A Generalization of Minimal Varieties

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(*Received* 14*th October*, 1946. *Read* 1*st November*, 1946.)

1. *Formulae for the first variation of the volume integral.*

I consider an n-dimensional generalized metric space[1] S_n with coordinates $x^i(h, i, j, k \ldots$ run from 1 to n throughout), with each point of which is associated a contravariant vector-density with components u^i and weight p, called the *element of support*. The unit vector in the direction of the element of support has components denoted by l^i.

Let S_ν be a ν-space in S_n with coordinates t^α ($\alpha, \beta, \gamma \ldots$ run from 1 to ν throughout), and let $S_{\nu+1}$ be any $(\nu+1)$-space containing S_ν, defined by equations of the form

$$x^i = x^i(t^1, t^2, \ldots t^\nu, v),$$

at each point of which the element of support is defined by equations of the form

$$u^i = u^i(t^1, t^2, \ldots t^\nu, v),$$

the coordinates t^α, v in $S_{\nu+1}$ being chosen so that S_ν is the surface $v = v_0$. Let $B_{\nu-1}$ be a given closed hypersurface of S_ν, bounding a region R. If points of R are displaced in $S_{\nu+1}$ by variation of v from v_0 to $v_0 + \delta v$, the region formed by the displaced points will be denoted by R'.

If the first fundamental form of S_ν is denoted by $g_{\alpha\beta} dt^\alpha dt^\beta$, the volume of R is given by

$$V = \int_R \sqrt{g}\,(dt)^\nu$$

where g is the determinant $| g_{\alpha\beta} |$ and $(dt)^\nu$ is an abbreviation for $dt^1\, dt^2 \ldots . dt^\nu$. The volume of R' is similarly given by

$$V' = \int_R \left\{ \sqrt{g} + \delta v \frac{\partial \sqrt{g}}{\partial v} + O(\delta v^2) \right\} (dt)^\nu$$

so that

$$V' - V = \delta V + O(\delta v^2)$$

where

$$\delta V = \delta v \int_R \frac{\partial \sqrt{g}}{\partial v} (dt)^\nu,$$

[1] Of the type treated by Schouten and Haantjes in (3). (See the list of references at the end of the paper.)

the *first variation of the volume integral.*

For brevity I shall put [1]

$$\partial x^i/\partial t^a = \lambda_{a)}{}^i, \quad \partial x^i/\partial v = \mu^i, \qquad g^{a\beta}\lambda_{a)}{}^i = \lambda^{\beta)i}, \quad \sqrt{g}\lambda^{\beta)i} = \zeta^{\beta)i},$$

calling μ^i the *displacement vector.*

To evaluate δV we have

$$d\sqrt{g} = \frac{1}{2\sqrt{g}}dg = \frac{\sqrt{g}}{2}g^{a\beta}dg_{a\beta}.$$

If D indicates absolute differentiation in S_n, $dg_{a\beta} = Dg_{a\beta}$ since the $g_{a\beta} = g_{ij}\lambda_{a)}{}^i\lambda_{\beta)}{}^j$ are scalar in S_n; hence

$$d\sqrt{g} = \sqrt{g}g^{a\beta}\lambda_{\beta)}{}^i D\lambda_{a)i},$$

i.e.

$$d\sqrt{g} = D\lambda_{a)}{}^i\zeta^{a)}{}_i. \tag{1.1}$$

Defining ν torsion vectors [2] $\Omega_{a)}{}^i$ by

$$\frac{D}{\partial v}\left(\frac{\partial x^i}{\partial t^a}\right) - \frac{D}{\partial t^a}\left(\frac{\partial x^i}{\partial v}\right) = \Omega_{a)}{}^i \tag{1.2}$$

so that

$$\frac{D\lambda_{a)}{}^i}{\partial v} - \frac{D\mu^i}{\partial t^a} = \left(\lambda_{a)}{}^j\frac{\varpi^k}{\partial v} - \mu^j\frac{\varpi^k}{\partial t^a}\right)A_j{}^i{}_k = \Omega_{a)}{}^i, \tag{1.3}$$

we obtain from (1.1), (1.3)

$$\frac{\partial\sqrt{g}}{\partial v} = \frac{D\lambda_{a)}{}^i}{\partial v}\zeta^{a)}{}_i = \left(\frac{D\mu^i}{\partial t^a} + \Omega_{a)}{}^i\right)\zeta^{a)}{}_i. \tag{1.4}$$

