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Colourings of Uniform Hypergraphs with Large Girth and Applications

ANDREY KUPAVSKII[1] and DMITRY SHABANOV[2†]

[1]Laboratory of Advanced Combinatorics and Network Applications, Moscow Institute of Physics and Technology, Moscow Region, Russia
and
Ecole Polytechnique Fédérale de Lausanne, Switzerland
(e-mail: kupavskii@ya.ru)

[2]Laboratory of Advanced Combinatorics and Network Applications, Moscow Institute of Physics and Technology, 141700, Institutskiy per. 9, Dolgoprudny, Moscow Region, Russia
and
Faculty of Computer Science, National Research University Higher School of Economics (HSE), 101000, Myasnitskaya Str. 20, Moscow, Russia
(e-mail: dm.shabanov.msu@gmail.com)

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This paper deals with a combinatorial problem concerning colourings of uniform hypergraphs with large girth. We prove that if H is an n-uniform non-r-colourable simple hypergraph then its maximum edge degree $\Delta(H)$ satisfies the inequality

$$\Delta(H) \geqslant c \cdot r^{n-1} \frac{n(\ln\ln n)^2}{\ln n}$$

for some absolute constant $c > 0$.

As an application of our probabilistic technique we establish a lower bound for the classical van der Waerden number $W(n,r)$, the minimum natural N such that in an arbitrary colouring of the set of integers $\{1,\ldots,N\}$ with r colours there exists a monochromatic arithmetic progression of length n. We prove that

$$W(n,r) \geqslant c \cdot r^{n-1} \frac{(\ln\ln n)^2}{\ln n}.$$

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1. Introduction

The work deals with an extremal problem on colourings of uniform hypergraphs with large girth. One of the basic facts of graph theory is that the chromatic number of an arbitrary graph with

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maximum vertex degree d is at most $d+1$. A natural generalization of this fact for edge degrees can be stated as follows: if $\Delta(G) \leqslant 2r-4$ then G is r-colourable. Both bounds are tight since in each case we have equality for the complete graph.

The situation in the case of uniform hypergraphs is much more complicated. In particular, the best known quantitative bounds for the chromatic number in terms of the maximum edge (or vertex) degree are not sharp. A fundamental result in this field was obtained by Erdős and Lovász in their classical paper [5]. They showed that if H is an arbitrary n-uniform hypergraph with maximum edge degree $\Delta(H)$ satisfying

$$\Delta(H) \leqslant \frac{1}{e} r^{n-1}, \tag{1.1}$$

then H is r-colourable. This result does not provide a tight bound for the maximum edge degree. The restriction (1.1) has been successively weakened for different values of the parameter r in a series of papers. The results were obtained by Radhakrishnan and Srinivasan [17] for $r=2$, Shabanov [20] for $r=3$, and Kostochka, Kumbhat and Rödl [11] for $r>3$. Recently, for fixed $r>2$, the last two bounds have been improved by Cherkashin and Kozik [4], whose approach is based on the algorithm proposed by Pluhár [16].

The first analogue of the Erdős–Lovász statement for simple hypergraphs was proved by Szabó [24] in 1990. Recall that a hypergraph is said to be *simple* if every two distinct edges do not share more than one common vertex. In fact he gave a lower bound for the maximum vertex degree in an arbitrary n-uniform non-2-colourable simple hypergraph. Later, using a similar technique, Kostochka and Kumbhat [10] established the following extension of his result.

Theorem 1.1 (Kostochka and Kumbhat [10]). *For any $\varepsilon>0$ and $r \geqslant 2$ there exists $n_0 = n_0(\varepsilon, r)$ such that, for any $n>n_0$, every n-uniform simple hypergraph H with maximum edge degree at most*

$$\Delta(H) \leqslant r^{n-1} n^{1-\varepsilon} \tag{1.2}$$

is r-colourable.

Since the parameter ε in Theorem 1.1 can be taken arbitrarily small, one can replace it in (1.2) with an infinitesimal positive function $\varepsilon(n,r)$ tending to 0 with growth of n for any fixed r. In their final comment in [10], Kostochka and Kumbhat asserted that for fixed r, the bound (1.2) holds for some function $\varepsilon(n,r) = \Theta(\ln\ln\ln n/\ln\ln n)$. However, the calculations in their proof allowed for a better choice of $\varepsilon(n,r)$: $\varepsilon(n,r) = \Omega(\sqrt[4]{\ln r/\ln n})$. Theorem 1.1 was refined by Shabanov [21], where he proved that, for fixed r, the function $\varepsilon(n,r)$ can be taken of order $\sqrt{\ln\ln n/\ln n}$. We emphasize that each of these results holds only when n is sufficiently large, and for r (and fixed n) they do not improve the classical bound of Erdős and Lovász.

The main result of the current paper is the following Erdős–Lovász-type theorem for simple hypergraphs.

Theorem 1.2. *Suppose $n \geqslant 9$, $r \geqslant 2$, and H is an n-uniform simple hypergraph. There exists an absolute constant $c > 0$ such that if*

$$\Delta(H) \leqslant c \cdot r^{n-1} \frac{n(\ln\ln n)^2}{\ln n} \tag{1.3}$$

then H is r-colourable.

The bound (1.3) improves the results of Kostochka and Kumbhat [10] and Shabanov [21] stated above, since the right-hand side of (1.3) is equal to $r^{n-1}n^{1-\varepsilon}$ for some $\varepsilon = \varepsilon(n,r) = (1+o(1))\ln\ln n/\ln n$. It is moreover within a factor of $n((\ln n)(\ln r)/(\ln\ln n)^2)$ of the best possible bound, since Kostochka and Rödl [12] proved that for any $n, r \geqslant 2$, there exists an n-uniform non-r-colourable simple hypergraph with

$$\Delta(H) \leqslant n^2 r^{n-1} \ln r.$$

Extremal results concerning colourings of uniform hypergraphs can often be applied to various problems of Ramsey theory. In the next section we shall discuss the application of our main result to the problem of determining the van der Waerden numbers.

2. Bounds for the van der Waerden numbers

In 1927, van der Waerden [25] proved his famous theorem on arithmetic progressions. It states that for any integers $n \geqslant 3$ and $r \geqslant 2$, there exists the minimum number $W(n,r)$ such that any colouring with r colours of the set of integers $\{1,\dots,W(n,r)\}$ contains a monochromatic arithmetic progression of length n. The values $W(n,r)$ from the van der Waerden theorem are called the *van der Waerden numbers*.

The best known upper bounds for $W(n,r)$ were derived from results concerning densities of sets of integers without an arithmetic progression of length n. For each $N > n$, let $r_n(N)$ denote the maximum density of a subset of $\{1,\dots,N\}$ without an arithmetic progression of length n, that is,

$$r_n(N) = \frac{1}{N} \max\{|A| : A \subset \{1,\dots,N\},\ A \text{ does not contain APs of length } n\}.$$

It is easy to understand the following relation between the van der Waerden numbers and $r_n(N)$:

$$\text{if } r_n(N) < \frac{1}{r}, \quad \text{then } W(n,r) \leqslant N.$$

Indeed, every r-colouring of $\{1,\dots,N\}$ has a colour class of size at least N/r. If $r_n(N) < 1/r$ then its density is greater than $r_n(N)$, and by the definition of $r_n(N)$ this colour class should contain an arithmetic progression of length n. So, the upper bounds for $r_n(N)$ imply upper bounds for $W(n,r)$.

The best known bounds on $r_n(N)$ were obtained by Sanders [18] in the case $n = 3$, by Green and Tao [9] in the case $n = 4$, and by Gowers [7] in general. These upper bounds have the following form: an exponent of $r(\ln r)^5$ for $n = 3$,

$$W(3,r) \leqslant \exp\{c\,r(\ln r)^5\};$$

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a double exponent of $(\ln r)^2$ for $W(4,r)$,

$$W(4,r) \leqslant e^{e^{c(\ln r)^2}};$$

a tower of six numbers in the general case,

$$W(n,r) \leqslant 2^{2^{r^{2^{2^{n+9}}}}}. \tag{2.1}$$

By using a simple probabilistic approach, a lower bound for the van der Waerden number $W(n,r)$ was obtained by Erdős and Rado [6] in 1952. By using the simple probabilistic approach they established that

$$W(n,r) \geqslant \sqrt{2(n-1)r^{n-1}}. \tag{2.2}$$

In 1960 this bound was improved for large values of r (in comparison to n) by Moser [14], who gave an explicit construction of an r-colouring of the integers containing no long monochromatic arithmetic progression, and derived that

$$W(n,r) \geqslant n \cdot r^{c \ln r} \tag{2.3}$$

for some absolute constant $c > 0$. The result (2.2) of Erdős and Rado was improved by Schmidt [19] in 1962. He showed that there exists an absolute constant $c > 0$ such that

$$W(n,r) \geqslant r^{n-c\sqrt{n \ln n}}. \tag{2.4}$$

In the case when p is a prime number and $r = 2$, Berlekamp [3] established the relation

$$W(p+1,2) > p2^p. \tag{2.5}$$

Further advances concerning lower bounds for $W(n,r)$ were made employing the results and methods of hypergraph colouring theory. How are the van der Waerden numbers connected with colourings of hypergraphs? For any integers $N > n$, consider a hypergraph $H_n(N) = ([N], E_n(N))$, where $[N] = \{1,2,\ldots,N\}$ and $E_n(N)$ denotes the collection of all arithmetic progressions of length n contained in $[N]$. Clearly, $H_n(N)$ is an n-uniform hypergraph. Note that $H_n(N)$ is an induced subhypergraph in $H_n(N+1)$ and so

$$\chi(H_n(N)) \leqslant \chi(H_n(N+1)).$$

It is easy to see that in terms of hypergraph colouring theory an equivalent definition of the van der Waerden numbers can be formulated as follows:

$$W(n,r) = \min\{N : \ \chi(H_n(N)) > r\}.$$

Thus, to establish the inequality $W(n,r) > N$ for some N, we have to show that the hypergraph of arithmetic progressions $H_n(N)$ is r-colourable. One of the natural ways to do this is to use quantitative relations between the chromatic number and other hypergraph characteristics. For example, applying the bound (1.1) of Erdős and Lovász, one can easily get the following bound:

$$W(n,r) \geqslant \frac{r^{n-1}}{en}\left(1 - \frac{1}{n}\right). \tag{2.6}$$

Indeed, for any $x \in [N]$, there are at most $n(N-1)/(n-1)$ arithmetic progressions of length n from $[N]$ containing x, since any such progression is uniquely defined by the position of x in the

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progression and by the difference of the progression. Thus, if

$$N \leqslant \frac{r^{n-1}}{en}\left(1-\frac{1}{n}\right)+1,$$

then the maximum edge degree of $H_n(N)$ does not exceed r^{n-1}/e, and by (1.1), $H_n(N)$ is r-colourable, so the bound (2.6) is proved. Note that it improves the result (2.4) of Schmidt mentioned above.

Proving Erdős–Lovász-type results for hypergraphs with large girth is of special interest for the studies of the van der Waerden numbers. In 1990 Szabó [24] proved the following lower bound for $W(n,2)$: for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that, for all $n \geqslant n_0$,

$$W(n,2) \geqslant 2^n n^{-\varepsilon}. \tag{2.7}$$

Szabó's approach was to think of the hypergraph of arithmetic progressions $H_n(N)$ as an 'almost simple' hypergraph. It is easy to see that for any arithmetic progression A of length n, there are at most n^4 other progressions sharing more than one integer with A. On the other hand, the number of edges of $H_n(N)$ intersecting A is $\Omega(N)$. Based on these simple observations, Szabó applied the probabilistic technique used for colourings of simple hypergraphs to the hypergraph of arithmetic progressions. Although $H_n(N)$ is not simple, the number of 2-cycles it contains is small, and allowed for the same proof.

