J. Appl. Prob. **54,** 603–626 (2017)
*doi:*10.1017/jpr.2017.20

A UNIFIED APPROACH FOR DRAWDOWN (DRAWUP) OF TIME-HOMOGENEOUS MARKOV PROCESSES

DAVID LANDRIAULT * ** AND

BIN LI,* *** *University of Waterloo*

HONGZHONG ZHANG,**** *Columbia University*

Abstract

Drawdown (respectively, drawup) of a stochastic process, also referred as the reflected process at its supremum (respectively, infimum), has wide applications in many areas including financial risk management, actuarial mathematics, and statistics. In this paper, for general time-homogeneous Markov processes, we study the joint law of the first passage time of the drawdown (respectively, drawup) process, its overshoot, and the maximum of the underlying process at this first passage time. By using short-time pathwise analysis, under some mild regularity conditions, the joint law of the three drawdown quantities is shown to be the unique solution to an integral equation which is expressed in terms of fundamental two-sided exit quantities of the underlying process. Explicit forms for this joint law are found when the Markov process has only one-sided jumps or is a Lévy process (possibly with two-sided jumps). The proposed methodology provides a unified approach to study various drawdown quantities for the general class of time-homogeneous Markov processes.

Keywords: Drawdown; integral equation; reflected process; time-homogeneous Markov process

2010 Mathematics Subject Classification: Primary 60G07
Secondary 60G40

1. Introduction

We consider a time-homogeneous, real-valued, nonexplosive, càdlàg Markov process $X = (X_t)_{t\geq 0}$ with state space $\mathbb{R}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a complete and right-continuous filtration. (The state space can sometimes be relaxed to an open interval of $\mathbb{R}$ (e.g. $(0, +\infty)$ for geometric Brownian motions). It is also possible to treat some general state space with complex boundary behaviors. However, for simplicity, we choose $\mathbb{R}$ as the state space of X in this paper.)

Throughout, we silently assume that X satisfies the strong Markov property (see Rogers and Williams [33, Section III.8,9]), and exclude Markov processes with monotone paths. The first

Received 19 January 2016; revision received 28 September 2016.
* Postal address: Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada.
** Email address: dlandria@uwaterloo.ca
*** Email address: bin.li@uwaterloo.ca
**** Postal address: Department of Industrial Engineering and Operations Research, Columbia University, New York, NY, 10027, USA. Email address: hz2244@columbia.edu

passage time of X above (below) a level $x \in \mathbb{R}$ is denoted by

$$T_x^{+(-)} = \inf\{t \geq 0 \colon X_t > (<) x\},$$

with the common convention that $\inf \varnothing = \infty$.

The drawdown process of X (also known as the reflected process of X at its supremum) is denoted by $Y = (Y_t)_{t \geq 0}$ with $Y_t = M_t - X_t$, where $M_t = \sup_{0 \leq s \leq t} X_t$. Let $\tau_a = \inf\{t > 0 \colon Y_t > a\}$ be the first time the magnitude of drawdowns exceeds a given threshold $a > 0$. Note that $(\sup_{0 \leq s \leq t} Y_s > a) = (\tau_a \leq t)$ $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). Hence, the distributional study of the maximum drawdown of X is equivalent to the study of the stopping time τ_a. Similarly, the drawup process of X is defined as $\hat{Y}_t = X_t - m_t$ for $t \geq 0$, where $m_t = \inf_{0 \leq s \leq t} X_t$. However, given that the drawup of X can be investigated via the drawdown of $-X$, we exclusively focus on the drawdown process Y in this paper.

Applications of drawdowns can be found in many areas. For instance, drawdowns are widely used by mutual funds and commodity trading advisers to quantify downside risks. Interested readers are referred to Schuhmacher and Eling [34] for a review of drawdown-based performance measures. An extensive body of literature exists on the assessment and mitigation of drawdown risks; see, e.g. [7], [8], [13], and [42]. Drawdowns are also closely related to many problems in mathematical finance, actuarial science, and statistics such as the pricing of Russian options (see, e.g. [2], [3], and [35]), De Finetti's dividend problem (see, e.g. [4] and [26]), loss-carry-forward taxation models (see, e.g. [22] and [25]), and change-point detection methods (see, e.g. [31]). More specifically, in De Finetti's dividend problem under a fixed dividend barrier $a > 0$, the underlying surplus process with dividend payments is a process obtained from reflecting X at a fixed barrier a (the reflected process' dynamics may be different than the drawdown process Y when the underlying process X is not spatial homogeneous). However, the distributional study of ruin quantities in De Finetti's dividend problem can be transformed to the study of drawdown quantities for the underlying surplus process; see Kyprianou and Palmowski [21] for a more detailed discussion. Similarly, ruin problems in loss-carry-forward taxation models can also be transformed to a generalized drawdown problem for classical models without taxation, where the generalized drawdown process is defined in the form of $Y_t = \gamma(M_t) - X_t$ for some measurable function $\gamma(\cdot)$.

The distributional study of drawdown quantities is not only of theoretical interest, but also plays a fundamental role in the aforementioned applications. Early distributional studies on drawdowns date back to Taylor [36] on the joint Laplace transform of τ_a and M_{τ_a} for Brownian motions. This result was later generalized by Lehoczky [24] to time-homogeneous diffusion processes. Douady *et al.* [9] and Magdon *et al.* [27] derived infinite series expansions for the distribution of τ_a for a standard Brownian motion and a drifted Brownian motion, respectively. For spectrally negative Lévy processes, Mijatovic and Pistorius [28] obtained a sextuple formula for the joint Laplace transform of τ_a and the last reset time of the maximum prior to τ_a, together with the joint distribution of the running maximum, the running minimum, and the overshoot of Y at τ_a. For some studies on the joint law of drawdown and drawup of spectrally negative Lévy processes or diffusion processes, we refer the reader to [30], [32], [40], and [41].

As mentioned above, Lévy processes (most often, one-sided Lévy processes (an exception to this is [5] for general Lévy processes)) and time-homogeneous diffusion processes are two main classes of Markov processes for which various drawdown problems have been extensively studied. The treatment of these two classes of Markov processes has typically been considered distinctly in the literature. For Lévy processes, Itô's excursion theory is a powerful approach to handle drawdown problems; see, e.g. [3], [28], and [30]. However, the excursion-theoretic approach is somewhat specific to the underlying model, and additional care is required when a more general class of Markov processes is considered. On the other hand, for time-homogeneous diffusion processes, Lehoczky [24] introduced an ingenious approach which has recently been generalized by many researchers; see, e.g. [25], [40], and [43]. Here again, Lehoczky's approach relies on the continuity of the sample path of the underlying model, and, hence, is not applicable for processes with upward jumps. Also, other general methodologies (such as the martingale approach in, e.g. [2] and the occupation density approach in, e.g. [14]) are well documented in the literature but they depend strongly on the specific structure of the underlying process. To the best of the authors' knowledge, no unified treatment of drawdowns (drawups) for general Markov processes has been proposed in the literature.

In this paper we propose a general and unified approach to study the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a})$ for time-homogeneous Markov processes with possibly two-sided jumps. Under mild regularity conditions, the joint law is expressed as the solution to an integral equation which involves two-sided exit quantities of the underlying process X. The uniqueness of the integral equation for the joint law is also investigated. In particular, the joint law possesses explicit forms when X has only one-sided jumps or is a Lévy process (possibly with two-sided jumps). In general, our main result reduces the drawdown problem to fundamental two-sided exit quantities.

The main idea of our proposed approach is briefly summarized below. By analyzing the evolution of sample paths over a short time period following time 0 and using renewal arguments, we first establish tight upper and lower bounds for the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a})$ in terms of the two-sided exit quantities. Then, under mild regularity conditions, we use a Fatou's lemma with varying measures to show that the upper and lower bounds converge when the length of the time interval approaches 0. This leads to an integro-differential equation satisfied by the desired joint law. Finally, we reduce the integro-differential equation to an integral equation. When X is a spectrally negative Markov process or a general Lévy process, the integral equation can be solved and the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a})$ is, hence, explicitly expressed in terms of two-sided exit quantities.

The rest of the paper is organized as follows. In Section 2 we introduce some fundamental two-sided exit quantities and present several preliminary results. In Section 3 we derive the joint law of $(\tau_a, Y_{\tau_a}, M_{\tau_a})$ for general time-homogeneous Markov processes. Several Markov processes for which the proposed regularity conditions are met are further discussed. Some numerical examples are investigated in more detail in Section 4. Some technical proofs are postponed to Appendices A–C.

2. Preliminary

For ease of notation, we adopt the following conventions throughout the paper. We denote by $\mathbb{P}_x$ the law of X given $X_0 = x \in \mathbb{R}$ and write $\mathbb{P} \equiv \mathbb{P}_0$ for brevity. We write $u \wedge v = \min\{u, v\}$, $\mathbb{R}_+ = [0, \infty)$, and $\int_x^y \mathrm{d}z$ for an integral on the open interval $z \in (x, y)$.

For $q, s \geq 0$, $u \leq x \leq v$, and $z > 0$, we introduce the following two-sided exit quantities of X:

$$B_1^{(q)}(x;\ u, v) := \mathbb{E}_x[\mathrm{e}^{-qT_v^+} \mathbf{1}_{\{T_v^+ < \infty,\ T_v^+ < T_u^-,\ X_{T_v^+} = v\}}],$$

$$B_2^{(q)}(x, \mathrm{d}z;\ u, v) := \mathbb{E}_x[\mathrm{e}^{-qT_v^+} \mathbf{1}_{\{T_v^+ < \infty,\ T_v^+ < T_u^-,\ X_{T_v^+} - v \in \mathrm{d}z\}}],$$

$$C^{(q,s)}(x;\ u, v) := \mathbb{E}_x[\mathrm{e}^{-qT_u^- - s(u - X_{T_u^-})} \mathbf{1}_{\{T_u^- < \infty,\ T_u^- < T_v^+\}}].$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

We also define the joint Laplace transform

$$B^{(q,s)}(x;\ u,v) := \mathbb{E}_x[\mathrm{e}^{-qT_v^+ - s(X_{T_v^+}-v)}\,\mathbf{1}_{\{T_v^+<\infty,\ T_v^+<T_u^-\}}] = B_1^{(q)}(x;\ u,v) + B_2^{(q,s)}(x;\ u,v), \tag{1}$$

where $B_2^{(q,s)}(x;\ u,v) := \int_0^\infty \mathrm{e}^{-sz} B_2^{(q)}(x,\mathrm{d}z;\ u,v)$.

The following pathwise inequalities are central to the construction of tight bounds for the joint law of the triplet $(\tau_a, M_{\tau_a}, Y_{\tau_a})$.