Thus

$$\delta V = \delta v\int_R \left(\frac{D\mu_i}{\partial t^a} + \Omega_{a)}{}^i\right)\zeta^{a)}{}_i(dt)^\nu. \tag{1.5}$$

Now

$$\frac{\partial}{\partial t^a}\left(\mu^i\zeta^{a)}{}_i\right) = \frac{D}{\partial t^a}\left(\mu^i\zeta^{a)}{}_i\right)$$

$$= \frac{D\mu^i}{\partial t^a}\zeta^{a)}{}_i + \mu^i\frac{D}{\partial t^a}\zeta^{a)}{}_i.$$

From (1.5) we now obtain

$$\delta V = \delta v\int_R \frac{\partial}{\partial t^a}\left(\mu^i\zeta^{a)}{}_i\right)(dt)^\nu - \delta v\int_R \left(\mu^i\frac{D\zeta^{a)}{}_i}{\partial t^a} - \Omega_{a)}{}^i\zeta^{a)}{}_i\right)(dt)^\nu. \tag{1.6}$$

On integrating $\frac{\partial}{\partial t^a}\left(\mu^i\zeta^{a)}{}_i\right)$, we obtain from (1.6)

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[1] In subspace theory it is customary to write B^i_a for $\partial x^i/\partial t^a$; I have written $\lambda_{a)}{}^i$ instead, in order to emphasise the similarity between the equations given herein for a minimal variety and those given in (2) for an extremal curve.

[2] A geometrical interpretation of a torsion vector is given in (2), § 2.

$$\delta V = \delta v \int_{B_{\nu-1}} \mu^i \zeta^{a)}{}_i (dt)^\nu_a - \delta v \int_R \left(\mu^i \frac{D\zeta^{a)}{}_i}{\partial t^a} - \Omega_{a)}{}^i \zeta^{a)}{}_i \right) (dt)^\nu \tag{1.7}$$

where $(dt)^\nu_a$ stands for $(dt)^\nu$ with the term dt^a omitted.[1]

2. *Conditions for a minimal variety.*

The ν-space S_ν will be said to be a *minimal variety* in S_n if, for any given $B_{\nu-1}$, $\delta V = 0$ for arbitrary displacement of points of S_ν and the element of support within $B_{\nu-1}$, and for $\mu^i = 0$ on $B_{\nu-1}$. From (1.3) and (1.7), when $\mu^i = 0$ on $B_{\nu-1}$

$$\delta V = -\delta v \int_R \left\{ \mu^i \left(\frac{D\zeta^{a)}{}_i}{\partial t^a} + \zeta^{a)j} \frac{\varpi^k}{\partial t^a} A_{ijk} \right) - \frac{\varpi^k}{\partial v} \zeta^{a)i} \lambda_{a)}{}^j A_{ijk} \right\} (dt)^\nu. \tag{2.1}$$

The conditions that $\delta V = 0$, for values of μ^i and $\frac{\varpi^k}{\partial v}$ arbitrary save for the latter satisfying $\frac{\varpi^k}{\partial v} l_k = 0$, are given by equating to zero the coefficients of μ^i, $\frac{\varpi^k}{\partial v}$ in (2.1), since $l^k A_{ijk} = 0$. Hence

S_ν is a minimal variety in S_n if and only if

$$\left.\begin{aligned} &\text{(i)} \quad \frac{D\zeta^{a)}{}_i}{\partial t^a} + \zeta^{a)j} \frac{\varpi^k}{\partial t^a} A_{ijk} = 0 \\ &\text{(ii)} \quad \lambda^{a)i} \lambda_{a)}{}^j A_{ijk} = 0 \end{aligned}\right\} \quad \textit{over } S_\nu. \tag{2.2}$$