The lower bound (2.7) obviously improves (2.6). Moreover, since $\varepsilon > 0$ is arbitrary in the right-hand side of (2.7), the bound is actually of the form

$$W(n,2) \geqslant 2^n n^{-\varepsilon(n)}, \tag{2.8}$$

where $\varepsilon(n) \to 0$ as $n \to \infty$. The calculations from Szabó's proof give $\varepsilon(n)$ of order $(\ln n)^{-1/4}$. More recently, this bound has been improved by Shabanov [22], who showed that one can take $\varepsilon(n) = \Theta(\sqrt{\ln\ln n/\ln n})$. The proof of [22] develops Szabó's approach and considers the hypergraph of arithmetic progressions $H_n(N)$ as a hypergraph almost without 2- and 3-cycles.

The second main result of the paper provides a new lower bound for the van der Waerden numbers.

Theorem 2.1. *There exists an absolute constant $c > 0$ such that, for any $n \geqslant 9$, $r \geqslant 2$,*

$$W(n,r) \geqslant c \cdot r^{n-1}\frac{(\ln\ln n)^2}{\ln n}. \tag{2.9}$$

We remark that the result of Theorem 2.1 improves (2.6), which until now was the best known general bound, and also the main result of [22], which was the best known bound in the case $r = 2$.

When the colour parameter r is large in comparison with n, one can combine the Hypergraph Symmetry Theorem (the reader is referred to the monograph [8] for details) and the lower bounds for the function $r_n(N)$ (see *e.g.* [15]) to obtain better bounds than those given above. In particular, for fixed n and large r, the following inequality can be deduced:

$$W(n,r) \geqslant \exp\{c(n)(\ln r)^m\}, \tag{2.10}$$

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for some $c(n) > 0$ and $m = \lceil \log_2 n \rceil$. However, the bound (2.10) becomes non-trivial only if $r \geqslant n^{\Omega(\sqrt{n})}$. We remark that if $n = 3$ then (2.10) coincides with (2.3), the old result of Moser stated earlier. We also remark that (2.9) is stronger than (2.10) whenever

$$\ln r = O(\sqrt{n} \ln n),$$

and hence provide a new lower bound for the van der Waerden numbers for all pairs in this range.

The proof of Theorem 1.2 is a reduction argument from a similar statement concerning colourings of uniform hypergraphs with girth at least 5. Recall that a *simple cycle of length k* (k-cycle) in the hypergraph H is a sequence $e_0, v_0, e_1, v_1, \ldots, e_{k-1}, v_{k-1}, e_k = e_0$ of k distinct edges $e_0, \ldots, e_{k-1}$ and k distinct vertices $v_0, \ldots, v_{k-1}$ such that $v_i \in e_i \cap e_{i+1}$ for all $i = 0, \ldots, k-1$. The length of the shortest simple cycle in a hypergraph is called *the girth* of the hypergraph, and is denoted by $g(H)$.

Theorem 2.2. *Suppose that $n \geqslant 9$, $r \geqslant 2$, and H is an n-uniform hypergraph with $g(H) > 5$. There exists an absolute constant $c > 0$ such that if*

$$\Delta(H) \leqslant c \cdot r^{n-1} \frac{n(\ln \ln n)^2}{\ln n} \tag{2.11}$$

then H is r-colourable.

In the next section we prove Theorem 2.2. In Section 4 we derive Theorem 2.1. In the final remarks we comment on the proof of Theorem 1.2.

3. Proof of Theorem 2.2

To prove that the hypergraph H is r-colourable we have to show the existence of a proper vertex r-colouring for it. We construct a random r-colouring of H and estimate the probability that this colouring is not proper. We show that certain bad 'local' edge configurations are responsible for the occurrence of the non-proper colouring; we will estimate their probabilities and deduce, using the Local Lemma of Erdős and Lovász [5], that all of them can be avoided with positive probability.

3.1. Construction of random colouring

The construction of a random colouring is based on the method of *random recolouring*. This technique was introduced by Beck [2] and then developed by Spencer [23] and by Radhakrishnan and Srinivasan [17] for 2–colourings. In our work we use the ideas of Radhakrishnan and Srinivasan [17] concerning colourings of sparse hypergraphs.

Without loss of generality, we assume that $V = \{1, \ldots, m\}$. The construction consists of two stages.

First stage: initial colouring. At this stage we randomly colour the vertices of the hypergraph with r colours uniformly and independently. Namely, let $\xi_1, \ldots, \xi_m$ be independent random variables, each taking values $1, 2, \ldots, r$ with equal probability $1/r$. The random vector $\xi = (\xi_1, \ldots, \xi_m)$ can be interpreted as a random r-colouring of the vertex set V (we assign the colour ξ_i to the vertex i).

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The random colouring ξ can contain monochromatic and almost monochromatic edges. We say that an edge $A \in E$ is *almost monochromatic* in ξ if there is a colour $u \in \{1, \ldots, r\}$ such that the number of vertices in A which are coloured with u in ξ is at least $n-s$ and at most $n-1$. In this case the colour u is called *dominating* in A. Here $s < n/2$ is a parameter of our construction. In our proof we will set $s \sim \ln n$, so s will be small in comparison with n.

Formally, for any $A \in E$ and every $u = 1, \ldots, r$, let $\mathcal{M}(A,u)$ and $\mathcal{AM}(A,u)$ denote the following events:

$$\mathcal{M}(A,u) = \bigcap_{j \in A} \{\xi_j = u\}, \quad \mathcal{AM}(A,u) = \left\{1 \leqslant \sum_{j \in A} I\{\xi_j \neq u\} \leqslant s\right\}. \tag{3.1}$$

Here $I\{\mathcal{B}\}$ denotes an indicator of the event $\mathcal{B}$. Thus $\mathcal{M}(A,u)$ denotes the event that A is monochromatic of colour u in ξ, and $\mathcal{AM}(A,u)$ denotes the event that A is almost monochromatic with dominating colour u in ξ.

Second stage: the process of random recolouring. The main principle of the random recolouring method is very simple: during the recolouring stage we would like to recolour some vertices from the monochromatic edges to make them non-monochromatic. The crucial idea provided by Radhakrishnan and Srinivasan is to pay special attention to the almost monochromatic edges during the recolouring procedure. Since they are very close to being monochromatic, we will not allow them to become monochromatic in the dominating colour.

To make a formal construction of the idea described above, consider the following set of random variables.

(1) $X_1, \ldots, X_m$ – independent identically distributed random variables (also independent of ξ) with uniform distribution on $[0,1]$, that is, for any $j = 1, \ldots, m$,

$$\mathbb{P}(X_j < x) = x, \; x \in [0,1].$$

(2) $\{\eta_1, \ldots, \eta_m\}$ – independent identically distributed random variables (also independent of $X_1, \ldots, X_m$) taking values $1, 2, \ldots, r$ with the following conditional distribution: for every $j = 1, \ldots, m$,

$$\mathbb{P}(\eta_j = u \,|\, \xi_j = a) = \frac{1}{r-1} \quad \text{for any } u \neq a \in \{1, \ldots, r\},$$

that is, η_j has uniform conditional distribution on the set $\{1, \ldots, r\} \setminus \{\xi_j\}$.

A continuous-time process of random recolouring goes as follows. Every vertex $v \in V$ is considered only at time X_v, and at this moment of the procedure we check the following two conditions.

Cond1 There is an edge A, $v \in A$, which is monochromatic in the colouring ξ and none of the vertices of A changed the initial colour up to time X_v.

Cond2 The recolouring with colour η_v is not blocked. We say that the recolouring of the vertex v with a colour u is *blocked* if there is an edge B, $v \in B$, such that B is almost monochromatic with dominating colour u in ξ, and at time X_v the vertex v is the last vertex in B which is not coloured with u.

If both conditions **Cond1** and **Cond2** hold then we recolour v with colour η_v. Otherwise, we do not recolour v, and the process continues.

For a vertex v and a time $t \geqslant 0$, let us define a random variable $\zeta_v(t)$, corresponding to the colour of v at time t:

$$\zeta_v(t) = \begin{cases} \xi_v & \text{if } t < X_v, \\ \xi_v & \text{if } t \geqslant X_v \text{ and one of the conditions } \textbf{Cond1}, \textbf{Cond2} \text{ does not hold}, \\ \eta_v & \text{if } t \geqslant X_v \text{ and both } \textbf{Cond1}, \textbf{Cond2} \text{ hold}. \end{cases}$$

Thus, for any $t \geqslant 0$, we have the random r-colouring $\zeta(t) = (\zeta_1(t), \ldots, \zeta_m(t))$ of the hypergraph H. Our aim is to show that for some $t \in (0,1)$ and s, the colouring $\zeta(t)$ is a proper r-colouring of H under the conditions of Theorem 2.2.

Remark. In fact we only need a random ordering of the vertex set to start the recolouring procedure. In our construction we order the vertices according to the values of the random variables $X_1, \ldots, X_m$. Nevertheless, using continuous-time helps to simplify the calculations.

3.2. Bad events

Consider the situation that the colouring $\zeta(t)$, $t > 0$, is not proper for H. Let us denote this event by $\mathcal{F}(t)$. Suppose an edge $A \in E$ is monochromatic in $\zeta(t)$. We have the following covering of $\mathcal{F}(t)$ by three different classes of events.

(1) A bad event of the first type occurs when there is an edge A satisfying the following conditions:

- A is monochromatic in the initial colouring ξ,
- A is still monochromatic of the same colour in $\zeta(t)$,
- up to time t we have already considered at least h vertices of A (h is another parameter, we will choose $h \sim \ln n / \ln\ln n$).

Let $\mathcal{E}(A,t)$ denote the described bad event.

(2) A bad event of the second type (denoted by $\mathcal{D}(A,t)$) occurs when there is an edge A satisfying the following conditions:

- A is monochromatic in ξ,
- it is still monochromatic of the same colour in $\zeta(t)$,
- up to time t we have considered at most $h-1$ vertices of A.

(3) A bad event of the third type (denoted by $\mathcal{G}(A,t)$) happens if there is an edge A such that

- A is monochromatic of colour u in the colouring $\zeta(t)$,
- it is not monochromatic of colour u in the initial colouring ξ.

It is easy to see that $\mathcal{F}(t)$ is a union of these bad events:

$$\mathcal{F}(t) = \bigcup_{A \in E} (\mathcal{E}(A,t) \cup \mathcal{D}(A,t) \cup \mathcal{G}(A,t)). \tag{3.2}$$

In the next few subsections we shall analyse the bad events more closely.

3.3. Bad events of the first type

Suppose that the event $\mathcal{E}(A,t)$ occurs for some edge $A \in E$. This event implies that the edge A is monochromatic in the initial colouring, and that after the consideration of its first h vertices

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(according to the ordering provided by the random variables $X_1,\dots,X_m$) it is still monochromatic of the same colour, *i.e.*, no recolouring has been made. We will show that a special hypertree configuration of depth at most 3 is responsible for this.

Suppose that $v_1,\dots,v_h$ are the first h vertices of A to be considered, that is,

$$X_{v_1} < X_{v_2} < \cdots < X_{v_h} \text{ and } X_{v_h} < X_{v'}, \quad \text{for any } v' \in A\setminus\{v_1,\dots,v_h\}.$$

At time X_{v_h} we have considered all of them, but no recolouring has been made. Why did we not change their colours? Since A is monochromatic in the initial colouring ξ, the condition **Cond1** holds for every v_i. So the second condition **Cond2** does not hold, that is, the recolouring of v_i with colour η_{v_i} is blocked by some almost monochromatic edge B_i. Due to our algorithm we have the following properties for the *blocking* edge B_i.

BL1 B_i is almost monochromatic in the initial colouring ξ with dominating colour η_{v_i}.
BL2 at time X_{v_i} the vertex v_i remains the only vertex of B_i which is not coloured with η_{v_i}.

Property **BL2** implies that any other vertex of B_i which is not coloured with η_{v_i} in the colouring ξ should be recoloured with η_{v_i} up to time X_{v_i}. Property **BL1** implies that the number of such vertices is at most $s-1$ (recall that v_i is not coloured with η_{v_i} in ξ).