Proposition 1. *For $q,s \geq 0$, $x \in \mathbb{R}$, and $\varepsilon \in (0,a)$, we have $\mathbb{P}_x$-a.s.*

$$\mathbf{1}_{\{T_{x+\varepsilon}^+<\infty,\ T_{x+\varepsilon}^+<T_{x+\varepsilon-a}^-\}} \leq \mathbf{1}_{\{T_{x+\varepsilon}^+<\infty,\ T_{x+\varepsilon}^+<\tau_a\}} \leq \mathbf{1}_{\{T_{x+\varepsilon}^+<\infty,\ T_{x+\varepsilon}^+<T_{x-a}^-\}}, \tag{2}$$

$$\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{\tau_a<\infty,\ \tau_a<T_{x+\varepsilon}^+\}} \geq \mathrm{e}^{-qT_{x-a}^- - s(x-a-X_{T_{x-a}^-})-s\varepsilon}\,\mathbf{1}_{\{T_{x-a}^-<\infty,\ T_{x-a}^-<T_{x+\varepsilon}^+\}}, \tag{3}$$

$$\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{\tau_a<\infty,\ \tau_a<T_{x+\varepsilon}^+\}} \leq \mathrm{e}^{-qT_{x+\varepsilon-a}^- - s(x-a-X_{T_{x+\varepsilon-a}^-})}\,\mathbf{1}_{\{T_{x+\varepsilon-a}^-<\infty,\ T_{x+\varepsilon-a}^-<T_{x+\varepsilon}^+\}}. \tag{4}$$

Proof. By analyzing the sample paths of X, it is easy to see that, for any path $\omega \in (T_{x+\varepsilon}^+ < \infty)$, we have $\mathbb{P}_x\{\tau_a \leq T_{x-a}^-\} = 1$, so $\mathbb{P}_x$-a.s.

$$(T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < \tau_a) = (T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < \tau_a \leq T_{x-a}^-) \subset (T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < T_{x-a}^-)$$

and, similarly, $\mathbb{P}_x$-a.s.

$$\begin{aligned}(T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < T_{x+\varepsilon-a}^-) &= (T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < T_{x+\varepsilon-a}^-, T_{x+\varepsilon}^+ < \tau_a)\\ &\subset (T_{x+\varepsilon}^+ < \infty, T_{x+\varepsilon}^+ < \tau_a),\end{aligned}$$

which immediately implies (2). On the other hand, by using the same argument, we have $\mathbb{P}_x$-a.s.

$$(T_{x-a}^- < \infty, T_{x-a}^- < T_{x+\varepsilon}^+) = (T_{x-a}^- < \infty, \tau_a \leq T_{x-a}^- < T_{x+\varepsilon}^+) \subset (\tau_a < \infty, \tau_a < T_{x+\varepsilon}^+) \tag{5}$$

and

$$\begin{aligned}(\tau_a < \infty, \tau_a < T_{x+\varepsilon}^+) &= (\tau_a < \infty, T_{x+\varepsilon-a}^- \leq \tau_a < T_{x+\varepsilon}^+)\\ &\subset (T_{x+\varepsilon-a}^- < \infty, T_{x+\varepsilon-a}^- < T_{x+\varepsilon}^+).\end{aligned} \tag{6}$$

For any path $\omega \in (T_{x-a}^- < \infty, T_{x-a}^- < T_{x+\varepsilon}^+)$, we know from (5) that $\omega \in (T_{x-a}^- < \infty, \tau_a \leq T_{x-a}^- < T_{x+\varepsilon}^+)$. This implies $M_{\tau_a}(\omega) \leq x+\varepsilon$ and $X_{\tau_a}(\omega) \geq X_{T_{x-a}^-}(\omega)$, which further entails that $Y_{\tau_a}(\omega) = M_{\tau_a}(\omega) - X_{\tau_a}(\omega) \leq x + \varepsilon - X_{T_{x-a}^-}(\omega)$. Therefore, by the above analysis and the second inequality of (2),

$$\mathrm{e}^{-qT_{x-a}^- - s(x+\varepsilon-X_{T_{x-a}^-})}\,\mathbf{1}_{\{T_{x-a}^-<\infty,\ T_{x-a}^-<T_{x+\varepsilon}^+\}} \leq \mathrm{e}^{-q\tau_a - sY_{\tau_a}}\,\mathbf{1}_{\{\tau_a<\infty,\ \tau_a<T_{x+\varepsilon}^+\}}, \qquad \mathbb{P}_x\text{-a.s.},$$

which naturally leads to (3).

Similarly, for any sample path $\omega \in (\tau_a < \infty, \tau_a < T_{x+\varepsilon}^+)$, we know from (6) that $\omega \in (\tau_a < \infty, T_{x+\varepsilon-a}^- \leq \tau_a < T_{x+\varepsilon}^+)$, which implies that $x - X_{T_{x+\varepsilon-a}^-}(\omega) \leq Y_{T_{x+\varepsilon-a}^-}(\omega) \leq Y_{\tau_a}(\omega)$. Therefore, by the first inequality of (2), we obtain

$$\mathrm{e}^{-q\tau_a - sY_{\tau_a}}\,\mathbf{1}_{\{\tau_a<\infty,\ \tau_a<T_{x+\varepsilon}^+\}} \leq \mathrm{e}^{-qT_{x+\varepsilon-a}^- - s(x-X_{T_{x+\varepsilon-a}^-})}\,\mathbf{1}_{\{T_{x+\varepsilon-a}^-<\infty,\ T_{x+\varepsilon-a}^-<T_{x+\varepsilon}^+\}}, \qquad \mathbb{P}_x\text{-a.s.}$$

This implies the second inequality of (4). □

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

By Proposition 1, we easily obtain the following useful estimates.

Corollary 1. *For $q, s \geq 0$, $x \in \mathbb{R}$, $z > 0$, and $\varepsilon \in (0, a)$,*

$$\begin{aligned}
B_1^{(q)}(x;\, x+\varepsilon-a, x+\varepsilon) &\leq \mathbb{E}_x[\mathrm{e}^{-qT_{x+\varepsilon}^+}\mathbf{1}_{\{T_{x+\varepsilon}^+<\infty,\, T_{x+\varepsilon}^+<\tau_a,\, X_{T_{x+\varepsilon}^+}=x+\varepsilon\}}] \\
&\leq B_1^{(q)}(x;\, x-a, x+\varepsilon), \\
B_2^{(q)}(x, \mathrm{d}z;\, x+\varepsilon-a, x+\varepsilon) &\leq \mathbb{E}_x[\mathrm{e}^{-qT_{x+\varepsilon}^+}\mathbf{1}_{\{T_{x+\varepsilon}^+<\infty,\, T_{x+\varepsilon}^+<\tau_a,\, X_{T_{x+\varepsilon}^+}-x-\varepsilon\in\mathrm{d}z\}}] \\
&\leq B_2^{(q)}(x, \mathrm{d}z;\, x-a, x+\varepsilon), \\
\mathrm{e}^{-s\varepsilon}C^{(q,s)}(x;\, x-a, x+\varepsilon) &\leq \mathbb{E}_x[\mathrm{e}^{-q\tau_a-s(Y_{\tau_a}-a)}\mathbf{1}_{\{\tau_a<\infty,\, \tau_a<T_{x+\varepsilon}^+\}}] \\
&\leq \mathrm{e}^{s\varepsilon}C^{(q,s)}(x;\, x+\varepsilon-a, x+\varepsilon).
\end{aligned}$$

Remark 1. It is not difficult to check that the results of Proposition 1 and Corollary 1 still hold if the first passage times and the drawdown times are only observed discretely or randomly (such as the Poisson observation framework in [1] for the latter). Further, explicit relationship between Poisson observed first passage times and Poisson observed drawdown times (similar as for Theorem 1 below) can be found by exploiting the same approach as laid out in this paper.

The later analysis involves the weak convergence of measures which is recalled here. Consider a metric space S with the Borel σ-algebra on it. We say a sequence of finite measures $\{\mu_n\}_{n\in\mathbb{N}}$ is weakly convergent to a finite measure μ as $n \to \infty$ if

$$\lim_{n\to\infty}\int_S \phi(z)\,\mathrm{d}\mu_n(z) = \int_S \phi(z)\,\mathrm{d}\mu(z)$$

for any bounded and continuous function $\phi(\cdot)$ on S.

In the next lemma, we show some forms of Fatou's lemma for varying measures under weak convergence. Similar results were proved in [10] for probability measures. For completeness, a proof for general finite measures is provided in Appendix A.

Lemma 1. *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of finite measures on S which is weakly convergent to a finite measure μ, and $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly bounded and nonnegative functions on S. Then*

$$\int_S \liminf_{n\to\infty, w\to z}\phi_n(w)\,\mathrm{d}\mu(z) \leq \liminf_{n\to\infty}\int_S \phi_n(z)\,\mathrm{d}\mu_n(z), \tag{7}$$

and

$$\int_S \limsup_{n\to\infty, w\to z}\phi_n(w)\,\mathrm{d}\mu(z) \geq \limsup_{n\to\infty}\int_S \phi_n(z)\,\mathrm{d}\mu_n(z). \tag{8}$$

3. Main results

In this section we study the joint law of $(\tau_a, M_{\tau_a}, Y_{\tau_a})$ for a general Markov process with possibly two-sided jumps. The following assumptions on the two-sided exit quantities of X are assumed to hold, which are sufficient (but not necessary) conditions for the applicability of our proposed methodology. Weaker assumptions might be assumed for special Markov processes; see, for instance, Remark 4 and Corollary 2 below.

Assumption 1. *For all $q, s \geq 0$, $z > 0$, and $x > X_0$, we assume the following limits exist and identities hold:*

(A1) $$b_{a,1}^{(q)}(x) := \lim_{\varepsilon \downarrow 0} \frac{1 - B_1^{(q)}(x;\ x - a, x + \varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1 - B_1^{(q)}(x;\ x + \varepsilon - a, x + \varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1 - B_1^{(q)}(x - \varepsilon;\ x - a, x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1 - B_1^{(q)}(x - \varepsilon;\ x - \varepsilon - a, x)}{\varepsilon},$$

and $\int_x^y b_{a,1}^{(q)}(w)\, \mathrm{d}w < \infty$ for any $x, y \in \mathbb{R}$;

(A2) $$b_{a,2}^{(q,s)}(x) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} B_2^{(q,s)}(x;\ x - a, x + \varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} B_2^{(q,s)}(x;\ x + \varepsilon - a, x + \varepsilon) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} B_2^{(q,s)}(x - \varepsilon;\ x - a, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} B_2^{(q,s)}(x - \varepsilon;\ x - \varepsilon - a, x),$$

and $s \mapsto b_{a,2}^{(q,s)}(x)$ is right continuous at $s = 0$;

(A3) $$c_a^{(q,s)}(x) := \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x;\ x - a, x + \varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x;\ x + \varepsilon - a, x + \varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x - \varepsilon;\ x - a, x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{C^{(q,s)}(x - \varepsilon;\ x - \varepsilon - a, x)}{\varepsilon}.$$

Under assumptions (A1) and (A2), it follows from (1) that

$$b_a^{(q,s)}(x) := \lim_{\varepsilon \downarrow 0} \frac{1 - B^{(q,s)}(x;\ x - a, x + \varepsilon)}{\varepsilon} = b_{a,1}^{(q)}(x) - b_{a,2}^{(q,s)}(x). \tag{9}$$

Remark 2. Due to the general structure of X, it is difficult to refine assumptions (A1)–(A3) unless a specific structure for X is given. A necessary condition for assumptions (A1)–(A3) to hold is that

$$T_x^+ = 0 \quad \text{and} \quad X_{T_x^+} = x, \quad \mathbb{P}_x\text{-a.s. for all } x \in \mathbb{R}.$$

In other words, X must be upward regular and creeping upward at every x; see [19, p. 142 and p. 197] for definitions of regularity and creeping for Lévy processes.

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

In the latter part of this section, we provide some examples of Markov processes which satisfy assumptions (A1)–(A3), including spectrally negative Lévy processes, linear diffusions, piecewise exponential Markov processes, and jump diffusions.

Remark 3. By Theorem 5.22 of [16] or Proposition 7.1 of [23], we know that assumption (A2) implies that the measures $(1/\varepsilon)B_2^{(q)}(x,\ \mathrm{d}z;\ x-a, x+\varepsilon)$, $(1/\varepsilon)B_2^{(q)}(x,\ \mathrm{d}z;\ x+\varepsilon-a, x+\varepsilon)$, $(1/\varepsilon)B_2^{(q)}(x-\varepsilon,\ \mathrm{d}z;\ x-a, x)$, and $(1/\varepsilon)B_2^{(q)}(x-\varepsilon, \mathrm{d}z;\ x-\varepsilon-a, x)$ weakly converge to the same measure on $\mathbb{R}_+$, denoted as $b_{a,2}^{(q)}(x,\ \mathrm{d}z)$, such that $\int_{\mathbb{R}_+}\mathrm{e}^{-sz}b_{a,2}^{(q)}(x,\ \mathrm{d}z) = b_{a,2}^{(q,s)}(x)$. We point out that it is possible that $b_{a,2}^{(q)}(x, \{0\}) > 0$, though the measure $B_2^{(q)}(x, \mathrm{d}z;\ u, v)$ is only defined on $z \in (0, \infty)$.

