Proceedings of the Edinburgh Mathematical Society (1994) 37, 471–476 ©

S_n-NORMAL SEMIGROUPS

by I. LEVI and R. B. McFADDEN

(Received 1st December 1992)

Certain subsemigroups of the full transformation semigroup T_n on a finite set of cardinality n are investigated, namely those subsemigroups S of T_n which are normalised by the symmetric group on n elements, the group of units of T_n. The S_n-normal closure of an element of T_n is determined, and the structure of the S_n-normal ideals consisting of the members of T_n whose image contains at most r elements is studied.

1991 *Mathematics subject classification*: 20M20.

Let T_n denote the full transformation semigroup on a set of finite cardinality n, and let S_n denote the symmetric group on n elements, the group of units of T_n. A subsemigroup S of T_n is defined to be S_n-*normal* if for each a in S and for each h in S_n, the element $h^{-1}ah$ is in S. Both T_n and S_n themselves are S_n-normal; so are the ideals $K(n,r)=\{a\in T_n:|im(a)|\leqq r\}$, $1\leqq r\leqq n$ [2].

Given $a\in T_n$, denote by $\langle a:S_n\rangle$ the smallest S_n-normal subsemigroup of T_n containing a. Thus $\langle a:S_n\rangle$ is the subsemigroup S of T_n generated by $\{g^{-1}ag:g\in S_n\}$. If a is a permutation then $\langle a:S_n\rangle$ is a normal subgroup of S_n and we know what that is. Assuming for the rest of this paper that a is not a permutation, associate with a the partition $\pi(a)$ of X such that x and y are in the same class of $\pi(a)$ if and only if $xa=ya$. Partitions P, Q of X are said to be of the same type (denoted by $P\equiv Q$) if they have the same number of classes of each size. We show that $\langle a:S_n\rangle$ is idempotent-generated and consists of all transformations b in T_n for which $\pi(b)$ contains a partition of the same type as $\pi(a)$.

The *idempotent rank* of an idempotent-generated semigroup S is the cardinality of a minimal generating set of idempotents of S [2]. It was shown in [2] that the idempotent rank of the S_n-normal semigroup $K(n,r)$, consisting of all transformations a with $|im(a)|\leqq r$, is $S(n,r)$, the Stirling number of the second kind. We define the S_n-*idempotent rank* of an S_n-normal semigroup S to be the cardinality of a minimal generating set A of idempotents of S such that $S=\langle A:S_n\rangle$ $(=\langle\{g^{-1}ag:a\in A,g\in S_n\}\rangle)$. Given $1\leqq r\leqq n$, let $T(n,r)$ denote the number of different types of partitions of an n-element set into r subsets. We present a recursive formula for $T(n,r)$ and show that the S_n-idempotent rank of $K(n,r)$ is $T(n,r)$. Moreover, we can choose a minimal S_n-generating set of idempotents in a single L-class of both T_n and S.

For each r such that $2\leqq r\leqq n$, the principal factor $K(n,r)/K(n,r-1)$ of T_n is denoted by P_r in [2]. Each P_r is a completely 0-simple semigroup whose non-zero elements may be thought of as the elements a of T_n having $|im(a)|=r$. Then P_r is a band of $T(n,r)$

https://doi.org/10.1017/S0013091500018927 Published online by Cambridge University Press

subsemigroups, each of which is a quotient semigroup of an S_n-normal semigroup of S_n-idempotent rank 1 (Theorem 8).

Recall that two elements of T_n are $\mathscr{R}$-equivalent if and only if they have the same partition, and $\mathscr{L}$-equivalent if and only if they have the same image. Given $a \in T_n$ and $h \in S_n$ denote by $\pi(a)h$ the partition $\{Ah: A \in \pi(a)\}$ of X. For any $a \in T_n$ and $h \in S_n$ we have that $(a, ah) \in \mathscr{R}$ and $(ha, a) \in \mathscr{L}$, and the proof of the first two parts of the following Lemma is obvious.

Lemma 1. (i) *if* $h \in S_n$ *and* $a \in T_n$, *then* $\operatorname{im}(ah) = \operatorname{im}(a)h = \operatorname{im}(h^{-1}ah)$ *and* $\pi(h^{-1}a) = \pi(a)h = \pi(h^{-1}ah)$.