Let $\{w_{i,1},\dots,w_{i,y_i}\}$, where $y_i \leqslant s-1$, denote this subset of vertices of $B_i\setminus\{v_i\}$. Since every vertex $w_{i,j}$ ($i=1,\dots,h$, $j=1,\dots,y_i$) has been recoloured with colour η_{v_i} up to time X_{v_i}, we have $\eta_{w_{i,j}} = \eta_{v_i}$ and, moreover, there is an edge $C_{i,j}$ containing $w_{i,j}$ such that

- $C_{i,j}$ is monochromatic in the initial colouring ξ,
- up to time $X_{w_{i,j}}$ no recolouring has been made in the edge $C_{i,j}$, that is, $w_{i,j}$ is the first vertex of $C_{i,j}$ which changes its colour during the recolouring process.

Thus, we get a hypertree with 'trunk' A, 'branches' $B_1,\dots,B_h$ and 'leaves' $C_{1,1},\dots,C_{1,y_1}$, $C_{2,1},\dots,C_{h,1},\dots,C_{h,y_h}$ ($C_{i,1},\dots,C_{i,y_i}$ are the 'leaves' of 'branch' B_i). Since the girth of hypergraph H is greater than 5, it is indeed a hypertree.

Let us summarize. We define *a first-type configuration* $(A,\Phi,\mathbf{y},\Lambda)$ as follows:

- A is an edge of H,
- $\Phi = (B_1,\dots,B_h)$ is an ordered collection of distinct edges of A such that $|B_i\cap A| = 1$ for any i (since H is simple), and the vertices $v_i = B_i \cap A$ are distinct,
- $\mathbf{y} = (y_1,\dots,y_h)$ is a vector from $\{0,1,\dots,s-1\}^h$,
- Λ is an unordered collection of edges $\Lambda = \{C_{1,1},\dots,C_{1,y_1},C_{2,1},\dots,C_{h,y_h}\}$ such that $|C_{i,j}\cap B_i| = 1$ for any i,j, and all the vertices $w_{i,j} = C_{i,j}\cap B_i$ are distinct,
- the set of edges A, $B_1,\dots,B_h$, $C_{1,1},\dots,C_{h,y_h}$ form a hypertree with 'trunk' A, 'branches' $B_1,\dots,B_h$ and 'leaves' $C_{1,1},\dots,C_{h,y_h}$.

We shall denote the set of first-type configurations by Υ_1. The above discussion shows that the event $\mathcal{E}(A,t)$ implies an event $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$ for some first-type configuration $(A,\Phi,\mathbf{y},\Lambda)$, where $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$ denotes the event that:

(1) A is monochromatic in ξ; no recolouring is made up to the consideration of its first h vertices $v_1,\dots,v_h$;

(2) every B_i is almost monochromatic in ξ; v_1, $w_{i,1},\dots,w_{i,y_i}$ are the vertices which are not coloured with dominating colour; at time X_{v_i} only v_i is not coloured with the dominating colour;

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(3) every $C_{i,j}$ is monochromatic in ξ; $w_{i,j}$ is its first recoloured vertex during the recolouring procedure.

Thus, we have

$$\mathcal{E}(A,t) \subset \bigcup_{\substack{\Phi,\mathbf{y},\Lambda:\\ (A,\Phi,\mathbf{y},\Lambda)\in \Upsilon_1}} \mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda). \tag{3.3}$$

Now we are going to analyse the event $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$ more closely. Since this event implies that any leaf-edge $C_{i,j} \in \Lambda$ is monochromatic, the recolouring of any vertex preceding $w_{i,j}$ in this edge is blocked (recall that $w_{i,j}$ is the first vertex of $C_{i,j}$ that changes its colour during the recolouring process). The number of such vertices is a random variable

$$\mu_{i,j} = \sum_{w' \in C_{i,j}\setminus\{w_{i,j}\}} I\{X_{w'} < X_{w_{i,j}}\}.$$

Let $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ denote the event $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$ with the additional condition that $\mu_{i,j}$ does not exceed $h-1$ for any i and j, that is, in any leaf-edge from Λ at most $h-1$ vertices had been considered before the successful recolouring was made. The following claim is a key step in the analysis of the event $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$.

Claim 1. *The event $\mathcal{A}_0(A,\Phi,\boldsymbol{y},\Lambda)$ satisfies the relation*

$$\mathcal{A}_0(A,\Phi,\boldsymbol{y},\Lambda) \subset \bigcup_{\substack{A',\Phi',\boldsymbol{y}',\Lambda':\\ (A',\Phi',\boldsymbol{y}',\Lambda')\in \Upsilon_1}} \mathcal{A}(A',\Phi',\boldsymbol{y}',\Lambda'). \tag{3.4}$$

Proof. Suppose the event $\mathcal{A}_0(A,\Phi,\mathbf{y},\Lambda)$ holds. It implies that the edge $A \in E$ was monochromatic initially and no recolouring was made after the consideration of its first h vertices. For each $C_{i,j} \in \Lambda$, if the inequality

$$\mu_{i,j} = \sum_{w' \in C_{i,j}\setminus\{w_{i,j}\}} I\{X_{w'} < X_{w_{i,j}}\} \geqslant h \tag{3.5}$$

holds, then $w_{i,j}$ does not belong to the first h vertices of $C_{i,j}$. Since $w_{i,j}$ is the first vertex of $C_{i,j}$ that was recoloured, we get the same property for $C_{i,j}$ as for A: monochromatic initially and no recolouring was made after the consideration of its first h vertices. Note that the construction of the recolouring procedure implies that $w_{i,j}$ (and consequently also the hth vertex of $C_{i,j}$) should be considered before the hth vertex of A. Thus, for some first-type configuration $(C_{i,j},\widetilde{\Phi},\widetilde{\mathbf{y}},\widetilde{\Lambda})$, the event $\mathcal{A}_0(C_{i,j},\widetilde{\Phi},\widetilde{\mathbf{y}},\widetilde{\Lambda})$ occurs. Now we apply the same argument to the event $\mathcal{A}_0(C_{i,j},\widetilde{\Phi},\widetilde{\mathbf{y}},\widetilde{\Lambda})$.

Since our hypergraph is finite and every time the first h vertices should be considered earlier than in the previous edge, after several repetitions of the described argument we will get an edge A' and a first-type configuration $(A',\Phi',\mathbf{y}',\Lambda')$ such that in every monochromatic leaf-edge from Λ' at most $h-1$ vertices had been considered before the successful recolouring was made (note that the set Λ' can be empty). This implies the event $\mathcal{A}(A',\Phi',\mathbf{y}',\Lambda')$ and the relation (3.4) is established. □

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It remains to estimate the probability of the event $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ for a given first-type configuration $(A,\Phi,\mathbf{y},\Lambda)$.

Claim 2.

$$\mathbb{P}(\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)) \leqslant r^{-(n-1)(1+h+\sum_{i=1}^{h} y_i)} \left(\frac{h}{n}\right)^{\sum_{i=1}^{h} y_i} (n-h)^{-h}. \tag{3.6}$$

Proof. Let us fix the colour u of an edge A in ξ, dominating colours $u_1,\dots,u_h$ for the edges $B_1,\dots,B_h$ and the colours $u_{i,j}$ for the monochromatic edges $C_{i,j}$. Our construction of the hypertree implies that, for given values of u, u_i for each $i \leqslant h$, and $u_{i,j}$ for each $i \leqslant h$ and $j \leqslant y_i$, the initial colours of all the vertices in the configuration are uniquely determined:

$$\xi_v = \begin{cases} u & \text{if } v \in A, \\ u_i & \text{if } v \in B_i \setminus (A \cup \bigcup_{j=1}^{y_i} C_{i,j}), \\ u_{i,j} & \text{if } v \in C_{i,j}. \end{cases}$$

The number of edges in the hypertree is equal to $1+h+\sum_{i=1}^{h} y_i$, so the number of vertices is equal to $1+(n-1)(1+h+\sum_{i=1}^{h} y_i)$, and therefore

$$\mathbb{P}\left(\bigcap_{v\in A}\{\xi_v = u\} \cap \bigcap_{i=1}^{h} \bigcap_{v \in B_i \setminus (A \cup \bigcup_{j=1}^{y_i} C_{i,j})} \{\xi_v = u_i\} \cap \bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\bigcap_{v\in C_{i,j}} \{\xi_v = u_{i,j}\}\right)$$
$$= r^{-1-(n-1)(1+h+\sum_{i=1}^{h} y_i)}. \tag{3.7}$$

The values of the random variables η are also uniquely determined for the node vertices of the hypertree:

$$\eta_v = \begin{cases} u_i & \text{if } v = v_i,\ i = 1,\dots,h, \\ u_i & \text{if } v = w_{i,j},\ i=1,\dots,h,\ j = 1,\dots,y_i. \end{cases}$$

Since η_v has uniform conditional distribution on $\{1,\dots,r\} \setminus \{\xi_v\}$ and we know that $\xi_{v_i} = u$, $\xi_{w_{i,j}} = u_{i,j}$ for any i, j, we get

$$\mathbb{P}\left(\bigcap_{i=1}^{h}\{\eta_{v_i} = u_i\} \cap \bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\{\eta_{w_{i,j}} = u_i\} \;\middle|\; \bigcap_{i=1}^{h}\{\xi_{v_i} = u\} \cap \bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\{\xi_{w_{i,j}} = u_{i,j}\}\right)$$
$$= (r-1)^{-(h+\sum_{i=1}^{h} y_i)}. \tag{3.8}$$

Furthermore, we know that the number of every vertex $w_{i,j}$ in the edge $C_{i,j}$ is at most h, and that all these events are independent since the edges $C_{i,j}$ do not intersect, so

$$\mathbb{P}\left(\bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\left\{\sum_{v\in C_{i,j}} I\{X_v \leqslant X_{w_{i,j}}\} \leqslant h\right\}\right) = \left(\frac{h}{n}\right)^{\sum_{i=1}^{h} y_i}. \tag{3.9}$$

Finally, we know that v_i has number i in the edge A, $i=1,\dots,h$. The probability of this event can be easily calculated:

$$\mathbb{P}\Bigg(\{X_{v_1}<\cdots<X_{v_h}\}\cap\bigcap_{v\in A\setminus\{v_1,\dots,v_h\}}\{X_{v_h}<X_v\}\Bigg)=\frac{1}{n(n-1)\cdots(n-h+1)}. \tag{3.10}$$

The event $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ is the intersection of the events in (3.7)–(3.10). The edge A does not intersect with the edges $C_{i,j}$, so the events in (3.9) and (3.10) are independent. Moreover, they are also independent of the events in (3.7) and (3.8) since the random variables $\{X_v, v\in V\}$ and $\{\xi_v,\eta_v,v\in V\}$ are independent. Finally,

$$\begin{aligned}
&\mathbb{P}(\mathcal{A}(A,\Phi,\mathbf{y},\Lambda))\\
&=\sum_{u=1}^{r}\sum_{\substack{u_1,\dots,u_h=1,\\ u_i\neq u,\, i=1,\dots,h}}^{r}\sum_{\substack{u_{1,i},\dots,u_{h,y_h}=1,\\ u_{i,j}\neq u_i,\, j=1,\dots,y_i}}^{r}\mathbb{P}\Bigg(\bigcap_{i=1}^{h}\{\eta_{v_i}=u_i\}\cap\bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\{\eta_{w_{i,j}}=u_i\}\\
&\qquad\cap\bigcap_{v\in A}\{\xi_v=u\}\cap\bigcap_{i=1}^{h}\bigcap_{v\in B_i\setminus(A\cup\bigcup_{j=1}^{y_i}C_{i,j})}\{\xi_v=u_i\}\cap\bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\bigcap_{v\in C_{i,j}}\{\xi_v=u_{i,j}\}\\
&\qquad\cap\bigcap_{i=1}^{h}\bigcap_{j=1}^{y_i}\Bigg\{\sum_{v\in C_{i,j}}I\{X_v\leqslant X_{w_{i,j}}\}\leqslant h\Bigg\}\cap\{X_{v_1}<\cdots<X_{v_h}\}\cap\bigcap_{v\in A\setminus\{v_1,\dots,v_h\}}\{X_{v_h}<X_v\}\Bigg)\\
&=\sum_{u=1}^{r}\sum_{\substack{u_1,\dots,u_h=1,\\ u_i\neq u,\, i=1,\dots,h}}^{r}\sum_{\substack{u_{1,i},\dots,u_{h,y_h}=1,\\ u_{i,j}\neq u_i,\, j=1,\dots,y_i}}^{r}r^{-1-(n-1)(1+h+\sum_{i=1}^h y_i)}(r-1)^{-(h+\sum_{i=1}^h y_i)}\\
&\qquad\times\left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i}\frac{1}{n(n-1)\cdots(n-h+1)}
\end{aligned}$$

(the colours can be chosen in $r(r-1)^{h+\sum_{i=1}^h y_i}$ ways)

$$=r^{-(n-1)(1+h+\sum_{i=1}^h y_i)}\left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i}\frac{(n-h)!}{n!}\leqslant r^{-(n-1)(1+h+\sum_{i=1}^h y_i)}\left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i}(n-h)^{-h}. \qquad\square$$

This deals with the first bad event and we proceed to the second one.