In the particular case in which $\nu = 1$, these equations reduce to those defining an extremal curve.[2] As a further special case we may consider that in which S_n is a Finsler space, $\nu = 2$, and the element of support is tangential to S_ν. If m^i are the components of the unit vector orthogonal to the element of support and tangential to S_ν, the $\lambda_{a)}{}^i$ are of the form $\lambda_{a)}{}^i = a_a l^i + b_a m^i$. Then condition (2.2) (ii) becomes

$$g^{a\beta} b_a b_\beta m^i m^j A_{ijk} = 0. \tag{2.3}$$

Now $g^{a\beta} b_a b_\beta$ does not vanish unless the b_a all vanish, and in this case the two vectors $\lambda_{a)}{}^i$ are in the same direction. In general, however, they are not, and (2.3) leads to

$$m^i m^j A_{ijk} = 0. \tag{2.4}$$

Now[3] equations of the form $\xi^i \xi^j A_{ijk} = 0$, where ξ^i is a unit vector,

[1] Equations (1.5), (1.7) are similar to those for the first variation of the length integral; see (2), (3.5), (3.6).

[2] See (2), (4.2).

[3] See (2), § 4 for proof.

are satisfied only by $\xi^i = \pm l^i$, unless a restriction is placed on the A_{ijk}. Hence

THEOREM 1. *A Finsler S_n can possess a two-dimensional minimal variety S_2 with tangential element of support only in the restricted case in which the equations $\xi^i\xi^j A_{ijk} = 0$ have a solution other than $\xi^i = \pm l^i$ at points of S_2.*

Returning to the general conditions (2.2) and putting

$$\lambda^{a)}{}_i\lambda_{a)}{}^j = B^j_i,$$

we may write condition (2.2) (ii)

$$B^i_j A_i{}^j{}_k = 0. \tag{2.6}$$

To evaluate $\dfrac{D}{\partial t^a}\left(\sqrt{g}\lambda^{a)}{}_i\right)$ in (2.2) (i) we have

$$\frac{D}{\partial t^a} g^{a\beta} = -g^{a\rho}g^{\beta\sigma}\frac{D}{\partial t^a} g_{\rho\sigma} = -g^{a\rho}g^{\beta\sigma}\left(\lambda_{\rho)i}\frac{D\lambda_{\sigma)}{}^i}{\partial t^a} + \lambda_{\sigma)i}\frac{D\lambda_{\rho)}{}^i}{\partial t^a}\right).$$

Therefore $$\frac{D}{\partial t^a} g^{a\beta} = -g^{\beta\sigma}\lambda^{a)}{}_h\frac{D\lambda_{\sigma)}{}^h}{\partial t^a} - g^{a\rho}\lambda^{\beta)}{}_h\frac{D\lambda_{\rho)}{}^h}{\partial t^a}. \tag{2.7}$$

Thus from (1.1) and (2.7)

$$\begin{aligned}
&\frac{D}{\partial t^a}\left(\sqrt{g}g^{a\beta}\lambda_{\beta)i}\right)\\
&= \frac{D\sqrt{g}}{\partial t^a} g^{a\beta}\lambda_{\beta)i} + \sqrt{g}\frac{Dg^{a\beta}}{\partial t^a}\lambda_{\beta)i} + \sqrt{g}g^{a\beta}\frac{D\lambda_{\beta)i}}{\partial t^a}\\
&= \sqrt{g}\lambda^{\gamma)}{}_j\frac{D\lambda_{\gamma)}{}^j}{\partial t^a} g^{a\beta}\lambda_{\beta)i} - \sqrt{g}\left\{\lambda^{\sigma)}{}_i\lambda^{a)}{}_h\frac{D\lambda_{\sigma)}{}^h}{\partial t^a} + g^{a\rho}B^h_i\frac{D\lambda_{\rho)h}}{\partial t^a}\right\} + \sqrt{g}g^{a\beta}\frac{D\lambda_{\beta)i}}{\partial t^a}\\
&\qquad = \sqrt{g}g^{a\beta}C^h_i\frac{D\lambda_{\beta)h}}{\partial t^a} + \sqrt{g}\lambda^{a)}{}_i\lambda^{\beta)}{}_h\left(\lambda_{\beta)}{}^j\frac{\varpi^k}{\partial t^a} - \lambda_{a)}{}^j\frac{\varpi^k}{\partial t^\beta}\right)A_j{}^h{}_k,
\end{aligned}$$

