oplus \bigoplus_{r\in\frac{1}{2}+\mathbb{N}} \mathbb{C}h_{\pm r}, \quad \mathfrak{D}^0 = \mathbb{C}d_0 \oplus \mathbb{C}\mathbf{c}_1 \oplus \mathbb{C}\mathbf{c}_2.$$

Moreover, $\mathfrak{D} = \oplus_{i\in\mathbb{Z}}\mathfrak{D}_i$ is $\mathbb{Z}$-graded with $\mathfrak{D}_i = \mathbb{C}d_i \oplus \mathbb{C}h_{i+\frac{1}{2}}$ for $i \in \mathbb{Z}^* \setminus \{-1\}$, $\mathfrak{D}_0 = \mathbb{C}d_0 \oplus \mathbb{C}h_{\frac{1}{2}} \oplus \mathbb{C}\mathbf{c}_1$ and $\mathfrak{D}_{-1} = \mathbb{C}d_{-1} \oplus \mathbb{C}h_{-\frac{1}{2}} \oplus \mathbb{C}\mathbf{c}_2$.

Definition 2.3. *Let $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ be a $\mathbb{Z}$-graded Lie algebra. A $\mathcal{G}$-module V is called the* **smooth** *module if for any $v \in V$ there exists $n \in \mathbb{N}$ such that $\mathcal{G}_i v = 0$, for $i > n$. The category of smooth modules over $\mathcal{G}$ will be denoted as $\mathcal{R}_{\mathcal{G}}$.*

Smooth modules for affine Kac–Moody algebras $\mathfrak{g}$ were introduced and studied by Kazhdan and Lusztig in [33].

Definition 2.4. *Let $\mathfrak{a}$ be a subalgebra of a Lie algebra $\mathcal{G}$, and V be a $\mathcal{G}$-module. We denote*

$$\mathrm{Ker}_V(\mathfrak{a}) = \{v \in V : \mathfrak{a}v = 0\}.$$

Definition 2.5. *Let $\mathcal{G}$ be a Lie algebra and V a $\mathcal{G}$-module and $x \in \mathcal{G}$.*

(1) *If for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $x^n v = 0$, then we say that the action of x on V is* **locally nilpotent**.

(2) *If for any $v \in V$ we have $\dim(\sum_{n\in\mathbb{N}} \mathbb{C}x^n v) < +\infty$, then the action of x on V is said to be* **locally finite**.

(3) *The action of $\mathcal{G}$ on V is said to be* **locally nilpotent** *if for any $v \in V$ there exists an $n \in \mathbb{Z}_+$ (depending on v) such that $x_1x_2\cdots x_n v = 0$ for any $x_1, x_2, \cdots, x_n \in L$.*

(4) *The action of $\mathcal{G}$ on V is said to be* **locally finite** *if for any $v \in V$ there is a finite-dimensional $\mathcal{G}$-submodule of V containing v.*

Definition 2.6. *If W is a $\mathfrak{D}$-module (resp. $\bar{\mathfrak{D}}$-module) on which $\mathbf{c}_1$ (resp. $\bar{\mathbf{c}}_1$) acts as complex scalar c, we say that W is of* **central charge** *c. If W is a $\mathfrak{D}$-module (resp. $\bar{\mathfrak{D}}$-module) on which $\mathbf{c}_2$ (resp. $\bar{\mathbf{c}}_3$) acts as complex scalar ℓ, we say that W is of* **level** *ℓ.*

Note that if V is a $\mathfrak{Vir}$-module, then V can be easily viewed as a $\mathfrak{D}$-module (resp. $\bar{\mathfrak{D}}$-module) by defining $\mathcal{H}V = 0$ (resp. $(\bar{\mathcal{H}} + \mathbb{C}\bar{\mathbf{c}}_2)V = 0$), the resulting module is denoted by $V^{\mathfrak{D}}$(resp. $V^{\bar{\mathfrak{D}}}$).

For any $H \in \mathcal{R}_{\mathcal{H}}$ with the action of $\mathbf{c}_2$ as a non-zero scalar ℓ, we can give H a $\mathfrak{D}$-module structure denoted by $H^{\mathfrak{D}}$ via the following map

$$d_n \mapsto L_n = \frac{1}{2\ell} \sum_{k\in\mathbb{Z}+\frac{1}{2}} h_{n-k}h_k, \quad \forall n \in \mathbb{Z}, n \neq 0, \tag{2.2}$$

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$$d_0 \mapsto L_0 = \frac{1}{2\ell} \sum_{k\in\mathbb{Z}+\frac{1}{2}} h_{-|k|}h_{|k|} + \frac{1}{16}, \tag{2.3}$$

$$h_r \mapsto h_r, \quad \forall r \in \frac{1}{2} + \mathbb{Z}, \quad \mathbf{c_1} \mapsto 1, \quad \mathbf{c_2} \mapsto \ell. \tag{2.4}$$

The above operators were defined on highest weight modules H over $\mathcal{H}$ in [22]. We find that they are valid for smooth $\mathcal{H}$-modules. This is crucial to our further discussion on determining smooth $\mathfrak{D}$-modules in § 4.

According to (9.4.13) and (9.4.15) in [22] which are also valid in our case, we know that for all $m, n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, we have

$$\begin{aligned}&[L_n, h_r] = -rh_{n+r},\\ &[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}.\end{aligned} \tag{2.5}$$

Moreover, since

$$\begin{aligned}[d_m, h_{n-k}h_k] &= [d_m, h_{n-k}]h_k + h_{n-k}[d_m, h_k] = [L_m, h_{n-k}]h_k + h_{n-k}[L_m, h_k]\\ &= [L_m, h_{n-k}h_k],\end{aligned}$$

we see that

$$[d_m, L_n] = [L_m, L_n] \tag{2.6}$$

By [43], for any $z \in \mathbb{C}$ and $H \in \mathcal{R}_{\bar{\mathcal{H}}}$ with the action of $\bar{\mathbf{c}}_3$ as a non-zero scalar ℓ, we can give H a $\bar{\mathfrak{D}}$-module structure (denoted by $H(z)^{\bar{\mathfrak{D}}}$) via the following map

$$d_n \mapsto \bar{L}_n = \frac{1}{2\ell}\sum_{k\in\mathbb{Z}} : h_{n-k}h_k : + \frac{(n+1)z}{\ell}h_n, \quad \forall n \in \mathbb{Z}, \tag{2.7}$$

$$h_r \mapsto h_r, \quad \forall r \in \mathbb{Z}, \quad \bar{\mathbf{c}}_1 \mapsto 1 - \frac{12z^2}{\ell}, \quad \bar{\mathbf{c}}_2 \mapsto z, \quad \bar{\mathbf{c}}_3 \mapsto \ell, \tag{2.8}$$

where the normal order is defined as

$$:h_r h_s: = :h_s h_r: = h_r h_s, \text{ if } r \leqslant s.$$

According to (8.7.9), (8.7.13) in [22] which are also valid in our case and by some simple computation, we deduce that for all $m, n, r \in \mathbb{Z}$,

$$\begin{aligned}&[\bar{L}_m, h_r] = -rh_{m+r} + \delta_{m+r,0}(m^2+m)z,\\ &[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\left(1 - \frac{12z^2}{\ell}\right).\end{aligned} \tag{2.9}$$

REMARK 2.7. The relations (2.9) can be obtained using commutator formula, similarly as in [3].

Moreover, since

$$
\begin{aligned}
[d_m, h_{n-k}h_k] &= [d_m, h_{n-k}]h_k + h_{n-k}[d_m, h_k] = [\bar{L}_m, h_{n-k}]h_k + h_{n-k}[\bar{L}_m, h_k] \\
&= [\bar{L}_m, h_{n-k}h_k],
\end{aligned}
$$

we see that

$$[d_m, \bar{L}_n] = [\bar{L}_m, \bar{L}_n]. \tag{2.10}$$

For convenience, we define the following subalgebras of $\mathfrak{D}$. For any $m \in \mathbb{N}$, $n \in \mathbb{Z}$, set

$$
\begin{aligned}
\mathfrak{D}^{(m,n)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \mathbb{C}h_{n+i+\frac{1}{2}} \oplus \mathbb{C}\mathbf{c}_1 \oplus \mathbb{C}\mathbf{c}_2, \\
\mathfrak{D}^{(m,-\infty)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \sum_{i\in\mathbb{Z}} \mathbb{C}h_{i+\frac{1}{2}} + \mathbb{C}\mathbf{c}_1 + \mathbb{C}\mathbf{c}_2, \\
\mathfrak{Vir}^{(m)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \mathbb{C}\mathbf{c}_1, \\
\mathfrak{Vir}_{\geqslant m} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i}, \\
\mathfrak{Vir}_{\leqslant 0} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{-i}, \\
\mathfrak{Vir}_{+} &= \mathrm{span}\{\mathbf{c}_1, d_i : i \geqslant 0\}, \\
\mathcal{H}^{(n)} &= \sum_{i\in\mathbb{N}} \mathbb{C}h_{n+i+\frac{1}{2}} \oplus \mathbb{C}\mathbf{c}_2, \\
\mathcal{H}_{\geqslant n} &= \sum_{i\in\mathbb{N}} \mathbb{C}h_{n+i+\frac{1}{2}}.
\end{aligned} \tag{2.11}
$$

Similarly, we define the subalgebras of $\bar{\mathfrak{D}}$ as following: for $m \in \mathbb{N}$, $n \in \mathbb{Z}$, set

$$
\begin{aligned}
\bar{\mathfrak{D}}^{(m,n)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \mathbb{C}h_{n+i} \oplus \mathbb{C}\bar{\mathbf{c}}_1 \oplus \mathbb{C}\bar{\mathbf{c}}_2 + \mathbb{C}\bar{\mathbf{c}}_3, \\
\bar{\mathfrak{D}}^{(m,-\infty)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \sum_{i\in\mathbb{Z}} \mathbb{C}h_{i} + \mathbb{C}\bar{\mathbf{c}}_1 \oplus \mathbb{C}\bar{\mathbf{c}}_2 + \mathbb{C}\bar{\mathbf{c}}_3, \\
\mathfrak{Vir}^{(m)} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i} \oplus \mathbb{C}\bar{\mathbf{c}}_1, \\
\mathfrak{Vir}_{\geqslant m} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{m+i}, \\
\mathfrak{Vir}_{\leqslant 0} &= \sum_{i\in\mathbb{N}} \mathbb{C}d_{-i}, \\
\mathfrak{Vir}_{+} &= \mathrm{span}\{\bar{\mathbf{c}}_1, d_i : i \geqslant 0\},
\end{aligned} \tag{2.12}
$$

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$$\bar{\mathcal{H}}^{(n)} = \sum_{i\in\mathbb{N}} \mathbb{C}h_{n+i} \oplus \mathbb{C}\bar{\mathbf{c}}_3,$$

$$\bar{\mathcal{H}}_{\geqslant n} = \sum_{i\in\mathbb{N}} \mathbb{C}h_{n+i}.$$

Note that we use the same notations $\mathfrak{Vir}^{(m)}$, $\mathfrak{Vir}_{\geqslant m}$, $\mathfrak{Vir}_{\leqslant 0}$, $\mathfrak{Vir}_{+}$ to denote the subalgebras of $\mathfrak{D}$ and of $\bar{\mathfrak{D}}$ since there will be no ambiguities in later contexts.

Denote by $\mathbb{M}$ the set of all infinite vectors of the form $\mathbf{i} := (\ldots, i_2, i_1)$ with entries in $\mathbb{N}$, satisfying the condition that the number of non-zero entries is finite. We can make $(\mathbb{M}, +)$ a monoid by

$$(\ldots, i_2, i_1) + (\ldots, j_2, j_1) = (\ldots, i_2 + j_2, i_1 + j_1).$$

Let $\mathbf{0}$ denote the element $(\ldots, 0, 0) \in \mathbb{M}$ and for $i \in \mathbb{Z}_+$ let $\epsilon_i = (\ldots, 0, 1, 0, \ldots, 0) \in \mathbb{M}$, where 1 is in the i'th position from right. For any $\mathbf{i} \in \mathbb{M}$ we define

$$\mathrm{w}(\mathbf{i}) = \sum_{n\in\mathbb{Z}_+} i_n \cdot n, \tag{2.13}$$

Let $\prec$ be the **reverse lexicographic** total order on $\mathbb{M}$, that is, for any $\mathbf{i}$, $\mathbf{j} \in \mathbb{M}$,

$$\mathbf{j} \prec \mathbf{i} \iff \text{there exists } r \in \mathbb{N} \text{ such that } j_r < i_r \text{ and } j_s = i_s, \forall\, 1 \leqslant s < r.$$

We extend the above total order on $\mathbb{M} \times \mathbb{M}$, that is, for all $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l} \in \mathbb{M}$,

$$(\mathbf{i}, \mathbf{j}) \prec (\mathbf{k}, \mathbf{l}) \iff \begin{array}{l} (\mathbf{j}, \mathrm{w}(\mathbf{j}), \mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{j})) \prec (\mathbf{l}, \mathrm{w}(\mathbf{l}), \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l})), \text{ or} \\ (\mathbf{j}, \mathrm{w}(\mathbf{j}), \mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{j})) = (\mathbf{l}, \mathrm{w}(\mathbf{l}), \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l})), \text{ and } \mathbf{i} \prec \mathbf{k}. \end{array}$$

Now we define another total order $\prec'$ on $\mathbb{M} \times \mathbb{M}$: for all $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l} \in \mathbb{M}$,

$$(\mathbf{i}, \mathbf{j}) \prec' (\mathbf{k}, \mathbf{l}) \iff (\mathbf{j}, \mathbf{i}) \prec (\mathbf{l}, \mathbf{k}).$$

The symbols $\preceq$ and $\preceq'$ have the obvious meanings.

It is not hard to verify that

$$(a, b) \preceq (c, d) \ \&\ (c', d') \prec (a', b') \implies (a - a', b - b') \prec (c - c', d - d'),$$

provided (a, b), (c, d) (c', d'), (a', b'), $(a - a', b - b')$, $(c - c', d - d') \in \mathbb{M} \times \mathbb{M}$, where the difference is the corresponding entry difference.

For $n \in \mathbb{Z}$, let V be a simple $\mathfrak{D}^{(0,-n)}$-module. According to the PBW theorem, every non-zero element $v \in \mathrm{Ind}_{\mathfrak{D}^{(0,-n)}}^{\mathfrak{D}}(V)$ can be uniquely written in the following form

$$v = \sum_{\mathbf{i},\mathbf{k}\in\mathbb{M}} h^{\mathbf{i}} d^{\mathbf{k}} v_{\mathbf{i},\mathbf{k}}, \tag{2.14}$$

where

$$h^{\mathbf{i}} d^{\mathbf{k}} = \ldots h_{-2-n+\frac{1}{2}}^{i_2} h_{-1-n+\frac{1}{2}}^{i_1} \ldots d_{-2}^{k_2} d_{-1}^{k_1} \in U(\mathfrak{D}^{-}), v_{\mathbf{i},\mathbf{k}} \in V,$$

and only finitely many $v_{\mathbf{i},\mathbf{k}}$ are non-zero. For any non-zero $v \in \mathrm{Ind}(V)$ as in (2.14), we will use the following notations for later use:

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(1) Denote by $\mathrm{supp}(v)$ the set of all $(\mathbf{i}, \mathbf{k}) \in \mathbb{M} \times \mathbb{M}$ such that $v_{\mathbf{i},\mathbf{k}} \neq 0$.

(2) Denote by

$$\mathrm{w}(v) = \max\{\mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{k}) : (\mathbf{i}, \mathbf{k}) \in \mathrm{supp}(v)\},$$

called the **length** of v.

(3) Denote by $\deg(v)$ to be the largest element in $\mathrm{supp}(v)$ with respect to the total order $\prec$.

(4) Denote by $\deg'(v)$ to be the largest element in $\mathrm{supp}(v)$ with respect to the total order $\prec'$.

We first recall from [**46**] the classification for simple smooth $\mathfrak{Vir}$-modules.

THEOREM 2.8. *Any simple smooth $\mathfrak{Vir}$-module is a highest weight module, or isomorphic to $\mathrm{Ind}_{\mathfrak{Vir}_+}^{\mathfrak{Vir}} V$ for a simple $\mathfrak{Vir}_+$-module V such that for some $k \in \mathbb{Z}_+$,*

(a) *d_k acts injectively on V;*

(b) *$d_i V = 0$ for all $i > k$.*

Simple smooth $\mathfrak{D}$-modules with level 0 are classified in [**38**] by the following two theorems.

THEOREM 2.9. *Let V be a simple $\mathfrak{D}^{(0,-n)}$-module for some $n \in \mathbb{Z}_+$ and $c \in \mathbb{C}$ such that $\mathbf{c}_1 v = cv$, $\mathbf{c}_2 v = 0$ for any $v \in V$. Assume that there exists an integer $k \geqslant -n$ satisfying the following two conditions:*

(a) *the action of $h_{k+\frac{1}{2}}$ on V is bijective;*

(b) *$h_{m+\frac{1}{2}} V = 0 = d_{m+n} V$ for all $m > k$.*

Then the induced $\mathfrak{D}$-module $\mathrm{Ind}_{\mathfrak{D}^{(0,-n)}}^{\mathfrak{D}}(V)$ is simple.

THEOREM 2.10. *Every simple smooth $\mathfrak{D}$-module S of level 0 is isomorphic to a smooth $\mathfrak{Vir}$-module with $\mathcal{H}S = 0$, or $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-n)}}^{\mathfrak{D}}(V)$ for some $n \in \mathbb{N}$ and a simple $\mathfrak{D}^{(0,-n)}$-module V as in theorem 2.9.*

Actually the simple $\mathfrak{D}^{(0,-n)}$-module V can be considered as a simple module over a finite dimensional solvable Lie algebra $\mathfrak{D}^{(0,-n)}/\mathfrak{D}^{(t+n+1,t-n)}$ for some $t \in \mathbb{Z}_+$ and injective action of $h_{t+\frac{1}{2}}$ on V.

For simple smooth $\bar{\mathfrak{D}}$-modules with level 0, we know the following results from [**14**].

THEOREM 2.11. *Let $n \in \mathbb{N}$ and V be a simple module over $\bar{\mathfrak{D}}^{(0,-n)}$ or over $\bar{\mathfrak{D}}^{(0,-\infty)}$ with $\ell = 0$, $h_0 = \mu$, $\bar{\mathbf{c}}_2 = z$. If there exists $k \in \mathbb{N}$ such that*

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(a)

$$\begin{cases} h_k \text{ acts injectively on } V, & \text{if} \quad k \neq 0, \\ \mu + (1-r)z \neq 0, \forall r \in \mathbb{Z} \setminus \{0\}, & \text{if} \quad k = 0; \end{cases}$$

(b) $h_i V = d_j V = 0$ *for all* $i > k$ *and* $j > k + n$.

then

(1) $\mathrm{Ind}(V)$ *is a simple* $\bar{\mathfrak{D}}$*-module;*

(2) h_i, d_j *act locally nilpotently on* $\mathrm{Ind}(V)$ *for all* $i > k$ *and* $j > k + n$.

Now we generalize Theorem 12 in [**43**] as follows.

Let $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ be a Lie algebra where $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Let M be a $\mathfrak{g}$-module with a $\mathfrak{b}$-submodule H so that the $\mathfrak{b}$-submodule structure on H can be extended to a $\mathfrak{g}$-module structure on H. We denote this $\mathfrak{g}$-module by $H^{\mathfrak{g}}$. For any $\mathfrak{a}$-module U, we can make it into a $\mathfrak{g}$-module by $\mathfrak{b}U = 0$. We denote this $\mathfrak{g}$-module by $U^{\mathfrak{g}}$.

THEOREM 2.12. *Let* $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ *be a countable dimensional Lie algebra where* $\mathfrak{a}$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{b}$ *is an ideal of* $\mathfrak{g}$. *Let* M *be a simple* $\mathfrak{g}$*-module with a simple* $\mathfrak{b}$*-submodule* H *so that an* $H^{\mathfrak{g}}$ *exists. Then* $M \cong H^{\mathfrak{g}} \otimes U^{\mathfrak{g}}$ *as* $\mathfrak{g}$*-modules for some simple* $\mathfrak{a}$*-module* U.