We are now ready to present the main result of this paper related to the joint law of $(\tau_a, Y_{\tau_a}, M_{\tau_a})$.

Theorem 1. *Consider a general time-homogeneous Markov process X satisfying assumptions (A1)–(A3). For $q, s \geq 0$ and $K \in \mathbb{R}$, let*

$$h(x) = \mathbb{E}_x[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{\tau_a<\infty,\, M_{\tau_a}\leq K\}}], \qquad x \leq K.$$

Then $h(\cdot)$ is differentiable in $x < K$ and solves the following integral equation:

$$h(x) = \int_x^K \exp\Big(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\Big) \times \Big(c_a^{(q,s)}(y) + \int_{[0,K-y)} h(y+z)b_{a,2}^{(q)}(y,\ \mathrm{d}z)\Big)\,\mathrm{d}y, \qquad x \leq K. \tag{10}$$

Proof. By the strong Markov property of X, for any $X_0 = x \leq y < K$ and $0 < \varepsilon < (K-y)\wedge a$, we have

$$\begin{aligned} h(y) &= \mathbb{E}_y[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{\tau_a<\infty,\, \tau_a<T_{y+\varepsilon}^+\}}] \\ &\quad + \mathbb{E}_y[\mathrm{e}^{-qT_{y+\varepsilon}^+}\,\mathbf{1}_{\{T_{y+\varepsilon}^+<\infty,\, T_{y+\varepsilon}^+<\tau_a,\, X_{T_{y+\varepsilon}^+}=y+\varepsilon\}}]h(y+\varepsilon) \\ &\quad + \int_0^{K-y-\varepsilon} \mathbb{E}_y[\mathrm{e}^{-qT_{y+\varepsilon}^+}\,\mathbf{1}_{\{T_{y+\varepsilon}^+<\infty,\, T_{y+\varepsilon}^+<\tau_a,\, X_{T_{y+\varepsilon}^+}-y-\varepsilon\in\mathrm{d}z\}}]h(y+\varepsilon+z). \end{aligned}$$

By Corollary 1, it follows that

$$\begin{aligned} h(y+\varepsilon) - h(y) \geq &-\mathrm{e}^{s\varepsilon}C^{(q,s)}(y;\ y+\varepsilon-a, y+\varepsilon) \\ &+ (1 - B_1^{(q)}(y;\ y-a, y+\varepsilon))h(y+\varepsilon) \\ &- \int_0^{K-y-\varepsilon} h(y+\varepsilon+z)B_2^{(q)}(y, \mathrm{d}z;\ y-a, y+\varepsilon), \end{aligned} \tag{11}$$

and

$$\begin{aligned} h(y+\varepsilon) - h(y) \leq &-\mathrm{e}^{-s\varepsilon}C^{(q,s)}(y;\ y-a, y+\varepsilon) \\ &+ (1 - B_1^{(q)}(y;\ y+\varepsilon-a, y+\varepsilon))h(y+\varepsilon) \\ &- \int_0^{K-y-\varepsilon} h(y+\varepsilon+z)B_2^{(q)}(y, \mathrm{d}z;\ y+\varepsilon-a, y+\varepsilon). \end{aligned} \tag{12}$$

By assumptions (A1)–(A3) and $h(\cdot) \in [0, 1]$, it is clear that both the lower bound of $h(y+\varepsilon) - h(y)$ in (11) and the upper bound in (12) vanish as $\varepsilon \downarrow 0$. Hence, $h(y)$ is right continuous for $y \in [x, K)$. Replacing y by $y - \varepsilon$ in (11) and (12), and using assumptions (A1)–(A3) again, it follows that $h(y)$ is also left continuous for $y \in (x, K]$ with $h(K) = 0$. Therefore, $h(y)$ is continuous for $y \in [x, K]$ (left continuous at x and right continuous at K).

To consecutively show the differentiability, we divide inequalities (11) and (12) by ε. It follows from assumptions (A1)–(A3), Remark 3, Lemma 1, and the continuity of h that

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0} & \frac{h(y+\varepsilon) - h(y)}{\varepsilon} \\ &\geq -c_a^{(q,s)}(y) + b_{a,1}^{(q)}(y)h(y) - \limsup_{\varepsilon\downarrow 0} \int_0^{K-y-\varepsilon} h(y+\varepsilon+z) \frac{B_2^{(q)}(y, \mathrm{d}z;\ y-a, y+\varepsilon)}{\varepsilon} \\ &\geq -c_a^{(q,s)}(y) + b_{a,1}^{(q)}(y)h(y) - \int_{[0,K-y)} h(y+z) b_{a,2}^{(q)}(y,\ \mathrm{d}z),\end{aligned}$$

and, similarly,

$$\limsup_{\varepsilon\downarrow 0} \frac{h(y+\varepsilon) - h(y)}{\varepsilon} \leq -c_a^{(q,s)}(y) + b_{a,1}^{(q)}(y)h(y) - \int_{[0,K-y)} h(y+z) b_{a,2}^{(q)}(y,\ \mathrm{d}z).$$

Since the two limits coincide, one concludes that $h(y)$ is right-differentiable for $y \in (x, K)$. Moreover, by replacing y by $y - \varepsilon$ in (11) and (12), and using similar arguments, we can show that $h(y)$ is also left differentiable for $y \in (x, K)$. Since the left and right derivatives coincide, we conclude that $h(y)$ is differentiable for any $y \in (x, K)$ and solves the following ordinary integro-differential equation:

$$h'(y) - b_{a,1}^{(q)}(y)h(y) = -c_a^{(q,s)}(y) - \int_{[0,K-y)} h(y+z) b_{a,2}^{(q)}(y,\ \mathrm{d}z). \tag{13}$$

Multiplying both sides of (13) by $\exp(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w)$, integrating the resulting equation (with respect to y) from x to K, and using $h(K) = 0$ complete the proof of Theorem 1. □

When the Markov process X is spectrally negative (i.e. with no upward jumps), the upward overshooting density $b_{a,2}^{(q)}(x,\ \mathrm{d}z)$ is trivially 0. Theorem 1 reduces to the following corollary.

Corollary 2. *Consider a spectrally negative time-homogeneous Markov process X satisfying assumptions (A1) and (A3). For $q, s \geq 0$ and $K > 0$, we have*

$$\begin{aligned}\mathbb{E}_x&[\exp(-q\tau_a - s(Y_{\tau_a} - a))\,\mathbf{1}_{\{\tau_a<\infty,\, M_{\tau_a}\leq K\}}] \\ &= \int_x^K \exp\biggl(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\biggr) c_a^{(q,s)}(y)\,\mathrm{d}y, \qquad x \leq K.\end{aligned}$$

When X is a general Lévy process (possibly with two-sided jumps), we have the following result for the joint Laplace transform of the triplet $(\tau_a, Y_{\tau_a}, M_{\tau_a})$. Note that Corollary 3 should be compared to Theorem 4.1 of [5], in which, under the Lévy framework, the resolvent density of Y is expressed in terms of the resolvent density of X using excursion theory.

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

Corollary 3. *Consider a Lévy process X satisfying assumptions (A1)–(A3). For $q, s, \delta \geq 0$, we have (for Lévy processes $\mathbb{P}\{\tau_a < \infty\} = 1$ as long as X is not monotone)*

$$\mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)-\delta M_{\tau_a}}] = \frac{c_a^{(q,s)}(0)}{\delta + b_a^{(q,\delta)}(0)}. \tag{14}$$

Proof. By the spatial homogeneity of the Lévy process X, (10) at $x = 0$ reduces to

$$h(0) = \frac{c_a^{(q,s)}(0)}{b_{a,1}^{(q)}(0)}(1 - \mathrm{e}^{-b_{a,1}^{(q)}(0)K}) + \int_0^K \mathrm{e}^{-b_{a,1}^{(q)}(0)y} \int_{[0,K-y)} h(y+z) b_{a,2}^{(q)}(0,\, \mathrm{d}z)\,\mathrm{d}y. \tag{15}$$

Let

$$\hat{h}(0) := \mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)-\delta M_{\tau_a}}] = \mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{M_{\tau_a}\leq e_\delta\}}],$$

where e_δ is an independent exponential random variable with finite mean $1/\delta > 0$. Multiplying both sides of (15) by $\delta\mathrm{e}^{-\delta K}$, integrating the resulting equation (with respect to K) from 0 to ∞, and using integration by parts, we obtain

$$\begin{aligned}
\hat{h}(0) &= \frac{c_a^{(q,s)}(0)}{\delta + b_{a,1}^{(q)}(0)} + \int_0^\infty \delta\mathrm{e}^{-\delta K} \int_0^K \mathrm{e}^{-b_{a,1}^{(q)}(0)y} \int_{[0,K-y)} h(y+z) b_{a,2}^{(q)}(0,\, \mathrm{d}z)\,\mathrm{d}y\,\mathrm{d}K \\
&= \frac{c_a^{(q,s)}(0)}{\delta + b_{a,1}^{(q)}(0)} \\
&\quad + \int_0^\infty \mathrm{e}^{-b_{a,1}^{(q)}(0)y}\,\mathrm{d}y \int_{\mathbb{R}_+} b_{a,2}^{(q)}(0,\, \mathrm{d}z) \int_{z+y}^\infty \delta\mathrm{e}^{-\delta K}\mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{M_{\tau_a}\leq K-y-z\}}]\,\mathrm{d}K \\
&= \frac{c_a^{(q,s)}(0)}{\delta + b_{a,1}^{(q)}(0)} + \hat{h}(0)\frac{\int_{\mathbb{R}_+} \mathrm{e}^{-\delta z} b_{a,2}^{(q)}(0,\, \mathrm{d}z)}{\delta + b_{a,1}^{(q)}(0)}.
\end{aligned}$$

Solving for $\hat{h}(0)$ and using (9), it follows that

$$\hat{h}(0) = \frac{c_a^{(q,s)}(0)}{\delta + b_{a,1}^{(q)}(0) - \int_{\mathbb{R}_+} \mathrm{e}^{-\delta z} b_{a,2}^{(q)}(0,\, \mathrm{d}z)} = \frac{c_a^{(q,s)}(0)}{\delta + b_a^{(q,\delta)}(0)}.$$

It follows from the monotone convergence theorem that (14) also holds for $\delta = 0$. □

Remark 4. We point out that assumptions (A1)–(A3) are not necessary to yield (14) in the Lévy framework. In fact, by the spatial homogeneity of X, similar to (11) and (12), we have

$$\frac{\mathrm{e}^{-(s+\delta)\varepsilon} C^{(q,s)}(0;\ -a, \varepsilon)}{1 - \mathrm{e}^{-\delta\varepsilon} B^{(q,\delta)}(0;\ \varepsilon - a, \varepsilon)} \leq \mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)-\delta M_{\tau_a}}] \leq \frac{\mathrm{e}^{s\varepsilon} C^{(q,s)}(0;\ \varepsilon - a, \varepsilon)}{1 - \mathrm{e}^{-\delta\varepsilon} B^{(q,\delta)}(0;\ -a, \varepsilon)}$$

for any $\varepsilon \in (0, a)$. Suppose that the following condition holds:

$$\lim_{\varepsilon\downarrow 0} \frac{C^{(q,s)}(0;\ -a, \varepsilon)}{1 - \mathrm{e}^{-\delta\varepsilon} B^{(q,\delta)}(0;\ \varepsilon - a, \varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{C^{(q,s)}(0;\ \varepsilon - a, \varepsilon)}{1 - \mathrm{e}^{-\delta\varepsilon} B^{(q,\delta)}(0;\ -a, \varepsilon)} := D_a^{(q,s,\delta)}.$$

Then $\mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)-\delta M_{\tau_a}}] = D_a^{(q,s,\delta)}$.