3.4. Bad events of the second type

Let us consider a bad event $\mathcal{D}(A,t)$ of the second type. By definition, this event implies that we have considered fewer than h vertices, that is, $\sum_{v\in A}I\{X_v\leqslant t\}\leqslant h-1$. We know also that A is monochromatic in ξ. Denoting the intersection of these events by $\mathcal{B}(A,t)$,

$$\mathcal{B}(A,t)=\Bigg(\bigcup_{u=1}^{r}\bigcap_{v\in A}\{\xi_v=u\}\Bigg)\cap\Bigg\{\sum_{v\in A}I\{X_v\leqslant t\}\leqslant h-1\Bigg\},$$

we get

$$\mathcal{D}(A,t)\subset\mathcal{B}(A,t). \tag{3.11}$$

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The random variables X_v are independent of the initial colouring ξ, so for $t > 2/n$ we obtain the following upper bound for the probability of the event $\mathcal{B}(A,t)$:

$$\begin{aligned}\mathbb{P}(\mathcal{B}(A,t)) &= \mathbb{P}\left(\bigcup_{u=1}^{r}\bigcap_{v\in A}\{\xi_v = u\}\cap\left\{\sum_{v\in A} I\{X_v \leqslant t\} \leqslant h-1\right\}\right)\\ &= \mathbb{P}\left(\bigcup_{u=1}^{r}\bigcap_{v\in A}\{\xi_v = u\}\right)\mathbb{P}\left(\sum_{v\in A} I\{X_v \leqslant t\} \leqslant h-1\right)\\ &= \sum_{u=1}^{r}\mathbb{P}\left(\bigcap_{v\in A}\{\xi_v = u\}\right)\mathbb{P}\left(\sum_{v\in A} I\{X_v \leqslant t\} \leqslant h-1\right)\end{aligned}$$

(since $\sum_{v\in A} I\{X_v \leqslant t\}$ is a binomial random variable)

$$\begin{aligned}&= r^{1-n}\sum_{i=0}^{h-1}\binom{n}{i}t^i(1-t)^{n-i} \leqslant r^{1-n}(1-t)^{n-h+1}\sum_{i=0}^{h-1}(nt)^i\\ &\leqslant r^{1-n}(1-t)^{n-h+1}(nt)^h. \end{aligned} \tag{3.12}$$

3.5. Bad events of the third type

The last part of the event $\mathcal{F}(t)$ consists of bad events $\mathcal{G}(A,t)$ of the third type. Recall that $\mathcal{G}(A,t)$ occurs if the edge A is monochromatic of some colour u in the colouring $\zeta(t)$, but in the initial colouring ξ it was not monochromatic of this colour. Since during the recolouring process we forbid almost monochromatic edges to become completely monochromatic of the dominating colour, A is not almost monochromatic with dominating colour u in ξ. Hence, the number of vertices in A which are not coloured u in ξ is at least $s+1$ (see (3.1)).

Suppose that $\{v_1,\dots,v_l\}$ is the set of all vertices of A which are not coloured with u in ξ. Since $\zeta_{v_i}(t) = u$ for any $i = 1,\dots,l$, all these vertices should be recoloured with u up to time t. Thus, for any v_i, at time X_{v_i} both conditions **Cond1** and **Cond2** hold. In this case our construction provides the existence of a set of l edges, $\{B_1,\dots,B_l\}$, with the following properties:

(1) $v_i \in B_i\cap A$ and $\eta_{v_i} = u$, $i = 1,\dots,l$,
(2) B_i is monochromatic in the initial colouring ξ of a colour other than u,
(3) v_i is the first recoloured vertex in B_i during the recolouring process.

Moreover, we can assume that, for every edge B_i, the event $\mathcal{E}(B_i,t)$ does not hold, since we have already analysed it in Section 3.3. The complement of this event implies that the number of vertices considered in B_i before v_i is at most $h-1$, otherwise the edge B_i would still be monochromatic after the consideration of its hth vertex. Hence, our configuration of edges also satisfies the following additional property:

(4) for any $i = 1,\dots,l$,

$$\sum_{v\in B_i\setminus\{v_i\}} I\{X_v < X_{v_i}\} \leqslant h-1,\ X_{v_i} \leqslant t.$$

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The described properties form an event which we denote by $\mathcal{C}(A,l,\Phi)$, where $\Phi=\{B_1,\dots,B_l\}$. Formally,

$$\mathcal{C}(A,l,\Phi,t)=\bigcup_{u=1}^{r}\bigcup_{u_1,\dots,u_l:u_j\neq u}\left(\bigcap_{i=1}^{l}\bigcap_{v\in B_i}\{\xi_v=u_i\}\cap\bigcap_{v\in A\setminus\{v_1,\dots,v_l\}}\{\xi_v=u\}\right)$$
$$\cap\bigcap_{i=1}^{l}\{\eta_{v_i}=u,X_{v_i}\leqslant t\}\cap\bigcap_{i=1}^{l}\left\{\sum_{v\in B_i\setminus\{v_i\}}I\{X_v<X_{v_i}\}\leqslant h-1\right\}. \quad (3.13)$$

Since the girth of hypergraph H is at least 6, the set of edges $A,B_1,\dots,B_l$ forms a hypertree with 'trunk' A and 'branches' $B_1,\dots,B_l$. Note that the edges $B_1,\dots,B_l$ are pairwise disjoint since all the vertices $v_1,\dots,v_l$ are distinct.

Let us define *a second-type configuration* (A,l,Φ) as follows:

- A is an edge of E,
- $l\in\{s+1,\dots,n\}$,
- Φ is an unordered collection of distinct edges $\Phi=\{B_1,\dots,B_l\}$, such that $|B_i\cap e|=1$ for any i, and the vertices $v_i=B_i\cap A$ are distinct,
- the set of edges $A,B_1,\dots,B_l$ forms a hypertree.

We denote the set of second-type configurations by Υ_2. The above discussion implies the following relation:

$$\mathcal{G}(A,t)\cap\overline{\bigcup_{B\in E}\mathcal{E}(B,t)}\subset\bigcup_{l,\Phi:\,(A,l,\Phi)\in\Upsilon_2}\mathcal{C}(A,l,\Phi,t). \quad (3.14)$$

Let us estimate the probability of the event $\mathcal{C}(A,l,\Phi,t)$ for a given second-type configuration (A,l,Φ,t).

Claim 3.

$$\mathbb{P}(\mathcal{C}(A,l,\Phi,t))\leqslant r^{-(n-1)(l+1)}\left(\frac{h}{n}\right)^{l}. \quad (3.15)$$

Proof. Let us fix a colour u of A in the final colouring and $u_1,\dots,u_l$ as colours of $B_1,\dots,B_l$ in the initial colouring. Thus, for given $u,u_1,\dots,u_l$, the initial colours of all the vertices in the configuration are uniquely defined:

$$\xi_v=\begin{cases}u & \text{if } v\in A\setminus(\bigcup_{i=1}^{l}B_i),\\ u_i & \text{if } v\in B_i.\end{cases}$$

The number of edges in the hypertree is equal to $1+l$, so the number of vertices is equal to $1+(n-1)(1+l)$, and thus

$$\mathbb{P}\left(\bigcap_{i=1}^{l}\bigcap_{v\in B_i}\{\xi_v=u_i\}\cap\bigcap_{v\in A\setminus(\bigcup_{i=1}^{l}B_i)}\{\xi_v=u\}\right)=r^{-1-(n-1)(1+l)}. \quad (3.16)$$

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The values of the random variables η are also uniquely determined for the vertices v_i: $\eta_{v_i} = u$. Thus

$$\mathbb{P}\left(\bigcap_{i=1}^{l}\{\eta_{v_i} = u\} \;\middle|\; \bigcap_{i=1}^{l}\{\xi_{v_i} = u_i\}\right) = (r-1)^{-l}. \tag{3.17}$$

Finally, every vertex v_i such that $v_{v_i} = u$ has one of the h smallest values of X_v among $v \in B_i$. Since the edges B_i do not intersect we get

$$\mathbb{P}\left(\bigcap_{i=1}^{l}\left\{\sum_{v\in B_i\setminus\{v_i\}} I\{X_v < X_{v_i}\} \leqslant h-1\right\}\right) = \left(\frac{h}{n}\right)^{l}. \tag{3.18}$$

Consequently, by (3.13) we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{C}(A,l,\Phi,t)) &\leqslant \sum_{u=1}^{r} \sum_{\substack{u_1,\dots,u_l=1,\\ u_i\neq u,\, i=1,\dots,l}}^{r} r^{-1-(n-1)(1+l)}(r-1)^{-l}\left(\frac{h}{n}\right)^{l} \\ &= r(r-1)^{l} r^{-1-(n-1)(1+l)}(r-1)^{-l}\left(\frac{h}{n}\right)^{l} = r^{-(n-1)(l+1)}\left(\frac{h}{n}\right)^{l}.\end{aligned}$$

Note that we discarded the conditions $X_{v_i} \leqslant t$, $i = 1,\dots,l$. □

3.6. Application of the Local Lemma

In the previous sections we have analysed a collection of bad events whose union contains the event $\mathcal{F}(t)$. Remember that $\mathcal{F}(t)$ is the event that the random colouring $\zeta(t)$ is not a proper r-colouring of hypergraph H. Recall equality (3.2):

$$\mathcal{F}(t) = \bigcup_{A\in E}(\mathcal{E}(A,t)\cup\mathcal{D}(A,t)\cup\mathcal{G}(A,t)).$$

It follows from the relations (3.3), (3.4), (3.11) and (3.14) that

$$\begin{aligned}\mathcal{F}(t) \subset \bigcup_{A\in E}\mathcal{B}(A,t) &\cup \bigcup_{(A,\Phi,\mathbf{y},\Lambda)\in\Upsilon_1}\mathcal{A}(A,\Phi,\mathbf{y}) \\ &\cup \bigcup_{(A,l,\Phi)\in\Upsilon_2}\mathcal{C}(A,l,\Phi,t).\end{aligned} \tag{3.19}$$

Our aim is to show that, for some parameters $h, s \in \mathbb{N}$ and $t \in (0,1)$, the probability of the event $\mathcal{F}(t)$ is strictly less than 1. To prove this we shall use a classical result, known as the Local Lemma, which was first obtained in the paper of Erdős and Lovász [5]. We shall formulate it in a special case convenient for later use.

Theorem 3.1. *Let events $\mathcal{Q}_1,\dots,\mathcal{Q}_M$ be given on some probability space. Let $S_1,\dots,S_M$ be subsets of $[M] = \{1,\dots,M\}$ such that, for any $i \in [M]$, the event $\mathcal{Q}_i$ is independent of the algebra generated by the events $\{\mathcal{Q}_j, j \in [M]\setminus(S_i \cup \{i\})\}$. If, for any $i \in [M]$, the following inequality holds,*

$$\sum_{j\in S_i\cup\{i\}} \mathbb{P}(\mathcal{Q}_j) \leqslant 1/4, \tag{3.20}$$

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then

$$\mathbb{P}\left(\bigcap_{j=1}^{M} \overline{\mathcal{Q}_j}\right) \geqslant \prod_{j=1}^{M}(1-2\mathbb{P}(\mathcal{Q}_j)) > 0.$$

The proof of the Local Lemma in the general case can be found in the monograph [1].