Proof. Define the one-dimensional $\mathfrak{b}$-module $\mathbb{C}v_0$ by $\mathfrak{b}v_0 = 0$. Then $H \cong H \otimes \mathbb{C}v_0$ as $\mathfrak{b}$-modules. Now from Lemma 8 in [**43**], we have

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} H \cong \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} (H \otimes \mathbb{C}v_0) \cong H^{\mathfrak{g}} \otimes \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}v_0.$$

Note that $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}v_0 \cong W^{\mathfrak{g}}$ for the universal $\mathfrak{a}$-module W. Since M is a simple quotient of $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} H$, from Theorem 7 in [**43**] we know that there is a simple quotient $\mathfrak{a}$-module U of W such that $M \cong H^{\mathfrak{g}} \otimes U^{\mathfrak{g}}$ as $\mathfrak{g}$-modules. Now the theorem follows. □

Remark. This theorem has particular meaning for $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ since $H^{\mathfrak{g}}$ automatically exists (see e.g. [**36**]). Also, theorem 2.12 holds for associative algebras.

Applying the above theorem to our mirror Heisenberg–Virasoro algebra $\mathfrak{D} = \mathfrak{Vir} \ltimes \mathcal{H}$ and twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}} = \mathfrak{Vir} \ltimes (\bar{\mathcal{H}} + \mathbb{C}\bar{\mathbf{c}}_2)$, we have the following results.

COROLLARY 2.13. *Let* V *be a simple* $\mathfrak{D}$*-module with non-zero action of* $\mathbf{c}_2$. *Then* $V \cong H^{\mathfrak{D}} \otimes U^{\mathfrak{D}}$ *as a* $\mathfrak{D}$*-module for some simple module* $H \in \mathcal{R}_{\mathcal{H}}$ *and some simple* $\mathfrak{Vir}$*-module* U *if and only if* V *contains a simple* $\mathcal{H}$*-submodule* $H \in \mathcal{R}_{\mathcal{H}}$.

Proof. The sufficiency follows from theorem 2.12; and the necessity follows from that $H \otimes u$ is a simple $\mathcal{H}$-submodule of $H^{\mathfrak{D}} \otimes U^{\mathfrak{D}}$ for any non-zero $u \in U$. □

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3. Induced modules over the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$

In this section, we construct some simple smooth $\mathfrak{D}$-modules induced from some simple ones over some subalgebras $\mathfrak{D}^{(0,-n)}$ for $n \in \mathbb{Z}_+$. For that, we need the following formulas in $U(\mathfrak{D})$ which can be shown by induction on t: let i, $j_s \in \mathbb{Z}$, $1 \leqslant s \leqslant t$ with $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t$,

$$\left[h_{i-\frac{1}{2}}, h_{j_1+\frac{1}{2}} h_{j_2+\frac{1}{2}} \cdots h_{j_t+\frac{1}{2}}\right] = \sum_{1\leqslant s\leqslant t} \delta_{i+j_s,0}\left(i-\frac{1}{2}\right)\mathbf{c}_2 h_{j_1+\frac{1}{2}} \cdots \hat{h}_{j_s+\frac{1}{2}} \cdots h_{j_t+\frac{1}{2}}, \tag{3.1}$$

$$\begin{aligned}\left[d_i, h_{j_1+\frac{1}{2}} h_{j_2+\frac{1}{2}} \cdots h_{j_t+\frac{1}{2}}\right] &= \sum_{1\leqslant s\leqslant t}\left(-j_s-\frac{1}{2}\right) h_{j_1+\frac{1}{2}} \cdots \hat{h}_{j_s+\frac{1}{2}} \cdots h_{j_t+\frac{1}{2}} h_{i+j_s+\frac{1}{2}} \\ &+ \sum_{1\leqslant s_1<s_2\leqslant t}\left(-j_{s_1}-\frac{1}{2}\right)\left(i+j_{s_1}+\frac{1}{2}\right)\delta_{i+j_{s_1}+j_{s_2}+1,0}\mathbf{c}_2 h_{j_1+\frac{1}{2}} \cdots \\ &\hat{h}_{j_{s_1}+\frac{1}{2}} \cdots \hat{h}_{j_{s_2}+\frac{1}{2}} \cdots h_{j_t+\frac{1}{2}},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\left[h_{i-\frac{1}{2}}, d_{j_1} d_{j_2} \cdots d_{j_t}\right] &= \sum_{1\leqslant s\leqslant t}\left(i-\frac{1}{2}\right) d_{j_1} \cdots \hat{d_{j_s}} \cdots d_{j_t} h_{i+j_s-\frac{1}{2}} \\ &+ \sum_{1\leqslant s_1<s_2\leqslant t} a_{s_1,s_2} d_{j_1} \cdots \hat{d}_{j_{s_1}} \cdots \hat{d}_{j_{s_2}} \cdots d_{j_t} h_{i+j_{s_1}+j_{s_2}-\frac{1}{2}} + \cdots \\ &+ a_{1,2,\cdots,t} h_{i+j_1+j_2+\cdots+j_t-\frac{1}{2}},\end{aligned} \tag{3.3}$$

$$\begin{aligned}[d_i, d_{j_1} d_{j_2} \cdots d_{j_t}] &= \sum_{1\leqslant s\leqslant t}(i-j_s) d_{j_1} \cdots \hat{d_{j_s}} \cdots d_{j_t} \tilde{d}_{i+j_s} \\ &+ \sum_{1\leqslant s_1<s_2\leqslant t} b_{s_1,s_2} d_{j_1} \cdots \hat{d}_{j_{s_1}} \cdots \hat{d}_{j_{s_2}} \cdots d_{j_t} \tilde{d}_{i+j_{s_1}+j_{s_2}} + \cdots \\ &+ b_{1,2,\cdots,t} \tilde{d}_{i+j_1+j_2+\cdots+j_t},\end{aligned} \tag{3.4}$$

where $\hat{h}_{j_s+\frac{1}{2}}$, $\hat{d}_{j_s}$ mean that $h_{j_s+\frac{1}{2}}$, d_{j_s} are deleted in the corresponding products, $a_{s_1,s_2}, \cdots, a_{1,2,\cdots,t}, b_{s_1,s_2}, \cdots, b_{1,2,\cdots,t} \in \mathbb{C}$, and $\tilde{d}_{i+j_1+\cdots+j_s} = d_{i+j_1+\cdots+j_s} + \frac{j_s^2-1}{24}\delta_{i+j_1+\cdots+j_s,0}\mathbf{c}_1$, $1 \leqslant s \leqslant t$.

We are now in the position to present the following main result in this section.

THEOREM 3.1. *Let $k \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$ with $k \geqslant n$. Let V be a simple $\mathfrak{D}^{(0,-n)}$-module with level $\ell \neq 0$ such that there exists $l \in \mathbb{N}$ satisfying both conditions:*

(a) *$h_{k-\frac{1}{2}}$ acts injectively on V;*

(b) *$h_{i-\frac{1}{2}}V = d_jV = 0$ for all $i > k$ and $j > l$.*

Then $\mathrm{Ind}_{\mathfrak{D}^{(0,-n)}}^{\mathfrak{D}}(V)$ is a simple $\mathfrak{D}$-module if one of the following conditions holds:

(1) *$k = n$, $l \geqslant 2n$ and d_l acts injectively on V;*

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(2) $k > n$, $k + n \geqslant 2$ and $l = n + k - 1$.

Theorem 3.1 follows from lemmas 3.2–3.5 directly.

LEMMA 3.2. *Let $n \in \mathbb{Z}_+$ and V be a $\mathfrak{D}^{(0,-n)}$-module such that $h_{n-\frac{1}{2}}$ acts injectively on V, and $h_{i-\frac{1}{2}}V = 0$ for all $i > n$. For any $v \in \mathrm{Ind}(V) \setminus V$, let $\deg(v) = (\mathbf{i}, \mathbf{j})$. If $\mathbf{i} \neq \mathbf{0}$, then $\deg(h_{p+n-\frac{1}{2}}v) = (\mathbf{i} - \epsilon_p, \mathbf{j})$ where $p = \min\{s : i_s \neq 0\}$.*

Proof. Write v in the form of (2.14) and let $(\mathbf{k}, \mathbf{l}) \in \mathrm{supp}(v)$.

Noticing that $h_{p+n-\frac{1}{2}}V = 0$, we have

$$h_{p+n-\frac{1}{2}}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = [h_{p+n-\frac{1}{2}}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}}[h_{p+n-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}.$$

First we consider the term $[h_{p+n-\frac{1}{2}}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ which is zero if $k_p = 0$. In the case that $k_p > 0$, since the level $\ell \neq 0$, it follows from (3.1) that $[h_{p+n-\frac{1}{2}}, h^{\mathbf{k}}] = \lambda h^{\mathbf{k}-\epsilon_p}$ for some $\lambda \in \mathbb{C}^*$. So

$$\deg([h_{p+n-\frac{1}{2}}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k} - \epsilon_p, \mathbf{l}) \preceq (\mathbf{i} - \epsilon_p, \mathbf{j}),$$

where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{j})$.

Now we consider the term $h^{\mathbf{k}}[h_{p+n-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$ which is by (3.3) a linear combination of some vectors in the form $h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}}$ with $j \in \mathbb{Z}_+$ and $\mathrm{w}(\mathbf{l}_j) = \mathrm{w}(\mathbf{l}) - j$. If $h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}} \neq 0$, we denote $\deg(h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}^*, \mathbf{l}^*)$. We will show that

$$(\mathbf{k}^*, \mathbf{l}^*) \prec (\mathbf{i} - \epsilon_p, \mathbf{j}). \tag{3.5}$$

We have four different cases to consider.

(a) $j < p$. Then $p + n - j > n$ and $h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}} = 0$. Hence $h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}} = 0$.

(b) $j = p$. Noting that $h_{n-\frac{1}{2}}$ acts injectively on V, we see $(\mathbf{k}^*, \mathbf{l}^*) = (\mathbf{k}, \mathbf{l}_p)$ and $\mathrm{w}(\mathbf{k}^*) + \mathrm{w}(\mathbf{l}^*) = (\mathbf{k}) + \mathrm{w}(\mathbf{l})$-$p$ with $\mathrm{w}(\mathbf{l}_p) = \mathrm{w}(\mathbf{l}) - p < \mathrm{w}(\mathbf{l})$.
If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{j})$, then $(\mathbf{k}^*, \mathbf{l}^*) \prec (\mathbf{i} - \epsilon_p, \mathbf{j})$.
If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{j})$, then there is $\tau \in \mathbb{M}$ such that $\mathrm{w}(\tau) = p$ and $\mathbf{l}_p = \mathbf{l} - \tau$. Since $(\epsilon_p, \mathbf{0}) \prec (\mathbf{0}, \tau)$ and $(\mathbf{k}, \mathbf{l}) \preceq (\mathbf{i}, \mathbf{j})$, we see that

$$(\mathbf{k}^*, \mathbf{l}^*) = (\mathbf{k}, \mathbf{l}) - (\mathbf{0}, \tau) \prec (\mathbf{i}, \mathbf{j}) - (\epsilon_p, \mathbf{0}) = (\mathbf{i} - \epsilon_p, \mathbf{j}).$$

In both cases, (3.5) holds.

(c) $p < j < 2n + p$. Then $h_{p+n-\frac{1}{2}-j} \in \mathfrak{D}^{(0,-n)}$ and $h_{p+n-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}} \in V$. So

$$\mathrm{w}(\mathbf{k}^*) + \mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - p$$

and (3.5) holds.

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(d) $j \geqslant 2n+p$. Then $p+n-\frac{1}{2}-j < -n+\frac{1}{2}$. Assume $p+n-\frac{1}{2}-j = -s-n+\frac{1}{2}$ for some $s \in \mathbb{Z}_+$, that is, $-j+s = -2n-p+1 < -p$. Clearly, the corresponding vector $h^{\mathbf{k}} d^{\mathbf{l}_j} h_{p+n-\frac{1}{2}-j} v_{\mathbf{k},\mathbf{l}}$ can be written in the form

$$h^{\mathbf{k}} h_{-s-n+\frac{1}{2}} d^{\mathbf{l}_j} v_{\mathbf{k},\mathbf{l}} + \text{lower terms},$$

which means

$$\mathrm{w}(\mathbf{k}^*) + \mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j + s < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - p,$$

and hence (3.5) holds.

In conclusion, $\deg(h_{p+n-\frac{1}{2}} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}}) \preceq (\mathbf{i}-\epsilon_p, \mathbf{j})$, where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{j})$, that is, $\deg(h_{p+n-\frac{1}{2}} v) = (\mathbf{i}-\epsilon_p, \mathbf{j})$. □

LEMMA 3.3. *Let* $n \in \mathbb{Z}_+$ *and* V *be a* $\mathfrak{D}^{(0,-n)}$*-module satisfying conditions (a), (b) and (1) in theorem 3.1. If* $v \in \mathrm{Ind}(V) \setminus V$ *with* $\deg(v) = (\mathbf{0}, \mathbf{j})$, *then* $\deg(d_{q+l} v) = (\mathbf{0}, \mathbf{j}-\epsilon_q)$ *where* $q = \min\{s : j_s \neq 0\}$.

Proof. Write v in the form of (2.14) and let $(\mathbf{k}, \mathbf{l}) \in \mathrm{supp}(v)$.

Since $d_{q+l} V = 0$, we have

$$d_{q+l} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} = [d_{q+l}, h^{\mathbf{k}}] d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}} [d_{q+l}, d^{\mathbf{l}}] v_{\mathbf{k},\mathbf{l}}.$$

We first consider the degree of $h^{\mathbf{k}}[d_{q+l}, d^{\mathbf{l}}] v_{\mathbf{k},\mathbf{l}}$ with $d_{q+l} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} \neq 0$. Clearly, by (3.4) we see that $h^{\mathbf{k}}[d_{q+l}, d^{\mathbf{l}}] v_{\mathbf{k},\mathbf{l}}$ is a linear combination of some vectors of the forms $h^{\mathbf{k}} d^{\mathbf{l}_j} d_{q+l-j} v_{\mathbf{k},\mathbf{l}}$, $j \in \mathbb{Z}_+$ and $h^{\mathbf{k}} d^{\mathbf{l}_{q+l}} v_{\mathbf{k},\mathbf{l}}$ where $\mathrm{w}(\mathbf{l}_j) = \mathrm{w}(\mathbf{l}) - j$. Clearly, $\deg(h^{\mathbf{k}} d^{\mathbf{l}_{q+l}} v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}, \mathbf{l}_{q+l})$ has weight

$$\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q - l < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q \leqslant \mathrm{w}(\mathbf{j}) - q,$$

so $\deg(h^{\mathbf{k}} d^{\mathbf{l}_{q+l}} v_{\mathbf{k},\mathbf{l}}) \prec (\mathbf{0}, \mathbf{j}-\epsilon_q)$. Then we need only to consider $h^{\mathbf{k}} d^{\mathbf{l}_j} d_{q+l-j} v_{\mathbf{k},\mathbf{l}}$. Denote $\deg(h^{\mathbf{k}} d^{\mathbf{l}_j} d_{q+l-j} v_{\mathbf{k},\mathbf{l}})$ by $(\mathbf{k}, \mathbf{l}^*)$. We will show that

$$(\mathbf{k}, \mathbf{l}^*) \preceq (\mathbf{0}, \mathbf{j}-\epsilon_q), \tag{3.6}$$

where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{0}, \mathbf{j})$. We have four different cases to consider.

(i) $j < q$. Then $q+l-j > l$ and $h^{\mathbf{k}} d^{\mathbf{l}_j} d_{q+l-j} v_{\mathbf{k},\mathbf{l}} = 0$.

(ii) $j = q$. Then $q+l-j = l$. Since d_l acts injectively on V, we see $(\mathbf{k}, \mathbf{l}^*) = (\mathbf{k}, \mathbf{l}_q)$ and $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q$. If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(< \mathrm{brc} > \mathbf{0}) + \mathrm{w}(\mathbf{j})$, then $(\mathbf{k}, \mathbf{l}^*) \prec (\mathbf{0}, \mathbf{j}-\epsilon_q)$. If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(< \mathrm{brc} > \mathbf{0}) + \mathrm{w}(\mathbf{j})$, there is $\tau \in \mathbb{M}$ such that $\mathrm{w}(\tau) = q$ and $\mathbf{l}_q = \mathbf{l} - \tau$. Then $(\mathbf{0}, \epsilon_q) \preceq (\mathbf{0}, \tau)$. Since $(\mathbf{k}, \mathbf{l}) \preceq (\mathbf{0}, \mathbf{j})$, we see that

$$(\mathbf{k}, \mathbf{l}^*) = (\mathbf{k}, \mathbf{l}) - (\mathbf{0}, \tau) \preceq (\mathbf{0}, \mathbf{j}) - (\mathbf{0}, \epsilon_q) = (\mathbf{0}, \mathbf{j}-\epsilon_q).$$

In both cases we have that

$$(\mathbf{k}, \mathbf{l}^*) \preceq (\mathbf{0}, \mathbf{j}-\epsilon_q),$$

where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{0}, \mathbf{j})$.

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(iii) $q+1 \leqslant j \leqslant q+l$. Then $0 \leqslant q+l-j \leqslant l-1$ and $d_{q+l-j}v_{\mathbf{k},\mathbf{l}} \in V$. So if $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+l-j}v_{\mathbf{k},\mathbf{l}} \neq 0$, then $\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-j < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q$.

(iv) $j > q+l$. Then $q+l-j < 0$. Clearly, $\mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{l}_j)+(j-q-l) = \mathrm{w}(\mathbf{l}) - q - l$, and hence

$$\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q-l < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q.$$

Therefore, we conclude that (3.6) holds, i.e. $\sum_{(\mathbf{k},\mathbf{l})} h^{\mathbf{k}}[d_{q+l}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$ has degree $(\mathbf{0}, \mathbf{j}-\epsilon_q)$.

Next we consider the degree of the non-zero vector $[d_{q+l}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. By (3.2) we can see that $[d_{q+l}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ is a linear combination of some vectors of the forms $h^{\mathbf{k}_s}h_{q+l-s-n+\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$, $s \in \mathbb{Z}_+$ and $h^{\mathbf{k}_{q+l+1-2n}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$, where $\mathrm{w}(\mathbf{k}_s) = \mathrm{w}(\mathbf{k})-s$. Noting that $l \geqslant 2n$, the degree of $h^{\mathbf{k}_{q+l+1-2n}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ has weight

$$\mathrm{w}(\mathbf{k})-(q+l+1-2n)+\mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q.$$

So

$$\deg(h^{\mathbf{k}_{q+l+1-2n}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec (\mathbf{0}, \mathbf{j}-\epsilon_q).$$

Next we will show that

$$\deg(h^{\mathbf{k}_s}h_{q+l-s-n+\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec (\mathbf{0}, \mathbf{j}-\epsilon_q). \tag{3.7}$$

We have two different cases to consider.

(a) $s > q+l$. The degree of $h^{\mathbf{k}_s}h_{q+l-s-n+\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ has weight

$$\mathrm{w}(\mathbf{k}_s)+(s-q-l)+\mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q-l < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q.$$

So, (3.7) holds in this case.

(b) $1 \leqslant s \leqslant q+l$. We have

$$h^{\mathbf{k}_s}h_{q+l-s-n+\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = h^{\mathbf{k}_s}[h_{q+l-s-n+\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}_s}d^{\mathbf{l}}h_{q+l-s-n+\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}.$$

Noting that $h_{q+l-s-n+\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} \in V$ (in particular, $h_{q+l-s-n+\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} = 0$ for $1 \leqslant s \leqslant q+l-2n$), we see that if $h^{\mathbf{k}_s}d^{\mathbf{l}}h_{q+l-s-n+\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} \neq 0$ for $q+l-2n+1 \leqslant s \leqslant q+l$, its degree has weight

$$\mathrm{w}(\mathbf{k}_s)+\mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-s < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q.$$

Now we consider $\deg(h^{\mathbf{k}_s}[h_{q+l-s-n+\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}})$ which is denoted by $(\tilde{\mathbf{k}}, \tilde{\mathbf{l}})$.