Theorem 1 shows that the joint law $\mathbb{E}_x[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\,\mathbf{1}_{\{M_{\tau_a}\leq K\}}]$ is a solution to (10). Further, the following theorem shows that (10) admits a unique solution.

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

Theorem 2. *Suppose that assumptions (A1)–(A3) hold. For* $q, s \geq 0$ *and* $K > 0$, *(10) admits a unique solution.*

Proof. From Theorem 1, we know that $h(x) := \mathbb{E}_x[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a}-a)}\mathbf{1}_{\{\tau_a<\infty,\, M_{\tau_a}\leq K\}}]$ is a solution of (10). We also note that any continuous solution to (10) must vanish when $x \uparrow K$. For any fixed $L \in (-\infty, K)$, we define a metric space $(\mathbb{A}_L, \boldsymbol{d}_L)$, where $\mathbb{A}_L = \{f \in C[L, K],\ f(K) = 0\}$ and the metric $\boldsymbol{d}_L(f, g) = \sup_{x\in[L,K]} |f(x) - g(x)|$ for $f, g \in \mathbb{A}_L$. We then define a mapping $\mathcal{L}$ on $\mathbb{A}_L$ by

$$\begin{aligned}\mathcal{L}f(x) = \int_x^K &\exp\bigg(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg)\\ &\times\bigg(c_a^{(q,s)}(y) + \int_{[0,K-y)} f(y+z)b_{a,2}^{(q)}(y,\,\mathrm{d}z)\bigg)\,\mathrm{d}y, \qquad x \in [L, K],\end{aligned}$$

where $f \in \mathbb{A}_L$. It is clear that $\mathcal{L}(\mathbb{A}_L) \subset \mathbb{A}_L$.

Next we show that $\mathcal{L}\colon \mathbb{A}_L \to \mathbb{A}_L$ is a contraction mapping. By the definitions of the two-sided exit quantities, for any $y \in \mathbb{R}$, it follows that

$$C^{(q,s)}(y;\ y-a, y+\varepsilon) + \int_{\mathbb{R}_+} B_2^{(q)}(y, \mathrm{d}z;\ y-a, y+\varepsilon) \leq 1 - B_1^{(q)}(y;\ y-a, y+\varepsilon). \tag{16}$$

Dividing each term in (16) by $\varepsilon \in (0, a)$ and letting $\varepsilon \downarrow 0$, it follows from assumptions (A1)–(A3) that

$$0 \leq c_a^{(q,s)}(y) + \int_{\mathbb{R}_+} b_{a,2}^{(q)}(y,\,\mathrm{d}z) \leq b_{a,1}^{(q)}(y), \qquad y \in \mathbb{R}. \tag{17}$$

By (17), we have, for any $f, g \in \mathbb{A}_L$,

$$\begin{aligned}\boldsymbol{d}_L(\mathcal{L}f, \mathcal{L}g) &\leq \sup_{t\in[L,K]} |f(t) - g(t)| \sup_{x\in[L,K]} \int_x^K \exp\bigg(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg)\int_{\mathbb{R}_+} b_{a,2}^{(q)}(y,\,\mathrm{d}z)\,\mathrm{d}y\\ &\leq \boldsymbol{d}_L(f, g) \sup_{L\leq x\leq K} \int_x^K \exp\bigg(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg)b_{a,1}^{(q)}(y)\,\mathrm{d}y\\ &\leq \boldsymbol{d}_L(f, g)\bigg(1 - \exp\bigg(-\int_L^K b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg)\bigg).\end{aligned}$$

Since $\int_L^K b_{a,1}^{(q)}(w)\,\mathrm{d}w < \infty$ by assumption (A1), we conclude that $\mathcal{L}\colon \mathbb{A}_L \to \mathbb{A}_L$ is a contraction mapping. By the Banach fixed point theorem, there exists a unique fixed point in $\mathbb{A}_L$. By a restriction of domain, it is easy to see that $\mathbb{A}_{L_1} \subset \mathbb{A}_{L_2}$ for $-\infty < L_1 < L_2 < K$. By the arbitrariness of L, the uniqueness holds for the space $\cap_{L<K}\mathbb{A}_L$. This completes the proof. □

For the remainder of this section, we state several examples of Markov processes satisfying assumptions (A1)–(A3). Note that the joint law of drawdown estimates for Examples 1 and 3 were solved by Mijatovic and Pistorius [28] and Lehoczky [24], respectively (using different approaches). Assumption verifications for Examples 4 and 5 are postponed to Appendices B and C, respectively.

Example 1. (*Spectrally negative Lévy processes.*) Consider a spectrally negative Lévy process X. Let $\psi(s) := (1/t)\log\mathbb{E}[\mathrm{e}^{sX_t}](s \geq 0)$ be the Laplace exponent of X. Further, let $W^{(q)}\colon \mathbb{R} \to [0, \infty)$ be the well-known q-scale function of X; see, for instance, [19, Chapter 8].

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

The second scale function is defined as $Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y$. Under some mild conditions (see, e.g. Lemma 2.4 of [18]), the scale functions are continuously differentiable which further implies that assumptions (A1) and (A3) hold with

$$b_{a,1}^{(q)}(0) = \frac{W^{(q)'}(a)}{W^{(q)}(a)} \quad \text{and} \quad c_a^{(q,s)}(0) = \mathrm{e}^{sa}\frac{Z_s^{(p)}(a)W_s^{(p)'}(a) - Z_s^{(p)'}(a)W_s^{(p)}(a)}{W_s^{(p)}(a)}, \tag{18}$$

where $p = q - \psi(s)$ and $W_s^{(p)}$ ($Z_s^{(p)}$) is the (second) scale function of X under a new probability measure $\mathbb{P}^s$ defined by the Radon–Nikodym derivative process $\mathrm{d}\mathbb{P}^s/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t} = \mathrm{e}^{sX_t - \psi(s)t}$ for $t \geq 0$. Therefore, by Corollary 3 and (18), we have

$$\mathbb{E}[\mathrm{e}^{-q\tau_a - s(Y_{\tau_a} - a) - \delta M_{\tau_a}}] = \frac{\mathrm{e}^{sa}W^{(q)}(a)}{\delta W^{(q)}(a) + W^{(q)'}(a)}\,\frac{Z_s^{(p)}(a)W_s^{(p)'}(a) - pW_s^{(p)}(a)^2}{W_s^{(p)}(a)},$$

which is consistent with Theorem 3.1 of [23] and Theorem 1 of [28].

Example 2. (*Refracted Lévy processes.*) Consider a refracted spectrally negative Lévy process X of the form

$$X_t = U_t - \lambda\int_0^t \mathbf{1}_{\{X_s > b\}}\,\mathrm{d}s, \tag{19}$$

where $\lambda \geq 0$, $b > 0$, and U is a spectrally negative Lévy process; see [20]. Let $W^{(q)}$ ($Z^{(q)}$) be the (second) q-scale function of U, and $\mathbb{W}^{(q)}$ be the q-scale function of the process $\{U_t - \lambda t\}_{t\geq 0}$. Similar to Example 1, all the scale functions are continuously differentiable under mild conditions.

For simplicity, we only consider the quantity $\mathbb{E}_x[\mathrm{e}^{-q\tau_a}\mathbf{1}_{\{\tau_a<\infty,\, M_{\tau_a}\leq K\}}]$ with $b > x - a$ (otherwise the problem reduces to Example 1 for $X_t = U_t - \lambda t$). By Theorem 4 of [20], we can verify that assumptions (A1) and (A3) hold. For $b > x$, from (18) with $s = 0$, we have

$$b_{a,1}^{(q)}(x) = \frac{W^{(q)'}(a)}{W^{(q)}(a)} \quad \text{and} \quad c_a^{(q,0)}(x) = \frac{Z^{(q)}(a)W^{(q)'}(a) - Z^{(q)'}(a)W^{(q)}(a)}{W^{(q)}(a)}.$$

For $x > b > x - a$,

$$b_{a,1}^{(q)}(x) = \frac{(1 + \lambda\mathbb{W}^{(q)}(0))W^{(q)'}(a) + \lambda\int_{b-x+a}^a \mathbb{W}^{(q)'}(a-y)W^{(q)'}(y)\,\mathrm{d}y}{W^{(q)}(a) + \lambda\int_{b-x+a}^a \mathbb{W}^{(q)}(a-y)W^{(q)'}(y)\,\mathrm{d}y}$$

and

$$c_a^{(q,0)}(x) = \frac{k_a^{(q)}(x)}{W^{(q)}(a) + \lambda\int_{b-x+a}^a \mathbb{W}^{(q)}(a-y)W^{(q)'}(y)\,\mathrm{d}y},$$

where

$$\begin{aligned} k_a^{(q)}(x) &= (1 + \lambda\mathbb{W}^{(q)}(0))(Z^{(q)}(a)W^{(q)'}(a) - qW^{(q)}(a)^2) \\ &\quad + \lambda q(1 + \lambda\mathbb{W}^{(q)}(0)) \\ &\qquad \times \int_{b-x+a}^a \mathbb{W}^{(q)}(a-y)(W^{(q)'}(a)W^{(q)}(y) - W^{(q)}(a)W^{(q)'}(y))\,\mathrm{d}y \end{aligned}$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

$$
\begin{aligned}
&- \lambda q\Big[W^{(q)}(a) \\
&\qquad + \lambda \int_{b-x+a}^{a} \mathbb{W}^{(q)}(a-y) W^{(q)\prime}(y)\,\mathrm{d}y\Big] \int_{b-x+a}^{a} \mathbb{W}^{(q)\prime}(a-y) W^{(q)}(y)\,\mathrm{d}y \\
&+ \lambda\Big[Z^{(q)}(a) \\
&\qquad + \lambda q \int_{b-x+a}^{a} \mathbb{W}^{(q)}(a-y) W^{(q)}(y)\,\mathrm{d}y\Big] \int_{b-x+a}^{a} \mathbb{W}^{(q)\prime}(a-y) W^{(q)\prime}(y)\,\mathrm{d}y.
\end{aligned}
$$

By Corollary 2, we obtain

$$
\mathbb{E}_x[\exp(-q\tau_a)\,\mathbf{1}_{\{M_{\tau_a}\le K\}}] = \int_x^K \exp\bigg(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg) c_a^{(q,0)}(y)\,\mathrm{d}y, \qquad x \le K,
$$

which is a new result for the refracted Lévy process (19).

Example 3. (*Linear diffusion processes.*) Consider a linear diffusion process X of the form

$$
\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t,
$$

where $(W_t)_{t\ge 0}$ is a standard Brownian motion, and the drift term $\mu(\cdot)$ and local volatility $\sigma(\cdot) > 0$ satisfy the usual Lipschitz continuity and linear growth conditions. As a special case of the jump diffusion process of Example 5, it will be shown later that assumptions (A1) and (A3) hold for linear diffusion processes. By Corollary 2, we obtain

$$
\mathbb{E}_x[\exp(-q\tau_a)\,\mathbf{1}_{\{\tau_a<\infty,\, M_{\tau_a}\le K\}}] = \int_x^K \exp\bigg(-\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w\bigg) c_a^{(q,0)}(y)\,\mathrm{d}y, \qquad x \le K,
$$

which is consistent with Equation (4) of [24].