Consider the system of events $\Psi(t)$ consisting of all the events $\mathcal{B}(A,t)$, $A \in E$, all the events $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$, where $(A,\Phi,\mathbf{y},\Lambda) \in \Upsilon_1$, and all the events $\mathcal{C}(A,l,\Phi,t)$, where $(A,l,\Phi) \in \Upsilon_2$. By (3.19) we have

$$\mathbb{P}(\mathcal{F}(t)) \leqslant \mathbb{P}\left(\bigcup_{\mathcal{Q}\in\Psi(t)} \mathcal{Q}\right) = 1-\mathbb{P}\left(\bigcap_{\mathcal{Q}\in\Psi(t)} \overline{\mathcal{Q}}\right). \tag{3.21}$$

We would like to show that the probability of $\bigcap_{\mathcal{Q}\in\Psi(t)} \overline{\mathcal{Q}}$ is greater than zero. Due to the Local Lemma it is sufficient to find, for every $\mathcal{Q} \in \Psi(t)$, a system of events $\Psi_{\mathcal{Q}} \subset \Psi(t)$ such that $\mathcal{Q} \in \Psi_{\mathcal{Q}}$, $\mathcal{Q}$ and the algebra generated by $\{\mathcal{J} \in \Psi(t)\backslash\Psi_{\mathcal{Q}}\}$ are independent, and

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}} \mathbb{P}(\mathcal{J}) \leqslant 1/4. \tag{3.22}$$

The event $\mathcal{Q} \in \Psi(t)$ can have one of the following three types:

(1) $\mathcal{Q} = \mathcal{B}(A,t)$ for some $A \in E$,
(2) $\mathcal{Q} = \mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ for some first-type configuration $(A,\Phi,\mathbf{y},\Lambda) \in \Upsilon_1$,
(3) $\mathcal{Q} = \mathcal{C}(A,l,\Phi,t)$ for some second-type configuration $(A,l,\Phi) \in \Upsilon_2$.

For any $\mathcal{Q} \in \Psi(t)$, we define *the domain* $D(\mathcal{Q})$ of the event $\mathcal{Q}$ and the edge set $E(\mathcal{Q})$ as follows:

$$D(\mathcal{Q}) = \begin{cases} A & \text{if } \mathcal{Q} = \mathcal{B}(A,t), \\ A \cup \bigcup_{B\in\Phi} B \cup \bigcup_{C\in\Lambda} C & \text{if } \mathcal{Q} = \mathcal{A}(A,\Phi,\mathbf{y},\Lambda), \\ A \cup \bigcup_{B\in\Phi} B & \text{if } \mathcal{Q} = \mathcal{C}(A,l,\Phi,t), \end{cases}$$

$$E(\mathcal{Q}) = \begin{cases} \{A\} & \text{if } \mathcal{Q} = \mathcal{B}(A,t), \\ \{A\} \cup \Phi \cup \Lambda & \text{if } \mathcal{Q} = \mathcal{A}(A,\Phi,\mathbf{y},\Lambda), \\ \{A\} \cup \Phi & \text{if } \mathcal{Q} = \mathcal{C}(A,l,\Phi,t). \end{cases}$$

Each of the events $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$, $\mathcal{B}(A,t)$ and $\mathcal{C}(A,l,\Phi,t)$ are defined based uniquely on the information about the random variables associated with the vertices in its domain. In other words, every $\mathcal{Q} \in \Psi(t)$ belongs to the algebra generated by the random variables $\{\xi_j, \eta_j, X_j : j \in D(\mathcal{Q})\}$. Therefore, this event is independent of the algebra generated by the random variables $\{\xi_j, \eta_j, X_j : j \in V \setminus D(\mathcal{Q})\}$. So, we can choose the system $\Psi_{\mathcal{Q}}$ to consist of all the events $\mathcal{J} \in \Psi(t)$ such that $D(\mathcal{J}) \cap D(\mathcal{Q}) \neq \emptyset$:

$$\Psi_{\mathcal{Q}} = \{\mathcal{J} : \mathcal{J} \in \Psi(t),\ D(\mathcal{J}) \cap D(\mathcal{Q}) \neq \emptyset\}.$$

Thus, the event $\mathcal{Q}$ is independent of the algebra generated by $\{\mathcal{J} \in \Psi(t)\backslash\Psi_{\mathcal{Q}}\}$. Moreover, $\mathcal{Q} \in \Psi_{\mathcal{Q}}$. It remains to check the inequality (3.22). By the choice of the set $\Psi_{\mathcal{Q}}$, for any $\mathcal{Q} \in \Psi(t)$ we

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have

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathbb{P}(\mathcal{J})\leqslant\sum_{J\in E(\mathcal{Q})}\Bigg[\sum_{A\in E:\,J\cap A\neq\emptyset}\mathbb{P}(\mathcal{B}(A,t))+\sum_{\substack{(A,\Phi,\mathbf{y},\Lambda)\in\Upsilon_1:\\ J\cap(A\cup\bigcup_{B\in\Phi}B\cup\bigcup_{C\in\Lambda}C)\neq\emptyset}}\mathbb{P}(\mathcal{A}(A,\Phi,\mathbf{y},\Lambda))$$
$$+\sum_{\substack{(A,l,\Phi)\in\Upsilon_2:\\ J\cap(A\cup\bigcup_{B\in\Phi}B)\neq\emptyset}}\mathbb{P}(\mathcal{C}(A,l,\Phi,t))\Bigg].\tag{3.23}$$

In what follows, we shall analyse the sums in the right-hand side of (3.23) separately. For convenience, we also use the notation Δ for the maximum edge degree $\Delta(H)$ of the hypergraph H. In fact, all we need to do is to estimate the number of different configurations intersecting a fixed edge A, that is, the number of edge configurations in which at least one edge has non-empty intersection with A.

For the first sum in the right-hand side of (3.23), the configuration consists of only one edge, so the number of intersecting edges is at most $\Delta+1$. So, we have to deal with the remaining two sums. Let us start with the second one.

Claim 4. *For a fixed edge J and $\mathbf{y}=(y_1,\dots,y_h)$, there are at most*

$$\Delta^{1+h+\sum_{i=1}^{h}y_i}\left(\frac{1+hs}{\prod_{i=1}^{h}(y_i!)}\right)$$

configurations $(A,\Phi,\mathbf{y},\Lambda)$ of the first type intersecting J.

Proof. If

$$J\cap\left(A\cup\bigcup_{B\in\Phi}B\cup\bigcup_{C\in\Lambda}C\right)\neq\emptyset,$$

then there are three possibilities.

- Suppose $J\cap A\neq\emptyset$. Then the edge A can be chosen in at most $\Delta+1$ ways, the ordered set Φ of h edges $\Phi=(B_1,\dots,B_h)$ can be chosen in at most $\Delta(\Delta-1)\cdots(\Delta-h+1)$ ways (every B_i should intersect A and not coincide with it) and the unordered set Λ of $y_1+\cdots+y_h$ edges $\Lambda=(C_{1,1},\dots,C_{1,y_1},\dots,C_{h,y_h})$ can be chosen in at most $\binom{\Delta-1}{y_1}\cdots\binom{\Delta-1}{y_h}$ ways (for any $i=1,\dots,h$, $j=1,\dots,y_h$, the edge $C_{i,j}$ should intersect B_i and not coincide with it and with A). Thus, the number of such configurations is at most
$$(\Delta+1)\Delta(\Delta-1)\cdots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right).\tag{3.24}$$
- Suppose $J\cap B_i\neq\emptyset$ for some $i=1,\dots,h$, but $J\cap A=\emptyset$. Thus, the number i and the edge B_i can be chosen in at most $h(\Delta+1)$ ways, the edge A can be chosen in at most $\Delta-1$ ways (among edges intersecting B_i we cannot choose B_i itself and J), the remaining set of $h-1$ edges $(B_1,\dots,B_{i-1},B_{i+1},\dots,B_h)$ can be chosen in at most $(\Delta-1)\cdots(\Delta-h+1)$ ways and, finally, the unordered set Λ of $y_1+\cdots+y_h$ edges as in the previous case can be chosen in at most $\binom{\Delta-1}{y_1}\cdots\binom{\Delta-1}{y_h}$ ways. Thus, the number of configurations in the second situation is at

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most

$$h(\Delta+1)(\Delta-1)(\Delta-1)\cdots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right). \tag{3.25}$$

- The last option is that $J\cap C_{i,j}\neq\emptyset$ for some $i=1,\dots,h$, $j=1,\dots,y_i$, but $J\cap(A\cup B_1\cup\cdots\cup B_h)=\emptyset$. In this case the edge $C_{i,j}$ can be chosen in at most $\Delta+1$ ways, the edge B_i can be chosen in at most $\Delta-1$ ways (among edges intersecting $C_{i,j}$ we cannot choose $C_{i,j}$ itself and J), the edge A can be chosen in at most $\Delta-1$ ways (we cannot choose B_i itself and $C_{i,j}$), the remaining set of $h-1$ edges $(B_1,\dots,B_{i-1},B_{i+1},\dots,B_h)$ can be chosen in at most $(\Delta-1)\cdots(\Delta-h+1)$ ways and the rest of the set Λ, that is, $(C_{1,1},\dots,C_{h,y_h})$ without $C_{i,j}$, can be chosen in at most

$$\binom{\Delta-1}{y_1}\cdots\binom{\Delta-2}{y_i-1}\cdots\binom{\Delta-1}{y_h}$$

ways. Thus, in this case the number of configurations is at most

$$\sum_{i=1}^{h}(\Delta+1)(\Delta-1)^2(\Delta-1)\cdots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)\frac{y_i}{\Delta-1}$$
$$=(\Delta+1)(\Delta-1)(\Delta-1)\cdots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)(y_1+\cdots+y_h). \tag{3.26}$$

Summarizing (3.24), (3.25) and (3.26), we obtain the following upper bound for the number of configurations of the first type with fixed $\mathbf{y}$:

$$(\Delta+1)(\Delta-1)\cdots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)\left((h+\sum_{i=1}^{h}y_i)(\Delta-1)+\Delta\right)$$
$$\leqslant \Delta^{1+h+\sum_{i=1}^{h}y_i}\left(\frac{1+h+\sum_{i=1}^{h}y_i}{\prod_{i=1}^{h}(y_i!)}\right)\leqslant \Delta^{1+h+\sum_{i=1}^{h}y_i}\left(\frac{1+hs}{\prod_{i=1}^{h}(y_i!)}\right).$$

The last inequality follows from the simple estimate $y_i\leqslant s-1$, $i=1,\dots,h$. □

Claim 5. *For a fixed edge J and $l\geqslant s$, there are at most*

$$\Delta^{1+l}\left(\frac{1+l}{l!}\right)$$

configurations (A,l,Φ) of the second type intersecting J.

Proof. If $J\cap A\neq\emptyset$, then A can be chosen in at most $\Delta+1$ ways and the unordered set of edges $\Phi=\{B_1,\dots,B_l\}$ in at most $\binom{\Delta}{l}$ ways. If $J\cap B_i\neq\emptyset$ for some $B_i\in\Phi$, then B_i can be chosen in at most $\Delta+1$ ways, the edge A in at most $\Delta-1$ ways and the remaining edges of Φ in at most $\binom{\Delta-1}{l-1}$ ways. So, the total number of configurations does not exceed

$$(\Delta+1)\binom{\Delta}{l}+(\Delta+1)(\Delta-1)\binom{\Delta-1}{l-1}\leqslant\frac{(l+1)\Delta^{l+1}}{l!}.$$ □

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Combining the inequality (3.23) with the estimates of probabilities (3.6), (3.12), (3.15) and Claims 4 and 5, we obtain that

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathbb{P}(\mathcal{J})\leqslant|E(\mathcal{Q})|\Bigg((\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h+n\sum_{y_1,\dots,y_h=0}^{s-1}\left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^h y_i}\times\left(\frac{\Delta}{r^{n-1}n}\right)^{h+1}\left(\frac{n}{n-h}\right)^h\frac{1+hs}{\prod_{i=1}^h(y_i!)}+\frac{n}{h}\sum_{l=s+1}^{n}\frac{(l+1)}{l!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{l+1}\Bigg). \tag{3.27}$$

To complete the proof of the theorem we have to show that the right-hand side of (3.27) is at most $1/4$ for some choice of parameters.