(b1) $1 \leqslant s \leqslant q$, that is, $q+l-s-n \geqslant n$. Then $q+l-s-n+\frac{1}{2} = n+p-\frac{1}{2}$ for some $p \in \mathbb{Z}_+$ and hence $s+p = q+l-2n+1 \geqslant q+1$. Thus, by the same arguments in the proof of lemma 3.2, we see

$$\begin{aligned}\mathrm{w}(\tilde{\mathbf{k}})+\mathrm{w}(\tilde{\mathbf{l}}) &\leqslant \mathrm{w}(\mathbf{k}_s)+\mathrm{w}(\mathbf{l})-p = \mathrm{w}(\mathbf{k})-s+\mathrm{w}(\mathbf{l})-p \\ &\leqslant \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q-1 < \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-q.\end{aligned}$$

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So, (3.7) holds in this case.

(b2) $q+1 \leqslant s \leqslant q+l$. Then by (3.3) and the same arguments in the proof of lemma 3.2, we see

$$\mathrm{w}(\tilde{\mathbf{k}}) + \mathrm{w}(\tilde{\mathbf{l}}) \leqslant \mathrm{w}(\mathbf{k}_s) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - s \leqslant \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q - 1$$
$$< \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q.$$

So, (3.7) holds in this case as well.
Therefore, $\deg(d_{q+l}v) = (\mathbf{0}, \mathbf{j} - \epsilon_q)$, as desired.

□

Lemma 3.4. *Let $k \in \mathbb{Z}_+$, $n \in \mathbb{Z}$ with $k \geqslant n$ and $k+n \geqslant 2$, and let V be a $\mathfrak{D}^{(0,-n)}$-module such that $h_{k-\frac{1}{2}}$ acts injectively on V, and $h_{i-\frac{1}{2}}V = 0$ for all $i > k$. If $v \in \mathrm{Ind}(V) \setminus V$ with $\deg'(v) = (\mathbf{i}, \mathbf{j})$ and $\mathbf{j} \neq \mathbf{0}$, then $\deg'(h_{p+k-\frac{1}{2}}v) = (\mathbf{i}, \mathbf{j} - \epsilon_p)$ where $p = \min\{s : j_s \neq 0\}$.*

Proof. As in (2.14), write $v = \sum_{(\mathbf{k},\mathbf{l})} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}}$. Consider $\deg'(h_{p+k-\frac{1}{2}} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}})$ if $h_{p+k-\frac{1}{2}} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} \neq 0$. Noting that $h_{p+k-\frac{1}{2}} V = 0$, we see

$$h_{p+k-\frac{1}{2}} h^{\mathbf{k}} d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} = [h_{p+k-\frac{1}{2}}, h^{\mathbf{k}}] d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}} [h_{p+k-\frac{1}{2}}, d^{\mathbf{l}}] v_{\mathbf{k},\mathbf{l}}.$$

First we consider the term $[h_{p+k-\frac{1}{2}}, h^{\mathbf{k}}] d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}}$ which is zero if $k_{p'} = 0$ for $p' := p+k-n$. In the case that $k_{p'} > 0$, since the level $\ell \neq 0$, it follows from (3.1) that $[h_{p+k-\frac{1}{2}}, h^{\mathbf{k}}] = \lambda h^{\mathbf{k}-\epsilon_{p'}}$ for some $\lambda \in \mathbb{C}^*$. Note that $(\mathbf{k}, \mathbf{l}) \preceq' (\mathbf{i}, \mathbf{j})$, $(\mathbf{0}, \epsilon_p) \prec' (\epsilon_{p'}, \mathbf{0})$. So

$$\deg'([h_{p+k-\frac{1}{2}}, h^{\mathbf{k}}] d^{\mathbf{l}} v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k} - \epsilon_{p'}, \mathbf{l}) = (\mathbf{k}, \mathbf{l}) - (\epsilon_{p'}, \mathbf{0}) \prec' (\mathbf{i}, \mathbf{j}) - (\mathbf{0}, \epsilon_p)$$
$$= (\mathbf{i}, \mathbf{j} - \epsilon_p).$$

Now we consider the term $h^{\mathbf{k}} [h_{p+k-\frac{1}{2}}, d^{\mathbf{l}}] v_{\mathbf{k},\mathbf{l}}$ which is by (3.3) a linear combination of some vectors in the form $h^{\mathbf{k}} d^{\mathbf{l}_j} h_{p+k-\frac{1}{2}-j} v_{\mathbf{k},\mathbf{l}}$ with $j \in \mathbb{Z}_+$ and $\mathrm{w}(\mathbf{l}_j) = \mathrm{w}(\mathbf{l}) - j$. We will show that

$$\deg'(h^{\mathbf{k}} d^{\mathbf{l}_j} h_{p+k-\frac{1}{2}-j} v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}^*, \mathbf{l}^*) \preceq' (\mathbf{i}, \mathbf{j} - \epsilon_p), \tag{3.8}$$

where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{j})$. We have four different cases to consider.

(a) $j < p$. Then $p + k - j > n$ and $h_{p+k-\frac{1}{2}-j} v_{\mathbf{k},\mathbf{l}} = 0$. Hence, $h^{\mathbf{k}} d^{\mathbf{l}_j} h_{p+k-\frac{1}{2}-j} v_{\mathbf{k},\mathbf{l}} = 0$.

(b) $j = p$. Noting that $h_{k-\frac{1}{2}}$ acts injectively on V, we see $(\mathbf{k}^*, \mathbf{l}^*) = (\mathbf{k}, \mathbf{l}_p)$ and $\mathrm{w}(\mathbf{k}^*) + \mathrm{w}(\mathbf{l}^*) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l})$-$p$.

If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{i}) + \mathrm{w}(\mathbf{j})$, then $(\mathbf{k}^*, \mathbf{l}^*) \preceq' (\mathbf{i}, \mathbf{j} - \epsilon_p)$.

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If $\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})=\mathrm{w}(\mathbf{i})+\mathrm{w}(\mathbf{j})$, then there is $\tau\in\mathbb{M}$ such that $\mathrm{w}(\tau)=p$ and $\mathbf{l}_p=\mathbf{l}-\tau$. Since $(\mathbf{0},\,\epsilon_p)\preceq'(\mathbf{0},\,\tau)$ and $(\mathbf{k},\,\mathbf{l})\preceq'(\mathbf{i},\,\mathbf{j})$, we see that

$$(\mathbf{k}^*,\mathbf{l}^*)=(\mathbf{k},\mathbf{l})-(0,\tau)\preceq'(\mathbf{i},\mathbf{j})-(\mathbf{0},\epsilon_p)=(\mathbf{i},\mathbf{j}-\epsilon_p),$$

where the equality holds if and only if $(\mathbf{k},\,\mathbf{l})=(\mathbf{i},\,\mathbf{j})$.

(c) $p<j<n+k+p$. Then $h_{p+k-\frac{1}{2}-j}\in\mathfrak{D}^{(0,-n)}$ and $h_{p+k-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}}\in V$. So

$$\mathrm{w}(\mathbf{k}^*)+\mathrm{w}(\mathbf{l}^*)=\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-j<\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-p$$

and $(\mathbf{k}^*,\,\mathbf{l}^*)\prec'(\mathbf{i},\,\mathbf{j}-\epsilon_p)$.

(d) $j\geqslant n+k+p$. Then $p+k-\frac{1}{2}-j<-n+\frac{1}{2}$. Assume $p+k-\frac{1}{2}-j=-s-n+\frac{1}{2}$ for some $s\in\mathbb{Z}_+$, that is, $-j+s=-n-k-p+1<-p$ since $k+n\geqslant 2$. Since the corresponding vector $h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+k-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}}=h^{\mathbf{k}}h_{-s-n+\frac{1}{2}}d^{\mathbf{l}_j}v_{\mathbf{k},\mathbf{l}}-h^{\mathbf{k}}[h_{-s-n+\frac{1}{2}},\,d^{\mathbf{l}_j}]v_{\mathbf{k},\mathbf{l}}$, by (3.3) and simple computations, we see $h^{\mathbf{k}}d^{\mathbf{l}_j}h_{p+k-\frac{1}{2}-j}v_{\mathbf{k},\mathbf{l}}$ can be written as a linear combination of vectors in the form $h^{\mathbf{k}}h_{-s'-s-n+\frac{1}{2}}d^{\mathbf{l}_{s'+j}}v_{\mathbf{k},\mathbf{l}}$ where $s'\in\mathbb{N}$ and $\deg'(h^{\mathbf{k}}h_{-s'-s-n+\frac{1}{2}}d^{\mathbf{l}_{s'+j}}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k})+s'+s+\mathrm{w}(\mathbf{l}_{s'+j})=\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})+s-j.$$

So

$$\mathrm{w}(\mathbf{k}^*)+\mathrm{w}(\mathbf{l}^*)=\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-j+s<\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-p\leqslant\mathrm{w}(\mathbf{i})+\mathrm{w}(\mathbf{j})-p,$$

and hence $(\mathbf{k}^*,\,\mathbf{l}^*)\prec'(\mathbf{i},\,\mathbf{j}-\epsilon_p)$.

In conclusion, $\deg'(h_{p+k-\frac{1}{2}}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}})\preceq'(\mathbf{i},\,\mathbf{j}-\epsilon_p)$, where the equality holds if and only if $(\mathbf{k},\,\mathbf{l})=(\mathbf{i},\,\mathbf{j})$, that is, $\deg'(h_{p+k-\frac{1}{2}}v)=(\mathbf{i},\,\mathbf{j}-\epsilon_p)$. ☐

Lemma 3.5. *Let $k\in\mathbb{Z}_+$, $n\in\mathbb{Z}$ such that $k>n$ and $k+n\geqslant 2$, and V be a $\mathfrak{D}^{(0,-n)}$-module such that $h_{k-\frac{1}{2}}$ acts injectively on V, and $h_{i-\frac{1}{2}}V=d_jV=0$ for all $i>k$, $j>k+n-1$. Assume that $v=\sum_{(\mathbf{k},\mathbf{l})}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}\in\mathrm{Ind}(V)\setminus V$ with $\deg'(v)=(\mathbf{i},\,\mathbf{0})$. Set $q=\min\{s:i_s\neq 0\}$.*

(1) *If the sum $\sum_{(\mathbf{k},\mathbf{l})}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ does not contain terms $h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ satisfying*

$$w(\mathbf{k})+w(\mathbf{l})=w(\mathbf{i}),w(\mathbf{i})-q\leqslant w(\mathbf{k})<w(\mathbf{i}),\tag{3.9}$$

then $\deg'(d_{q+k+n-1}v)=(\mathbf{i}-\epsilon_q,\,\mathbf{0})$;

(2) *Assume that the sum $\sum_{(\mathbf{k},\mathbf{l})}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ contains terms $h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ satisfying (3.9). Let $v'=v-\sum_{\mathrm{w}(\mathbf{k})=\mathrm{w}(\mathbf{i})}h^{\mathbf{k}}v_{\mathbf{k},\mathbf{0}}$ and $\deg'(v')=(\mathbf{k}^*,\,\mathbf{l}^*)$ with $t=min\{s:l^*_s\neq 0\}$. Then $\deg'(h_{k+t-\frac{1}{2}}v)=(\mathbf{k}^*,\,\mathbf{l}^*-\epsilon_t)$.*

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Proof. Consider $\deg'(d_{q+k+n-1}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}})$ with $d_{q+k+n-1}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} \neq 0$. Noting that $d_{q+k+n-1}V = 0$, we see that

$$d_{q+k+n-1}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = [d_{q+k+n-1}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}}[d_{q+k+n-1}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}.$$

First we consider the term $[d_{q+k+n-1}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. It follows from (3.2) that $[d_{q+k+n-1}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ is a linear combination of vectors in the forms $h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ and $h^{\mathbf{s}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ where $\mathbf{k}_j = \mathbf{k} - \epsilon_j$, $\mathrm{w}(\mathbf{s}) = \mathrm{w}(\mathbf{k}) - (k+q-n)$. If $\mathbf{l} = 0$, it is not hard to see that $\deg'(d_{q+k+n-1}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \preceq' (\mathbf{i} - \epsilon_q, \mathbf{0})$ where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{0})$.

Next we assume that $\mathbf{l} \neq \mathbf{0}$, and continue to consider the term $[d_{q+k+n-1}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. We first consider the term $h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. We break the arguments into four different cases next.

(a) $j < q$. In this case, we have $h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = h^{\mathbf{k}_j}[h_{(q-j)+k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$. Then it follows from (3.3) that $h^{\mathbf{k}_j}[h_{(q-j)+k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$ is a linear combination of vectors in the form $h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}$ where $\mathrm{w}(\mathbf{l}_s) = \mathrm{w}(\mathbf{l}) - s$.

(a1) If $s < q-j$, then $h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} = 0$.

(a2) If $s = q-j$, then $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k}_j) + \mathrm{w}(\mathbf{l}_s) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j - s = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q.$$

If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{i})$, or $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{i})$ and $\mathrm{w}(\mathbf{k}) < \mathrm{w}(\mathbf{i}) - q$, then $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i} - \epsilon_q, \mathbf{0})$. We will discuss the remaining cases that $(\mathbf{k}, \mathbf{l})$ satisfies (3.9) in case (2) later.

(a3) If $q-j < s \leqslant q+k+n-1-j$, then $h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} \in V$ and $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k}_j) + \mathrm{w}(\mathbf{l}_s) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j - s < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q \leqslant \mathrm{w}(\mathbf{i}) - q.$$

So $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i} - \epsilon_q, \mathbf{0})$.

(a4) If $s > q+k+n-1-j$, then $q-j-s+k-\frac{1}{2} = -s'-n+\frac{1}{2}$ for some $s' \in \mathbb{Z}_+$. It is easy to see $h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}$ can be written as a linear combination of vectors of the form $h^{\mathbf{k}_j}h_{-s'-s''-n+\frac{1}{2}}d^{\mathbf{l}_{s+s''}}v_{\mathbf{k},\mathbf{l}}$, $0 \leqslant s'' \leqslant \mathrm{w}(\mathbf{l}_s)$. Note that both $\deg'(h^{\mathbf{k}_j}h_{-s'-s''-n+\frac{1}{2}}d^{\mathbf{l}_{s+s''}}v_{\mathbf{k},\mathbf{l}})$ and $\deg'(h^{\mathbf{k}_j}h_{-s'-n+\frac{1}{2}}d^{\mathbf{l}_s}v_{\mathbf{k},\mathbf{l}})$ have the same weight and $-j-s+s' = -q-k-n+1 < -q$, we see $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k}_j) + \mathrm{w}(\mathbf{l}_s) + s' = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j - s + s' < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q \leqslant \mathrm{w}(\mathbf{i}) - q.$$

So $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}_s}h_{(q-j-s)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i} - \epsilon_q, \mathbf{0})$.

(b) $j = q$. In this case, we have $h^{\mathbf{k}_q}h_{k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = h^{\mathbf{k}_q}d^{\mathbf{l}}h_{k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}_q}[h_{k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$. Clearly, $\deg'(h^{\mathbf{k}_q}d^{\mathbf{l}}h_{k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}_q, \mathbf{l}) \prec' (\mathbf{i} - \epsilon_q, \mathbf{0})$ since $\mathbf{l} \neq \mathbf{0}$. By (3.3)

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and the similar arguments in cases (a3) and (a4) we can deduce that $\deg'(h^{\mathbf{k}_q}[h_{k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$. Hence, $\deg'(h^{\mathbf{k}_q}h_{k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$.

(c) $q < j \leqslant q+k+n-1$. In this case, we have $h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = h^{\mathbf{k}_j}d^{\mathbf{l}}h_{(q-j)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}} + h^{\mathbf{k}_j}[h_{(q-j)+k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$. Clearly, $\deg'(h^{\mathbf{k}_j}d^{\mathbf{l}}h_{(q-j)+k-\frac{1}{2}}v_{\mathbf{k},\mathbf{l}}) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j < \mathrm{w}(\mathbf{i}) - q$. Then by (3.3) and the similar arguments in cases (a3) and (a4) we can deduce that $\deg'(h^{\mathbf{k}_q}[h_{(q-j)+k-\frac{1}{2}}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$. Hence, $\deg'(h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$.

(d) $j > q+k+n-1$. In this case, we have $h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}} = h^{\mathbf{k}_j}h_{-(j-(q+k+n-1))-n+\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. Then $\deg'(h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}^*, \mathbf{l})$ with weight $\mathrm{w}(\mathbf{k}^*) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - (q+k+n-1) < \mathrm{w}(\mathbf{i}) - q$. Hence, $\deg'(h^{\mathbf{k}_j}h_{(q-j)+k-\frac{1}{2}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$.

Next consider the term $h^{\mathbf{s}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$. Since $\mathrm{w}(\deg'(h^{\mathbf{s}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}})) = \mathrm{w}(\mathbf{s}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l})\text{-}q \leqslant \mathrm{w}(\mathbf{i})\text{-}q$, it follows that $\deg'(h^{\mathbf{s}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$.

Thus, if $h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ does not satisfy (3.9) we have

$$\deg'([d_{q+k+n-1}, h^{\mathbf{k}}]d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}) \preceq' (\mathbf{i}-\epsilon_q, \mathbf{0})$$

where the equality holds if and only if $(\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{0})$.

Now, consider the term $h^{\mathbf{k}}[d_{q+k+n-1}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$ where we still assume that $\mathbf{l} \neq \mathbf{0}$. By (3.4) we see $h^{\mathbf{k}}[d_{q+k+n-1}, d^{\mathbf{l}}]v_{\mathbf{k},\mathbf{l}}$ is a linear combination of vectors $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}}$ and $h^{\mathbf{k}}d^{\mathbf{l}_{q+k+n-1}}v_{\mathbf{k},\mathbf{l}}$ where $\mathrm{w}(\mathbf{l}_{\mathrm{j}}) = \mathrm{w}(\mathbf{l}) - \mathrm{j}$, $\mathrm{j} \in \mathbb{N}$. Since $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_{q+k+n-1}}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - (q+k+n-1) < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q \leqslant \mathrm{w}(\mathbf{i}) - q,$$

we see $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_{q+k+n-1}}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$. So we need only to consider the vectors $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}}$. There are four different cases.

(i) $j < q$. Then $q+k+n-1-j > k+n-1$ and $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}} = 0$. In particular, for $\mathrm{w}(\mathbf{l}) < q$ we have $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}} = 0$.

(ii) $j = q$. Then $q+k+n-1-q = k+n-1$ and hence $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_q}d_{k+n-1}v_{\mathbf{k},\mathbf{l}}) = (\mathbf{k}, \mathbf{l}_q)$ (in the case $d_{k+n-1}v_{\mathbf{k},\mathbf{l}} \neq 0$) with $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}_{\mathrm{q}}) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q$. If $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) < \mathrm{w}(\mathbf{i})$, or $\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) = \mathrm{w}(\mathbf{i})$ and $\mathrm{w}(\mathbf{k}) < \mathrm{w}(\mathbf{i}) - q$, then $(\mathbf{k}, \mathbf{l}_q) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$. We will discuss the remaining cases that $(\mathbf{k}, \mathbf{l})$ satisfies (3.9) in case (2) later.

(iii) $q < j \leqslant q+k+n-1$. Then $d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}} \in V$ and $h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}} = 0$ or $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}_j) = \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - j < \mathrm{w}(\mathbf{k}) + \mathrm{w}(\mathbf{l}) - q \leqslant \mathrm{w}(\mathbf{i}) - q,$$

so $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q, \mathbf{0})$.

(iv) $j > q+k+n-1$. Then $q+k+n-1-j<0$. Assume $q+k+n-1-j=-j'$, $j'\in\mathbb{Z}_+$. Then $-j+j'=-(q+k+n-1)<-q$. So $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}})$ has weight

$$\begin{aligned}\mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l}_j)+j' &= \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-j+j' = \mathrm{w}(\mathbf{k})+\mathrm{w}(\mathbf{l})-(q+k+n-1)\\ &< \mathrm{w}(\mathbf{i})-q,\end{aligned}$$

which means $\deg'(h^{\mathbf{k}}d^{\mathbf{l}_j}d_{q+k+n-1-j}v_{\mathbf{k},\mathbf{l}}) \prec' (\mathbf{i}-\epsilon_q,\,\mathbf{0})$.