Example 4. (*Piecewise exponential Markov processes.*) Consider a piecewise exponential Markov process (PEMP) X of the form

$$
\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \mathrm{d}Z_t, \tag{20}
$$

where $\mu > 0$ is the drift coefficient and $Z = (Z_t)_{t\ge 0}$ is a compound Poisson process given by $Z_t = \sum_{i=1}^{N_t} J_i$. Here, $(N_t)_{t\ge 0}$ is a Poisson process with intensity $\lambda > 0$ and the J_i are independent and identically distributed copies of a real-valued random variable J with cumulative distribution function F. We also assume the initial value $X_0 \ge a$ which ensures that $X_t \ge 0$ for all $t < \tau_a$. In this case, as discussed in Remark 2, X is upward regular and creeps upward before τ_a. The first passage times of X have been extensively studied in area of applied probability; see, e.g. [17] and [37]. For the PEMP (20), semiexplicit expressions for the two-sided exit quantities $B_1^{(q)}(\cdot)$, $B_2^{(q)}(\cdot,\cdot)$, and $C^{(q,s)}(\cdot)$ are given in [15, Section 6]. As will be shown in Appendix B, assumptions (A1)–(A3) and Theorem 1 hold for the PEMP X with a continuous jump size distribution F.

Example 5. (*Jump diffusion.*) Consider a jump diffusion process X of the form

$$
\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t + \int_{-\infty}^{\infty} \gamma(X_{t-}, z) N(\mathrm{d}t,\, \mathrm{d}z), \tag{21}
$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

where $\mu(\cdot)$ and $\sigma(\cdot) > 0$ are functions on $\mathbb{R}$, $(W_t)_{t\geq 0}$ is a standard Brownian motion, $\gamma(\cdot,\cdot)$ is a real-valued function on $\mathbb{R}^2$ modeling the jump size, and $N(\mathrm{d}t,\ \mathrm{d}z)$ is an independent Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ with a finite intensity measure $\mathrm{d}t\times\nu(\mathrm{d}z)$. For specific $\mu(\cdot)$ and $\sigma(\cdot)$, the jump diffusion (21) can be used to model the surplus process of an insurer with investment in risky assets; see, e.g. [12] and [39]. We assume the same conditions as Theorem 1.19 of [29] so that (21) admits a unique càdlàg adapted solution. Under this setup, we show in Appendix C that assumptions (A1)–(A3) and, thus, Theorem 1 hold for the jump diffusion (21).

4. Numerical examples

The main results of Section 3 rely on the analytic tractability of the two-sided exit quantities. To further illustrate their applicability, we now consider the numerical evaluation of the joint law of (Y_{τ_a}, M_{τ_a}) for two particular spatial-inhomogeneous Markov processes with (positive) jumps through Theorem 1. For simplicity, we assume that the discount rate $q=0$ throughout this section.

4.1. PEMP

In this section, we consider the PEMP X in Example 4 with $\mu=1$, $\lambda=3$, and the generic jump size J with density

$$p(x)=\begin{cases}\frac{1}{3}\mathrm{e}^{-x}, & x>0,\\ \frac{1}{3}(\mathrm{e}^{x}+2\mathrm{e}^{2x}), & x<0.\end{cases} \tag{22}$$

We follow Jacobsen and Jensen [15, Section 6] to first solve for the two-sided exit quantities. Define the integral kernel

$$\psi_0(z):=\frac{1}{z(z+1)(z-1)(z-2)},\qquad z\in\mathbb{C},$$

and the linearly independent functions

$$\begin{aligned}
g_1(x)&:=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_1}\psi_0(z)\mathrm{e}^{-xz}\,\mathrm{d}z=\frac{1}{6}\mathrm{e}^{-2x},\\
g_2(x)&:=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_2}\psi_0(z)\mathrm{e}^{-xz}\,\mathrm{d}z=-\frac{1}{2}\mathrm{e}^{-x},\\
g_3(x)&:=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_3}\psi_0(z)\mathrm{e}^{-xz}\,\mathrm{d}z=\frac{1}{2},\\
g_4(x)&:=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_4}\psi_0(z)\mathrm{e}^{-xz}\,\mathrm{d}z=-\frac{1}{6}\mathrm{e}^{x},
\end{aligned}$$

for $x>0$, where Γ_i $(i=1,2,3,4)$ is a small counterclockwise circle centered at the pole $\mu_i=3-i$ of $\psi_0(z)$. Moreover, for $0<u<v$, we consider the matrix-valued function

$$(M_{i,k}(u,v))_{1\leq i,\,k\leq 4}:=\begin{pmatrix}-\frac{1}{3}\mathrm{e}^{-2u}\big(u+\frac{11}{6}\big) & \frac{1}{6}\mathrm{e}^{-2u} & \frac{1}{18}\mathrm{e}^{-2v} & g_1(v)\\ \mathrm{e}^{-u} & \frac{1}{2}\mathrm{e}^{-u}\big(u+\frac{1}{2}\big) & -\frac{1}{4}\mathrm{e}^{-v} & g_2(v)\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & g_3(v)\\ \frac{1}{9}\mathrm{e}^{u} & \frac{1}{12}\mathrm{e}^{u} & \frac{1}{6}\mathrm{e}^{v}\big(v-\frac{11}{6}\big) & g_4(v)\end{pmatrix},$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

where the matrix M entries are chosen according to

$$M_{i,k}(u,v) = \frac{\mu_k}{2\pi\sqrt{-1}} \int_{\Gamma_i} \frac{\psi_0(z)}{z-\mu_k} \mathrm{e}^{-uz}\,\mathrm{d}z, \qquad 1 \le i \le 4,\ k = 1, 2,$$

$$M_{i,3}(u,v) = \frac{|\mu_4|}{2\pi\sqrt{-1}} \int_{\Gamma_i} \frac{\psi_0(z)}{z-\mu_4} \mathrm{e}^{-vz}\,\mathrm{d}z, \qquad 1 \le i \le 4.$$

Let $(N_{k,j}(u,v))_{1\le k,\,j\le 4}$ be the inverse of $(M_{i,k}(u,v))_{1\le i,\,k\le 4}$. Combining Equation (46) and a generalized Equation (48) of [15] (with $\zeta = s \ge 0$ and $\rho \ge 0$), we obtain the linear system of equations

$$(c_1, c_2, c_3, c_4)(M_{i,k}) = \bigg(-\frac{2\underline{C}}{s+2}, -\frac{\underline{C}}{s+1}, \frac{\overline{C}}{\rho+1}, f(v)\bigg), \tag{23}$$

where $\underline{C}$ and $\overline{C}$ are constants specified later, and $f(x)$ could stand for any of $B_1^{(0)}(x;\ u,v)$, $B_2^{(0,\rho)}(x;\ u,v)$, or $C^{(0,s)}(x;\ u,v)$ and has the representation $f(x) = \sum_{i=1}^{4} c_i g_i(x),\ x \in [u,v]$.

To solve for $B_1^{(0)}(x;\ u,v)$, $B_2^{(0,\rho)}(x;\ u,v)$, or $C^{(0,s)}(x;\ u,v)$, we only need to solve (23) with different assigned values of $\underline{C}$, $\overline{C}$, and $f(v)$ according to Equation (45) of [15]. By letting $\underline{C} = \overline{C} = 0$ and $f(v) = 1$, we obtain

$$B_1^{(0)}(x;\ u,v) = \sum_{i=1}^{4} N_{4,i}(u,v) g_i(x).$$

Similarly, by letting $\underline{C} = f(v) = 0$ and $\overline{C} = 1$, for $\rho \ge 0$, we obtain

$$B_2^{(0,\rho)}(x;\ u,v) = \frac{1}{1+\rho} \sum_{i=1}^{4} N_{3,i}(u,v) g_i(x).$$

A Laplace inversion with respect to ρ yields, for $z > 0$,

$$B_2^{(0)}(x, \mathrm{d}z;\ u,v) = \mathrm{e}^{-z} \sum_{i=1}^{4} N_{3,i}(u,v) g_i(x)\,\mathrm{d}z.$$

By letting $\underline{C} = 1$ and $\overline{C} = f(v) = 0$, for $s \ge 0$, we obtain

$$C^{(0,s)}(x;\ u,v) = \sum_{i=1}^{4} \bigg(\frac{-2}{s+2} N_{1,i}(u,v) + \frac{-1}{s+1} N_{2,i}(u,v)\bigg) g_i(x).$$

By the definitions, we have

$$b_{a,1}^{(0)}(x) = -\sum_{i=1}^{4} D_{4,i}(x-a,x) g_i(x), \qquad b_{a,2}^{(0)}(x,\ \mathrm{d}z) = \mathrm{e}^{-z} \bigg(\sum_{i=1}^{4} D_{3,i}(x-a,x) g_i(x)\bigg)\,\mathrm{d}z,$$

$$c_a^{(0,s)}(x) = \sum_{i=1}^{4} \bigg(\frac{-2}{s+2} D_{1,i}(x-a,x) + \frac{-1}{s+1} D_{2,i}(x-a,x)\bigg) g_i(x),$$

where we denote $D_{k,j}(u,v) := (\partial/\partial v) N_{k,j}(u,v)$.

In Figure 1, we use MATHEMATICA® to numerically solve the integral equation (10).

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

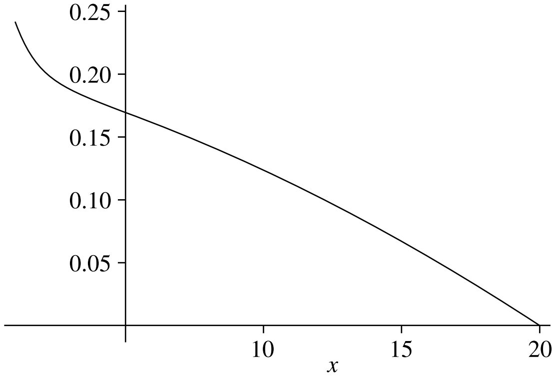

FIGURE 1: Plot of the probability $h(x) = \mathbb{P}_x\{M_{\tau_a} \leq K\}$ for PEMP (20) with $q = 0$, $\mu = 1$, $\lambda = 3$, $a = 1$, $K = 20$, and jump size distribution given in (22).

4.2. A jump diffusion model

In this section we consider a generalized PEMP $(X_t)_{t\geq 0}$ with diffusion whose dynamics are governed by

$$\mathrm{d}X_t = X_t\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t + \mathrm{d}Z_t, \qquad t > 0, \tag{24}$$

where the initial value $X_0 = x \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a standard Brownian motion, and $(Z_t)_{t\geq 0}$ is an independent compound Poisson process with a unit jump intensity and a unit mean exponential jump distribution. The two-sided exit quantities of this generalized PEMP can also be solved using the approach described in [15, Sections 6 and 7].

We define an integral kernel

$$\psi_1(z) = \frac{\mathrm{e}^{z^2/2}}{z(z+1)}, \qquad z \in \mathbb{C}.$$

Let Γ_i $(i = 1, 2)$ be small counterclockwise circles around the simple poles $\mu_1 = 0$ and $\mu_2 = -1$, respectively, and define the linearly independent functions

$$g_1(x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \psi_1(z)\mathrm{e}^{-xz}\,\mathrm{d}z = 1,$$
$$g_2(x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_2} \psi_1(z)\mathrm{e}^{-xz}\,\mathrm{d}z = -\mathrm{e}^{x+1/2},$$

for $x \in \mathbb{R}$. To find another linearly independent partial eigenfunction, we consider the vertical line $\Gamma_3 = \{1 + t\sqrt{-1},\ t \in \mathbb{R}\}$ and define

$$g_3(x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_3} \psi_1(z)\mathrm{e}^{-xz}\,\mathrm{d}z. \tag{25}$$

Next we derive an explicit expression for $g_3(x)$. We know from (25) that $\lim_{x\to\infty} g_3(x) = 0$ and g_3 is continuously differentiable with

$$g_3'(x) = -\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_3} \frac{\mathrm{e}^{z^2/2}}{z+1}\mathrm{e}^{-xz}\,\mathrm{d}z. \tag{26}$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

Note that the bilateral Laplace transform functions (see, e.g. [38, Chapter VI]) of a standard normal random variable U_1 and an independent unit mean exponential random variable U_2 are given respectively by

$$\int_{-\infty}^{\infty} \mathrm{e}^{-zy} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2} \,\mathrm{d}y = \mathrm{e}^{z^2/2}, \qquad \int_0^{\infty} \mathrm{e}^{-zy} \mathrm{e}^{-y} \,\mathrm{d}y = \frac{1}{z+1},$$

for all complex z such that $\mathrm{Re}(z) \geq 0$. Hence, the bilateral Laplace transform of the density function of $U_1 + U_2$, i.e.

