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ON A PERTURBATION IN A TWO-PARAMETER ORDINARY DIFFERENTIAL EQUATION OF THE SECOND ORDER

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1. **Introduction.** Let us consider the linear system in the two parameters λ and μ; i.e.,

(1.1) $$y''(x)+(\lambda+\mu b(x)+q(x))y(x) = 0, \qquad 0 \leq x \leq 1, \quad ' = \frac{d}{dx},$$

(1.2) $$\begin{aligned} \cos\alpha y(0)-\sin\alpha y'(0) &= 0, \quad 0 \leq \alpha < \pi, \\ \cos\beta y(1)-\sin\beta y'(1) &= 0, \quad 0 < \beta \leq \pi, \end{aligned}$$

and where for the moment we shall assume both $b(x)$ and $q(x)$ are real-valued, continuous functions in [0, 1]. Then for each real μ, the eigenvalues of (1.1) and (1.2) are real and form a countably infinite set denoted by $\{\lambda_n(\mu)\}_{n=0}^{\infty}$, with $\lambda_0(\mu) < \lambda_1(\mu) < \ldots, \lim_{n\to\infty}\lambda_n(\mu) = \infty$, and where an eigenfunction corresponding to $\lambda_n(\mu)$ has precisely n zeros in (0, 1); in fact in the real (μ, λ) plane $\{(\mu, \lambda_n(\mu))\}_{n=0}^{\infty}$ determine a countably infinite set of disjoint analytic curves called the eigenvalue curves. For further information regarding the eigenvalue curves we refer to ([1], [2], [3]) and the references listed therein.

From now on we shall further assume that $b(x) \in C_4[0, 1]$ and attains its absolute maximum in [0, 1] at precisely the finite set of points $\{c_i\}_{i=1}^{p}, p \geq 1$, where $0 < c_1 < c_2 < \cdots < c_p < 1$, and $b''(c_i) < 0$, $i = 1, \ldots, p$. Now let us put for $p > 1$,

$$\begin{aligned} J_1 &= \{x \mid 0 \leq x \leq \tfrac{1}{2}(c_1+c_2)\}, \\ J_i &= \{x \mid \tfrac{1}{2}(c_{i-1}+c_i) \leq x \leq \tfrac{1}{2}(c_i+c_{i+1})\}, \quad i = 2, \ldots, (p-1), \\ J_p &= \{x \mid \tfrac{1}{2}(c_{p-1}+c_p) \leq x \leq 1\}, \end{aligned}$$

while if $p = 1$,

$$J_1 = \{x \mid 0 \leq x \leq 1\},$$

and introduce the perturbed equation

(1.3) $$y''(x)+(\lambda+\mu b(x)-t\mu b^{\dagger}(x)+q(x))y(x) = 0, \quad 0 \leq x \leq 1,$$

where $t > 0$, and for $p > 1$,

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$$b^\dagger(x) = \left(\frac{c_1+c_2}{2}-x\right)^5 (x-c_1)^2 b_1, \quad x \in J_1,$$
$$= \left(x-\frac{c_i+c_{i-1}}{2}\right)^5\left(\frac{c_{i+1}+c_i}{2}-x\right)^5 (x-c_i)^2 b_i, \quad x \in J_i, \quad i = 2, \ldots, (p-1),$$
$$= \left(x-\frac{c_p+c_{p-1}}{2}\right)^5 (x-c_p)^2 b_p, \quad x \in J_p,$$

while if $p=1$,

$$b^\dagger(x) = (x-c_1)^2 b_1, \quad x \in J_1,$$

and the b_i, $i=1,\ldots,p$, are positive constants. It is then the purpose of this paper to investigate some effects of this perturbation on our eigenvalue curves.

2. **Preliminary Results.** For $\mu>0$ put $\Lambda_n(\mu)=\mu^{-1/2}(\lambda_n(\mu)+\mu B)$, where $B=\sup_{0\le x\le 1} b(x)$. Then for each $n\ge 0$ there are numbers $\mu_n^\dagger$ and $\Lambda_n^\dagger$, both greater than one, such that for $\mu\ge\mu_n^\dagger$, $0<\Lambda_n(\mu)<\Lambda_n^\dagger$ ([1, Ch. 3, p. 102], [2, p. 135]); in fact $\lim_{\mu\to\infty}\Lambda_n(\mu)$ exists [3, Theorem 3.2], and, denoting this limit by Λ_n, we have $\lim_{n\to\infty}\Lambda_n=\infty$ [3]. Let us now denote the eigenvalues of (1.3), (1.2) by $\{\lambda_n(\mu, t)\}_{n=0}^\infty$, and for $n=0, 1, \ldots,$ and $\mu>0$ put

$$\Lambda_n(\mu, t) = \mu^{-1/2}(\lambda_n(\mu, t)+\mu B), \qquad \Lambda_n(t) = \lim_{\mu\to\infty} \Lambda_n(\mu, t).$$