(1) If $v=\sum_{(\mathbf{k},\mathbf{l})}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ does not contain a term $h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ satisfying (3.9), then by the above arguments we see $\deg'(d_{q+k+n-1}v)=(\mathbf{i}-\epsilon_q,\,\mathbf{0})$.

(2) If $v=\sum_{(\mathbf{k},\mathbf{l})}h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ contains terms $h^{\mathbf{k}}d^{\mathbf{l}}v_{\mathbf{k},\mathbf{l}}$ satisfying (3.9), then we see $\deg'(v')=(\mathbf{k}^*,\,\mathbf{l}^*)$ with

$$\mathrm{w}(\mathbf{k}^*)+\mathrm{w}(\mathbf{l}^*)=\mathrm{w}(\mathbf{i}), \mathrm{w}(\mathbf{k}^*)\geqslant \mathrm{w}(\mathbf{i})-q,\, 1\leqslant \mathrm{w}(\mathbf{l}^*)\leqslant q.$$

Then by lemma 3.4 we see $\deg'(h_{t+k-\frac{1}{2}}v')=(\mathbf{k}^*,\,\mathbf{l}^*-\epsilon_t)$.

Noticing that $k>n$, by (3.1) we see $h_{t+k-\frac{1}{2}}h^{\mathbf{k}}v_{\mathbf{k},\mathbf{0}}=\mathbf{0}$ or $\lambda h^{\mathbf{k}_{t'}}v_{\mathbf{k},\mathbf{0}}$, $\lambda\in\mathbb{C}^*$ with $t'=t+k-n>t$ and $\mathrm{w}(\mathbf{k}_{t'})=\mathrm{w}(\mathbf{k})-t'$, so $\deg'(h_{t+k-\frac{1}{2}}(h^{\mathbf{k}}v_{\mathbf{k},\mathbf{0}}))=(\mathbf{k}_{t'},\,\mathbf{0})$ has weight $\mathrm{w}(\mathbf{k}_{t'})=\mathrm{w}(\mathbf{k})-t'<\mathrm{w}(\mathbf{k}^*)+\mathrm{w}(\mathbf{l}^*)-t=\mathrm{w}(\mathbf{k}^*)+\mathrm{w}(\mathbf{l}^*-\epsilon_t)$. Hence

$$\deg'(h_{t+k-\frac{1}{2}}v)=\deg'\left(h_{t+k-\frac{1}{2}}\left(v-\sum_{\mathrm{w}(\mathbf{k})=\mathrm{w}(\mathbf{i})}h^{\mathbf{k}}v_{\mathbf{k},\mathbf{0}}\right)\right)=(\mathbf{k}^*,\mathbf{l}^*-\epsilon_t).$$

□

4. Simple smooth $\mathfrak{D}$-modules

In this section, we will determine all simple smooth $\mathfrak{D}$-modules. Based on theorem 2.10, we only need to determine all simple smooth $\mathfrak{D}$-modules S of level $\ell\neq 0$.

For a given simple smooth $\mathfrak{D}$-module S with level $\ell\neq 0$, we define the following invariants of S as follows:

$$S(r)=\mathrm{Ker}_S(\mathcal{H}_{\geqslant r}), n_S=\min\{r\in\mathbb{Z}: S(r)\neq 0\}, W_0=S(n_S),$$

and

$$U(r)=\mathrm{Ker}_{W_0}(\mathfrak{Vir}_{\geqslant r}), m_S=\min\{r\in\mathbb{Z}: U(r)\neq 0\}, U_0=U(m_S).$$

LEMMA 4.1. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell\neq 0$.*

(i) $h_{n_S-\frac{1}{2}}$ *acts injectively on* W_0, d_{m_S-1} *acts injectively on* U_0.

(ii) $n_S,\, m_S\in\mathbb{N}$.

(iii) W_0 *is a non-zero* $\mathfrak{D}^{(0,-n_S)}$*-module, and is invariant under the action of the operators* L_n *defined in* (2.2)–(2.4) *for* $n\in\mathbb{N}$.

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(iv) *If $m_S \geqslant 2n_S$, then U_0 is a non-zero $\mathfrak{D}^{(0,-n_S)}$-submodule of W_0, and is invariant under the action of the operators L_n defined in* (2.2)–(2.4) *for* $n \in \mathbb{N}$.

Proof. (i) follows from the definitions of n_S and m_S.

(ii) Suppose $n_S < 0$, take any non-zero $v \in W_0$, we then have

$$h_{\frac{1}{2}}v = 0 = h_{-\frac{1}{2}}v.$$

This implies that $\frac{1}{2}\ell v = [h_{\frac{1}{2}}, h_{-\frac{1}{2}}]v = 0$, a contradiction. Hence, $n_S \in \mathbb{N}$. Suppose $m_S < 0$. Take any non-zero $v \in U_0$, we then have $d_{-1}v = 0 = h_{n_S+\frac{1}{2}}v$. Then

$$-(n_S + \frac{1}{2})h_{n_S-\frac{1}{2}}v = [d_{-1}, h_{n_S+\frac{1}{2}}]v = 0,$$

a contradiction with (1). Hence, $m_S \in \mathbb{N}$.

(iii) It is obvious that $W_0 \neq 0$ by definition. For any $w \in W_0$, i, j, $k \in \mathbb{N}$, we have

$$h_{k+n_S+\frac{1}{2}}d_i w = d_i h_{k+n_S+\frac{1}{2}}w + \left(k + n_S + \frac{1}{2}\right) h_{i+k+n_S+\frac{1}{2}}w = 0,$$

and

$$h_{k+n_S+\frac{1}{2}}h_{j-n_S+\frac{1}{2}}w = h_{j-n_S+\frac{1}{2}}h_{k+n_S+\frac{1}{2}}w = 0.$$

Hence, $d_i u \in W_0$ and $h_{j-n_S+\frac{1}{2}}u \in W_0$, i.e. W_0 is a non-zero $\mathfrak{D}^{(0,-n_S)}$-module. For $n \in \mathbb{N}$, $i \in \mathbb{N}$, $w \in W_0$, by (2.5) we have

$$h_{i+n_S+\frac{1}{2}}L_n w = \Big(L_n h_{i+n_S+\frac{1}{2}} - \left(i + n_S + \frac{1}{2}\right) h_{n+i+n_S+\frac{1}{2}}\Big)w = 0.$$

This implies that $L_i w \in W_0$ for $i \in \mathbb{N}$, that is, W_0 is invariant under the action of the operators L_i for $i \in \mathbb{N}$.

(iv) It is obvious that $0 \neq U_0 \subseteq W_0$. Suppose that $m_S \geqslant 2n_S$. For any $u \in U_0$, i, j, $k \in \mathbb{N}$, it follows from (iii) that $d_i u \in W_0$ and $h_{j-n_S+\frac{1}{2}}u \in W_0$. Furthermore,

$$d_{k+m_S}d_i u = d_i d_{k+m_S}u + (k - i - m_S)d_{k+i+m_S}u = 0,$$

and

$$d_{k+m_S}h_{j-n_S+\frac{1}{2}}u = h_{j-n_S+\frac{1}{2}}d_{k+m_S}u - \left(j - n_S + \frac{1}{2}\right) h_{k+j+m_S-n_S+\frac{1}{2}}u = 0.$$

Hence, $d_i u \in U_0$ and $h_{j-n_S+\frac{1}{2}}u \in U_0$, i.e. U_0 is a non-zero $\mathfrak{D}^{(0,-n_S)}$ submodule of W_0.

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Furthermore, if in addition $m_S > 0$, then for n, $i \in \mathbb{N}$, $u \in U_0$, it follows from (iii) that $L_n u \in W_0$. Moreover, for $n \in \mathbb{N}$, using (2.2–2.6) we have

$$\begin{aligned} d_{i+m_S} L_n u &= L_n d_{i+m_S} u + [d_{i+m_S}, L_n]u = [d_{i+m_S}, L_n]u \\ &= (n - i - m_S) L_{i+n+m_S} u = 0. \end{aligned}$$

This implies that $L_i u \in U_0$ for $i \in \mathbb{N}$, that is, U_0 is invariant under the action of the operators L_i for $i \in \mathbb{N}$.

□

PROPOSITION 4.2. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$.*

(i) *If $n_S = 0$, then $S \cong H^{\mathfrak{D}} \otimes U^{\mathfrak{D}}$ as $\mathfrak{D}$-modules for some simple modules $H \in \mathcal{R}_{\mathcal{H}}$ and $U \in \mathcal{R}_{\mathfrak{Vir}}$.*

(ii) *If $m_S > 2n_S > 0$, then $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}}(U_0)$ and U_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module.*

(iii) *If $m_S < 2n_S$, then U_0 is a non-zero $\mathfrak{D}^{(0,-(m_S-n_S))}$-submodule of W_0. Moreover,*

(iii-1) *If $m_S \geqslant 2$, then $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-(m_S-n_S))}}^{\mathfrak{D}}(U_0)$ and U_0 is a simple $\mathfrak{D}^{(0,-(m_S-n_S))}$-module.*

(iii-2) *If $m_S = 0$ or 1, and $n_S > 1$, then $U(2)$ is a simple $\mathfrak{D}^{(0,-(2-n_S))}$-module, and $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-(2-n_S))}}^{\mathfrak{D}}(U(2))$.*

Proof. (i) Since $n_S = 0$, we take any non-zero $v \in W_0$. Then $\mathbb{C}v$ is a trivial $\mathcal{H}^{(0)}$-module. Let $H = U(\mathcal{H})v$, the $\mathcal{H}$-submodule of S generated by v. It follows from representation theory of Heisenberg algebras (or from the same arguments as in the proof of lemma 3.2) that $\mathrm{Ind}_{\mathcal{H}^{(0)}}^{\mathcal{H}}(\mathbb{C}v)$ is a simple $\mathcal{H}$-module. Consequently, the following surjective $\mathcal{H}$-module homomorphism

$$\begin{aligned} \varphi : \mathrm{Ind}_{\mathcal{H}^{(0)}}^{\mathcal{H}}(\mathbb{C}v) &\longrightarrow H \\ \sum_{\mathbf{i} \in \mathbb{M}} a_{\mathbf{i}} h^{\mathbf{i}} \otimes v &\mapsto \sum_{\mathbf{i} \in \mathbb{M}} a_{\mathbf{i}} h^{\mathbf{i}} v \end{aligned}$$

is an isomorphism, that is, H is a simple $\mathcal{H}$-module, which is certainly smooth. Then the desired assertion follows directly from corollary 2.13.

(ii) By taking $V = U_0$, $k = n = n_S$ and $l = m_S - 1$ in theorem 3.1(1) we see that any non-zero $\mathfrak{D}$-submodule of $\mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}}(U_0)$ has a non-zero intersection with U_0. Consequently, the surjective $\mathfrak{D}$-module homomorphism

$$\begin{aligned} \varphi : \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}}(U_0) &\longrightarrow S \\ \sum_{\mathbf{i},\mathbf{k} \in \mathbb{M}} h^{\mathbf{i}} d^{\mathbf{k}} \otimes v_{\mathbf{i},\mathbf{k}} &\mapsto \sum_{\mathbf{i},\mathbf{k} \in \mathbb{M}} h^{\mathbf{i}} d^{\mathbf{k}} v_{\mathbf{i},\mathbf{k}} \end{aligned}$$

is an isomorphism, i.e. $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}}(U_0)$. Since S is simple, we see U_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module.

(iii) Suppose that $m_S < 2n_S$. For any $u \in U_0$, $i,\ j,\ k \in \mathbb{N}$, it follows from lemma 4.1 (iii) that $d_i u \in W_0$ and $h_{j-(m_S-n_S)+\frac{1}{2}} u \in W_0$. Furthermore,

$$d_{k+m_S} d_i u = d_i d_{k+m_S} u + (k - i + m_S) d_{k+i+m_S} u = 0,$$

and

$$d_{k+m_S} h_{j-(m_S-n_S)+\frac{1}{2}} u = h_{j-(m_S-n_S)+\frac{1}{2}} d_{k+m_S} u - \left(j - (m_S - n_S) + \frac{1}{2}\right) h_{k+j+n_S+\frac{1}{2}} u = 0.$$

Hence, $d_i u \in U_0$ and $h_{j-(m_S-n_S)+\frac{1}{2}} u \in U_0$, i.e. U_0 is a non-zero $\mathfrak{D}^{(0,-(m_S-n_S))}$ submodule of W_0.

Now suppose $m_S \geqslant 2$. Then it follows from theorem 3.1(2) that any non-zero $\mathfrak{D}$-submodule of $\mathrm{Ind}_{\mathfrak{D}^{(0,-(m_S-n_S))}}^{\mathfrak{D}}(U_0)$ has a non-zero intersection with U_0 by taking $k = n_S$, $n = m_S - n_S$ and $l = m_S - 1$ therein. Consequently, $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-(m_S-n_S))}}^{\mathfrak{D}}(U_0)$ by similar arguments as in (ii). Since S is simple, we see U_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module.

Suppose that $m_S = 0$ or 1, and $n_S > 1$. Then $\mathfrak{D}^{(0,-(2-n_S))} \subseteq \mathfrak{D}^{(0,-n_S)}$. Hence, W_0 is a $\mathfrak{D}^{(0,-(2-n_S))}$-module. Moreover, for any $u \in U(2)$, $i,\ j \in \mathbb{N}$, we have

$$d_{j+2} d_i u = d_i d_{j+2} u = 0,$$

and

$$d_{j+2} h_{i-(2-n_S)+\frac{1}{2}} u = h_{i-(2-n_S)+\frac{1}{2}} d_{j+2} u + \left(2 - n_S - i - \frac{1}{2}\right) h_{i+j+n_S+\frac{1}{2}} u = 0.$$

Therefore, $U(2)$ is a $\mathfrak{D}^{(0,-(2-n_S))}$-module. Then it follows from theorem 3.1(2) that any non-zero $\mathfrak{D}$-submodule of $\mathrm{Ind}_{\mathfrak{D}^{(0,-(2-n_S))}}^{\mathfrak{D}}(U(2))$ has a non-zero intersection with $U(2)$ by taking $V = U(2)$, $k = n_S$, $n = 2 - n_S$ and $l = 1$ therein. Consequently, $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-(2-n_S))}}^{\mathfrak{D}}(U(2))$ by similar arguments as in (ii). In particular, $U(2)$ is a simple $\mathfrak{D}^{(0,-(2-n_S))}$-module. □

From proposition 4.2, what remains to consider are the following two cases: (1) $m_S = 2n_S > 0$, (2) $m_S = 0$ or 1, and $n_S = 1$.

Now we first consider case (1): $m_S = 2n_S > 0$. For that, we define the operators $d'_n = d_n - L_n$ on S for $n \in \mathbb{Z}$. Since S is a smooth $\mathfrak{D}$-module, then d'_n is well-defined for any $n \in \mathbb{Z}$. By (2.5) and (2.6), we have

$$[d'_m, \mathbf{c}_1] = 0, [d'_m, d'_n] = (m-n) d'_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}(c-1), m, n \in \mathbb{Z}, \tag{4.1}$$

where $\mathbf{c}'_1 = \mathbf{c}_1 - \mathrm{id}_S$ and c is the central charge of S. So the operator algebra

$$\mathfrak{Vir}' = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d'_n \oplus \mathbb{C}\mathbf{c}'_1$$

is isomorphic to the Virasoro algebra $\mathfrak{Vir}$. Since $[d_n,\ h_{k+\frac{1}{2}}] = [L_n,\ h_{k+\frac{1}{2}}] = -(k+\frac{1}{2}) h_{n+k+\frac{1}{2}}$, we have $[d'_n,\ h_{k+\frac{1}{2}}] = 0,\ n,\ k \in \mathbb{Z}$ and hence $[\mathfrak{Vir}',\ \mathcal{H}] = 0$.

Clearly, the operator algebra $\mathfrak{D}' = \mathfrak{Vir}' \oplus \mathcal{H}$ is a direct sum, and $S = \mathcal{U}(\mathfrak{D})v = \mathcal{U}(\mathfrak{D}')v$, $0 \neq v \in S$. Similar to (2.11) we can define its subalgebras, $\mathfrak{D}'^{(m,n)}$ and the likes.

Let

$$Y_n = \bigcap_{p \geqslant n} \mathrm{Ker}_{U_0}(d'_p), r_S = \min\{n \in \mathbb{Z} : Y_n \neq 0\}, K_0 = Y_{r_S}.$$

If $Y_n \neq 0$ for any $n \in \mathbb{Z}$, we define $r_S = -\infty$. Denote by $K = U(\mathcal{H})K_0$.

Lemma 4.3. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. Assume that $m_S = 2n_S > 0$. Then the following statements hold.*

(i) $-1 \leqslant r_S \leqslant m_S$ *or* $r_S = -\infty$.

(ii) K_0 *is a* $\mathfrak{D}^{(0,-n_S)}$*-module and* $h_{n_S-\frac{1}{2}}$ *acts injectively on* K_0.

(iii) K *is a* $\mathfrak{D}^{(0,-\infty)}$*-module and* $K^{\mathfrak{D}}$ *has a* $\mathfrak{D}$*-module structure by* (2.2)-(2.4).

(iv) K_0 *and* K *are invariant under the action of* d'_n *for* $n \in \mathbb{N}$.

(v) *If* $r_S \neq -\infty$*, then* d'_{r_S-1} *acts injectively on* K_0 *and* K.

Proof. (i) Since $m_S = 2n_S > 0$, the operators d_m and $L_m = \frac{1}{2\ell}\sum_{k \in \mathbb{Z}+\frac{1}{2}} h_{m-k}h_k$ act trivially on U_0 for any $m \geqslant m_S$. This implies that $Y_{m_S} = U_0 \neq 0$. Consequently, $r_S \leqslant m_S$ by the definition of r_S.
If $Y_{-2} \neq 0$, then $d'_{-2}K_0 = d'_{-1}K_0 = 0$. We deduce that $\mathfrak{Vir}'K_0 = 0$ and hence $r_S = -\infty$.
If $Y_{-2} = 0$, then $r_S \geqslant -1$ and hence $-1 \leqslant r_S \leqslant m_S$.

(ii) For any $0 \neq v \in K_0$ and $x \in \mathfrak{D}^{(0,-n_S)}$, it follows from lemma 4.1(iv) that $xv \in U_0$. We need to show that $d'_p xv = 0$, $p \geqslant r_S$. Indeed, $d'_p h_{k+\frac{1}{2}}v = h_{k+\frac{1}{2}}d'_p v = 0$ by (2.5) for any $k \geqslant -n_S$. Moreover, it follows from (2.6) and (4.1) that

$$d'_p d_n v = d_n d'_p v + [d'_p, d_n]v = (p-n)d'_{p+n}v = 0.$$

Hence, $d'_p xv = 0$, $p \geqslant r_S$, that is, $xv \in K_0$, as desired.
Since $0 \neq K_0 \subseteq U_0 \subseteq W_0$, we see that $h_{n_S-\frac{1}{2}}$ acts injectively on K_0 by lemma 4.1(i).

(iii) follows from (ii).

(iv) It follows from lemma 4.1(iv) that U_0 is invariant under the action of d'_n for $n \in \mathbb{N}$, so is K_0 by (4.1). Moreover, since $[\mathfrak{Vir}', \mathcal{H}] = 0$, K is also invariant under the action of d'_n for $n \in \mathbb{N}$.

(v) follows directly from the definition of r_S and K.