3.7. Choosing the parameters and completing the proof

The cardinality of the set $E(\mathcal{Q})$ can be easily calculated:

$$|E(\mathcal{Q})|=\begin{cases}1 & \text{if } \mathcal{Q}=\mathcal{B}(A,t),\\ 1+h+y_1+\cdots+y_h & \text{if } \mathcal{Q}=\mathcal{A}(A,\Phi,\mathbf{y},\Lambda),\\ 1+l & \text{if } \mathcal{Q}=\mathcal{C}(A,l,\Phi,t).\end{cases} \tag{3.28}$$

Let us make the following choice of parameters s and h:

$$h=\left\lfloor\frac{3\ln n}{\ln\ln n}\right\rfloor,\quad s=\lfloor\ln n\rfloor. \tag{3.29}$$

This choice of parameters satisfies the required conditions: $s<n/2$ for any $n\geqslant 3$, and $h<n$ for any $n\geqslant 9$.

Since the parameter l does not exceed n, and every y_i is at most $s-1$ (hence $1+h+y_1+\cdots+y_h\leqslant 1+h+h(s-1)\leqslant 4(\ln n)^2$), it follows from (3.28) that $|E(\mathcal{Q})|\leqslant n+1$ for any $\mathcal{Q}\in\Psi(t)$.

Using the initial restriction (2.11) on the maximum edge degree Δ of the hypergraph H (recall that $\Delta\leqslant cr^{n-1}n(\ln\ln n)^2/\ln n$) and our choice of parameters (3.29), we get the following upper bounds for the second and the third summands of the right-hand side of (3.27):

$$\begin{aligned}&n\sum_{y_1,\dots,y_h=0}^{s-1}\left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^h y_i}\left(\frac{\Delta}{r^{n-1}n}\right)^{h+1}\left(\frac{n}{n-h}\right)^h\frac{1+hs}{\prod_{i=1}^h(y_i!)}\\&\leqslant n\sum_{y_1,\dots,y_h=0}^{s-1}(3c\ln\ln n)^{\sum_{i=1}^h y_i}\left(\frac{c(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}}\left(\frac{n}{n-h}\right)^h\frac{4(\ln n)^2}{\prod_{i=1}^h(y_i!)}\\&=n\left(\frac{c(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}}\left(\frac{n}{n-h}\right)^h4(\ln n)^2\left(\sum_{y=0}^{s-1}\frac{(3c\ln\ln n)^y}{y!}\right)^h\end{aligned}$$

(assuming $0<c<1$)

$$\begin{aligned}&\leqslant nc\left(\frac{(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}}\left(\frac{n}{n-h}\right)^h4(\ln n)^2\,(e^{3c\ln\ln n})^h\\&\leqslant n\cdot c\cdot e^{-3\ln n+O(\ln n\ln\ln\ln n/\ln\ln n)}\,e^{O((\ln n)^2/n)}n^{O(\ln\ln n/\ln n)}e^{3c\ln n}=c\cdot n^{(3c-2)(1+o(1))}\end{aligned} \tag{3.30}$$

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and

$$\frac{n}{h}\sum_{l=s+1}^{n}\frac{l+1}{l!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{l+1}\leqslant\frac{2\Delta}{r^{n-1}}\sum_{l=s+1}^{n}\frac{1}{(l-1)!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{l}$$

(since $\Delta/r^{n-1}<cn$ and $(s+j)!>s!j!$)

$$\leqslant 2cn\frac{1}{s!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{s+1}\sum_{l=0}^{n-s-1}\frac{1}{l!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{l}$$

(since $h\Delta/nr^{n-1}\leqslant 3c\ln\ln n$ and $s!>(s/e)^s$)

$$\leqslant\frac{2cns}{e}\left(\frac{3ec\ln\ln n}{s}\right)^{s+1}\sum_{l=0}^{n-s-1}\frac{(3c\ln\ln n)^l}{l!}\leqslant\frac{2cns}{e}\left(\frac{3ec\ln\ln n}{s}\right)^{s+1}e^{3c\ln\ln n}$$

(since $s=\lfloor\ln n\rfloor$)

$$=c\cdot n^{1+O(\ln\ln n/\ln n)}e^{-(\ln\ln n)(\ln n)(1+o(1))}e^{O(\ln\ln n)}=c\cdot n^{-(1+o(1))\ln\ln n}. \tag{3.31}$$

We are finally ready to complete the proof of Theorem 2.2. The established estimates (3.30) and (3.31) imply the following bound for the desired sum in the left-hand side of (3.27):

$$\begin{aligned}\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathbb{P}(\mathcal{J})\leqslant|E(\mathcal{Q})|\Bigg((\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h+n\sum_{y_1,\dots,y_h=0}^{s-1}\left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^h y_i}\\ \times\left(\frac{\Delta}{r^{n-1}n}\right)^{h+1}\left(\frac{n}{n-h}\right)^h\frac{1+hs}{\prod_{i=1}^h(y_i!)}+\frac{n}{h}\sum_{l=s+1}^{n}\frac{(l+1)}{l!}\left(\frac{h\Delta}{nr^{n-1}}\right)^{l+1}\Bigg)\\ \leqslant(n+1)(\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h+c\cdot n^{1+(1+o(1))(3c-2)}+c\cdot n^{1-(1+o(1))\ln\ln n}.\end{aligned}$$

There exists an absolute constant $c\in(0,1)$ such that, for any $n\geqslant 9$,

$$c\cdot n^{1+(1+o(1))(3c-2)}+c\cdot n^{1-(1+o(1))\ln\ln n}<1/4.$$

Having chosen such a constant c, we can take t very close to 1 satisfying

$$(n+1)(\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h+c\cdot n^{1+(1+o(1))(3c-2)}+c\cdot n^{1-(1+o(1))\ln\ln n}<1/4. \tag{3.32}$$

Let us make the final conclusions. With this choice of parameters we establish that the required inequality (3.22) holds for any $\mathcal{Q}\in\Psi(t)$. This relation implies that the Local Lemma can be applied to the system of the events $\Psi(t)$. The Local Lemma states that

$$\mathbb{P}\Bigg(\bigcap_{\mathcal{Q}\in\Psi(t)}\overline{\mathcal{Q}}\Bigg)>0.$$

So, by (3.21) we get that the probability of the event $\mathcal{F}(t)$ is strictly less than 1, and, consequently, the hypergraph H is r-colourable. Theorem 2.2 is proved. □

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4. Proof of Theorem 2.1

Suppose that

$$N \leqslant c \cdot r^{n-1} \frac{(\ln \ln n)^2}{\ln n}.$$

We have to establish r-colourability of the hypergraph of arithmetic progressions

$$H_n(N) = ([N], E_n(N)).$$

For this purpose, we shall apply the same random colouring $\zeta(t)$ from the proof of Theorem 2.2 to $H_n(N)$ and show that it is a proper r-colouring with positive probability. However, $H_n(N)$ does not satisfy the conditions of Theorem 2.2 (it does not have large girth), so it will require some additional analysis.

Before starting the analysis, let us make some useful observations concerning arithmetic progressions.

Observation 1. *For any two integers x, y, there at most m^2 arithmetic progressions of length m containing both x and y.*

Observation 2. *For any arithmetic progression of length n, there are at most n^2m^2 arithmetic progressions of length m having at least two common vertices with it.*

For any arithmetic progression $A \in E_n(N)$, $A = \{a, a+d, \dots, a+d(n-1)\}$, let us introduce a larger progression $l(A)$ of length $2n$ as follows:

$$l(A) = \left\{ a - d\left\lfloor \frac{n}{2} \right\rfloor, \dots, a, a+d, \dots, a+d(n-1) + d\left\lceil \frac{n}{2} \right\rceil \right\}.$$

Observation 3. *If A and B are arithmetic progressions of length n such that $|A \cap B| \geqslant n/2$, then $B \subset l(A)$.*

4.1. Short cycles in the hypergraph of arithmetic progressions

Now we apply the randomized colouring algorithm from Section 3 and construct the final colouring $\zeta(t)$. The analysis in Section 3 shows that the existence of a monochromatic edge implies one of the bad events which can be one of three types:

(1) $\mathcal{A}(A, \Phi, \mathbf{y}, \Lambda)$ for some first-type configuration $(A, \Phi, \mathbf{y}, \Lambda) \in \Upsilon_1$,
(2) $\mathcal{B}(A, t)$ for some $A \in E$,
(3) $\mathcal{C}(A, l, \Phi, t)$ for some second-type configuration $(A, l, \Phi) \in \Upsilon_2$.

However, in Section 3 the probabilities of these events were estimated for a hypergraph with girth greater than 5. In our hypergraph $H_n(N)$ there are cycles of smaller length. So, we have to consider additional cases with short cycles in the configurations $(A, \Phi, \mathbf{y}, \Lambda)$ and (A, l, Φ).

To simplify the analysis, we introduce the notion of a generalized cycle. An ordered set of $j \geqslant 2$ edges $(A_1, \dots, A_j)$ of $H_n(N)$ is said to form a *generalized j-cycle* if, for $j > 2$,

- $|A_i \cap A_{i+1}| = 1$ for $i = 1, \dots, j-1$,

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- $|A_i \cap A_k| = 0$ for $|k-i| > 1$, except the pair $(1, j)$,
- $1 \leqslant |A_1 \cap A_j| < n/2$ or $|A_1 \cap A_j| = 0$ and $|A_1 \cap l(A_j)| \geqslant 1$.

If $1 \leqslant |A_1 \cap A_j| < n/2$ then the generalized cycle is a usual cycle in a hypergraph. For $j = 2$, we assume that $|l(A_1) \cap l(A_2)| \geqslant 2$ and $|A_1 \cap A_2| < n/2$.

For a given generalized j-cycle $(A_1, \ldots, A_j)$, we define a bad event $\mathcal{L}_j(A_1, \ldots, A_j)$ as follows: every edge A_i is monochromatic or almost monochromatic in the initial colouring ξ. The probability of this event is roughly estimated in the following claim.

Claim 6.

$$\mathbb{P}(\mathcal{L}_j(A_1, \ldots, A_j)) \leqslant r^j \binom{n}{s}^j r^{js - |A_1 \cup \cdots \cup A_j|} \leqslant n^{js} r^{(j+2)s - (j-1/2)n}. \tag{4.1}$$

Proof. Since all the edges $A_1, \ldots, A_j$ are monochromatic or almost monochromatic, in every edge there is a set of s vertices such that all the remaining $n - s$ vertices are coloured with the chosen dominating colour. The dominating colours can be chosen in r^j ways, the uncoloured set of vertices can be chosen in at most $\binom{n}{s}^j$ ways. The number of the remaining vertices is at least $|A_1 \cup \cdots \cup A_j| - js$, so the probability that all of them are coloured with the chosen colours is at most $r^{js - |A_1 \cup \cdots \cup A_j|}$. The last thing to do is to note that for generalized j-cycles,

$$|A_1 \cup \cdots \cup A_j| \geqslant jn - \frac{n}{2} - j. \qquad \square$$

We will show that every situation in the additional analysis of the first bad event implies the event $\mathcal{L}_j(A_1, \ldots, A_j)$ for some $j \leqslant 5$, so it would be sufficient to avoid only these events. To apply the Local Lemma, we will also need to estimate the number of generalized j-cycles in the hypergraph $H_n(N)$, intersecting a fixed edge J.

Claim 7. *The number of generalized j-cycles intersecting any fixed edge does not exceed*

$$32 j n^4 \Delta^{j-1}, \tag{4.2}$$

where $\Delta = \Delta(H_n(N))$ is the maximum edge degree of $H_n(N)$.