Also let

$$a_i = -\frac{b''(c_i)}{2}, \quad i = 1,\ldots,p, \qquad a = \min_{1\le i\le p} a_i,$$
$$a^\dagger = \max_{1\le i\le p} a_i, \qquad Q = \sup_{0\le x\le 1} |q(x)|,$$

and for $i=1,\ldots,p$,

$$\theta_i(x) = \frac{B-b(x)}{a_i(x-c_i)^2}, \qquad x\in J_i-\{c_i\}, \quad \theta_i(c_i) = 1,$$

and where it is clear that $\theta_i(x)\in C_2[J_i]$ and $0<\theta=\min_{1\le i\le p}\inf_{x\in J_i}\theta_i(x)$. From now on in this section we shall fix an integer $m\ge 0$ and always assume that

$$\mu \ge \max\left\{\mu_m^\dagger, \left[\left(\frac{\Lambda_m^\dagger+Q}{\theta}\right)^{32}\left(\frac{1}{a^2d^4}+\left(\frac{8a^\dagger}{a^2}\right)^{16}+(4ad^4)^{-8/7}\right)\right]\right\},$$

where

$$d = \min_{0\le i\le p}\frac{c_{i+1}-c_i}{8}, \quad \text{and} \quad c_0 = 0, \quad c_{p+1} = 1;$$

hence if

$$\delta(\mu) = \mu^{-1/4}\left(\frac{\Lambda_m^\dagger+Q}{a\theta}\right)^{1/2}$$

and, for $i=1,\ldots,p$,

$$I_i(\mu) = \{x \mid |x-c_i| \le \delta(\mu)\},$$

then $\lambda_m(\mu)+\mu b(x)+q(x)<0$ in $[0, 1]-\bigcup_{i=1}^p I_i(\mu)$. For $i=1,\ldots,p$, we shall also put

$$\nu_i(\mu) = \left(\frac{\Lambda_m(\mu)}{2\sqrt{a_i}}-\frac{1}{2}\right), \qquad \nu_i = \lim_{\mu\to\infty}\nu_i(\mu).$$

THEOREM 2.1. *Let* $y_m(x, \mu)$ *be the eigenfunction of* (1.1), (1.2) *corresponding to* $\lambda_m(\mu)$ *and satisfying* $y_m(0, \mu)=\sin \alpha$, $y_m'(0, \mu)=\cos \alpha$. *Then for all sufficiently large values of* μ, *the absolute maximum of* $|y_m(x, \mu)|$ *in* $[0, 1]$ *is assumed in* $\bigcup_{i=1}^{p} I_i(\mu)$.

Proof. In the intervals $(0, c_1-\delta(\mu))$, $(c_i+\delta(\mu), c_{i+1}-\delta(\mu))$, $i=1,\ldots,(p-1)$, and $(c_p+\delta(\mu), 1)$, it is clear that $y_m(x, \mu)$ is convex downward wherever it is positive and concave downward wherever it is negative, and hence the absolute maximum of $|y_m(x, \mu)|$ in $[0, 1]$ cannot be assumed in these intervals. Also, if $0\le\alpha\le\pi/2$, then $y_m^2(x, \mu)$ is strictly increasing in the interval $(0, c_1-\delta(\mu)]$; and if $\pi/2\le\beta\le\pi$, then $y_m^2(x, \mu)$ strictly decreases in the interval $[c_p+\delta(\mu), 1)$. By use of a Prüfer transformation [4, p. 209], and arguing as in ([1, Ch. 3, pp. 85–87], [4, p. 213]), it is easy to show for $\alpha>\pi/2$ and all sufficiently large values of μ, $y_m^2(x, \mu)$ strictly decreases in the interval $[0, x_1(\mu))$ and strictly increase in the interval $(x_1(\mu), c_1-\delta(\mu)]$ and where $y_m'(x_1(\mu), \mu)=0$ and $x_1(\mu)=O(\mu^{-1})$ as $\mu\to\infty$; similarly, if $\beta<\pi/2$, then for all sufficiently large values of μ, $y_m^2(x, \mu)$ strictly decreases in the interval $[c_p+\delta(\mu), x_2(\mu))$ and strictly increases in the interval $(x_2(\mu), 1]$ and where $y_m'(x_2(\mu), \mu)=0$ and $(1-x_2(\mu))=O(\mu^{-1})$ as $\mu\to\infty$. From [3, §2.2 and Theorem 3.2] we also have for all μ sufficiently large and $\alpha\neq 0$,