□

Proposition 4.4. *Let S be a simple smooth $\mathfrak{D}$-module with central charge c and level $\ell \neq 0$. Assume that $m_S = 2n_S > 0$. If $r_S = -\infty$, then $c = 1$. Moreover, $S = K^{\mathfrak{D}}$ and K is a simple $\mathcal{H}$-module.*

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Proof. Since $r_S = -\infty$, we see that $\mathfrak{Vir}' K_0 = 0$. This together with (4.1) implies that $c = 1$. Noting that $[\mathfrak{Vir}', \mathcal{H}] = 0$, we further obtain that $\mathfrak{Vir}' K = 0$, that is, $d_n v = L_n v \in K$ for any $v \in K$ and $n \in \mathbb{Z}$. Hence, $K^{\mathfrak{D}}$ is a $\mathfrak{D}$-submodule of S, yielding that $S = K^{\mathfrak{D}}$. In particular, K is a simple $\mathcal{H}$-module. □

PROPOSITION 4.5. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. If $r_S \geqslant 2$, then K_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module and $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}} K_0$.*

Proof. We first show that $\mathrm{Ind}_{\mathfrak{D}^{(0,-n_s)}}^{\mathfrak{D}^{(0,-\infty)}} K_0 \cong K$ as $\mathfrak{D}^{(0,-\infty)}$ modules. For that, let

$$\begin{aligned} \phi :\ \mathrm{Ind}_{\mathfrak{D}^{(0,-n_s)}}^{\mathfrak{D}^{(0,-\infty)}} K_0 &\longrightarrow K \\ \sum_{\mathbf{k}\in\mathbb{M}} h^{\mathbf{k}} \otimes v_{\mathbf{k}} &\mapsto \sum_{\mathbf{k}\in\mathbb{M}} h^{\mathbf{k}} v_{\mathbf{k}}, \end{aligned}$$

where $h^{\mathbf{k}} = \cdots h^{k_2}_{-2-n_S+\frac{1}{2}} h^{k_1}_{-1-n_S+\frac{1}{2}}$. Then ϕ is a $\mathfrak{D}^{(0,-\infty)}$-module epimorphism and $\phi|_{K_0}$ is one-to-one. By similar arguments in the proof of lemma 3.2 we see that any non-zero submodule of $\mathrm{Ind}_{\mathfrak{D}^{(0,-n_s)}}^{\mathfrak{D}^{(0,-\infty)}} K_0$ contains non-zero vectors of K_0, which forces that the kernel of ϕ must be zero and hence ϕ is an isomorphism.

By lemma 4.3(v), we see that d'_{r_S-1} acts injectively on K.

As $\mathfrak{D}$-modules,

$$\mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}} K_0 \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-\infty)}}^{\mathfrak{D}} (\mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{(0,-\infty)} K_0) \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-\infty)}}^{\mathfrak{D}} K.$$

And we further have $\mathrm{Ind}_{\mathfrak{D}^{(0,-\infty)}}^{\mathfrak{D}} K \cong \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$ as vector spaces. Moreover, we have the following $\mathfrak{D}$-module epimorphism

$$\begin{aligned} \pi : \mathrm{Ind}_{\mathfrak{D}^{(0,-\infty)}}^{\mathfrak{D}} K = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K &\to S, \\ \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} \otimes v_{\mathbf{l}} &\mapsto \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} v_{\mathbf{l}}, \end{aligned}$$

where $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2} (d'_{-1})^{l_1}$. We see that π is also a $\mathfrak{Vir}'$-module epimorphism. By the proof of Theorem 2.1 in [**46**] we know that any non-zero $\mathfrak{Vir}'$-submodule of $\mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$ contains non-zero vectors of K. Note that $\pi|_K$ is one-to-one, we see that the image of any non-zero $\mathfrak{D}$-submodule (and hence $\mathfrak{Vir}'$-submodule) of $\mathrm{Ind}_{\mathfrak{D}^{(0,-\infty)}}^{\mathfrak{D}} K$ must be a non-zero $\mathfrak{D}$-submodule of S and hence be the whole module S, which forces that the kernel of π must be 0. Therefore, π is an isomorphism. Since S is simple, we see K_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module. □

As a direct consequence of proposition 4.5, we have

COROLLARY 4.6. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. If $m_S \leqslant 1$ and $n_S = 1$, then K_0 is a simple $\mathfrak{D}^{(0,-1)}$-module and $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-1)}}^{\mathfrak{D}} K_0$.*

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Proof. For any non-zero $u \in U_0$, since $m_S \leqslant 1$ and $n_S = 1$, it follows from the definitions of m_S, n_S and lemma 4.1(i) that

$$d_1 u = 0, \quad L_1 u = \frac{1}{2\ell} \sum_{k \in \mathbb{Z}+\frac{1}{2}} h_{1-k} h_k u = \frac{1}{2\ell} (h_{\frac{1}{2}})^2 u \neq 0.$$

This implies that $d_1' u \neq 0$, i.e. d_1' acts injectively on U_0. Hence, $r_S \geqslant 2$. More precisely, since

$$d_{2+i} v = L_{2+i} v = 0, \quad \forall\, i \in \mathbb{N}, v \in U_0,$$

we see that $r_S = 2$. Now the desired assertion follows directly from proposition 4.5. □

Remark 4.7. From corollary 4.6, we have dealt with case (2).

What remains to consider for case (1) is that $m_S = 2n_S \geqslant 2$ and $r_S \leqslant 1$. In this case, we will show that K is a simple $\mathcal{H}$-module.

For the Verma module $M_{\mathfrak{Vir}}(c, h)$ over $\mathfrak{Vir}$, it is well-known from [**6**, **20**] that there exist two homogeneous elements $P_1, P_2 \in \mathcal{U}(\mathfrak{Vir}^-)\mathfrak{Vir}^-$ such that $\mathcal{U}(\mathfrak{Vir}^-)P_1 w_1 + \mathcal{U}(\mathfrak{Vir}^-)P_2 w_1$ is the unique maximal proper $\mathfrak{Vir}$-submodule of $M_{\mathfrak{Vir}}(c, h)$, where P_1, P_2 are allowed to be zero and w_1 is the highest weight vector in $M_{\mathfrak{Vir}}(c, h)$.

Lemma 4.8. *Let $d = 0, -1$. Suppose M is a $\mathfrak{Vir}^{(d)}$-module on which d_0 acts as multiplication by a given scalar λ. Then there exists a unique maximal submodule N of $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$ with $N \cap M = 0$. More precisely, N is generated by $P_1 M$ and $P_2 M$, i.e. $N = \mathcal{U}(\mathfrak{Vir}^-)(P_1 M + P_2 M)$.*

Proof. Note that d_0 acts diagonalizably on $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$ and its submodules, and

$$M = \{u \in \mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M \mid d_0 u = \lambda u\},$$

i.e. M is the highest weight space of $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$. Let N be the sum of all $\mathfrak{Vir}$-submodules of $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$ which intersect with M trivially. Then N is the desired unique maximal $\mathfrak{Vir}$-submodule of $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$ with $N \cap M = 0$.

Let N' be the $\mathfrak{Vir}$-submodule generated by $P_1 M$ and $P_2 M$, i.e. $N' = \mathcal{U}(\mathfrak{Vir}^-)(P_1 M + P_2 M)$. Then $N' \cap M = 0$. Hence, $N' \subseteq N$. Suppose there is a proper submodule U of $\mathrm{Ind}_{\mathfrak{Vir}^{(d)}}^{\mathfrak{Vir}} M$ that is not contained in N'. There is a non-zero homogeneous $v = \sum_{i=1}^{r} u_i v_i \in U \setminus N'$ where $u_i \in \mathcal{U}(\mathfrak{Vir}^-)$ and $v_1, \dots v_r \in M$ are linearly independent. Note that all u_i have the same weight. Then some $u_i v_i \notin N'$, say $u_1 v_1 \notin N'$. There is a homogeneous $u \in \mathcal{U}(\mathfrak{Vir})$ such that $u u_1 v_1 = v_1$. Noting that all $u u_i$ has weight 0, so $u u_i v_i \in \mathbb{C} v_i$. Thus, $uv \in M \setminus \{0\}$. This implies that $N \subseteq N'$. Hence, $N = N'$, as desired. □

Proposition 4.9. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. If $m_S = 2n_s \geqslant 2$, and $r_S = 0$ or -1, then K is a simple $\mathcal{H}$-module and $S \cong U^{\mathfrak{D}} \otimes K^{\mathfrak{D}}$ for some simple $U \in \mathcal{R}_{\mathfrak{Vir}}$.*

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Proof. By lemma 4.3 (iii), we see that $K^{\mathfrak{D}}$ is a $\mathfrak{D}$-module, and hence $K^{\mathfrak{D}'}$ is a $\mathfrak{D}'$-module with $d'_n K = 0$ for any $n \in \mathbb{Z}$. Let $\mathbb{C}v_0$ be a one-dimensional $\mathfrak{D}'^{(r_S,-\infty)}$-module with module structure defining by $d'_n v_0 = h_{k+\frac{1}{2}} v_0 = \mathbf{c}_2 v_0 = 0$, $n \geqslant r_S$, $k \in \mathbb{Z}$, $\mathbf{c}'_1 v_0 = (c-2)v_0$. Then $\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}$ is a $\mathfrak{D}'^{(r_S,-\infty)}$-module with central charge $c-1$ and level ℓ. It is easy to see that we have the following $\mathfrak{D}'^{(r_S,-\infty)}$-module homomorphism

$$\begin{aligned} \tau_K : \mathbb{C}v_0 \otimes K^{\mathfrak{D}'} &\longrightarrow S, \\ v_0 \otimes u &\mapsto u, \forall u \in K. \end{aligned}$$

Clearly, τ_K is an injective map and can be extended to a $\mathfrak{D}'$-module epimorphism

$$\begin{aligned} \tau : \mathrm{Ind}_{\mathfrak{D}'^{(r_S,-\infty)}}^{\mathfrak{D}'}(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) &\longrightarrow S, \\ x(v_0 \otimes u) &\mapsto xu, x \in \mathcal{U}(\mathfrak{D}'), u \in K. \end{aligned}$$

By Lemma 8 in [43] we know that

$$\mathrm{Ind}_{\mathfrak{D}'^{(r_S,-\infty)}}^{\mathfrak{D}'}(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) \cong (\mathrm{Ind}_{\mathfrak{D}'^{(r_S,-\infty)}}^{\mathfrak{D}'}\mathbb{C}v_0) \otimes K^{\mathfrak{D}'} = (\mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}\mathbb{C}v_0)^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}.$$

Then we have the following $\mathfrak{D}'$-module epimorphism

$$\begin{aligned} \tau' : (\mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}\mathbb{C}v_0)^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'} &\longrightarrow S, \\ xv_0 \otimes u &\mapsto xu, x \in \mathcal{U}(\mathfrak{Vir}'), u \in K. \end{aligned}$$

Note that $(\mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}\mathbb{C}v_0)^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'} \cong \mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'})$ as $\mathfrak{Vir}'$-modules, and τ' is also a $\mathfrak{Vir}'$-module epimorphism, $\tau'|_{\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}}$ is one-to-one, and $(\mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}\mathbb{C}v_0)^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$ is a highest weight $\mathfrak{Vir}'$-module.

Let $V = \mathrm{Ind}_{\mathfrak{Vir}'^{(r_S)}}^{\mathfrak{Vir}'}\mathbb{C}v_0$ and $\mathfrak{K} = \mathrm{Ker}(\tau')$. It should be noted that

$$\mathbb{C}v_0 \otimes K^{\mathfrak{D}'} = \{u \in V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'} \mid d'_0 u = 0\}.$$

We see that $(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) \cap \mathfrak{K} = 0$. Let $\mathfrak{K}'$ be the sum of all $\mathfrak{Vir}'$-submodules W of $V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$ with $(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) \cap W = 0$, that is, the unique maximal $\mathfrak{Vir}'$-submodule of $V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$ with trivial intersection with $(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'})$. It is obvious that $\mathfrak{K} \subseteq \mathfrak{K}'$. Next we further show that $\mathfrak{K} = \mathfrak{K}'$. For that, take any $\mathfrak{Vir}'$-submodule W of $V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$ such that $(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) \cap W = 0$. Then for any weight vector $w = \sum_{\mathbf{l} \in \mathbb{M}} d'^{\mathbf{l}} v_0 \otimes u_{\mathbf{l}} \in W$, where $u_{\mathbf{l}} \in K^{\mathfrak{D}'}$, $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2}(d'_{-1})^{l_1}$ if $r_S = 0$, or $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2}$ if $r_S = -1$, and all $\mathrm{w}(\mathbf{l}) \geqslant 1$ are equal. Note that $h_{k+\frac{1}{2}} w = \sum_{\mathbf{l} \in \mathbb{M}} d'^{\mathbf{l}} v_0 \otimes h_{k+\frac{1}{2}} u_{\mathbf{l}}$ either equals to 0 or has the same weight as w under the action of d'_0. So $U(\mathfrak{D}')\mathfrak{K}' \cap (\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) = 0$. The maximality of $\mathfrak{K}'$ forces that $\mathfrak{K}' = U(\mathfrak{D}')\mathfrak{K}'$ is a proper $\mathfrak{D}'$-submodule of $V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$. Since $\mathfrak{K}$ is a maximal proper $\mathfrak{D}'$-submodule of $V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$, it follows that $\mathfrak{K} = \mathfrak{K}'$.

By lemma 4.8 we know that $\mathfrak{K}$ is generated by $P_1(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) = \mathbb{C}P_1 v_0 \otimes K^{\mathfrak{D}'}$ and $P_2(\mathbb{C}v_0 \otimes K^{\mathfrak{D}'}) = \mathbb{C}P_2 v_0 \otimes K^{\mathfrak{D}'}$. Let V' be the maximal submodule of V

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generated by P_1v_0 and P_2v_0, then $\mathfrak{K} = V'^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}$. Therefore,

$$S \cong (V^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'})/(V'^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'}) \cong (V/V')^{\mathfrak{D}'} \otimes K^{\mathfrak{D}'},$$

which forces that $K^{\mathfrak{D}'}$ is a simple $\mathfrak{D}'$-module and hence a simple $\mathcal{H}$-module. So S contains a simple $\mathcal{H}$-module K. By corollary 2.13 we know there exists a simple $\mathfrak{Vir}$-module $U \in \mathcal{R}_{\mathfrak{Vir}}$ such that $S \cong U^{\mathfrak{D}} \otimes K^{\mathfrak{D}}$, as desired. □

Lemma 4.10. *Let M be a $\mathfrak{Vir}^{(0)}$-module on which $\mathfrak{Vir}^{(1)}$ acts trivially. If any finitely generated $\mathbb{C}[d_0]$-submodule of M is a free $\mathbb{C}[d_0]$-module, then any non-zero submodule of $\mathrm{Ind}_{\mathfrak{Vir}^{(0)}}^{\mathfrak{Vir}} M$ intersects with M non-trivially.*

Proof. Let V be a non-zero submodule of $\mathrm{Ind}_{\mathfrak{Vir}^{(0)}}^{\mathfrak{Vir}} M$. Take a non-zero $u \in V$. If $u \in M$, there is nothing to do. Now assume $u \in V \backslash M$. Write $u = \sum_{i=1}^{n} a_i u_i$ where $a_i \in \mathcal{U}(\mathfrak{Vir}_{\leqslant 0})$, $u_i \in M$. Since $M_1 = \sum_{1 \leqslant i \leqslant n} \mathbb{C}[d_0]u_i$ (a $\mathfrak{Vir}^{(0)}$-submodule of M) is a finitely generated $\mathbb{C}[d_0]$-module, we see M_1 is a free module over $\mathbb{C}[d_0]$ by the assumption. Without loss of generality, we may assume that $M_1 = \oplus_{1 \leqslant i \leqslant n} \mathbb{C}[d_0]u_i$ with basis $u_1, \cdots, u_n$ over $\mathbb{C}[d_0]$. Note that each a_i can be expressed as a sum of eigenvalue subspaces of $\mathrm{ad}\, d_0$ for $1 \leqslant i \leqslant n$. Assume that a_1 has a maximal eigenvalue among all a_i for $1 \leqslant i \leqslant n$. Then $a_1 u_1 \notin M$. For any $\lambda \in \mathbb{C}$, let $M_1(\lambda)$ be the $\mathbb{C}[d_0]$-submodule of M_1 generated by $u_2, u_3, \cdots, u_n, d_0u_1 - \lambda u_1$. Then $M_1/M_1(\lambda)$ is a one-dimensional $\mathfrak{Vir}^{(0)}$-module with $d_0(u_1 + M_1(\lambda)) = \lambda u_1 + M_1(\lambda)$. By the Verma module theory for Virasoro algebra, we know that there exists some $0 \neq \lambda_0 \in \mathbb{C}$ such that the corresponding Verma module $\mathfrak{V} = \mathrm{Ind}_{\mathfrak{Vir}^{(0)}}^{\mathfrak{Vir}}(M_1/M_1(\lambda_0))$ is irreducible. We know that $u = a_1u_1 \neq 0$ in $\mathfrak{V}$. Hence, we can find a homogeneous $w \in \mathcal{U}(\mathfrak{Vir}^+)$ such that $wa_1u_1 = f_1(d_0)u_1$ in $\mathrm{Ind}_{\mathfrak{Vir}^{(0)}}^{\mathfrak{Vir}} M$, where $0 \neq f_1(d_0) \in \mathbb{C}[d_0]$. So $wu = \sum_{i=1}^{n} wa_iu_i = \sum_{i=1}^{n} f_i(d_0)u_i$ for $f_i(d_0) \in \mathbb{C}[d_0]$, $1 \leqslant i \leqslant n$. Therefore, $0 \neq wu \in V \cap M_1 \subset V \cap M$, as desired. □

Proposition 4.11. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. If $m_S = 2n_s \geqslant 2$, $r_S = 1$, then d_0' has an eigenvector in K.*

Proof. Suppose first that any finitely generated $\mathbb{C}[d_0']$-submodule of $K = \mathrm{Ind}_{\mathcal{H}^{(-n_S)}}^{\mathcal{H}} K_0$ is a free $\mathbb{C}[d_0']$-module. By lemma 4.10 we see that the following $\mathfrak{D}'$-module homomorphism

$$\tau : \mathrm{Ind}_{\mathfrak{D}'^{(0,-\infty)}}^{\mathfrak{D}'} K = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K \longrightarrow S,$$
$$x \otimes u \mapsto xu, x \in \mathcal{U}(\mathfrak{Vir}'), u \in K$$

is an isomorphism. So $S = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$, and consequently, K is an irreducible $\mathfrak{D}'^{(0,-\infty)}$-module. Since $\mathfrak{Vir}'^{(1)}K = 0$, we consider K as an irreducible module over the Lie algebra $\mathcal{H} \oplus \mathbb{C}d_0'$. Since d_0' is the centre of the Lie algebra $\mathcal{H} \oplus \mathbb{C}d_0'$, we see that the action of d_0' on K is a scalar, a contradiction. So this case does not occur.

Now there exists some finitely generated $\mathbb{C}[d_0']$-submodule M of K that is not a free $\mathbb{C}[d_0']$-module. Since $\mathbb{C}[d_0']$ is a principal ideal domain, by the structure theorem of finitely generated modules over a principal ideal domain, there exists a monic polynomial $f(d_0') \in \mathbb{C}[d_0']$ with positive degree and non-zero element

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$u \in M$ such that $f(d'_0)u = 0$. Furthermore, we can write $f(d'_0) = (d'_0 - \lambda_1)(d'_0 - \lambda_2) \cdots (d'_0 - \lambda_p)$ for some $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$. Then there exists some $s \leqslant p$ such that $w := \Pi_{i=s+1}^{p}(d'_0 - \lambda_j)u \neq 0$ and $d'_0 w = \lambda_s w$, where we make convention that $w = u$ if $s = p$. Then w is a desired eigenvector of d'_0. □

PROPOSITION 4.12. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. If $m_S = 2n_s \geqslant 2$, $r_S = 1$, then K is a simple $\mathcal{H}$-module and $S \cong U^{\mathfrak{D}} \otimes K^{\mathfrak{D}}$ for some simple $U \in \mathcal{R}_{\mathfrak{Vir}}$.*

Proof. We see that S is a weight $\mathfrak{D}'$-module since S is a simple $\mathfrak{D}'$-module and d'_0 has an eigenvector. From lemma 4.3(iii), K and K_0 are weight $\mathfrak{D}'$-modules as well. We can take some $0 \neq u_0 \in K$ such that $d'_0 u_0 = \lambda u_0$ for some $\lambda \neq 0$ by proposition 4.11. Set $K' = U(\mathcal{H})u_0$, which is an $\mathcal{H}$ submodule of K. Then we have the $\mathfrak{D}'$-module $K'^{\mathfrak{D}'}$, on which $\mathfrak{Vir}'$ acts trivially by definition for any $n \in \mathbb{Z}$. Let $\mathbb{C}v_0$ be the one-dimensional $\mathfrak{D}'^{(0,-\infty)}$-module defined by $d'_0 v_0 = \lambda v_0$, $d'_n v_0 = h_{k+\frac{1}{2}} v_0 = \mathbf{c}_2 v_0 = 0$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $\mathbf{c}'_1 v_0 = (c-2)v_0$. Then $\mathbb{C}v_0 \otimes K'^{\mathfrak{D}'}$ is a $\mathfrak{D}'^{(0,-\infty)}$-module with central charge $c - 1$ and level ℓ. There is a $\mathfrak{D}'^{(0,-\infty)}$-module homomorphism

$$\begin{aligned} \tau_{K'} : \mathbb{C}v_0 \otimes K'^{\mathfrak{D}'} &\longrightarrow S, \\ v_0 \otimes u &\mapsto u, \forall u \in K', \end{aligned}$$

which is injective and can be extended to be the following $\mathfrak{D}'$-module homomorphism

$$\begin{aligned} \tau : \mathrm{Ind}_{\mathfrak{D}'^{(0,-\infty)}}^{\mathfrak{D}'}(\mathbb{C}v_0 \otimes K'^{\mathfrak{D}'}) &\longrightarrow S, \\ x(v_0 \otimes u) &\mapsto xu, x \in \mathcal{U}(\mathfrak{D}'), u \in K'. \end{aligned}$$

Since S is a simple $\mathfrak{D}'$ module and $\tau \neq 0$, we see that τ is surjective. By similar arguments in the proof of proposition 4.9, we can obtain that K' is a simple $\mathcal{H}$-module. By corollary 2.13 we know there exists a simple $\mathfrak{Vir}$-module $U \in \mathcal{R}_{\mathfrak{Vir}}$ such that $S \cong U^{\mathfrak{D}} \otimes K'^{\mathfrak{D}}$, as desired. Now it is clear that $K = K'$. □

We are now in a position to present the following main result on a classification of simple smooth $\mathfrak{D}$-modules with non-zero level.