Proof. Recall that for a progression $A = \{a, a+d, \ldots, a+d(n-1)\}$, $l(A)$ is a larger progression of length $2n$,

$$l(A) = \left\{ a - d\left\lfloor \frac{n}{2} \right\rfloor, \ldots, a + d(n-1) + d\left\lceil \frac{n}{2} \right\rceil \right\}.$$

Let $(A_1, \ldots, A_j)$ be a generalized cycle intersecting an edge J. Suppose that $A_i \cap J \neq \emptyset$. Then

- A_i can be chosen in at most $\Delta = \Delta(H_n(N))$ ways,
- its position i in the cycle can be chosen in at most j ways,
- the remaining edges except A_j, if $i < j$, or A_1 for $i = j$ can be chosen in at most Δ^{j-2} ways,
- for $j > 2$, $l(A_j)$ (or $l(A_1)$) should intersect A_1 and A_{j-1} (or A_2 and $l(A_j)$), so it can be chosen (due to Observation 1) in at most $(2n)^4$ ways,

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- for $j=2$, we have $|l(A_1)\cap l(A_2)|\geqslant 2$, hence Observation 2 says that here we also have at most $(2n)^4$ variants.

Finally, we obtain that the total number of generalized j-cycles intersecting a fixed edge does not exceed

$$j(\Delta+1)\Delta^{j-2}(2n)^4\leqslant 32jn^4\Delta^{j-1}. \qquad \square$$

The choice of parameters of the probabilistic construction will be almost the same as in the proof of Theorem 2.2, but we shall not take the time parameter t very close to 1:

$$h=\left\lfloor\frac{3\ln n}{\ln\ln n}\right\rfloor,\quad s=\lfloor\ln n\rfloor,\quad t=\frac{2(\ln n)^2}{n}. \tag{4.3}$$

Note that the choice $t=\Theta((\ln n)^2/n)$ is also sufficient to satisfy (3.32), but small t ensures that, for every vertex v, the probability that it can be recoloured during the process is also small, because X_v should be less than t. This observation is highly useful for avoiding bad configurations with short cycles.

Now we are ready to analyse bad events caused by short cycles in $H_n(N)$.

4.2. Additional analysis of the first bad event

Recall that in the first bad event $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ we have a 'tree' with 'trunk' A, 'branches' $\Phi=(B_1,\dots,B_h)$ and 'leaves' $\Lambda=(C_{1,1},\dots,C_{1,y_1},C_{2,1},\dots,C_{h,1},\dots,C_{h,y_h})$, where $C_{i,1},\dots,C_{i,y_i}$ are the 'leaves' of the 'branch' B_i. This event implies that A is monochromatic in the initial colouring ξ, that B_i is almost monochromatic for any i, and that leaf-edges $C_{i,j}$ are also monochromatic in ξ. In Theorem 2.2 the induced hypergraph on the set of edges A, Φ and Λ did not contain any cycles due to the condition on girth. So, for $H_n(N)$, we have to analyse the situations when the configuration contains short cycles.

Claim 8. *If the configuration $(A,\Phi,\mathbf{y},\Lambda)$ contains cycles of length at most 5, then the event $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ implies one of the events $\mathcal{L}_j(A_1,\dots,A_j)$ for some generalized j-cycle $(A_1,\dots,A_j)$ and some $j\leqslant 5$.*

Proof. Suppose that $(A_1,\dots,A_j)$ is the shortest cycle in the configuration $(A,\Phi,\mathbf{y},\Lambda)$. Since the tree (A,Φ,Λ) has depth equal to 2, the cycle $(A_1,\dots,A_j)$ has length at most 5. Further, we may assume that

- $|A_i\cap A_{i+1}|=1$ for $1\leqslant i<j$,
- $|A_i\cap A_k|=0$ for $|i-k|>1$, except the pair A_1,A_j,
- $|A_1\cap A_j|>0$.

The event $\mathcal{A}(A,\Phi,\mathbf{y},\Lambda)$ implies that every edge in the configuration $(A,\Phi,\mathbf{y},\Lambda)$ is either monochromatic or almost monochromatic in the initial colouring ξ. If $|A_1\cap A_j|<n/2$ then $(A_1,\dots,A_j)$ is a generalized j-cycle and the event $\mathcal{L}_j(A_1,\dots,A_j)$ holds. If $|A_1\cap A_j|\geqslant n/2$ then we have to consider different values of j.

Case $j>2$**.** In this case $A_1\subset l(A_j)$ and therefore, $A_2\cap l(A_j)\neq\emptyset$. Hence $(A_2,\dots,A_j)$ forms a generalized $(j-1)$-cycle and the event $\mathcal{L}_{j-1}(A_2,\dots,A_j)$ holds.

Case $j = 2$**.** Since $|A_1 \cap A_2| \geqslant n/2$, these edges cannot be neighbours in the tree. Indeed, in this situation one of them is monochromatic in ξ and the other should be almost monochromatic with a different colour. Thus, they can have at most $s < n/2$ common vertices and form only a generalized 2-cycle. Consequently, we may also assume that any tree neighbours do not form a 2-cycle.

Hence, A_1 and A_2 are not neighbours in the tree and we can construct direct paths from both of them to the root A. For example, if $A_1 = C_{i,k}, A_2 = C_{u,z}$ for some i,k,u,z, then we obtain a 5-cycle $(C_{i,k}, B_i, A, B_u, C_{u,z})$. After that we have a few possibilities:

- if $|B_i \cap B_u| \geqslant n/2$ then we obtain a generalized 2-cycle (A, B_i),
- if $0 < |B_i \cap B_u| < n/2$ then we obtain a generalized 3-cycle (A, B_i, B_u),
- if $|B_i \cap B_u| = 0$ and $|C_{i,k} \cap B_u| < n/2$ then we obtain a generalized 4-cycle $(C_{i,k}, B_i, A, B_u)$,
- if $|B_i \cap B_u| = 0$ and $|C_{i,k} \cap B_u| \geqslant n/2$ then we again obtain a generalized 3-cycle (B_i, A, B_u).

The other situations are considered similarly. □

We have finished the additional analysis of the first bad event $\mathcal{A}(A, \Phi, \mathbf{y}, \Lambda)$. The second one, $\mathcal{B}(A,t)$, depends only on one edge, so there is no need to deal with short cycles here. Now we proceed to the third bad event.

4.3. Additional analysis of the third bad event

The third bad event $\mathcal{C}(A, l, \Phi, t)$ (see (3.14)) contains a configuration (A, l, Φ) of the second type, where A and $\Phi = (B_1, \dots, B_l)$ form a tree with the 'trunk' A and the 'branches' $B_1, \dots, B_l$. For the hypergraph of arithmetic progressions, this set of edges does not necessarily form a real hypertree, so we have to make an additional analysis. In comparison with the first event here we shall make use of the time parameter t, as it helps to make the probability of the event sufficiently small.

Indeed, for simple hypergraphs we know that v_i is the only vertex in the intersection of A and B_i. In the non-simple case we do not know exactly which vertex in $A \cap B_i$ will be the first to be recoloured during the procedure. But for every $w \in A \cap B_i$, the inequality $X_w \leqslant t$ should hold, because the edge B_i is monochromatic initially and the edge A is monochromatic in the final colouring of another colour. All the vertices from $A \cap B_i$ should be recoloured during the procedure. Thus, we get the following statement.

Claim 9.

(1) *For any* $B_i \in \Phi$,

$$\begin{aligned}\mathcal{C}(A, l, \Phi, t) &\subset \mathcal{L}_6(A, B_i, t)\\ &= \bigcup_{b=1}^{r} \bigcup_{a=1: a \neq b}^{r} \Bigg(\bigcap_{v \in B_i} \{\xi_v = b\} \cap \bigcap_{w \in A \cap B_i} \{\eta_v = a, X_v \leqslant t\}\\ &\qquad \cap \bigcap_{v \in A \setminus B_i} (\{\xi_v = a\} \cup \{\xi_v \neq a, \eta_v = a, X_v \leqslant t\}) \Bigg).\end{aligned}$$

(2) *For any* $B_i, B_k \in \Phi$,

$$\begin{aligned}\mathcal{C}(A,l,\Phi,t) &\subset \mathcal{L}_7(A,B_i,B_k,t)\\ &= \bigcup_{c,b=1}^{r} \bigcup_{a=1:a\neq b,c}^{r} \Bigg(\bigcap_{v\in B_i}\{\xi_v = b\} \cap \bigcap_{v\in B_k}\{\xi_v = c\} \cap \bigcap_{w\in A\cap(B_i\cup B_k)} \{\eta_v = a, X_v \leqslant t\} \\ &\qquad\qquad \cap \bigcap_{v\in A\setminus(B_i\cup B_k)} (\{\xi_v = a\}\cup\{\xi_v \neq a, \eta_v = a, X_v \leqslant t\})\Bigg).\end{aligned}$$

Proof. **(1)** The proof of the relation follows easily from the definition of $\mathcal{C}(A,l,\Phi,t)$ (see (3.13)). It says that the edge B_i should be monochromatic in ξ (first event $\bigcap_{v\in B_i}\{\xi_v = b\}$). Moreover, every vertex of A should be coloured with colour a in ζ, so for every $v \in A$ either $\xi_v = a$ or $\xi_v \neq a, \eta_v = a, X_v \leqslant t$ (third event $\bigcap_{v\in A\setminus B_i}\{\xi_v = a\}\cup\{\xi_v \neq a, \eta_v = a, X_v \leqslant t\}$). But for $v \in A\cap B_i$ we already know that $\xi_v = b \neq a$, so we have only the second alternative.

(2) The argument is exactly the same as for the first statement. □

Next, we give the estimates for the probabilities of the events $\mathcal{L}_6$ and $\mathcal{L}_7$.

Claim 10.

(1) *If* $|A\cap l(B_i)| \geqslant 2$, *then*

$$\mathbb{P}(\mathcal{L}_6(A,B_i,t)) \leqslant r^{2-n}\max\{(t/(r-1))^{n/2}, r^{-n/2}e^{tn/2}\}. \tag{4.4}$$

(2) *If* $|B_i\cap A| = 1$, $|B_k\cap A| = 1$, $0 < |B_i\cap B_k| \leqslant n/2$ *and* $|B_i\cap B_k\cap A| = 0$, *then*

$$\mathbb{P}(\mathcal{L}_7(A,B_i,B_k,t)) = r(r-1)\left(\frac{1}{r}+\frac{t}{r}\right)^{n-2}\left(\frac{t}{r-1}\right)^{2} r^{-|B_i\cup B_k|} \leqslant r^{4-5n/2}e^{tn}t^2. \tag{4.5}$$

Proof. **(1)** The probabilities of the events in the intersections are as follows:

$$\mathbb{P}\Bigg(\bigcap_{v\in B_i}\{\xi_v = b\}\Bigg) = r^{-n},$$

$$\mathbb{P}\Bigg(\bigcap_{w\in A\cap B_i}\{\eta_v = a, X_v \leqslant t\} \,\Bigg|\, \bigcap_{v\in B_i}\{\xi_v = b\}\Bigg) = \left(\frac{t}{r-1}\right)^{|A\cap B_i|},$$

$$\mathbb{P}\Bigg(\bigcap_{v\in A\setminus B_i}(\{\xi_v = a\}\cup\{\xi_v \neq a, \eta_v = a, X_v \leqslant t\})\Bigg) = \left(\frac{1}{r}+\frac{t}{r}\right)^{n-|A\cap B_i|}.$$

The third event is independent of the first two, so taking the sum over a and b we obtain

$$\mathbb{P}(\mathcal{L}_6(A,B_i,t)) = r(r-1)r^{-n}\left(\frac{t}{r-1}\right)^{|A\cap B_i|}\left(\frac{1}{r}+\frac{t}{r}\right)^{n-|A\cap B_i|}.$$

If $|A\cap B_i| \leqslant n/2$ then

$$\left(\frac{1}{r}+\frac{t}{r}\right)^{n-|A\cap B_i|} \leqslant r^{-n/2}(1+t)^{n/2} \leqslant r^{-n/2}e^{tn/2}.$$

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Otherwise $(t/(r-1))^{|A\cap B_i|} \leqslant (t/(r-1))^{n/2}$. Hence, in the general case we obtain the upper bound (4.4) for the probability of $\mathcal{L}_6(A,B_i,t)$:

$$\mathbb{P}(\mathcal{L}_6(A,B_i,t)) \leqslant r^{2-n}\max\{(t/(r-1))^{n/2}, r^{-n/2}e^{tn/2}\}.$$

(2) The proof repeats the argument for the first statement of Claim 10. Note that, since $B_i \cap B_k \neq \emptyset$, the colours b and c should be equal, and instead of the factor r^{-n} we have $r^{-|B_i\cup B_k|}$. Finally, the probability of the event $\mathcal{L}_7(A,B_j,B_k,t)$ is equal to

$$\mathbb{P}(\mathcal{L}_7(A,B_j,B_k,t)) = r(r-1)\left(\frac{1}{r}+\frac{t}{r}\right)^{n-2}\left(\frac{t}{r-1}\right)^2 r^{-|B_j\cup B_k|} \leqslant r^{4-5n/2}e^{tn}t^2,$$

and the relation (4.5) follows. □

Finally, we show that the cases considered cover all the situations.