(ii) *For any subset* A *and partition* P *of* X *such that* $|A| = |\operatorname{im}(a)|$, $P \equiv \pi(a)$, *there exist* $b, c \in \langle a:S_n \rangle$ *with* $\operatorname{im}(b) = A$ *and* $\pi(c) = P$.

(iii) *Let* e, f *be idempotents with* $\pi(e) \equiv \pi(f)$. *Then there exists a permutation* h *of* X *such that* $e = h^{-1}fh$.

Proof of (iii). Noting that the image of an idempotent e is a transversal of the partition of e, we can choose h such that $\pi(f)h = \pi(e)$ and $\operatorname{im}(f)h = \operatorname{im}(e)$. Then for any $x \in X$ and $B \in \pi(e)$ containing x there exists $A \in \pi(f)$ such that $B = Ah$, $B \cap \operatorname{im}(e) = (A \cap \operatorname{im}(f))h$ and so $xh^{-1}fh = Afh = B \cap \operatorname{im}(e) = xe$.

Since for all $a, b \in T_n$, $h \in S_n$, $\pi(a) \equiv \pi(h^{-1}ah)$ and $\pi(a) \subseteq \pi(ab)$, we have that $\langle a:S_n \rangle \subseteq \{c \in T_n: \pi(c) \text{ contains } P \equiv \pi(a)\}$. The reverse inclusion is proved in Lemmas 2, 3 and Proposition 4 below. We note that a variation of this result may be found in [4]. However, the present proofs are in a completely different vein and are much shorter than those in [4].

It is clear that for each $a \in T_n$, every conjugate of a is $\mathscr{D}$-equivalent to a and is in a group $\mathscr{H}$class if and only if a itself is in a group $\mathscr{H}$-class. It is not obvious that if a is not in a group $\mathscr{H}$-class then $\langle a:S_n \rangle$ contains even one idempotent in the $\mathscr{D}$-class of a. But we do have Lemma 2.

Lemma 2. *The semigroup* $\langle a:S_n \rangle$ *contains all idempotents* e *with* $\pi(e) \equiv \pi(a)$.

Proof. Observe that for transformations b and c with $|\operatorname{im}(b)| = |\operatorname{im}(c)|$, we have that $\pi(bc) = \pi(b)$ if and only if $\operatorname{im}(b)$ is a transversal of $\pi(c)$. Let $a = a_0$, and consider all products of the form

$$a_0, a_0a_1, a_0a_1a_2, a_0a_1a_2a_3, \ldots$$

where for each $i = 1, 2, 3, \ldots, a_i$ is a conjugate of a such that $\operatorname{im}(a_{i-1})$ is a transversal of $\pi(a_i)$. Since $\langle a:S_n \rangle$ is finite, there exist $i < j$ such that

$$a_0a_1a_2 \ldots a_i = a_0a_1a_2 \ldots a_ia_{i+1} \ldots a_j.$$

Define $u = a_0a_1a_2 \ldots a_i$, $v = a_{i+1} \ldots . a_j$. Then

$$u = uv, \quad \pi(u) = \pi(a) \equiv \pi(v),$$

so $\operatorname{im}(u) = \operatorname{im}(v)$ and $\operatorname{im}(v)$ is a transversal of $\pi(v)$, thus v is the identity on its image, and so v is an idempotent. The result follows from Lemma 1 (iii).

For transformations a and b, let $D(a,b) = \{x \in X : xa \neq xb\}$.

Lemma 3. *Let a, $b \in T_n$ with $\pi(b) = \pi(a)$, and let E_a be the set of all idempotents e in T_n with $\pi(e) \equiv \pi(a)$. Then $b \in \langle \{a\} \cup E_a \rangle \subseteq \langle a : S_n \rangle$.*

Proof. Let $S = \langle a : S_n \rangle$ and take $b \in T_n$ satisfying $\pi(b) = \pi(a)$. To show that $b \in S$, it suffices to prove that if $b \neq a$ then we can enlarge the set on which a and b agree by finding $c \in S$ with $|D(b,c)| < |D(a,b)|$ and observing that $S = \langle a : S_n \rangle \supseteq \langle c : S_n \rangle$. The result follows by induction on $|D(a,b)|$.