THEOREM 4.13. *Let S be a simple smooth $\mathfrak{D}$-module with level $\ell \neq 0$. The invariants m_S, n_S, r_S of S, U_0, $U(2)$, K_0, K are defined as before. Then one of the following cases occurs.*

Case 1*: $n_S = 0$.*

In this case, $S \cong H^{\mathfrak{D}} \otimes U^{\mathfrak{D}}$ as $\mathfrak{D}$-modules for some simple modules $H \in \mathcal{R}_{\mathcal{H}}$ and $U \in \mathcal{R}_{\mathfrak{Vir}}$.

Case 2*: $n_S > 0$.*

In this case, we further have the following three subcases.

Subcase 2.1*: $m_S > 2n_S$.*

In this subcase, $S \cong \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}}(U_0)$.

Subcase 2.2*: $m_S = 2n_S$.*

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In this subcase, we have

$$S \cong \begin{cases} K^{\mathfrak{D}}, & \text{if } r_S = -\infty, \\ U^{\mathfrak{D}} \otimes K^{\mathfrak{D}}, & \text{if } -1 \leqslant r_S \leqslant 1, \\ \mathrm{Ind}_{\mathfrak{D}^{(0,-n_S)}}^{\mathfrak{D}} K_0, & \text{otherwise,} \end{cases}$$

where $U \in \mathcal{R}_{\mathfrak{Vir}}$*.*

Subcase 2.3*:* $m_S < 2n_S$*.*

In this subcase, we have

$$S \cong \begin{cases} \mathrm{Ind}_{\mathfrak{D}^{(0,-(m_S-n_S))}}^{\mathfrak{D}}(U_0), & \text{if } m_S \geqslant 2, \\ \mathrm{Ind}_{\mathfrak{D}^{(0,-(2-n_S))}}^{\mathfrak{D}}(U(2)), & \text{if } m_S < 2, n_S > 1, \\ \mathrm{Ind}_{\mathfrak{D}^{(0,-1)}}^{\mathfrak{D}} K_0, & \text{otherwise.} \end{cases}$$

Proof. The assertion follows directly from proposition 4.2, proposition 4.4, proposition 4.5, corollary 4.6, proposition 4.9 and proposition 4.12. □

Remark 4.14. By theorems 2.10 and 4.13, we know that any simple smooth module S is a highest weight $\mathfrak{Vir}$-module with trivial action of $\mathcal{H}$, or a tensor product of a simple smooth $\mathfrak{Vir}$-module and a simple smooth $\mathcal{H}$-module, or an induced module from some simple module M over certain subalgebra of $\mathfrak{D}$. Moreover, M can be viewed as a simple module over some finite-dimensional solvable Lie algebra. This reduces the study of such $\mathfrak{D}$-modules to the study of simple modules over the corresponding finite-dimensional solvable Lie algebras.

5. Simple smooth $\bar{\mathfrak{D}}$-modules with non-zero level

In this section, we will determine all simple smooth $\bar{\mathfrak{D}}$-modules M of level $\ell \neq 0$. The main method we will use is similar to the one used in § 4.

For a given simple smooth $\bar{\mathfrak{D}}$-module M with level $\ell \neq 0$, we define the following invariants of M as follows:

$$M(r) = \mathrm{Ker}_M(\bar{\mathcal{H}}^{(r)}), n_M = \min\{r \in \mathbb{Z} : M(r) \neq 0\}, M_0 = M(n_M).$$

Lemma 5.1. *Let M be an irreducible smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$.*

(i) $n_M \in \mathbb{N}$*, and* h_{n_M-1} *acts injectively on* M_0*.*

(ii) M_0 *is a non-zero* $\bar{\mathfrak{D}}^{(0,-(n_M-1))}$*-module, and is invariant under the action of the operators* $\bar{L}_n$ *defined in (2.7) for* $n \in \mathbb{N}$*.*

Proof. (i) Assume that $n_M < 0$. Take any non-zero $v \in M_0$, we then have

$$h_1 v = 0 = h_{-1} v.$$

This implies that $v = \frac{1}{\ell}[h_1, h_{-1}]v = 0$, a contradiction. Hence, $n_M \in \mathbb{N}$. The definition of n_M means that h_{n_M-1} acts injectively on M_0.

(ii) It is obvious that $M_0 \neq 0$ by definition. For any $w \in M_0$, i, j, $k \in \mathbb{N}$, we have

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$$h_{k+n_M} d_i w = d_i h_{k+n_M} w + (k+n_M) h_{i+k+n_M} w = 0,$$

and

$$h_{k+n_M} h_{j-n_M+1} w = h_{j-n_M+1} h_{k+n_M} w = 0.$$

Hence, $d_i w$, $h_{j-n_M+1} w \in M_0$, i.e. M_0 is a non-zero $\mathfrak{D}^{(0,-(n_M-1))}$-module.

For i, $n \in \mathbb{N}$, $w \in M_0$, noticing $n_M \geqslant 0$ by (i), it follows from (2.4) that

$$h_{i+n_M} \bar{L}_n w = \Big(\bar{L}_n h_{i+n_M} + (i+n_M) h_{n+i+n_M}\Big) w = 0.$$

This implies that $\bar{L}_n w \in M_0$ for $n \in \mathbb{N}$, that is, M_0 is invariant under the action of the operators $\bar{L}_n$ for $n \in \mathbb{N}$. □

Proposition 5.2. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. If $n_M = 0, 1$, then $M \cong H^{\bar{\mathfrak{D}}} \otimes U^{\bar{\mathfrak{D}}}$ as $\bar{\mathfrak{D}}$-modules for some simple modules $H \in \mathcal{R}_{\bar{\mathcal{H}}}$ and $U \in \mathcal{R}_{\mathfrak{Vir}}$.*

Proof. Since $n_M = 0, 1$, we take any non-zero $v \in M_0$. Then $\mathbb{C}v$ is a $\bar{\mathcal{H}}^{(0)}$-module. Let $H = \mathcal{U}(\bar{\mathcal{H}})v$, the $\bar{\mathcal{H}}$-submodule of M generated by v. It follows from representation theory of Heisenberg algebras that $\mathrm{Ind}_{\bar{\mathcal{H}}^{(0)}}^{\bar{\mathcal{H}}}(\mathbb{C}v)$ is a simple $\bar{\mathcal{H}}$-module. Consequently, the following surjective $\bar{\mathcal{H}}$-module homomorphism

$$\begin{aligned} \varphi : \mathrm{Ind}_{\bar{\mathcal{H}}^{(0)}}^{\bar{\mathcal{H}}}(\mathbb{C}v) &\longrightarrow H \\ \sum_{\mathbf{i} \in \mathbb{M}} a_{\mathbf{i}} h^{\mathbf{i}} \otimes v &\mapsto \sum_{\mathbf{i} \in \mathbb{M}} a_{\mathbf{i}} h^{\mathbf{i}} v \end{aligned}$$

is an isomorphism, that is, H is a simple $\bar{\mathcal{H}}$-module, which is certainly smooth. Then the desired assertion follows directly from [**43**, Theorem 12]. □

Next we assume that $n_M \geqslant 2$.

We define the operators $d'_n = d_n - \bar{L}_n$ on M for $n \in \mathbb{Z}$. Since M is a smooth $\bar{\mathfrak{D}}$-module, then d'_n is well-defined for any $n \in \mathbb{Z}$. By (2.4) and (2.10), we have

$$[d'_m, \bar{\mathbf{c}}'_1] = 0, [d'_m, d'_n] = (m-n) d'_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} \bar{\mathbf{c}}'_1, m, n \in \mathbb{Z}, \tag{5.1}$$

where $\bar{\mathbf{c}}'_1 = c - (1 - \frac{12z^2}{\ell}) \mathrm{id}_M$ and c is the central charge of M. So the operator algebra

$$\mathfrak{Vir}' = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} d'_n \oplus \mathbb{C} \bar{\mathbf{c}}'_1$$

is isomorphic to the Virasoro algebra $\mathfrak{Vir}$. Since $[d_n, h_k] = [\bar{L}_n, h_k] = -k h_{n+k} + \delta_{n+k,0}(n^2+n)\bar{\mathbf{c}}_2$, we have

$$[d'_n, h_k] = 0, n, k \in \mathbb{Z} \tag{5.2}$$

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and hence $[\mathfrak{Vir}', \bar{\mathcal{H}} + \mathbb{C}\bar{\mathbf{c}}_2] = 0$. Clearly, the operator algebra $\bar{\mathfrak{D}}' = \mathfrak{Vir}' \oplus (\bar{\mathcal{H}} + \mathbb{C}\bar{\mathbf{c}}_2)$ is a direct sum, and $M = \mathcal{U}(\bar{\mathfrak{D}})v = \mathcal{U}(\bar{\mathfrak{D}}')v$ for any $v \in M \setminus \{0\}$. Let

$$Y_n = \bigcap_{p \geqslant n} \mathrm{Ker}_{M_0}(d'_p), r_M = \min\{n \in \mathbb{Z} : Y_n \neq 0\}, K_0 = Y_{r_M}.$$

Noting that M is a smooth $\bar{\mathfrak{D}}$-module, we know that $r_M < +\infty$. If $Y_n \neq 0$ for any $n \in \mathbb{Z}$, we define $r_M = -\infty$. Denote by $K = \mathcal{U}(\bar{\mathcal{H}})K_0$.

LEMMA 5.3. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. Then the following statements hold.*

(i) *$r_M \geqslant -1$ or $r_M = -\infty$.*

(ii) *If $r_M \geqslant -1$, then K_0 is a $\bar{\mathfrak{D}}^{(0,-(n_M-1))}$-module and h_{n_M-1} acts injectively on K_0.*

(iii) *K is a $\bar{\mathfrak{D}}^{(0,-\infty)}$-module and $K(z)^{\bar{\mathfrak{D}}}$ has a $\bar{\mathfrak{D}}$-module structure by (2.7)–(2.8).*

(iv) *K_0 and K are invariant under the actions of $\bar{L}_n$ and d'_n for $n \in \mathbb{N}$.*

(v) *If $r_M \neq -\infty$, then d'_{r_M-1} acts injectively on K_0 and K.*

Proof. (i) If $Y_{-2} \neq 0$, then $d'_p K_0 = 0$, $p \geqslant -2$. We deduce that $\mathfrak{Vir}' K_0 = 0$ and hence $r_M = -\infty$.
If $Y_{-2} = 0$, then $r_M \geqslant -1$.

(ii) For any $0 \neq v \in K_0$ and $x \in \bar{\mathfrak{D}}^{(0,-(n_M-1))}$, it follows from lemma 5.1(ii) that $xv \in M_0$. We need to show that $d'_p xv = 0$, $p \geqslant r_M$. Indeed, $d'_p h_k v = h_k d'_p v = 0$ by (5.2) for any $k \geqslant -(n_M-1)$. Moreover, it follows from (2.10) and (5.1) that

$$d'_p d_n v = d_n d'_p v + [d'_p, d_n]v = (n-p)d'_{p+n}v = 0, \forall n \in \mathbb{N}.$$

Hence, $d'_p xv = 0$, $p \geqslant r_M$, that is, $xv \in K_0$, as desired.
Since $0 \neq K_0 \subseteq M_0$, we see that h_{n_M-1} acts injectively on K_0 by lemma 5.1(i).

(iii) follows from (ii).

(iv) Note that if $n_M = 0$, then $\bar{L}_n K_0 = 0$ for any $n \in \mathbb{N}$. For $n_M > 0$ we compute that

$$\begin{aligned}\bar{L}_n &= \frac{1}{2\ell}\sum_{k \in \mathbb{Z}} : h_{n-k}h_k : + \frac{(n+1)z}{\ell}h_n \\ &= \frac{1}{2\ell}\sum_{-(n_M-1) \leqslant k \leqslant n_M-1} : h_{n-k}h_k : + \frac{(n+1)z}{\ell}h_n, n \in \mathbb{N}.\end{aligned}$$

We see $\bar{L}_n K_0 \subset K_0$ and $\bar{L}_n K \subset K$ by (ii), and hence $d'_n K_0 \subset K_0$ and $d'_n K \subset K$.

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(v) follows directly from the definitions of r_M and K. □

We first consider the case $r_M = -\infty$.

Proposition 5.4. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with central charge c and level $\ell \neq 0$. If $r_M = -\infty$, then $M = K(z)^{\bar{\mathfrak{D}}}$ for some $z \in \mathbb{C}$. Hence, $c = 1 - \frac{12z^2}{\ell}$ and K is a simple $\bar{\mathcal{H}}$-module.*

Proof. Since $r_M = -\infty$, we see that $\mathfrak{Vir}' K_0 = 0$. This together with (5.1) implies that $c = 1 - \frac{12z^2}{\ell}$. Noting that $[\mathfrak{Vir}', \bar{\mathcal{H}} + \mathbb{C}\bar{\mathbf{c}}_2] = 0$, we further obtain that $\mathfrak{Vir}' K = 0$, that is, $d_n v = \bar{L}_n v \in K$ for any $v \in K$ and $n \in \mathbb{Z}$. Hence, $K(z)^{\bar{\mathfrak{D}}}$ is a $\bar{\mathfrak{D}}$-submodule of M, yielding that $M = K(z)^{\bar{\mathfrak{D}}}$. In particular, K is a simple $\bar{\mathcal{H}}$-module. □

Proposition 5.5. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. If $r_M \geqslant 2$ and $n_M \geqslant 2$, then K_0 is a simple $\bar{\mathfrak{D}}^{(0,-(n_M-1))}$-module and $M \cong \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}} K_0$.*

Proof. We first show that $\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}^{(0,-\infty)}} K_0 \cong K$ as $\bar{\mathfrak{D}}^{(0,-\infty)}$ modules. For that, let

$$\phi : \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}^{(0,-\infty)}} K_0 \longrightarrow K$$
$$\sum_{\mathbf{k} \in \mathbb{M}} h^{\mathbf{k}} \otimes v_{\mathbf{k}} \mapsto \sum_{\mathbf{k} \in \mathbb{M}} h^{\mathbf{k}} v_{\mathbf{k}},$$

where $h^{\mathbf{k}} = \cdots h_{-2-(n_M-1)}^{k_2} h_{-1-(n_M-1)}^{k_1} \in \mathcal{U}(\bar{\mathcal{H}})$. Then ϕ is a $\bar{\mathfrak{D}}^{(0,-\infty)}$-module epimorphism and $\phi|_{K_0}$ is one-to-one.

Claim. Any non-zero submodule V of $\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}^{(0,-\infty)}} K_0$ does not intersect with K_0 trivially.

Assume $V \cap K_0 = 0$. Let $v = \sum_{\mathbf{k} \in \mathbb{M}} h^{\mathbf{k}} \otimes v_{\mathbf{k}} \in V \setminus K_0$ with minimal degree $\mathbf{i}$. Then $\mathbf{0} \prec \mathbf{i}$.

Let $p = \min\{s : i_s \neq 0\}$. Since $h_{p+n_M-1} v_{\mathbf{k}} = 0$, we have $h_{p+n_M-1} h^{\mathbf{k}} v_{\mathbf{k}} = [h_{p+n_M-1}, h^{\mathbf{k}}] v_{\mathbf{k}}$. The following equality

$$[h_i, h_{j_1} h_{j_2} \cdots h_{j_t}] = \sum_{1 \leqslant s \leqslant t} \delta_{i+j_s,0}\, i \bar{\mathbf{c}}_3 h_{j_1} \cdots \hat{h}_{j_s} \cdots h_{j_t},\ i, j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t \in \mathbb{Z}$$

implies that if $k_p = 0$ then $h_{p+n_M-1} h^{\mathbf{k}} v_{\mathbf{k}} = 0$; and if $k_p \neq 0$, noticing the level $\ell \neq 0$, then $[h_{p+n}, h^{\mathbf{k}}] = \lambda h^{\mathbf{k}-\epsilon_p}$ for some $\lambda \in \mathbb{C}^*$ and hence

$$\deg([h_{p+n_M-1}, h^{\mathbf{k}}] v_{\mathbf{k}}) = \mathbf{k} - \epsilon_p \preceq \mathbf{i} - \epsilon_p,$$

where the equality holds if and only if $\mathbf{k} = \mathbf{i}$. Hence, $\deg(h_{p+n_M-1} v) = \mathbf{i} - \epsilon_p \prec \mathbf{i}$ and $h_{p+n_M-1} v \in V$, contrary to the choice of v. Thus, the claim holds.

From the claim we know that the kernel of ϕ must be zero and hence ϕ is an isomorphism.

By lemma 5.3(v), we see that d'_{r_M-1} acts injectively on K.

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As $\bar{\mathfrak{D}}$-modules,

$$\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}} K_0 \cong \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-\infty)}}^{\bar{\mathfrak{D}}} (\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}^{(0,-\infty)}} K_0) \cong \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-\infty)}}^{\bar{\mathfrak{D}}} K.$$

And we further have $\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-\infty)}}^{\bar{\mathfrak{D}}} K \cong \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$ as vector spaces. Moreover, we have the following $\bar{\mathfrak{D}}$-module epimorphism

$$\begin{aligned} \pi : \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-\infty)}}^{\bar{\mathfrak{D}}} K = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K & \qquad\qquad \to M, \\ \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} \otimes v_{\mathbf{l}} \mapsto \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} v_{\mathbf{l}}, & \end{aligned}$$

where $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2}(d'_{-1})^{l_1}$. We see that π is also a $\mathfrak{Vir}'$-module epimorphism. By the proof of Theorem 2.1 in [**46**] we know that any non-zero $\mathfrak{Vir}'$-submodule of $\mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$ contains non-zero vectors of K. Note that $\pi|_K$ is one-to-one, we see that the image of any non-zero $\bar{\mathfrak{D}}$-submodule (and hence $\mathfrak{Vir}'$-submodule) of $\mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-\infty)}}^{\bar{\mathfrak{D}}} K$ must be a non-zero $\bar{\mathfrak{D}}$-submodule of M and hence be the whole module M, which forces that the kernel of π must be 0. Therefore, π is an isomorphism. Since M is simple, we see K_0 is a simple $\bar{\mathfrak{D}}^{(0,-(n_M-1))}$-module. □

PROPOSITION 5.6. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. If $r_M = 1$, then d'_0 has an eigenvector in K.*

Proof. Lemma 5.3 (iv) means that K is a $\bar{\mathfrak{D}}'^{(0,-\infty)}$-module. Assume that any finitely generated $\mathbb{C}[d'_0]$-submodule of K is a free $\mathbb{C}[d'_0]$-module. By lemma 4.10 we see that the following $\bar{\mathfrak{D}}'$-module homomorphism

$$\begin{aligned} \tau : \mathrm{Ind}_{\bar{\mathfrak{D}}'^{(0,-\infty)}}^{\bar{\mathfrak{D}}'} K = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K \longrightarrow M, & \\ x \otimes u \mapsto xu, x \in \mathcal{U}(\mathfrak{Vir}'), u \in K & \end{aligned}$$

is an isomorphism. So $M = \mathrm{Ind}_{\mathfrak{Vir}'^{(0)}}^{\mathfrak{Vir}'} K$, and consequently, K is a simple $\bar{\mathfrak{D}}'^{(0,-\infty)}$-module. Since $r_M = 1$ and $\mathfrak{Vir}'^{(1)} K = 0$, K can be seen as a simple module over the Lie algebra $\mathcal{H} \oplus \mathbb{C}\mathbf{c}_2 \oplus \mathbb{C}d'_0$ where $\mathbb{C}d'_0$ lies in the centre of the Lie algebra. Schur's lemma tells us that d'_0 acts as a scalar on K, a contradiction. So this case will not occur.