Claim 11. *If the configuration* (A,Φ) *contains a* 2- *or* 3-*cycle then the event* $\mathcal{C}(A,l,\Phi,t)$ *implies either the event* $\mathcal{L}_6(A,B_i,t)$ *with* $|A\cap l(B_i)| \geqslant 2$ *for some* $i=1,\ldots,l$, *or the event* $\mathcal{L}_7(A,B_j,B_k,t)$ *with* $|B_i\cap A|=1$, $|B_k\cap A|=1$, $0<|B_i\cap B_k|\leqslant n/2$, $|B_i\cap B_k\cap A|=0$ *for some* $i\neq k$.

Proof. If $|A\cap l(B_i)| \geqslant 2$ for some i then we immediately apply Claim 9, case (1).

If $|A\cap B_i|=1$ for every $i=1,\ldots,l$, then there should be a pair of edges B_i and B_k with non-empty intersection (otherwise there will be no 2- and 3-cycles in the configuration). If $|B_i\cap B_k| \geqslant n/2$ then $B_k\subset l(B_i)$. Therefore $|A\cap l(B_i)|\geqslant 2$ and we apply Claim 9, case (1). In the other case we apply Claim 9, case (2) to get the event $\mathcal{L}_7(A,B_j,B_k,t)$. □

Now we proceed to the application of the Local Lemma.

4.4. Completion of the proof

In comparison with Theorem 2.2 for the hypergraph of arithmetic progressions $H_n(N)$, the set of bad events $\Psi(t)$ is larger since we add to it the events of the types $\mathcal{L}_i$, $i=2,\ldots,7$. For any such new event $\mathcal{Q}$, the domain $D(\mathcal{Q})$ and the edge-set $E(\mathcal{Q})$ are defined similarly to Section 3.6. All we have to do is to check the condition (3.22) required for the application of the Local Lemma. By analogy with (3.23), for any $\mathcal{Q}\in\Psi(t)$, we have

$$\begin{aligned}\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathbb{P}(\mathcal{J}) \leqslant \sum_{U\in E(\mathcal{Q})}\Bigg[&\sum_{A\in E_n(N):\,U\cap A\neq\emptyset}\mathbb{P}(\mathcal{B}(A,t)) + \sum_{\substack{(A,\Phi,\mathbf{y},\Lambda)\in\Upsilon_1:\\ U\cap(A\cup\bigcup_{B\in\Phi}B\cup\bigcup_{C\in\Lambda}C)\neq\emptyset}}\mathbb{P}(\mathcal{A}(A,\Phi,\mathbf{y},\Lambda))\\ &+\sum_{\substack{(A,l,\Phi)\in\Upsilon_2:\\ U\cap(A\cup\bigcup_{B\in\Phi}B)\neq\emptyset}}\mathbb{P}(\mathcal{C}(A,l,\Phi,t)) + \sum_{j=2}^{5}\sum_{\substack{\text{gen. } j\text{-cycle } (A_1,\ldots,A_j):\\ U\cap(A_1\cup\cdots\cup A_j)\neq\emptyset}}\mathbb{P}(\mathcal{L}_j(A_1,\ldots,A_j))\\ &+\sum_{\substack{(A,B):\,|A\cap l(B)|\geqslant 2,\\ U\cap(A\cup B)\neq\emptyset}}\mathbb{P}(\mathcal{L}_6(A,B,t)) + \sum_{\substack{\text{gen. 3-cycle } (A,B,C):\\ U\cap(A\cup B\cup C)\neq\emptyset}}\mathbb{P}(\mathcal{L}_7(A,B,C,t))\Bigg]. \qquad (4.6)\end{aligned}$$

Before starting the analysis of the sums in the right-hand side of (4.6), let us make a preliminary observation. Let $\Delta = \Delta(H_n(N))$ denote the maximum edge degree of the hypergraph $H_n(N)$. Since any integer from $[N]$ is contained in at most $n(N-1)/(n-1)$ arithmetic progressions of length n from $E_n(N)$, we get

$$\Delta \leqslant \left(n\frac{N-1}{n-1}-1\right)n \leqslant Nn\,\frac{n}{n-1} \leqslant \frac{9}{8}c\,r^{n-1}\frac{n(\ln\ln n)^2}{\ln n}. \tag{4.7}$$

Here we use the restriction on N and the condition $n \geqslant 9$.

(1) The first sum in (4.6) is estimated in the same way as in (3.12) (the bound is correct since the choice of the parameter t implies the required inequality $nt > 2$):

$$\sum_{A\in E:\,U\cap A\neq\emptyset} \mathbb{P}(\mathcal{B}(A,t)) \leqslant (\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h$$

(using (4.7) and (4.3))

$$\leqslant \frac{9}{4}cn\,e^{-t(n-h+1)}(2(\ln n)^2)^h = cn\,e^{-2(\ln n)^2(1+o(1))}e^{2h\ln\ln n(1+o(1))} = c\,e^{-2(\ln n)^2(1+o(1))}. \tag{4.8}$$

(2) The second and the third sums were analysed in Section 3.7, so the bounds (3.30) and (3.31) hold with c replaced by $(9/8)c$.

(3) In the fourth sum the number of summands does not exceed the bound (4.2). Thus, using the estimate (4.1), we obtain

$$\sum_{j=2}^{5}\sum_{\substack{\text{gen. } j\text{-cycle } (A_1,\dots,A_j):\\ U\cap(A_1\cup\dots\cup A_j)\neq\emptyset}} \mathbb{P}(\mathcal{L}_j(A_1,\dots,A_j)) \leqslant \sum_{j=2}^{5} 32\,j\,n^4\Delta^{j-1}n^{js}r^{(j+2)s-(j-1/2)n}$$

(since $j \leqslant 5$, $s \leqslant \ln n$ and $\Delta \leqslant (9c/8)nr^{n-1}$)

$$\leqslant \sum_{j=2}^{5} 160\,(9c/8)^{j-1}n^{5\ln n+4}r^{7\ln n-1/2n} \leqslant c\,e^{5(\ln n)^2(1+o(1))}r^{7\ln n-1/2n}$$

(since $r \geqslant 2$)

$$\leqslant c\,e^{5(\ln n)^2(1+o(1))}2^{-n/2(1+o(1))} = c\,2^{-n/2(1+o(1))}. \tag{4.9}$$

(4) In the fifth sum we have to calculate the number of pairs (A,B) intersecting a fixed progression U. The intersecting edge can be chosen in at most $2(\Delta+1)$ ways, but the second edge in at most $(2n)^4$ ways since $|A\cap l(B)| \geqslant 2$. Hence, the estimate (4.4) implies that

$$\sum_{\substack{(A,B):\,|A\cap l(B)|\geqslant 2,\\ U\cap(A\cup B)\neq\emptyset}} \mathbb{P}(\mathcal{L}_6(A,B,t)) \leqslant 2(\Delta+1)(2n)^4r^{2-n}\max\{(t/(r-1))^{n/2}, r^{-n/2}e^{tn/2}\}$$

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(since $t = 2(\ln n)^2/n$ and $\Delta \leqslant (9c/8)n\,r^{n-1}$)

$$\leqslant 64\cdot(9c/8)\,n^5\,r\cdot\max\left\{\left(\frac{2(\ln n)^2}{n(r-1)}\right)^{n/2}, r^{-n/2}e^{(\ln n)^2}\right\}$$
$$\leqslant 72\,c\,n^5\,r\cdot\max\left\{\left(\frac{4(\ln n)^2}{nr}\right)^{n/2}, r^{-n/2}e^{(\ln n)^2}\right\}$$

(it is easy to see that the second value is always maximal)

$$\leqslant 72\,c\,n^5\,r^{1-n/2}e^{(\ln n)^2} \leqslant 72\,c\,n^5\,2^{1-n/2}e^{(\ln n)^2} = c\,2^{-n/2(1+o(1))}. \tag{4.10}$$

(5) Finally, for the last sum we know the estimate for the number of intersecting generalized 3-cycles (4.2) and the bound for the probability of the event (4.5). Thus,

$$\sum_{\substack{\text{gen. 3-cycle } (A,B,C):\\ U\cap(A\cup B\cup C)\neq\emptyset}} \mathbb{P}(\mathcal{L}_7(A,B,C)) \leqslant 96\Delta^2 n^4\,r^{4-5n/2}e^{tn}t^2$$

(applying (4.3), (4.7))

$$\leqslant \frac{96\cdot 9^2}{8^2}c^2n^6\,r^{2-n/2}\left(\frac{2\ln n}{n}\right)^2 = c^2n^{4+o(1)}r^{2-n/2}$$
$$\leqslant c^2n^{4+o(1)}2^{2-n/2} \leqslant c^2\cdot 2^{-n/2(1+o(1))}. \tag{4.11}$$

Let us complete the proof. For the application of the Local Lemma it is sufficient to show that the right-hand side of (4.6) does not exceed $1/4$. Since for any $\mathcal{Q}\in\Psi(t)$ we have $|E(\mathcal{Q})|\leqslant n+1$ (see (3.28)), by using the obtained estimates (4.8)–(4.11) we have

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathbb{P}(\mathcal{J}) \leqslant (n+1)\big(c\,e^{-2(\ln n)^2(1+o(1))} + (9/8)c\cdot n^{(27c/8-2)(1+o(1))} + (9/8)c\cdot n^{-\ln\ln n(1+o(1))}$$
$$+ c\cdot 2^{-n/2(1+o(1))} + c\cdot 2^{-n/2(1+o(1))} + c^2\cdot 2^{-n/2(1+o(1))}\big).$$

It is easy to see that there exists $c\in(0,1)$ such that the given function of n is strictly less than 1/4 for all $n\geqslant 9$. Theorem 2.1 is proved.

5. Final remarks

(1) In the previous section we used a reduction argument to establish r-colourability of the hypergraph of arithmetic progressions. For a simple hypergraph H, this reduction is almost the same. There are no 2-cycles in H, so the bad events of the type $\mathcal{L}_2$ and $\mathcal{L}_6$ do not happen. The analysis of the remaining bad events in Section 4 used only the following properties:

- the maximum edge degree of the hypergraph is $O(r^{n-1}(n(\ln\ln n)^2/\ln n))$ ((1.3) for H),
- the size of edge intersections is at most $n/2$ (at most 1 for a simple hypergraph),
- the maximum codegree is at most n^2 (1 for a simple hypergraph).

All these properties hold for simple hypergraphs, so the same proof argument shows that it is r-colourable. Theorem 1.2 is proved.

(2) After we submitted this paper, similar results concerning colourings of simple hypergraphs and van der Waerden numbers were independently obtained by Kozik [13]. Our bounds are greater by a factor of $(\ln\ln n)^2$ than those in [13]. The general proof approach in [13] (random recolouring method) is the same, but the recolouring technique is quite different.

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