We may assume without loss of generality that $\operatorname{im}(a) \neq \operatorname{im}(b)$. For if $\operatorname{im}(a) = \operatorname{im}(b)$ we may replace a by af, where $f \in S$ is an idempotent chosen as follows to ensure that $D(af, b) = D(a,b)$. Let $v \in \operatorname{im}(a)$ be such that $va^{-1} \neq vb^{-1}$, and $w \in X - \operatorname{im}(a)$. Choose f with $\operatorname{im}(a)$ being a transversal of $\pi(f) \equiv \pi(a)$, $vf = wf = w$, and $uf = u$ for all $u \in \operatorname{im}(a) - \{v\}$. Observe that $\pi(af) = \pi(a) = \pi(b)$ while $w = vf \in \operatorname{im}(af) - \operatorname{im}(a) = \operatorname{im}(af) - \operatorname{im}(b)$, and $D(af, b) = D(a,b)$.

Now we show that for any $z \in \operatorname{im}(b) - \operatorname{im}(a)$ and $A \in \pi(af) = \pi(a)$ such that $Ab = z$, there exists $c \in S$ satisfying $Ac = Ab$ and $xc = xa$ for all $x \in X - A$. Let $Aa = y$. Choose an idempotent $e \in S$ such that $\pi(e) \equiv \pi(a)$, $ye = ze = z$, and $ue = u$, for all $u \in \operatorname{im}(a) - \{y\}$. Then $c = ae$ is the required mapping.

Let us illustrate the proof of Lemma 3 by the following example.

Example 1. Let $a = 333112$ (by which is meant $1a = 2a = 3a = 3, 4a = 5a = 1, 6a = 2$), $b = 222113$. We have that $\operatorname{im}(a) = \operatorname{im}(b) = \{1,2,3\}$, $D(a,b) = \{1,2,3,6\}$. Let $v = 3, 3a^{-1} = \{1,2,3\}$, $3b^{-1} = \{6\}$, and we take $w = 4$. Then a possible f is $f = 124422$, giving $af = 444112$, with $\operatorname{im}(af) = \{1,2,4\} \neq \operatorname{im}(b)$, $D(af, b) = \{1,2,3,6\} = D(a,b)$. Replace a by af, so that $a = 444112$. Take $v = 3$, $A = \{6\}$, $y = 2$. Then a possible e is $e = 133444$, with $c = ae = 444113$, $|D(b,c)| = |\{1,2,3\}| = 3 < 4 = |D(a,b)|$.

Proposition 4. *Let $a \in T_n$. Then $\langle a : S_n \rangle = \{b \in T_n : \pi(b)$ contains $P \equiv \pi(a)\}$.*

Proof. We show that for any transformation b of X such that $\pi(b)$ contains $\pi(a)$ and $|\operatorname{im}(b)| = |\operatorname{im}(a)| - 1$, there exist transformations c, d with $\pi(c) \equiv \pi(d) \equiv \pi(a)$ and $b = cd$. The result then follows from Lemmas 3 and 1, using an inductive argument. Let $\pi(a) = \{A_1, A_2, \ldots, A_{r-1}, A_r\}$ $\pi(b) = \{A_1, A_2, \ldots, A_{r-1} \cup A_r\}$, and $A_i b = x_i$, $i = 1, 2, \ldots, r-1$. Choose an idempotent c with $\pi(c) = \pi(a)$ and let $y_i = A_i c$, $i = 1, 2, \ldots, r$. Choose a partition $P \equiv \pi(a)$ such that $\{y_i : i = 1, 2, \ldots, r-1\}$ is a partial transversal of P, and y_{r-1}, y_r are in the same class of P. Choose a transformation d with $\pi(d) = P$, and $y_i d = x_i$, $i = 1, 2, \ldots, r-1$. Then $b = cd$, as required.

It follows from the description of $\langle a : S_n \rangle$ above and Lemma 1 that $\langle a : S_n \rangle$ is actually the complement of the symmetric group in the semigroup generated by a and S_n. As the example below demonstrates, this surprising result generally does not hold for the infinite analog of S_n-normal semigroups, the $\mathscr{G}_X$-normal semigroups on an infinite set X. (The symmetric group on an infinite set X is denoted by $\mathscr{G}_X$, and a semigroup of transformations of X is said to be $\mathscr{G}_X$-normal if it is invariant under conjugation by elements of $\mathscr{G}_X$).