Therefore, there exists some finitely generated $\mathbb{C}[d'_0]$-submodule W of K that is not a free $\mathbb{C}[d'_0]$-module. Since $\mathbb{C}[d'_0]$ is a principal ideal domain, by the structure theorem of finitely generated modules over a principal ideal domain, there exists a monic polynomial $f(d'_0) \in \mathbb{C}[d'_0]$ with minimal positive degree and non-zero element $u \in W$ such that $f(d'_0)u = 0$. Write $f(d'_0) = \Pi_{1\leqslant i\leqslant s}(d'_0 - \lambda_i)$, $\lambda_1, \cdots, \lambda_s \in \mathbb{C}$. Denote $w := \prod_{i=1}^{s-1}(d'_0 - \lambda_i)u \neq 0$, we see $(d'_0 - \lambda_s)w = 0$ where we make convention that $w = u$ if $s = 1$. Then w is a desired eigenvector of d'_0. □

PROPOSITION 5.7. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. If $r_M = 0$, ± 1, then K is a simple $\mathcal{H}$-module and $M \cong K(z)^{\bar{\mathfrak{D}}} \otimes U^{\bar{\mathfrak{D}}}$ for some simple module $U \in \mathcal{R}_{\mathfrak{Vir}}$ and some $z \in \mathbb{C}$.*

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Proof. If $r_M = 1$, then by proposition 5.6 we know that there exists $0 \neq u \in K$ such that $d'_0 u = \lambda u$ for some $\lambda \neq 0$; if $r_M = 0, -1$, then $d'_0 K = 0$. In summary, for all the three cases, d'_0 has an eigenvector in K. Since M is a simple $\bar{\mathfrak{D}}'$-module, Schur's lemma implies that h_0, $\bar{\mathbf{c}}'_1$, $\bar{\mathbf{c}}_2$, $\bar{\mathbf{c}}_3$ act as scalars on M. So M is a weight $\bar{\mathfrak{D}}'$-module, and K is a weight module for $\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}$. Take a weight vector $u_0 \in K$ with $d'_0 u_0 = \lambda_0 u_0$ for some $\lambda_0 \in \mathbb{C}$.

Set $K' = \mathcal{U}(\bar{\mathcal{H}})u_0$, which is an $\bar{\mathcal{H}}$ submodule of K. Now we define the $\bar{\mathfrak{D}}'$-module $K'^{\bar{\mathfrak{D}}'}$ with trivial action of $\mathfrak{Vir}'$. Let $\mathbb{C}v_0$ be the one-dimensional $\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}$-module defined by

$$\bar{\mathbf{c}}'_1 v_0 = (c - 1 + \frac{12z^2}{\ell})v_0,$$
$$d'_0 v_0 = \lambda_0 v_0,\ d'_n v_0 = h_k v_0 = \bar{\mathbf{c}}_2 v_0 = \bar{\mathbf{c}}_3 v_0 = 0,\ 0 \neq n \geqslant r_M, k \in \mathbb{Z}.$$

Then $\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}$ is a $\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}$-module with central charge $c - 1 + \frac{12z^2}{\ell}$ and level ℓ. There is a $\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}$-module homomorphism

$$\tau_{K'} : \mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'} \longrightarrow M,$$
$$v_0 \otimes u \mapsto u, \forall u \in K',$$

which is injective and can be extended to be the following $\bar{\mathfrak{D}}'$-module epimorphism

$$\tau : \mathrm{Ind}^{\bar{\mathfrak{D}}'}_{\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}}(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) \longrightarrow M,$$
$$x(v_0 \otimes u) \mapsto xu, x \in \mathcal{U}(\bar{\mathfrak{D}}'), u \in K'.$$

By Lemma 8 in [43] we know that

$$\mathrm{Ind}^{\bar{\mathfrak{D}}'}_{\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}}(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) \cong (\mathrm{Ind}^{\bar{\mathfrak{D}}'}_{\bar{\mathfrak{D}}'^{(r_M-\delta_{r_M},1,-\infty)}}\mathbb{C}v_0) \otimes K'^{\bar{\mathfrak{D}}'}$$
$$= (\mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}\mathbb{C}v_0)^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}.$$

Then we have the following $\bar{\mathfrak{D}}'$-module epimorphism

$$\tau' : (\mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}\mathbb{C}v_0)^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'} \longrightarrow M,$$
$$xv_0 \otimes u \mapsto xu, x \in \mathcal{U}(\mathfrak{Vir}'), u \in K'.$$

Note that $(\mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}\mathbb{C}v_0)^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'} \cong \mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'})$ as $\mathfrak{Vir}'$-modules, and τ' is also a $\mathfrak{Vir}'$-module epimorphism, $\tau'|_{\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}}$ is one-to-one, and $(\mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}\mathbb{C}v_0)^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$ is a highest weight $\mathfrak{Vir}'$-module. Let $V = \mathrm{Ind}^{\mathfrak{Vir}'}_{\mathfrak{Vir}'^{(r_M-\delta_{r_M},1)}}\mathbb{C}v_0$ and $\mathfrak{K} = \mathrm{Ker}(\tau')$. It should be noted that

$$\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'} = \{u \in V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'} \mid d'_0 u = \lambda_0 u\}.$$

We see that $(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) \cap \mathfrak{K} = 0$. Let $\mathfrak{K}'$ be the sum of all $\mathfrak{Vir}'$-submodules W of $V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$ with $(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) \cap W = 0$, that is, the unique maximal (weight)

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$\mathfrak{Vir}'$-submodule of $V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$ with trivial intersection with $(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'})$. It is obvious that $\mathfrak{K} \subseteq \mathfrak{K}'$. Next we further show that $\mathfrak{K} = \mathfrak{K}'$. For that, take any $\mathfrak{Vir}'$-submodule W of $V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$ such that $(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) \cap W = 0$. Then for any weight vector $w = \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} v_0 \otimes u_{\mathbf{l}} \in W$, where $u_{\mathbf{l}} \in K'^{\bar{\mathfrak{D}}'}$, $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2}(d'_{-1})^{l_1}$ if $r_M = 1, 0$, or $d'^{\mathbf{l}} = \cdots (d'_{-2})^{l_2}$ if $r_M = -1$, and all $\mathrm{w}(\mathbf{l}) \geqslant 1$ are equal. Note that $h_k w = \sum_{\mathbf{l}\in\mathbb{M}} d'^{\mathbf{l}} v_0 \otimes h_k u_{\mathbf{l}}$ either equals to 0 or has the same weight as w under the action of d'_0. So $\mathcal{U}(\bar{\mathfrak{D}}')W \cap (\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) = 0$, i.e. $\mathcal{U}(\bar{\mathfrak{D}}')W \subset \mathfrak{K}'$. Hence, $\mathcal{U}(\bar{\mathfrak{D}}')\mathfrak{K}' \cap (\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) = 0$. The maximality of $\mathfrak{K}'$ forces that $\mathfrak{K}' = \mathcal{U}(\bar{\mathfrak{D}}')\mathfrak{K}'$ is a proper $\bar{\mathfrak{D}}'$-submodule of $V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$. Since $\mathfrak{K}$ is a maximal proper $\bar{\mathfrak{D}}'$-submodule of $V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$, it follows that $\mathfrak{K} = \mathfrak{K}'$.

By lemma 4.8 we know that $\mathfrak{K}$ is generated by $P_1(\mathbb{C}v_0 \otimes K^{\bar{\mathfrak{D}}'}) = \mathbb{C}P_1v_0 \otimes K'^{\bar{\mathfrak{D}}'}$ and $P_2(\mathbb{C}v_0 \otimes K'^{\bar{\mathfrak{D}}'}) = \mathbb{C}P_2v_0 \otimes K'^{\bar{\mathfrak{D}}'}$. Let V' be the maximal submodule of V generated by P_1v_0 and P_2v_0, then $\mathfrak{K} = V'^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}$. Therefore,

$$M \cong (V^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'})/(V'^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}) \cong (V/V')^{\bar{\mathfrak{D}}'} \otimes K'^{\bar{\mathfrak{D}}'}, \tag{5.3}$$

which forces that $K'^{\bar{\mathfrak{D}}'}$ is a simple $\bar{\mathfrak{D}}'$-module and hence a simple $\bar{\mathcal{H}}$-module. So K' is a simple $\bar{\mathcal{H}}$-module. By [**43**, Theorem 12] we know there exists a simple $\mathfrak{Vir}$-module $U \in \mathcal{R}_{\mathfrak{Vir}}$ such that $M \cong K'^{\bar{\mathfrak{D}}} \otimes U^{\bar{\mathfrak{D}}}$. From this isomorphism and some computations we see that $K_0 \subseteq K'^{\bar{\mathfrak{D}}} \otimes v_0$ where v_0 is a highest weight vector. So $K = K'$. □

We are now in a position to present the following main result on characterization of simple smooth $\bar{\mathfrak{D}}$-modules with non-zero level.

Theorem 5.8. *Let M be a simple smooth $\bar{\mathfrak{D}}$-module with level $\ell \neq 0$. The invariants n_M, r_M of M, K_0, K are defined as before. Then*

$$M \cong \begin{cases} K(z)^{\bar{\mathfrak{D}}}, & \text{if } r_M = -\infty,\\ K(z)^{\bar{\mathfrak{D}}} \otimes U^{\bar{\mathfrak{D}}}, & \text{if } -1 \leqslant r_M \leqslant 1 \text{ or } n_M = 0, 1,\\ \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(n_M-1))}}^{\bar{\mathfrak{D}}} K_0, & \text{otherwise,} \end{cases}$$

for some $U \in \mathcal{R}_{\mathfrak{Vir}}$ and some $z \in \mathbb{C}$.

Proof. The assertion follows directly from proposition 5.2, proposition 5.4, proposition 5.5 and proposition 5.7. □

The following result characterizes simple Whittaker modules over the twisted Heisenberg–Virasoro algebra $\bar{\mathfrak{D}}$.

Theorem 5.9. *Let M be a $\bar{\mathfrak{D}}$-module (not necessarily weight) on which the algebra $\bar{\mathfrak{D}}^+$ acts locally finitely. Then the following statements hold.*

(i) *The module M contains a non-zero vector v such that $\bar{\mathfrak{D}}^+ v \subseteq \mathbb{C}v$.*

(ii) *If M is simple, then M is a Whittaker module or a highest weight module.*

Proof. (i) Let (M_1, ρ) be a finite dimensional $\bar{\mathfrak{D}}^+$-submodule of M. Then M_1 is also a finite dimensional $\mathfrak{Vir}_{\geqslant 1}$-module. Let $\mathfrak{a} := \ker(\rho|_{\mathfrak{Vir}_{\geqslant 1}})$ be the kernel of the representation map of $\mathfrak{Vir}_{\geqslant 1}$ on M_1. Then $\mathfrak{a}$ is an ideal of $\mathfrak{Vir}_{\geqslant 1}$ of finite codimension. We claim that $d_n \in \mathfrak{a}$ for some $n \in \mathbb{Z}_+$. If this is not true, then there exists a minimal $m \in \mathbb{Z}_+$ such that $\mathfrak{a}$ contains an element of the form $a_{i_1}d_{i_1} + a_{i_2}d_{i_2} + \cdots + a_{i_{m+1}}d_{i_{m+1}}$ for positive integers $i_1 < i_2 < \cdots < i_{m+1}$ and non-zero complex numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_{m+1}}$. We further see that $\mathfrak{a}$ contains

$$[d_{i_1}, a_{i_1}d_{i_1} + a_{i_2}d_{i_2} + \cdots + a_{i_{m+1}}d_{i_{m+1}}]$$
$$= a_{i_2}(i_2 - i_1)d_{i_1+i_2} + a_{i_3}(i_3 - i_1)d_{i_1+i_3} + \cdots + a_{i_{m+1}}(i_{m+1} - i_1)d_{i_1+i_{m+1}},$$

which contradicts with the minimality of m. Hence, the claim follows. Consequently,

$$\widetilde{\mathfrak{Vir}}_{\geqslant n} := \sum_{i \geqslant n,\, i \neq 2n} \mathbb{C}d_i = \mathbb{C}d_n + [d_n, \mathfrak{Vir}_{\geqslant 1}] \subseteq \mathfrak{a}.$$

Then

$$\widetilde{\mathfrak{Vir}}_{\geqslant n} + \bar{\mathcal{H}}_{\geqslant n+1} = \widetilde{\mathfrak{Vir}}_{\geqslant n} + [\bar{\mathcal{H}}_{\geqslant 1}, \widetilde{\mathfrak{Vir}}_{\geqslant n}] \subseteq \ker(\rho).$$

This implies that M_1 is a finite dimensional module over a finite dimensional solvable Lie algebra $\bar{\mathfrak{D}}^+/(\widetilde{\mathfrak{Vir}}_{\geqslant n} + \bar{\mathcal{H}}_{\geqslant n+1})$. The desired assertion follows directly from Lie theorem.

(ii) follows directly from (i) and [**46**].

□

REMARK 5.10. From theorem 5.9 we know that if M is a simple Whittaker module over $\bar{\mathfrak{D}}$ with non-zero level, and $\bar{\mathfrak{D}}^+ v \subset \mathbb{C}v$ for some non-zero vector $v \in M$, then $K = \mathcal{U}(\bar{\mathcal{H}})v = \mathcal{U}(\oplus_{r \in -\mathbb{Z}_+} \mathbb{C}h_r)v$ is a simple Whittaker module over $\bar{\mathcal{H}}$. Therefore, [**43**, Theorem 12] implies that $M \cong U^{\bar{\mathfrak{D}}} \otimes K(z)^{\bar{\mathfrak{D}}}$ for some $U \in \mathcal{R}_{\mathfrak{Vir}}$. Clearly, U is a simple Whittaker module or a simple highest weight module over $\mathfrak{Vir}$.

6. Application: characterization of simple highest weight modules and Whittaker modules over the mirror Heisenberg–Virasoro algebra

Based on the results on the structure of simple smooth modules over the mirror Heisenberg–Virasoro algebra $\mathfrak{D}$ given in theorems 2.10 and 4.13, we give characterization of simple highest weight $\mathfrak{D}$-modules and simple Whittaker $\mathfrak{D}$-modules in this section.

We first have the following result characterizing simple highest weight modules over the mirror Heisenberg–Virasoro algebra.

THEOREM 6.1. *Let $\mathfrak{D}$ be the mirror Heisenberg–Virasoro algebra with the triangular decomposition $\mathfrak{D} = \mathfrak{D}^+ \oplus \mathfrak{D}^0 \oplus \mathfrak{D}^-$. Let S be a $\mathfrak{D}$-module (not necessarily weight) on which every element in the algebra $\mathfrak{D}^+$ acts locally nilpotently. Then the following statements hold.*

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(i) *The module S contains a non-zero vector v such that $\mathfrak{D}^+ v = 0$.*

(ii) *If S is simple, then S is a highest weight module.*

Proof. (i) It follows from [**45**, Theorem 1] that there exists a non-zero vector $v \in S$ such that $d_i v = 0$ for any $i \in \mathbb{Z}_+$. If $h_{\frac{1}{2}} v = 0$, then $\mathfrak{D}^+ v = 0$ as d_1, d_2 and $h_{\frac{1}{2}}$ generate $\mathfrak{D}^+$. Assume that $w := h_{\frac{1}{2}} v \neq 0$. Then

$$d_1 w = d_1 h_{\frac{1}{2}} v = h_{\frac{1}{2}} d_1 v + [d_1, h_{\frac{1}{2}}] v = -\frac{1}{2} h_{\frac{3}{2}} v.$$

Similar arguments yield that the element $d_1^j w = \lambda h_{j+\frac{1}{2}} v$ for some $\lambda \in \mathbb{C}^*$ and $j \in \mathbb{Z}_+$. As d_1 acts locally nilpotently on S, it follows that there exists some $n \in \mathbb{Z}_+$ such that $h_{j+\frac{1}{2}} v = 0$ for $j \geqslant n$.

We now show that for every $m \in \mathbb{N}$ there exists some non-zero element $u \in S$ such that $d_i u = h_{k+\frac{1}{2}} u = 0$ for $i \in \mathbb{Z}_+$ and $k \geqslant m$ by a backward induction on m. The above arguments imply that the assertion is true for $m \geqslant n$. Assume that $0 \neq u \in S$ satisfies that $d_i u = h_{k+\frac{1}{2}} u = 0$ for $i \in \mathbb{Z}_+$ and $k \geqslant m > 0$. If $h_{m-\frac{1}{2}} u = 0$, then the induction step is proved. Otherwise, $h_{m-\frac{1}{2}} u \neq 0$, and there exists some $l \in \mathbb{N}$ such that $u' := h_{m-\frac{1}{2}}^l u \neq 0$ and $h_{m-\frac{1}{2}} u' = h_{m-\frac{1}{2}}^{l+1} u = 0$. Moreover, $d_i u' = h_{k+\frac{1}{2}} u' = 0$ for $i \in \mathbb{Z}_+$ and $k \geqslant m-1$. The induction step follows.

(ii) By (i), we know that S is a simple smooth $\mathfrak{D}$-module with $n_S = 0$ and $m_S \leqslant 1$. From theorem 2.10 and case 1 of theorem 4.13 we know that $S \cong H^{\mathfrak{D}} \otimes U^{\mathfrak{D}}$ as $\mathfrak{D}$-modules for some simple modules $H \in \mathcal{R}_{\mathcal{H}}$ and $U \in \mathcal{R}_{\mathfrak{Vir}}$. Moreover, $H = \mathrm{Ind}_{\mathcal{H}^{(0)}}^{\mathcal{H}}(\mathbb{C}v)$ is a simple highest weight module over $\mathfrak{D}$. Note that every element in the algebra $\mathfrak{Vir}^{(1)}$ acts locally nilpotently on $\mathbb{C}v \otimes U$ by the assumption. This implies that the same property also holds on U. From [**45**, Theorem 1] we know that U is a simple highest weight $\mathfrak{Vir}$-module. This completes the proof.

□

As a direct consequence of theorem 6.1, we have

Corollary 6.2. *Let S be a simple smooth $\mathfrak{D}$-module with $m_S \leqslant 1$ and $n_S = 0$. Then S is a highest weight module.*

Proof. The assumption that $m_S \leqslant 1$ and $n_S = 0$ implies that there exists a non-zero vector $v \in M$ such that $\mathfrak{D}^+ v = 0$. Then $M = \mathcal{U}(\mathfrak{D}^- + \mathfrak{D}^0)v$. It follows that each element in $\mathfrak{D}^+$ acts locally nilpotently on M. Consequently, the desired assertion follows directly from theorem 6.1. □

The following result characterizes simple Whittaker modules over the mirror Heisenberg–Virasoro algebra.