writing $$\delta^h_i - B^h_i = C^h_i \tag{2.8}$$

and using $$\frac{D\lambda_{\beta)}{}^h}{\partial t^a} - \frac{D\lambda_{a)}{}^h}{\partial t^\beta} = \left(\lambda_{\beta)}{}^j\frac{\varpi^k}{\partial t^a} - \lambda_{a)}{}^j\frac{\varpi^k}{\partial t^\beta}\right)A_j{}^h{}_k. \tag{2.9}$$

Now (2.2) (i) may be written

$$\sqrt{g}g^{a\beta}C^h_i\frac{D\lambda_{\beta)h}}{\partial t^a} + \sqrt{g}\lambda^{a)}{}_i B^j_h\frac{\varpi^k}{\partial t^a}A_j{}^h{}_k + \sqrt{g}\lambda^{\beta)}{}_h C^j_i\frac{\varpi^k}{\partial t^\beta}A_j{}^h{}_k = 0,$$

in which the middle term vanishes on account of (2.6). Finally (2.2) becomes

THEOREM 2. *S_ν is a minimal variety in S_n if and only if*

$$\left.\begin{aligned}&\text{(i)}\quad g^{\alpha\beta}C^h_i\left(\frac{D\lambda_{\beta)h}}{\partial t^\alpha}+\lambda_{\alpha)j}\frac{\varpi^k}{\partial t^\beta}A_h{}^j{}_k\right)=0\\ &\text{(ii)}\quad B^h_i A_h{}^i{}_k=0\end{aligned}\right\}\ \textit{over } S_\nu.$$

3. *Mean curvature of a minimal variety.*

Let C be a curve on any subspace S_ν (not necessarily a minimal variety of S_n), with unit tangent vector at a given point P having S_n, S_ν components $dx^i/ds=\xi^i$, $dx^\alpha/ds=\xi^\alpha$ respectively, s being the arc-length of C.

If ρ is the radius of first curvature of C in S_n at P, $\rho D\xi^i/ds$ is the unit vector in the direction of its principal normal in S_n; hence if θ is the angle between this principal normal and any unit vector X^i normal to S_ν at P,

$$\cos\theta=\rho\,\frac{D\xi^i}{ds}\,X_i.$$

Writing $1/R$ for $(\cos\theta)/\rho$, we have

$$\frac{1}{R}=\frac{D\xi^i}{ds}X_i.$$

I call $1/R$ *the normal curvature of S_ν for the normal X^i corresponding to the curve C.* Now since $\xi^i=\lambda_{\alpha)}{}^i\xi^\alpha$,

$$\frac{D\xi^i}{ds}=\frac{D\lambda_{\alpha)}{}^i}{ds}\xi^\alpha+\lambda_{\alpha)}{}^i\frac{d\xi^\alpha}{ds}$$

and
$$\frac{1}{R}=\frac{D\lambda_{\alpha)}{}^i}{ds}X_i\xi^\alpha\quad\text{for}\quad\lambda_{\alpha)}{}^iX_i=0.$$

Since
$$\begin{aligned}\frac{D\lambda_{\alpha)}{}^i}{ds}&=\frac{d\lambda_{\alpha)}{}^i}{ds}+\lambda_{\alpha)}{}^j\frac{dx^k}{ds}\Gamma_j{}^i{}_k+\lambda_{\alpha)}{}^j\frac{du^k}{ds}C_j{}^i{}_k\\ &=\left(\frac{\partial\lambda_{\alpha)}{}^i}{\partial t^\beta}+\lambda_{\alpha)}{}^j\lambda_{\beta)}{}^k\Gamma_j{}^i{}_k\right)\xi^\beta+\lambda_{\alpha)}{}^j\frac{du^k}{ds}C_j{}^i{}_k\end{aligned}$$