Example 2. Let X be the set of all integers, and let a be the transformation of X defined by $xa=x+1$, for $x\geqq 0$, and $xa=x$, if $x<0$. Note that a is a one-to-one transformation with $|X-\mathrm{im}(a)|=1$. Let h be the permutation of X given by $xh=x+1$, for all $x\in X$. Then $ah\in\langle\{a\},\mathscr{G}_X\rangle-\langle a:\mathscr{G}_X\rangle$. Indeed, for all one-to-one transformations b and c, $|X-\mathrm{im}(bc)|=|X-\mathrm{im}(b)|+|X-\mathrm{im}(c)|$. Therefore if $ah\in\langle a:\mathscr{G}_X\rangle$. then ah has to be a conjugate of a. However, this is impossible since ah fixes no element of X but any conjugate $p^{-1}ap$ of a fixes infinitely many points of X (for each $x\in X$ such that $xp^{-1}<0$, we have that $xp^{-1}ap=xp^{-1}p=x$).

It is easy to see that the intersection of two $S_n-(\mathscr{G}_X-)$ normal semigroups is again an $S_n-(\mathscr{G}_X-)$ normal semigroup. In [3], the first author described the $\mathscr{G}_{\mathfrak{X}}$-normal semigroups of total one-to-one transformation of an infinite set X. It follows from this description that a union of two G_X-normal semigroups does not have to be a semigroup. However for any $a,b\in T_n-S_n$,

$$\langle a:S_n\rangle\cup\langle b:S_n\rangle=\langle a,b:S_n\rangle,$$

an S_n-normal semigroup (this is a direct consequence of Proposition 4 and the observation that $\pi(a)\subseteq\pi(ab)$). Therefore a union of two S_n-normal semigroups is again an S_n-normal semigroup and so the following is true.

Proposition 5. *Let S be an S_n-normal semigroup. Then the set $S(\cup,\cap)$ of the S_n-normal subsemigroups of S forms a modular lattice.*

It follows from Proposition 4 that if a is any transformation of X, and e is an idempotent with $\pi(e)\equiv\pi(a)$, then $\langle a:S_n\rangle=\langle e:S_n\rangle$, and so the following is true.

Theorem 6. *An S_n-normal semigroup is generated by its idempotents.*

Recall that for $1\leqq r\leqq n$, $T(n,r)$ denotes the number of different types of partitions of an n-element set into r subsets. Let P be a partition of X, and let $t_1<t_2<\cdots<t_k$ be the sizes of classes of P, and suppose that P contains exactly m_i classes of size t_i. We say that P is a partition of type $\tau=[(m_i,t_i):i=1,2,\ldots,k]$.

Lemma 7. $T(n,r)=\sum_{k=1}^{\min\{r,n-r\}}T(n-r,k)$.

It is possible to deduce Lemma 7 using classical partition generating functions—see [1]. To avoid introducing extraneous formulae not needed in the sequel, we offer instead the following direct proof.

Proof. Assume P is a partition of X of type $\tau=[(m_i,t_i):i=1,2,\dots,k]$ hacing r classes, that is $m_1+m_2+\cdots+m_k=r$. Let Y be a transversal of P; then the restriction of P to $X-Y$ is a partition of $X-Y$ of type $\tau_1=[(m_i,t_i-1):i=1,2,\dots,k]$ if $t_1>1$, and $\tau_2=[(m_i,t_i-1):i=2,\dots,k]$ if $t_1=1$. Observe that τ_1 and τ_2 are partition types of an $(n-r)$-elemement set having r and $t-m_1$ classes respectively. Therefore with each τ we may associate uniquely a type of a partition of an $(n-r)$-element set into k classes, $k\leqq r, k\leqq n-r$. Therefore

$$T(n,r)\leqq\sum_{k=1}^{\min\{r,n-r\}}T(n-r,k).$$

Conversely, let Q be a partition of an $(n-r)$-element subset Z of X of a type $\tau_3=[(m_i',t_i'):i=1,2,\dots,\ell]$ consisting of k classes, $1\leqq k\leqq\min\{r,n-r\}$. Let g be a one-to-one function from the classes of τ_3 into $X-Z$. Then $Q'=\{\{x\}\cup xg^{-1}:x\in X-Z\}$ is a partition of X of type $[(m_i',t_i'+1):i=1,2,\dots,\ell']$, if $k=r$, and $[(m_1,1),(m_1',t_1'+1),\dots,(m_\ell',t_\ell')]$, if $k<r$, where $m_1=r-k$. The equality follows.

Recall that for each $\mathscr{L}$-class L of T_n, there exists an r, $1\leqq r\leqq n$ such that $L\subseteq K(n,r)-K(n,r-1)$, where $K(n,0)$ is the empty set.