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x-y)^2/2} \mathrm{e}^{-y} \,\mathrm{d}y$$

is given by $\mathrm{e}^{z^2/2}/(z+1)$ for all complex z such that $\mathrm{Re}(z) \geq 0$. Since the right-hand side of (26) is just the Bromwich integral for the inversion of the bilateral Laplace transform $-\mathrm{e}^{z^2/2}/(z+1)$, evaluated at $-x$, we deduce that

$$g_3'(x) = -\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x+y)^2/2} \mathrm{e}^{-y} \,\mathrm{d}y.$$

It follows that

$$g_3(x) = -\int_x^{\infty} g_3'(y) \,\mathrm{d}y = 1 - \int_0^{\infty} N(x+y) \mathrm{e}^{-y} \,\mathrm{d}y,$$

where $N(\cdot)$ is the cumulative distribution function of standard normal distribution.

For any fixed $-\infty < u < v < \infty$, we define a matrix-valued function

$$(M_{i,k}(u,v))_{1 \leq i,\, k \leq 3} := \begin{pmatrix} 1 & g_1(v) & g_1(u) \\ v\mathrm{e}^{v+1/2} & g_2(v) & g_2(u) \\ 1 - \int_0^{\infty} N(v+y) y \mathrm{e}^{-y} \,\mathrm{d}y & g_3(v) & g_3(u) \end{pmatrix},$$

where the first row is computed according to

$$M_{i,1}(u,v) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_i} \frac{\psi_0(z)}{z+1} \mathrm{e}^{-vz} \,\mathrm{d}z.$$

Note that $M_{3,1}(u,v)$ can be calculated in the same way as $g_3(x)$. We also denote the inverse of $(M_{i,k}(u,v))_{1 \leq i,\, k \leq 3}$ by $(N_{k,j}(u,v))_{1 \leq k,\, j \leq 3}$.

By Equation (46) and a generalized Equation (48) of [15] (with $\zeta = s = 0$ and $\rho \geq 0$), we obtain the linear system of equations

$$(c_1, c_2, c_3)(M_{i,k}) = \left(\frac{\overline{C}}{\rho+1}, f(v), f(u) \right), \tag{27}$$

where $\overline{C}$ is a constant to be specified later, and $f(x)$ could stand for any of $B_1^{(0)}(x;\, u, v)$, $B_2^{(0,\rho)}(x;\, u, v)$, or $C^{(0,0)}(x;\, u, v)$ and has the representation $f(x) = \sum_{i=1}^{3} c_i g_i(x)$, $x \in [u, v]$. By letting

- $\overline{C} = f(u) = 0$ and $f(v) = 1$,
- $\overline{C} = 1$ and $f(v) = f(u) = 0$,
- $\overline{C} = f(v) = 0$ and $f(u) = 1$,

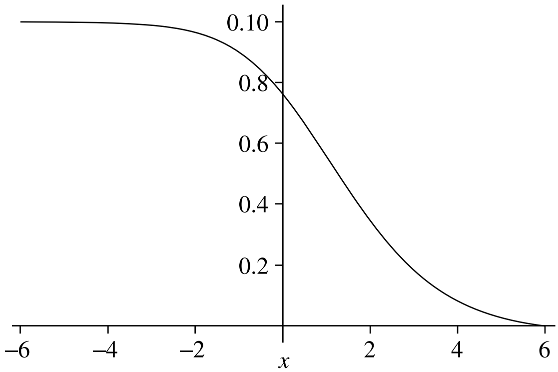

FIGURE 2: Plot of the probability $h(x) = \mathbb{P}_x\{M_{\tau_a} \leq K\}$ for the jump diffusion in (24) with $K = 6$ and $a = 1$.

for any $\rho \geq 0$ and $z > 0$, and solving the linear system (27), we respectively obtain

$$B_1^{(0)}(x;\ u, v) = \sum_{i=1}^{3} N_{2,i}(u, v) g_i(x), \qquad B_2^{(0,\rho)}(x;\ u, v) = \frac{1}{1+\rho} \sum_{i=1}^{3} N_{1,i}(u, v) g_i(x),$$

$$B_2^{(0)}(x, \mathrm{d}z;\ u, v) = \mathrm{e}^{-z} \sum_{i=1}^{3} N_{1,i}(u, v) g_i(x)\, \mathrm{d}z, \qquad C^{(0,0)}(x;\ u, v) = \sum_{i=1}^{3} N_{3,i}(u, v) g_i(x).$$

Further, this implies that

$$b_{a,1}^{(0)}(x) = -\sum_{i=1}^{3} D_{2,1}(x-a, x) g_i(x), \qquad b_{a,2}^{(0)}(x,\ \mathrm{d}z) = \mathrm{e}^{-z}\bigg(\sum_{i=1}^{3} D_{1,i}(x-a, x) g_i(x)\bigg),$$

$$c_a^{(0,0)}(x) = \sum_{i=1}^{3} D_{3,i}(x-a, x) g_i(x),$$

where we denote $D_{k,j}(u, v) = (\partial/\partial v) N_{k,j}(u, v)$.

In Figure 2, we plot $h(x) = \mathbb{P}_x\{M_{\tau_a} \leq K\}$ by numerically solving the integral equation (10) using MATHEMATICA.

Appendix A. Proof of Lemma 1

We define $\psi_n(z) = \inf_{m \geq n} \phi_m(z)$ for $z \in S$. Further, we define $\underline{\psi}_n(z) = \liminf_{w \to z} \psi_n(w)$ which is lower semicontinuous; see, e.g. Lemma 5.13.4 of [6]. Note that $\underline{\psi}_n$ is increasing in n, and by the definition of $\underline{\psi}_n$, we have

$$\begin{aligned} \lim_{n\to\infty} \underline{\psi}_n(z) &= \lim_{n\to\infty} \lim_{r\downarrow 0} \inf_{w\in(z-r,z+r)} \inf_{m\geq n} \phi_m(w) \\ &= \lim_{n\to\infty} \lim_{r\downarrow 0} \inf_{m\geq n,\ w\in(z-r,z+r)} \phi_m(w) \\ &\equiv \liminf_{n\to\infty, w\to z} \phi_n(w), \end{aligned}$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

where the second equality is because there is no ambiguity in switching the order of two infimums. By the monotone convergence theorem, we have

$$\int_S \liminf_{n\to\infty, w\to z} \phi_n(w)\, \mathrm{d}\mu(z) = \lim_{n\to\infty} \int_S \underline{\psi}_n(z)\, \mathrm{d}\mu(z). \tag{A.28}$$

By the Portmanteau theorem of weak convergence and the fact that $\underline{\psi}_n(z)$ is nonnegative and lower semicontinuous, it follows that

$$\int_S \underline{\psi}_n(z)\, \mathrm{d}\mu(z) \le \liminf_{m\to\infty} \int_S \underline{\psi}_n(z)\, \mathrm{d}\mu_m(z) \quad \text{for any } n \in \mathbb{N}. \tag{A.29}$$

Moreover, since $\psi_n(z)$ is monotone increasing in n, we have

$$\liminf_{m\to\infty} \int_S \underline{\psi}_n(z)\, \mathrm{d}\mu_m(z) \le \liminf_{m\to\infty} \int_S \underline{\psi}_m(z)\, \mathrm{d}\mu_m(z). \tag{A.30}$$

By (A.28)–(A.30),

$$\int_S \liminf_{n\to\infty, w\to z} \phi_n(w)\, \mathrm{d}\mu(z) \le \liminf_{m\to\infty} \int_S \underline{\psi}_m(z)\, \mathrm{d}\mu_m(z) \le \liminf_{m\to\infty} \int_S \phi_m(z)\, \mathrm{d}\mu_m(z),$$

where the last inequality is due to $\underline{\psi}_m(z) \le \psi_m(z) \le \phi_m(z)$.

Suppose that $\{\phi_n\}_{n\in\mathbb{N}}$ is uniformly bounded by $K > 0$, by applying (7) to $\{K - \phi_n\}_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned} K\mu(S) - \int_S \limsup_{n\to\infty, w\to z} \phi_n(w)\, \mathrm{d}\mu(z) &= \int_S \liminf_{n\to\infty, w\to z} (K - \phi_n(w))\, \mathrm{d}\mu(z) \\ &\le \liminf_{n\to\infty} \int_S (K - \phi_n(z))\, \mathrm{d}\mu_n(z) \\ &= K \liminf_{n\to\infty} \mu_n(S) - \limsup_{n\to\infty} \int_S \phi_n(z)\, \mathrm{d}\mu_n(z). \end{aligned}$$

Therefore, (8) follows immediately by the weak convergence of μ_n and $\mu(S) < \infty$.

Appendix B. Assumption verification for Example 4

Lemma B.1. *Consider the PMEP (20) with a continuous jump size distribution $F(\cdot)$. For $q, s \ge 0$ and $0 < u_0 < x_0 < v_0$, we have*

$$\lim_{(u,v)\downarrow(u_0,v_0)} g(x_0;\ u, v) = \lim_{(x,u)\uparrow(x_0,u_0)} g(x;\ u, v_0) = g(x_0, u_0, v_0),$$

where the function $g(x;\ u, v)$ is any of the following three functions:

$$B_1^{(q)}(x;\ u, v), \qquad B_2^{(q,s)}(x;\ u, v), \qquad C^{(q,s)}(x;\ u, v).$$

Proof. Note that the condition $0 < u_0 < x_0 < v_0$ is to ensure the process X remains positive before exiting these finite intervals, which further implies X is upward regular and creeps upward. We limit our proof to

$$\lim_{(u,v)\downarrow(u_0,v_0)} B_1^{(q)}(x_0;\ u, v) = B_1^{(q)}(x_0;\ u_0, v_0). \tag{B.1}$$

The other results can be proved in a similar manner. By the relationship $v > v_0 > u > u_0$, we have

$$\begin{aligned}|B_1^{(q)}(x_0;\, u_0, v_0) - B_1^{(q)}(x_0;\, u, v)| \\ &\le |\mathbb{E}_{x_0}[\mathrm{e}^{-qT_{v_0}^+}\mathbf{1}_{\{T_{v_0}^+<T_{u_0}^-,\, X_{T_{v_0}^+}=v_0\}}] - \mathbb{E}_{x_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v,\, X_{T_{v_0}^+}=v_0\}}]| \\ &\quad + \mathbb{P}_{x_0}\{v_0 < X_{T_{v_0}^+} \le v\}. \end{aligned} \tag{B.2}$$

It is clear that the last term of (B.2) vanishes as $v \downarrow v_0$ by the right-continuity of the distribution function of $X_{T_{v_0}^+}$. Also,

$$\begin{aligned}&|\mathbb{E}_{x_0}[\mathrm{e}^{-qT_{v_0}^+}\mathbf{1}_{\{T_{v_0}^+<T_{u_0}^-,\, X_{T_{v_0}^+}=v_0\}}] - \mathbb{E}_{x_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v,\, X_{T_{v_0}^+}=v_0\}}]| \\ &\quad = \mathbb{E}_{x_0}[\mathrm{e}^{-qT_{v_0}^+}\mathbf{1}_{\{T_{v_0}^+<T_u^-,\, X_{T_{v_0}^+}=v_0\}}] - \mathbb{E}_{x_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v,\, X_{T_{v_0}^+}=v_0\}}] \\ &\qquad + \mathbb{E}_{x_0}[\mathrm{e}^{-qT_{v_0}^+}\mathbf{1}_{\{T_u^-<T_{v_0}^+<T_{u_0}^-,\, X_{T_{v_0}^+}=v_0\}}] \\ &\quad \le 1 - \mathbb{E}_{v_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v\}}] + \mathbb{P}_{x_0}\{T_u^- < T_{v_0}^+ < T_{u_0}^-\}. \end{aligned} \tag{B.3}$$

Let ζ be the time of the first jump of the compound Poisson process Z with jump rate $\lambda > 0$. Note that X will increase continuously up to time ζ as long as the initial value is positive. Since $v > v_0 > 0$, we have

$$1 - \mathbb{E}_{v_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v\}}] \le 1 - \mathbb{E}_{v_0}[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{\zeta>T_v^+\}}] = 1 - \left(\frac{v}{v_0}\right)^{-(q+\lambda)/\mu}. \tag{B.4}$$

By conditioning on $X_{T_u^- -}$, we obtain

$$\begin{aligned}\mathbb{P}_{x_0}\{T_u^- < T_{v_0}^+ < T_{u_0}^-\} &\le \int_u^{v_0} \mathbb{P}_{x_0}\{X_{T_u^- -} \in \mathrm{d}y\}\mathbb{P}\{y-u < J \le y-u_0\} \\ &\le \max_{u_0\le y\le v_0}(F(y-u_0) - F(y-u)). \end{aligned} \tag{B.5}$$

Since $F(\cdot)$ is continuous, and, hence, uniformly continuous for $y \in [0, v_0 - u_0]$, it follows that the right-hand side of (B.5) vanishes as $u \downarrow u_0$. From (B.2)–(B.5), we conclude that (B.1) holds.