it follows that $1/R$ depends not only on ξ^α but also on du^k/ds and will therefore, in general, have different values corresponding to different curves having the same tangent at P. If, however, S_ν is a minimal variety in S_n, the u^k are supposed functions of the t^β and $du^k/ds=\xi^\beta\,\partial u^k/\partial t^\beta$; then $D\lambda_{\alpha)}{}^i/ds=\xi^\beta\,D\lambda_{\alpha)}{}^i/\partial t^\beta$ and $1/R$ is now of the form

$$\frac{1}{R}=X_{\alpha\beta}\xi^\alpha\xi^\beta,\tag{3.1}$$

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where $$X_{\alpha\beta} = \tfrac{1}{2}\left(\frac{D\lambda_{\alpha)}{}^i}{\partial t^\beta} + \frac{D\lambda_{\beta)}{}^i}{\partial t^\alpha}\right)X_i, \tag{3.2}$$

which is symmetrical in α, β.

Defining then *the mean curvature of* S_ν *for the normal* X^i as the sum of the values of $1/R$ stationary for variation of ξ^α, we have for this mean curvature

$$K_m(X) = g^{\alpha\beta}X_{\alpha\beta}. \tag{3.3}$$

If we multiply equation (i) of THEOREM 2 by X^i, and use

$$X^iC^h_i = X^i(\delta^h_i - B^h_i) = X^h$$

we obtain $$K_m(X) + \lambda^{\beta)}{}_jX^h\frac{\varpi^k}{\partial t^\beta}A_h{}^j{}_k = 0.$$

Hence the following necessary, but not sufficient, condition for a minimal variety:

THEOREM 3. *If* S_ν *is a minimal variety in* S_n, *its mean curvature for a normal* X^i *is given by*

$$K_m(X) = -X^h\lambda^{\beta)}{}_j\frac{\varpi^k}{\partial t^\beta}A_h{}^j{}_k.$$

When S_n is Riemannian and $A_h{}^j{}_k = 0$, this reduces to the well-known theorem: *The mean curvature for every normal of a minimal variety in a Riemannian space vanishes.* In this case the condition is sufficient as well as necessary.[1]

If in particular the element of support of the generalised S_n is normal to S_ν,

$$l^h\lambda^{\beta)}{}_j\frac{\varpi^k}{\partial t^\beta}A_h{}^j{}_k = \lambda^{\beta)}{}_j\frac{\varpi^k}{\partial t^\beta}pl^jA_k = 0$$

since $\lambda^{\beta)}{}_j\, l^j = 0$. Hence

THEOREM 4. *If* S_ν *is a minimal variety to which the element of support is normal, its mean curvature for the element of support vanishes.*

4. *Conditions for the vanishing of the first variation of the volume integral.*

From (1.5) follows

[1] Proved in (1), § 52.

https://doi.org/10.1017/S0013091500024809 Published online by Cambridge University Press

THEOREM 5. *The first variation vanishes if*

$$\left(\frac{Du^i}{\partial t^\alpha} + \Omega_{\alpha)}{}^i\right) \zeta^{\alpha)}{}_i = 0 \text{ over } R.$$

Also, from (1.7) we have

THEOREM 6. *The first variation vanishes if R is a region of a minimal variety in S_n and the displacement vector is normal to R on its boundary.*

For if R is a region of a minimal variety in S_n the second integral in (1.7) vanishes identically. Finally, from (1.7) we have also

THEOREM 7. *The first variation vanishes if the displacement vector satisfies $\mu^i D\zeta^{\alpha)}{}_i/\partial t^\alpha = \Omega_{\alpha)}{}^i\zeta^{\alpha)}{}_i$ at points of R, and either vanishes or is normal to R on its boundary.*

REFERENCES.

1. L. P. Eisenhart, "Riemannian Geometry" (1926).
2. J. G. Freeman, "First and Second Variations of the Length Integral in a Generalized Metric Space," *Quart. Journ. Math.* (Oxford) 15 (1944), 70-83.
3. J. A. Schouten and J. Haantjes, "Über die Festlegung von allgemeinen Massbestimmungen und Übertragungen in Bezug auf ko- und kontravariante Vektordichten," *Monats. für Math. und Phys.*, 43 (1936), 161-76.

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