Theorem 8. (i) *For each* $r, 1\leqq r\leqq n-1$, *and each* $\mathscr{L}$*-class* L *of* T_n, *such that* $L\subseteq K(n,r)-K(n,r-1)$, *there exists a subset* E *of idempotents in* L *such that* $\langle E:S_n\rangle=K(n,r)$.

(ii) *The* S_n-idempotent rank of $K(n,r)$ *is* $T(n,r)$.

(iii) *For each* r, $1\leqq r\leqq n$, P_r *is a band of* $T(n,r)$ *subsemigroups, each of which is a quotient semigroup of an* S_n*-normal semigroup of* S_n*-idempotent rank* 1.

Proof. (i) Let r and L be as stated. Let $A\subseteq X$ be the image of a transformation in L. It suffices to show that given a partition type $\tau=[(m_i,t_i):i=1,2,\dots,\ell]$ consisting of r classes, there exists an idempotent $e\in T_n$ with $\operatorname{im}(e)=A$, $\pi(e)\equiv\tau$. Let Q be a partition of $X-A$ of type $[(m_i,t_i-1):i=j,\dots,\ell]$, where $j=1$ if $t_1>1$, and $j=2$ if $t_1=1$. Let g be a one-to-one function from the classes of Q into A. Define e to be the identify on A, and for $x\in X-A$ let $xe=Bg$, where B is the class of Q containing x.

(ii) It follows from the above that the S_n-idempotent rank of $K(n,r)$ is at most $T(n,r)$. Also if C is any set of idempotents in T_n with $\langle C:S_n\rangle=K(n,r)$, then $|\operatorname{im}(f)|<r+1$ for each $f\in C$. If $a\in K(n,r)$, $|\operatorname{im}(a)|=r$, there exists $t\in C$, $h\in S_n$, $s\in T_n$ with $a=h^{-1}ths$, so $\pi(t)\equiv\pi(h^{-1}th)\subseteq\pi(a)$. Since $\pi(t)$ and $\pi(a)$ consist of r classes each we have that $\pi(t)\equiv\pi(a)$. Therefore the S_n-idempotent rank of $K(n,r)$ is at least $T(n,r)$.

(iii) Let E be the S_n-generating set of $K(n,r)$ constructed in (i). For each $e\in E$, let $S(e)=\langle e:S_n\rangle/(\langle e:S_n\rangle\cap K(n,r-1))$. Then $S(e)$ is a subsemigroup of P_r. If e and f are distinct elements of E, then $\pi(e)\not\equiv\pi(f)$, and so for any $b\in\langle e:S_n\rangle\cap K(n,r)$, $c\in\langle f:S_n\rangle\cap K(n,r)$, we have that $\pi(b)\not\equiv\pi(c)$. Therefore $S(e)\cap S(f)$ is zero. Moreover $S(e)S(f)=S(e)$. Indeed, since for any $u\in\langle e:S_n\rangle$, $v\in\langle f:S_n\rangle$, we have that $\pi(u)\subseteq\pi(uv)$, so $S(e)S(f)\subseteq S(e)$. Also since $\operatorname{im}(e)=\operatorname{im}(f)$ we have that $ef=e$, so $S(e)\subseteq S(e)S(f)$.

Our last result asserts that Green's relations on an S_n-normal subsemigroup S of T_n coincide with the restrictions of the corresponding relations on T_n to S.

Proposition 9. *Let S be an S_n-normal semigroup. Then*

(i) $a\mathscr{R}b$ *if and only if* $\pi(a)=\pi(b)$;

(ii) $a\mathscr{L}b$ *if and only if* $\mathrm{im}\,(a)=\mathrm{im}\,(b)$;

(iii) $a\mathscr{D}b$ *if and only if* $|\mathrm{im}\,(a)|=|\mathrm{im}\,(b)|$;

(iv) $\mathscr{D}=\mathscr{J}$;

(v) *S is regular.*

REFERENCES

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition (Oxford University Press, 1960).

2. J. M. Howie and R. B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Roy. Soc. Edinburgh* **114A** (1990), 161–167.

3. I. Levi, Normal semigroups of one-to-one transformations, *Proc. Edinburgh Math. Soc.* **34** (1991), 65–76.

4. J. S. V. Symons, Normal transformation semigroups, *J. Austral. Math. Soc. Ser. A* **22** (1976), 385–390.

Mathematics Department
University of Louisville
Louisville, KY 40292
USA