Note that although (B.5) only uses the continuity of F on $[0, \infty)$, the proof for $C^{(q,s)}(x;\, u, v)$ will use the continuity of F on $(-\infty, 0]$. □

Proposition B.1. *Assumptions (A1)–(A3) hold for the piecewise exponential Markov process (20) with a continuous jump size distribution $F(\cdot)$ and initial value $X_0 \ge a$.*

Proof. For $0 < u < x < v$, by the strong Markov property, we have

$$\begin{aligned}B_1^{(q)}(x;\, u, v) &= \mathbb{E}_x[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v,\, \zeta>T_v^+\}}] + \mathbb{E}_x[\mathrm{e}^{-qT_v^+}\mathbf{1}_{\{T_v^+<T_u^-,\, X_{T_v^+}=v,\, \zeta<T_v^+\}}] \\ &= \left(\frac{v}{x}\right)^{-(q+\lambda)/\mu} \\ &\quad + \lambda\int_0^{(1/\mu)\ln v/x} \mathrm{e}^{-(q+\lambda)t}\,\mathrm{d}t \int_{u-x\mathrm{e}^{\mu t}}^{v-x\mathrm{e}^{\mu t}} B_1^{(q)}(x\mathrm{e}^{\mu t}+w;\, u, v)F(\mathrm{d}w). \end{aligned} \tag{B.6}$$

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

By Lemma B.1, (B.6), and the dominated convergence theorem, it is straightforward to verify that assumption (A1) holds and, for $x > a$,

$$b_{a,1}^{(q)}(x) = \frac{q+\lambda}{\mu x} - \frac{\lambda}{\mu x}\int_{-a}^{0} B_1^{(q)}(x+w;\ x-a, x)F(\mathrm{d}w).$$

Note that we require $x > a$ as otherwise $x + w$ in the above equation could be negative for $w \in (-a, 0)$, and then Lemma B.1 does not apply. Obviously, $\int_x^y b_{a,1}^{(q)}(w)\,\mathrm{d}w < \infty$ for all $0 < x < y < \infty$. Similarly, by conditioning on the first jump of Z, for $0 < u < x < v$,

$$\begin{aligned} B_2^{(q)}(x, \mathrm{d}z;\ u, v) &= \lambda \int_0^{(1/\mu)\ln v/x} \mathrm{e}^{-(q+\lambda)t} F(v - x\mathrm{e}^{\mu t} + \mathrm{d}z)\,\mathrm{d}t \\ &\quad + \lambda \int_0^{(1/\mu)\ln v/x} \mathrm{e}^{-(q+\lambda)t}\,\mathrm{d}t \int_{u-x\mathrm{e}^{\mu t}}^{v-x\mathrm{e}^{\mu t}} B_2^{(q)}(x\mathrm{e}^{\mu t}+w,\ \mathrm{d}z;\ u, v)F(\mathrm{d}w) \end{aligned}$$

and

$$C^{(q,s)}(x;\ u, v) = \lambda \int_0^{(1/\mu)\ln v/x} \mathrm{e}^{-(q+\lambda)t}\,\mathrm{d}t \int_{-\infty}^{v-x\mathrm{e}^{\mu t}} C^{(q,s)}(x\mathrm{e}^{\mu t}+w;\ u, v)F(\mathrm{d}w),$$

where it is understood that $C^{(q,s)}(x\mathrm{e}^{\mu t}+w;\ u, v) = \exp(s(x\mathrm{e}^{\mu t}+w-u))$ for $w < u - x\mathrm{e}^{\mu t}$. One can verify from Lemma B.1 and the dominated convergence theorem that assumptions (A2) and (A3) hold, and, for $x > a$,

$$b_{a,2}^{(q)}(x,\ \mathrm{d}z) = \frac{\lambda}{\mu x}F(\mathrm{d}z) + \frac{\lambda}{\mu x}\int_{-a}^{0} B_2^{(q)}(x+w, \mathrm{d}z;\ x-a, x)F(\mathrm{d}w)$$

and

$$c_a^{(q,s)}(x) = \frac{\lambda}{\mu x}\int_{-\infty}^{0} C^{(q,s)}(x+w;\ x-a, x)F(\mathrm{d}w).$$

This completes the proof. □

Appendix C. Assumption verification for Example 5

Let U be the continuous component of X, which is a linear diffusion process with the infinitesimal generator

$$\mathcal{L}_U = \frac{1}{2}\sigma^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mu(y)\frac{\mathrm{d}}{\mathrm{d}y}.$$

It is well known that, for any $q > 0$, there exist two independent and positive solutions, denoted as $\phi_q^{\pm}(y)$, to the Sturm–Liouville equation

$$\mathcal{L}_U\phi_q^{\pm}(y) = q\phi_q^{\pm}(y), \qquad \text{(C.1)}$$

where $\phi_q^{+}(\cdot)$ is strictly increasing and $\phi_q^{-}(\cdot)$ is strictly decreasing. By the Lipschitz assumption on $\mu(\cdot)$ and $\sigma(\cdot)$, it follows from the Schauder estimates (see, e.g. Theorem 6.14 of [11]) of (C.1) that $\phi_q^{\pm}(\cdot) \in C^{2,\alpha}(\bar{\Omega})$ for any $\alpha \in (0, 1]$ and any compact set $\bar{\Omega} \subset \mathbb{R}$. We refer the interested reader to [11, Section 4.1] for more detail on the Hölder space $C^{2,\alpha}(\bar{\Omega})$.

We denote the first hitting time of U to level $z \in \mathbb{R}$ by $H_z = \inf\{t > 0 \colon U_t = z\}$. It is well known that, for $u \le x \le v$,

$$\mathbb{E}_x[\mathrm{e}^{-qH_u}\mathbf{1}_{\{H_u<H_v\}}] = \frac{f_q(x,v)}{f_q(u,v)} \quad \text{and} \quad \mathbb{E}_x[\mathrm{e}^{-qH_v}\mathbf{1}_{\{H_v<H_u\}}] = \frac{f_q(u,x)}{f_q(u,v)}, \tag{C.2}$$

where $f_q(x,y) := \phi_q^+(x)\phi_q^-(y) - \phi_q^+(y)\phi_q^-(x)$. Note that $f_q(x,y)$ is strictly decreasing in x and strictly increasing in y with $f_q(x,x) = 0$. In particular, for $u \le x \le v$, we have

$$\mathbb{E}_x[\mathrm{e}^{-qH_u}] = \frac{\phi_q^-(x)}{\phi_q^-(u)} \quad \text{and} \quad \mathbb{E}_x[\mathrm{e}^{-qH_v}] = \frac{\phi_q^+(x)}{\phi_q^+(v)}. \tag{C.3}$$

For $\boldsymbol{e}_q$ an independent exponential random variable with mean $1/q < \infty$, the q-potential measure of U is given by

$$r_q(x,y) := \frac{(1/q)\mathbb{P}_x\{U_{\boldsymbol{e}_q} \in \mathrm{d}y\}}{\mathrm{d}y} = \begin{cases} \dfrac{2}{q\sigma^2(y)} \dfrac{\phi_q^+(x)\phi_q^-(y)}{f_{q,1}(y,y)}, & x \le y, \\ \dfrac{2}{q\sigma^2(y)} \dfrac{\phi_q^+(y)\phi_q^-(x)}{f_{q,1}(y,y)}, & x > y, \end{cases}$$

where $f_{q,1}(x,y) := (\partial/\partial x) f_q(x,y)$. Further, the q-potential measure of U killed on exiting the interval $[u,v]$, for $u \le x, y \le v$, is given by

$$\begin{aligned} \theta^{(q)}(x,y;\, u,v) &:= \frac{(1/q)\mathbb{P}_x\{U_{\boldsymbol{e}_q} \in \mathrm{d}y, \boldsymbol{e}_q < H_u \wedge H_v\}}{\mathrm{d}y} \\ &= r_q(x,y) - \frac{f_q(x,v)}{f_q(u,v)} r_q(u,y) - \frac{f_q(u,x)}{f_q(u,v)} r_q(v,y). \end{aligned} \tag{C.4}$$

The next lemma is an analogy of Lemma B.1. Thanks to the diffusion term in the jump diffusion model (21), we now allow for the presence of atoms in the jump intensity measure $\nu(\cdot)$.

Lemma C.1. *Consider the jump diffusion model (21). For $q, s \ge 0$ and $u_0 < x_0 < v_0$, we have*

$$\lim_{(u,v)\downarrow(u_0,v_0)} g(x_0;\, u,v) = \lim_{(x,u)\uparrow(x_0,u_0)} g(x;\, u,v_0) = g(x_0,u_0,v_0),$$

where $g(x;\, u,v)$ is any of the following functions:

$$B_1^{(q)}(x;\, u,v), \qquad B_2^{(q,s)}(x;\, u,v), \qquad C^{(q,s)}(x;\, u,v).$$

Proof. We can follow the same proof as Lemma B.1 with the exception that the probability term $\mathbb{P}_{x_0}\{T_u^- < T_{v_0}^+ < T_{u_0}^-\}$ in (B.5), which will be handled distinctly here. We have $X_t = U_t$ a.s. for $t < \zeta$, where ζ is the first time a jump occurs which follows an exponential distribution with mean $1/\lambda = 1/\nu(\mathbb{R}) > 0$. For any $u_0 < u < x_0 < v_0$, by (C.2) and (C.3), we have

$$\begin{aligned} \mathbb{P}_{x_0}\{T_u^- < T_{v_0}^+ < T_{u_0}^-\} &\le \mathbb{P}_u\{T_{v_0}^+ < T_{u_0}^-\} \\ &= \mathbb{P}_u\{T_{v_0}^+ < T_{u_0}^-, \xi > T_{v_0}^+\} + \mathbb{P}_u\{\xi \le T_{v_0}^+ < T_{u_0}^-\} \\ &\le \mathbb{E}_u[\mathrm{e}^{-\lambda H_{v_0}}\mathbf{1}_{\{H_{v_0}<H_{u_0}\}}] + 1 - \mathbb{E}_u[\mathrm{e}^{-\lambda H_{u_0}}] \\ &= \frac{f_q(u_0,u)}{f_q(u_0,v_0)} + 1 - \frac{\phi_q^-(u)}{\phi_q^-(u_0)}. \end{aligned}$$

Therefore, it follows that $\lim_{u\downarrow u_0} \mathbb{P}_{x_0}\{T_u^- < T_{v_0}^+ < T_{u_0}^-\} = 0$ by $f_q(u_0,u_0) = 0$. □

https://doi.org/10.1017/jpr.2017.20 Published online by Cambridge University Press