Theorem 6.3. *Let M be a $\mathfrak{D}$-module (not necessarily weight) on which the algebra $\mathfrak{D}^+$ acts locally finitely. Then the following statements hold.*

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(i) *The module M contains a non-zero vector v such that $\mathfrak{D}^+ v \subseteq \mathbb{C}v$.*

(ii) *If M is simple, then M is a Whittaker module or a highest weight module.*

Proof. (i) Let (M_1, ρ) be a finite dimensional $\mathfrak{D}^+$-submodule of M. Then M_1 is also a finite dimensional $\mathfrak{Vir}_{\geqslant 1}$-module. Let $\mathfrak{a} := \ker(\rho|_{\mathfrak{Vir}_{\geqslant 1}})$ be the kernel of the representation map of $\mathfrak{Vir}_{\geqslant 1}$ on M_1. Then $\mathfrak{a}$ is an ideal of $\mathfrak{Vir}_{\geqslant 1}$ of finite codimension. We claim that $d_n \in \mathfrak{a}$ for some $n \in \mathbb{Z}_+$. If this is not true, then there exists a minimal $m \in \mathbb{Z}_+$ such that $\mathfrak{a}$ contains an element of the form $a_{i_1}d_{i_1} + a_{i_2}d_{i_2} + \cdots + a_{i_{m+1}}d_{i_{m+1}}$ for positive integers $i_1 < i_2 < \cdots < i_{m+1}$ and non-zero complex numbers $a_{i_1}, a_{i_2}, \cdots, a_{i_{m+1}}$. We further see that $\mathfrak{a}$ contains

$$\begin{aligned}&[d_{i_1}, a_{i_1}d_{i_1} + a_{i_2}d_{i_2} + \cdots + a_{i_{m+1}}d_{i_{m+1}}]\\ &\quad = a_{i_2}(i_1 - i_2)d_{i_1+i_2} + a_{i_3}(i_1 - i_3)d_{i_1+i_3} + \cdots + a_{i_{m+1}}(i_1 - i_{m+1})d_{i_1+i_{m+1}},\end{aligned}$$

which contradicts with the minimality of m. Hence, the claim follows. Consequently,

$$\widetilde{\mathfrak{Vir}}_{\geqslant n} := \sum_{i \geqslant n,\, i \neq 2n} \mathbb{C}d_i = \mathbb{C}d_n + [d_n, \mathfrak{Vir}_{\geqslant 1}] \subseteq \mathfrak{a}.$$

Then

$$\widetilde{\mathfrak{Vir}}_{\geqslant n} + \mathcal{H}_{\geqslant n} = \widetilde{\mathfrak{Vir}}_{\geqslant n} + [\mathbb{C}h_{\frac{1}{2}} + \mathbb{C}h_{\frac{3}{2}}, \widetilde{\mathfrak{Vir}}_{\geqslant n}] \subseteq \ker(\rho).$$

This implies that M_1 is a finite dimensional module over a finite dimensional solvable Lie algebra $\mathfrak{D}^+/(\widetilde{\mathfrak{Vir}}_{\geqslant n} + \mathcal{H}_{\geqslant n})$. The desired assertion follows directly from Lie theorem.

(ii) follows directly from (i).

□

7. Examples

In this section, we will give a few examples of simple smooth $\bar{\mathfrak{D}}$- and $\mathfrak{D}$-modules, which are also weak (simple) untwisted and twisted $\mathcal{V}^c$-modules.

Example 7.1. For any $n \in \mathbb{Z}_+$, let $\mathcal{W}_0 = \mathbb{C}[x_1, \cdots, x_n]$ be the polynomial algebra in indeterminates $x_1, \cdots, x_n$. Define the $\mathcal{H}^{(-n)}$-module structure on $\mathcal{W}_0$ by

$$\begin{aligned}&h_{i-\frac{1}{2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = \lambda_i f(x_1, \cdots, x_i - 1, \cdots, x_n),\\ &h_{-i+\frac{1}{2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = -\frac{\ell(i-\frac{1}{2})}{\lambda_i}(x_i + a_i) f(x_1, \cdots, x_i + 1, \cdots, x_n),\\ &h_{n+j+\frac{1}{2}} \cdot f(x_1, \cdots, x_i, \cdots, x_n) = 0,\\ &\mathbf{c}_2 \cdot f(x_1, \cdots, x_n) = \ell f(x_1, \cdots, x_n)\end{aligned} \tag{7.1}$$

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where $\ell,\,\lambda_i\in\mathbb{C}^*$, $a_i\in\mathbb{C}$, $j\in\mathbb{N}$, $1\leqslant i\leqslant n$. It is not hard to check that $\mathcal{W}_0$ is a simple $\mathcal{H}^{(-n)}$-module. Then the induced $\mathcal{H}$-module $K=\mathrm{Ind}_{\mathcal{H}^{(-n)}}^{\mathcal{H}}\mathcal{W}_0$ is a simple smooth $\mathcal{H}$-module. So $K^{\mathfrak{D}}$ is a simple smooth $\mathfrak{D}$-module with central charge 1 and level ℓ. We may denote $K^{\mathfrak{D}}=K^{\mathfrak{D}}(\ell,\,\Lambda_n,\,\mathfrak{a}_n)$ for any $\ell\in\mathbb{C}^*$, $\Lambda_n=(\lambda_1,\,\cdots,\,\lambda_n)\in(\mathbb{C}^*)^n$, $\mathfrak{a}_n=(a_1,\,\cdots,\,a_n)\in\mathbb{C}^n$.

Let U be a simple smooth $\mathfrak{Vir}$-module (theorem 2.8 classified all simple smooth $\mathfrak{Vir}$-modules). From corollary 2.13, then $S=U^{\mathfrak{D}}\otimes K^{\mathfrak{D}}(\ell,\,\Lambda_n,\,\mathfrak{a}_n)$ is a simple smooth $\mathfrak{D}$-module.

If we replace (7.1) by

$$\begin{aligned}
&h_i\cdot f(x_1,\cdots,x_i,\cdots,x_n)=\lambda_i f(x_1,\cdots,x_i-1,\cdots,x_n),\\
&h_{-i}\cdot f(x_1,\cdots,x_i,\cdots,x_n)=-\frac{\ell i}{\lambda_i}(x_i+a_i)f(x_1,\cdots,x_i+1,\cdots,x_n),\\
&h_{n+j+1}\cdot f(x_1,\cdots,x_i,\cdots,x_n)=0,\\
&\bar{\mathbf{c}}_3\cdot f(x_1,\cdots,x_n)=\ell f(x_1,\cdots,x_n)
\end{aligned}$$

for $\ell,\,\lambda_i\in\mathbb{C}^*$, $a_i\in\mathbb{C}$, $j\in\mathbb{N}$, $1\leqslant i\leqslant n$, then $\mathcal{W}_0$ is a simple $\bar{\mathcal{H}}^{(-n)}$-module, and the induced $\bar{\mathcal{H}}$-module $\bar{K}=\mathrm{Ind}_{\bar{\mathcal{H}}^{(-n)}}^{\bar{\mathcal{H}}}\mathcal{W}_0$ is a simple smooth $\bar{\mathcal{H}}$-module. Hence, for any $z\in\mathbb{C}$, we have the simple $\bar{\mathfrak{D}}$-module $\bar{K}(z)^{\bar{\mathfrak{D}}}=\bar{K}(z)^{\bar{\mathfrak{D}}}(\ell,\,\Lambda_n,\,\mathfrak{a}_n)$ for any $\ell\in\mathbb{C}^*$, $\Lambda_n=(\lambda_1,\,\cdots,\,\lambda_n)\in(\mathbb{C}^*)^n$, $\mathfrak{a}_n=(a_1,\,\cdots,\,a_n)\in\mathbb{C}^n$. For any simple $\mathfrak{Vir}$-module $U\in\mathcal{R}_{\mathfrak{Vir}}$, the tensor product $M=U^{\bar{\mathfrak{D}}}\otimes\bar{K}(z)^{\bar{\mathfrak{D}}}(\ell,\,\Lambda_n,\,\mathfrak{a}_n)$ is a simple smooth $\bar{\mathfrak{D}}$-module.

For characterizing simple induced smooth $\mathfrak{D}$- and $\bar{\mathfrak{D}}$-module which are not tensor product modules, we need the following

Lemma 7.2. *Let $S=U^{\mathfrak{D}}\otimes V^{\mathfrak{D}}$ be a simple smooth $\mathfrak{D}$-module with $n_S>0$ and non-zero level, where $U\in\mathcal{R}_{\mathfrak{Vir}}$ and $V\in\mathcal{R}_{\mathcal{H}}$. Let $V_0=Ker_V(\mathcal{H}^{(n_S)})$ and $W_0=Ker_S(\mathcal{H}^{(n_S)})$. Then V_0 is a simple $\mathfrak{D}^{(0,-n_S)}$-module, and $W_0=U\otimes V_0$. Hence W_0 contains a simple $\mathcal{H}^{(-n_S)}$ submodule.*

Proof. This is clear. □

We also have the $\bar{\mathfrak{D}}$-module version of lemma 7.2:

Lemma 7.3. *Let $M=H(z)^{\bar{\mathfrak{D}}}\otimes U^{\bar{\mathfrak{D}}}$ be a simple smooth $\bar{\mathfrak{D}}$-module with $n_M>1$ and non-zero level, where $z\in\mathbb{C}$, $H\in\mathcal{R}_{\bar{\mathcal{H}}}$ and $U\in\mathcal{R}_{\mathfrak{Vir}}$. Let $H_0=Ker_H(\bar{\mathcal{H}}^{(n_M)})$ and $M_0=Ker_M(\bar{\mathcal{H}}^{(n_M)})$. Then H_0 is a simple $\bar{\mathfrak{D}}^{(0,-n_M+1)}$-module, and $M_0=H_0\otimes U$. Hence, M_0 contains a simple $\bar{\mathcal{H}}^{(-n_M+1)}$ submodule.*

Lemma 7.2 (resp. lemma 7.3) means that if $S\in\mathcal{R}_{\mathfrak{D}}$ (resp. $M\in\mathcal{R}_{\bar{\mathfrak{D}}}$) is not a tensor product module, then W_0 (resp. M_0) contains no simple $\mathcal{H}^{(-n_S)}$-submodule (resp. $\bar{\mathcal{H}}^{(-n_M+1)}$-submodules).

Here we will first consider the case $n_S=1$ (resp. $n_M=2$). Let $\mathfrak{b}=\mathbb{C}h+\mathbb{C}e$ be the 2-dimensional solvable Lie algebra with basis h, e and subject to Lie bracket $[h,\,e]=e$. The following concrete example using [**40**, Example 13] tells us how

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to construct induced smooth $\mathfrak{D}$-module (resp. $\bar{\mathfrak{D}}$-module) from a $\mathbb{C}[e]$-torsion-free simple $\mathfrak{b}$-module.

EXAMPLE 7.4 Simple induced smooth module, $n_S = 1/n_M = 2$. Let $c_1, c_2 \in \mathbb{C}$ with $c_2 \neq 0$. Let $W' = (t-1)^{-1}\mathbb{C}[t, t^{-1}]$. From [40, Example 13] we know that W' is a simple $\mathfrak{b}$-module whose structure is given by

$$h \cdot f(t) = t\frac{d}{dt}(f(t)) + \frac{f(t)}{t^2(t-1)},\ e \cdot f(t) = tf(t), \forall f(t) \in W'.$$

We can make W' into a $\mathfrak{D}^{(0,0)}$-module by

$$\begin{aligned}&\mathbf{c}_1 \cdot f(t) = c_1 f(t),\ \ \mathbf{c}_2 \cdot f(t) = c_2 f(t),\\ &d_0 \cdot f(t) = -\frac{1}{2}h \cdot f(t),\ h_{\frac{1}{2}} \cdot f(t) = e \cdot f(t),\ d_i \cdot f(t) = h_{\frac{1}{2}+i} \cdot f(t) = 0,\ \ i \in \mathbb{Z}_+.\end{aligned}$$

Then W' is a simple $\mathfrak{D}^{(0,0)}$-module. Clearly, the action of $h_{\frac{1}{2}}$ on W' implies that W' contains no simple $\mathcal{H}^{(0)}$-module. Then $W_0 = \mathrm{Ind}_{\mathfrak{D}^{(0,0)}}^{\mathfrak{D}^{(0,-1)}} W'$ is a simple $\mathfrak{D}^{(0,-1)}$-module and contains no simple $\mathcal{H}^{(-1)}$-module. So W_0 is not a tensor product $\mathfrak{D}^{(0,-1)}$-module. Let $S = \mathrm{Ind}_{\mathfrak{D}^{(0,-1)}}^{\mathfrak{D}} W_0$. It is easy to see $n_S = 1$, $m_S = 2 = r_S$ and $W_0 = U_0 = K_0$. The proof of proposition 4.5 implies that S is a simple smooth $\mathfrak{D}$-module. And lemma 7.2 means that S is not a tensor product $\mathfrak{D}$-module.

For $c, z, z' \in \mathbb{C}$, $\ell \in \mathbb{C}^*$, we also can make W' into a $\bar{\mathfrak{D}}^{(0,0)}$-module by

$$\begin{aligned}&d_0 \cdot f(t) = h \cdot f(t), \quad h_1 \cdot f(t) = e \cdot f(t),\\ &h_0 \cdot f(t) = z' f(t), h_{1+i} \cdot f(t) = d_i \cdot f(t) = 0, \quad i \in \mathbb{Z}_+,\\ &\bar{\mathbf{c}}_1 \cdot f(t) = cf(t), \quad \bar{\mathbf{c}}_2 \cdot f(t) = zf(t), \quad \bar{\mathbf{c}}_3 \cdot f(t) = \ell f(t),\end{aligned}$$

where $f(t) \in W'$. Then W' is a simple $\bar{\mathfrak{D}}^{(0,0)}$-module. Clearly, the action of h_1 on W' implies that W' contains no simple $\bar{\mathcal{H}}^{(0)}$-module. Then $M_0 = \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,0)}}^{\bar{\mathfrak{D}}^{(0,-1)}} W'$ is a simple $\bar{\mathfrak{D}}^{(0,-1)}$-module and contains no simple $\bar{\mathcal{H}}^{(-1)}$-module. Let $M = \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-1)}}^{\bar{\mathfrak{D}}} M_0$. It is easy to see $n_M = 2$, $r_M = 3$. The proof of proposition 5.5 implies that M is a simple smooth $\bar{\mathfrak{D}}$-module. And lemma 7.3 means that M is not a tensor product $\bar{\mathfrak{D}}$-module.

EXAMPLE 7.5 Simple induced modules of semi-Whittaker type, $n_S \geqslant 2$, $n_M \geqslant 3$. Take $p, q \in \mathbb{Z}_+$, $\mathbf{a} = (a_1, \ldots, a_q) \in (\mathbb{C}^*)^q$, $\mathbf{b} = (b_1, \ldots, b_p) \in (\mathbb{C}^*)^p$, $c, \ell \in \mathbb{C}$ with $\ell \neq 0$. Define the 1-dimensional $\mathfrak{D}^{(p,q)}$-module $\mathbb{C}_{\mathbf{a},\mathbf{b}} = \mathbb{C}v_0$ with

$$\begin{aligned}&\mathbf{c}_1 \cdot v_0 = cv_0,\ \ \mathbf{c}_2 \cdot v_0 = \ell v_0,\\ &d_p v_0 = a_1 v_0, \cdots, d_{p+q-1} v_0 = a_q v_0,\ d_i v_0 = 0 \text{ for } i > p+q-1,\\ &h_{q+\frac{1}{2}} v_0 = b_1 v_0, \cdots h_{p+q-\frac{1}{2}} v_0 = b_p v_0,\ h_{i-\frac{1}{2}} v_0 = 0 \text{ for } i > p+q.\end{aligned} \tag{7.2}$$

It is not hard to show that $U(\mathbf{a}, \mathbf{b}) := \mathrm{Ind}_{\mathfrak{D}^{(p,q)}}^{\mathfrak{D}^{(0,-1)}} \mathbb{C}_{\mathbf{a},\mathbf{b}}$ is a simple $\mathfrak{D}^{(0,-1)}$-module. Then in theorem 3.1 (2) we have $V = U(\mathbf{a}, \mathbf{b})$, $n = 1$, $k = p + q = l$, and so $S =$

$\widehat{U}(\mathbf{a},\,\mathbf{b}) := \mathrm{Ind}_{\mathfrak{D}^{(0,-1)}}^{\mathfrak{D}} U(\mathbf{a},\,\mathbf{b})$ is a simple smooth $\mathfrak{D}$-module. In lemma 7.2, $n_S = p+q$, and $W_0 = \mathrm{Ind}_{\mathcal{H}^{(q)}}^{\mathcal{H}^{(-(p+q))}}(\mathrm{Ind}_{\mathfrak{D}^{(p,q)}}^{\mathfrak{D}^{(0,q)}}\mathbb{C}_{\mathbf{a},\,\mathbf{b}})$ does not contain any simple $\mathcal{H}^{(-(p+q))}$-module (for $h_{\pm 1/2}$ acts freely on W_0). Hence, by lemma 7.2, $\widehat{U}(\mathbf{a},\,\mathbf{b})$ is not a tensor product $\mathfrak{D}$-module.

If we, in the above example, replace (7.2) by

$$\begin{aligned}
&\bar{\mathbf{c}}_1 \cdot v_0 = cv_0,\; \bar{\mathbf{c}}_2 \cdot v_0 = zv_0,\; \bar{\mathbf{c}}_3 v_0 = \ell v_0,\\
&d_p v_0 = a_1 v_0, \cdots, d_{p+q-1} v_0 = a_q v_0,\;\; d_i v_0 = 0 \text{ for } i > p+q-1,\\
&h_{q+1} v_0 = b_1 v_0, \cdots h_{p+q} v_0 = b_p v_0,\;\; h_i v_0 = 0 \text{ for } i > p+q,
\end{aligned}$$

where $z \in \mathbb{C}$ and leave other parts invariant, then for any $z' \in \mathbb{C}$, the induced $\bar{\mathfrak{D}}^{(0,-(p+q))}$-module

$$\bar{V} = \mathrm{Ind}_{\bar{\mathfrak{D}}^{(p,q+1)}}^{\bar{\mathfrak{D}}^{(0,-(p+q))}} \mathbb{C}_{\mathbf{a},\mathbf{b}} \Big/ \Big(\mathcal{U}(\bar{\mathfrak{D}}^{(0,-(p+q))})(h_0 - z')(1 \otimes v_0)\Big)$$

is a simple $\bar{\mathfrak{D}}^{(0,-(p+q))}$-module. Let $M = \mathrm{Ind}_{\bar{\mathfrak{D}}^{(0,-(p+q))}}^{\bar{\mathfrak{D}}} \bar{V}$. The proof of theorem 5.5 implies that M is a simple smooth $\bar{\mathfrak{D}}$-module where $n_M = p+q+1$, $r_M = 2(p+q)+1$ and $K_0 = \bar{V} = M_0$. Since $\bar{V}$ contains no simple $\bar{\mathcal{H}}^{(-n_M+1)}$-module, we see, by lemma 7.3, that M is not a tensor product $\bar{\mathfrak{D}}$-module.

Remark 7.6. From theorem 4.13 (resp. theorem 5.2) we know that if $n_S = 0$ (resp. $n_M = 0,\,1$), then simple smooth $\mathfrak{D}$-modules(resp. $\bar{\mathfrak{D}}$-modules) must be tensor product modules. And Examples 7.4–7.5 mean that for any $n_S > 0$ (resp. $n_M > 1$), there do exist simple smooth $\mathfrak{D}$-modules (resp. $\bar{\mathfrak{D}}$-modules) which are not tensor product modules. Clearly, the $\bar{\mathfrak{D}}$-modules here are simple smooth $\widetilde{\mathfrak{D}}$-modules for $z = 0$.

Remark 7.7. A connection between smooth modules over the Heisenberg–Virasoro algebra and vertex algebra modules in untwisted cases was considered by Guo and Wang in [**27**]. It is a routine to extend this correspondence for smooth modules for the mirror Heisenberg–Virasoro algebra, so that smooth modules of non-zero level for the mirror Heisenberg–Virasoro algebra can be treated as weak twisted modules for the Heisenberg–Virasoro vertex algebras, and smooth modules of non-zero level for the twisted Heisenberg–Virasoro algebra can be treated as weak modules for the Heisenberg–Virasoro vertex algebras.

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