Proposition C.1. *Assumptions (A1)–(A3) hold for the jump diffusion model (21).*

Proof. By the strong Markov property, (C.2), and (C.4), for $u < x < v$, it follows that

$$\begin{aligned} B_1^{(q)}(x;\, u, v) &= \mathbb{E}_x[\mathrm{e}^{-qT_v^+}\,\mathbf{1}_{\{T_v^+<T_u^-,\, T_v^+=v,\, \zeta>T_v^+\}}] + \mathbb{E}_x[\mathrm{e}^{-qT_v^+}\,\mathbf{1}_{\{T_v^+<T_u^-,\, T_v^+=v,\, \zeta<T_v^+\}}] \\ &= \mathbb{E}_x[\mathrm{e}^{-(q+\lambda)H_v}\,\mathbf{1}_{\{H_v<H_u\}}] \\ &\quad + \int_u^v \mathbb{E}_x[\mathrm{e}^{-q\zeta}\,\mathbf{1}_{\{\zeta<H_u\wedge H_v,\, U_\zeta\in \mathrm{d}y\}}] \int_{\mathbb{R}} B_1^{(q)}(y+\gamma(y,w);\, u, v)\frac{\nu(\mathrm{d}w)}{\lambda} \\ &= \frac{f_{q+\lambda}(u,x)}{f_{q+\lambda}(u,v)} + \int_u^v \theta^{(q+\lambda)}(x,y;\, u,v)\,\mathrm{d}y \int_{\mathbb{R}} B_1^{(q)}(y+\gamma(y,w);\, u,v)\nu(\mathrm{d}w), \end{aligned}$$

where it is understood that $B_1^{(q)}(y+\gamma(y,w);\, u,v) = 0$ if $\gamma(y,w) > v-y$ or $\gamma(y,w) < u-y$. By Lemma C.1, the dominated convergence theorem, and the identity $f_{q+\lambda}(u,v) = -f_{q+\lambda}(v,u)$, we can verify that assumption (A1) holds with

$$b_{a,1}^{(q)}(x) = \frac{-f_{q+\lambda,1}(x-a,x)}{f_{q+\lambda}(x-a,x)} - \int_{x-a}^{x} \tilde{\theta}_a^{(q+\lambda)}(x,y)\,\mathrm{d}y \int_{\mathbb{R}} B_1^{(q)}(y+\gamma(y,w);\, x-a,x)\nu(\mathrm{d}w),$$

where we write

$$\begin{aligned} &\tilde{\theta}_a^{(q+\lambda)}(x,y) \\ &\quad := -\frac{f_{q+\lambda,1}(x-a,x)}{f_{q+\lambda}(x-a,x)} r_{q+\lambda}(x,y) - r_{q+\lambda,1}(x,y) + \frac{f_{q+\lambda,1}(x,x)}{f_{q+\lambda}(x-a,x)} r_{q+\lambda}(x-a,y) \end{aligned}$$

and $r_{q+\lambda,1}(x,y) := (\partial/\partial x) r_{q+\lambda}(x,y)$. The integrability of $b_{a,1}^{(q)}(\cdot)$ follows from the continuity of the $\phi_q^+(\cdot)$ and $\phi_q^-(\cdot)$.

Similarly, by the strong Markov property of X, (C.2), and (C.4), we have

$$B_2^{(q)}(x,\mathrm{d}z;\, u,v) = \int_u^v \theta^{(q+\lambda)}(x,y;\, u,v)\,\mathrm{d}y \int_{\mathbb{R}} B_2^{(q)}(y+\gamma(y,w),\mathrm{d}z;\, u,v)\nu(\mathrm{d}w),$$

$$C^{(q,s)}(x;\, u,v) = \frac{f_{q+\lambda}(x,v)}{f_{q+\lambda}(u,v)} + \int_u^v \theta^{(q+\lambda)}(x,y;\, u,v)\,\mathrm{d}y \int_{\mathbb{R}} C^{(q,s)}(y+\gamma(y,w);\, u,v)\nu(\mathrm{d}w).$$

One can verify from Lemma C.1 that assumptions (A2) and (A3) hold with

$$b_{2,a}^{(q)}(x,\, \mathrm{d}z) = \int_{x-a}^{x} \tilde{\theta}_a^{(q+\lambda)}(x,y)\,\mathrm{d}y \int_{\mathbb{R}} B_2^{(q)}(y+\gamma(y,w),\mathrm{d}z;\, x-a,x)\nu(\mathrm{d}w),$$

$$c_a^{(q,s)}(x) = \frac{-f_{q+\lambda,1}(x,x)}{f_{q+\lambda}(x-a,x)} + \int_{x-a}^{x} \tilde{\theta}^{(q+\lambda)}(x,y)\,\mathrm{d}y \int_{\mathbb{R}} C^{(q,s)}(y+\gamma(y,z);\, x-a,x)\nu(\mathrm{d}w).$$

This completes the proof. □

Acknowledgements

The authors would like to thank two anonymous referees for their helpful comments and suggestions. Support from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by David Landriault and Bin Li (grant numbers 341316 and 05828, respectively). Support from a start-up grant from the University of Waterloo is gratefully acknowledged by Bin Li, as is support from the Canada Research Chair Program by David Landriault.

References

[1] Albrecher, H., Ivanovs, J. and Zhou, X. (2016). Exit identities for Lévy processes observed at Poisson arrival times. *Bernoulli* **22,** 1364–1382.

[2] Asmussen, S., Avram, F. and Pistorius, M. R. (2014). Russian and American put options under exponential phase-type Lévy models. *Stoch. Process. Appl.* **109,** 79–111.

[3] Avram, F., Kyprianou, A. E. and Pistorius, M. R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Prob.* **14,** 215–238.

[4] Avram, F., Palmowski, Z. and Pistorius, M. R. (2007). On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Prob.* **17,** 156–180.

[5] Baurdoux, E. J. (2009). Some excursion calculations for reflected Lévy processes. *ALEA Lat. Amer. J. Prob. Math. Statist.* **6,** 149–162.

[6] Berberian, S. K. (1999). *Fundamentals of Real Analysis*. Springer, New York.

[7] Carr, P., Zhang, H. and Hadjiliadis, O. (2011). Maximum drawdown insurance. *Internat. J. Theoret. Appl. Finance* **14,** 1195–1230.

[8] Cherny, V. and Obłoj, J. (2013). Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model. *Finance Stoch.* **17,** 771–800.

[9] Douady, R., Shiryaev, A. N. and Yor, M. (2000). On probability characteristics of "drop" variables in standard Brownian motion. *Theory Prob. Appl.* **44,** 29–38.

[10] Feinberg, E. A., Kasyanov, P. O. and Zadoianchuk N. V. (2014). Fatou's lemma for weakly converging probabilities. *Theory Prob. Appl.* **58,** 683–689.

[11] Gilbarg, D. and Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin.

[12] Gjessing, H. K. and Paulsen, J. (1997). Present value distributions with applications to ruin theory and stochastic equations. *Stoch. Process. Appl.* **71,** 123–144.

[13] Grossman, S. J. and Zhou, Z. (1993). Optimal investment strategies for controlling drawdowns. *Math. Finance* **3,** 241–276.

[14] Ivanovs, J. and Palmowski, Z. (2012). Occupation densities in solving exit problems for Markov additive processes and their reflections. *Stoch. Process. Appl.* **122,** 3342–3360.

[15] Jacobsen, M. and Jensen, A. T. (2007). Exit times for a class of piecewise exponential Markov processes with two-sided jumps. *Stoch. Process. Appl.* **117,** 1330–1356.

[16] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd edn. Springer, New York.

[17] Kella, O. and Stadje, W. (2001). On hitting times for compound Poisson dams with exponential jumps and linear release rate. *J. Appl. Prob.* **38,** 781–786.

[18] Kuznetsov, A., Kyprianou, A. E. and Rivero, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II* (Lecture Notes Math. **2061**), Springer, Heidelberg, pp. 97–186.

[19] Kyprianou, A. E. (2014). *Fluctuations of Lévy Processes with Applications*, 2nd edn. Springer, Heidelberg.

[20] Kyprianou, A. E. and Loeffen, R. L. (2010). Refracted Lévy processes. *Ann. Inst. H. Poincaré Prob. Statist.* **46,** 24–44.

[21] Kyprianou, A. E. and Palmowski, Z. (2007). Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process. *J. Appl. Prob.* **44,** 428–443.

[22] Kyprianou, A. E. and Zhou, X. (2009). General tax structures and the Lévy insurance risk model. *J. Appl. Prob.* **46,** 1146–1156.

[23] Landriault, D., Li, B. and Zhang, H. (2017). On magnitude, asymptotics and duration of drawdowns for Lévy models. *Bernoulli* **23,** 432–458.

[24] Lehoczky, J. P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Prob.* **5,** 601–607.

[25] Li, B., Tang, Q. and Zhou, X. (2013). A time-homogeneous diffusion model with tax. *J. Appl. Prob.* **50,** 195–207.

[26] Loeffen, R. L. (2008). On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *Ann. Appl. Prob.* **18,** 1669–1680.

[27] Magdon-Ismail, M., Atiya, A. F., Pratap, A. and Abu-Mostafa, Y. (2004). On the maximum drawdown of a Brownian motion. *J. Appl. Prob.* **41,** 147–161.

[28] Mijatovic, A. and Pistorius, M. R. (2012). On the drawdown of completely asymmetric Lévy processes. *Stoch. Process. Appl.* **122,** 3812–3836.

[29] Øksendal, B. and Sulem, A. (2007). *Applied Stochastic Control of Jump Diffusions*, 2nd edn. Springer, Berlin.

[30] Pistorius, M. R. (2004). On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *J. Theoret. Prob.* **17,** 183–220.

[31] Poor, H. V. and Hadjiliadis, O. (2009). *Quickest Detection*. Cambridge University Press.

[32] Pospisil, L., Vecer, J. and Hadjiliadis, O. (2009). Formulas for stopped diffusion processes with stopping times based on drawdowns and drawups. *Stoch. Process. Appl.* **119,** 2563–2578.

[33] Rogers, L. C. G. and Williams, D. (2000). *Diffusions, Markov Processes, and Martingales*, Vol. 1, 2nd edn. Cambridge University Press.
[34] Schuhmacher, F. and Eling, M. (2011). Sufficient conditions for expected utility to imply drawdown-based performance rankings. *J. Banking Finance* **35,** 2311–2318.
[35] Shepp, L. and Shiryaev, A. N. (1993). The Russian option: reduced regret. *Ann. Appl. Prob.* **3,** 631–640.
[36] Taylor, H. M. (1975). A stopped Brownian motion formula. *Ann. Prob.* **3,** 234–246.
[37] Tsurui, A. and Osaki, S. (1976). On a first-passage problem for a cumulative process with exponential decay. *Stoch. Process. Appl.* **4,** 79–88.
[38] Widder, D. V. (1946). *The Laplace Transform*. Princeton University Press.
[39] Yuen, K. C., Wang, G. and Ng, K. W. (2004). Ruin probabilities for a risk process with stochastic return on investments. *Stoch. Process. Appl.* **110,** 259–274.
[40] Zhang, H. (2015). Occupation time, drawdowns, and drawups for one-dimensional regular diffusions. *Adv. Appl. Prob.* **47,** 210–230.
[41] Zhang, H. and Hadjiliadis, O. (2010). Drawdowns and rallies in a finite time-horizon: drawdowns and rallies. *Methodology Comput. Appl. Prob.* **12,** 293–308.
[42] Zhang, H., Leung, T. and Hadjiliadis, O. (2013). Stochastic modeling and fair valuation of drawdown insurance. *Insurance Math. Econom.* **53,** 840–850.
[43] Zhou, X. (2007). Exit problems for spectrally negative Lévy processes reflected at either the supremum or the infimum. *J. Appl. Prob.* **44,** 1012–1030.