$$y_m(c_1-\mu^{-3/16}(4a_1)^{-1/4}, \mu) = \sin \alpha g_1(\mu)\mu^{-3\nu_1(\mu)/16} \exp\left\{\mu^{1/2}\int_0^{c_1}(B-b(x))^{1/2}\,dx-\frac{\mu^{1/8}}{4}\right\},$$

where $g\le g_1(\mu)\le G$ and g, G are positive constants; and if $Y_m(x, \mu)$ is the eigenfunction of (1.1), (1.2) corresponding to $\lambda_m(\mu)$ and satisfying $Y_m(1, \mu)=\sin\beta$, $Y_m'(1, \mu)=\cos\beta$, then for all sufficiently large values of μ and $\beta\neq\pi$,

$$Y_m(c_p+\mu^{-3/16}(4a_p)^{-1/4}, \mu) = \sin \beta g_2(\mu)\mu^{-3\nu_p(\mu)/16} \exp\left\{\mu^{1/2}\int_{c_p}^{1}(B-b(x))^{1/2}\,dx-\frac{\mu^{1/8}}{4}\right\},$$

where $g\le g_2(\mu)\le G$. This completes the proof of the theorem. Incidentally we have also shown that $y_m(x, \mu)$ has no zeros in the intervals $(0, c_1-\delta(\mu)]$ and $[c_p+\delta(\mu), 1)$ for all sufficiently large values of μ.

THEOREM 2.2. *Suppose for a sequence of values of* μ, $\{\mu_l\}_{l=1}^{\infty}$, $\mu_1<\mu_2<\cdots$, $\lim_{l\to\infty}\mu_l=\infty$, *there is a* j, $1\le j\le p$, *such that*

$$\sup_{x\in I_j(\mu_l)}|y_m(x, \mu_l)| = \sup_{0\le x\le 1}|y_m(x, \mu_l)|;$$

then $\nu_j=\lim_{\mu\to\infty}\{\Lambda_m(\mu)/2\sqrt{a_j}-\frac{1}{2}\}$ *is a nonnegative integer.*

Proof. For $l=1, 2, \ldots$, select $x_l\in I_j(\mu_l)$ so that $|y_m(x_l, \mu_l)|=\sup_{0\le x\le 1}|y_m(x, \mu_l)|$, and introduce the substitution $s=(4\mu a_j)^{1/4}(x-c_j)$, $x\in J_j$; then with $\mu=\mu_l$, denote by $J_j^\dagger(\mu_l)$ the image of J_j on the s-axis, s_l the image of x_l, and let $I_j^\dagger$ be the interval

$|s| \leq \delta_j = (4a_j)^{1/4}[(\Lambda_m^\dagger + Q)/a\theta]^{1/2}$, which is just the image of $I_j(\mu_l)$ under our transformation. Then for $l=1, 2, \ldots$, we see that if $z_{j,l}(s) = v(x, \mu_l) = [y_m(x, \mu_l)/y_m(x_l, \mu_l)]$, $x \in J_j$, then $z_{j,l}(s)$ satisfies the differential equation

$$(2.1)\qquad Y''(s) + \left(\nu_j(\mu_l) + \frac{1}{2} - \frac{s^2}{4}\right) Y(s) = f_l(s) Y(s), \qquad s \in J_j^\dagger(\mu_l), \quad ' = \frac{d}{ds},$$

with $z_{j,l}(s_l) = 1$, $z'_{j,l}(s_l) = 0$, and where

$$f_l(s) = -\frac{q_{1,l}(s)}{2\sqrt{a_j}} \mu_l^{-1/2} + \frac{s^2}{4}(\phi_{j,l}(s) - 1),$$

$q_{1,l}(s) = q(x)$, $x \in J_j$, and $\phi_{j,l}(s) = \theta_j(x)$, $x \in J_j$. We also observe that for $l=1, 2, \ldots$, $\nu_j(\mu_l) + \frac{1}{2} - s^2/4 < 0$ in $J_j^\dagger(\mu_l) - I_j^\dagger$ and $\nu_j + \frac{1}{2} - s^2/4 < 0$ if $s \notin I_j^\dagger$. By a selection of a subsequence of $\{\mu_l\}_{l=1}^\infty$ if necessary and relabelling suitably we may assume that $\lim_{l\to\infty} s_l = s_0 \in I_j^\dagger$. We therefore see that in any compact subset of the s-axis, $\lim_{l\to\infty} z_{j,l}(s) = w(s)$, uniformly, where $w(s)$ is the solution of Weber's equation $Y''(s) + (\nu_j + \frac{1}{2} - s^2/4) Y(s) = 0$, [5, p. 347], with $w(s_0) = 1$, $w'(s_0) = 0$. Now if ν_j is not a nonnegative integer then we know from [5, pp. 347–348] that there is an s^*, $s^* \notin I_j^\dagger$, such that $|w(s^*)| > 2 \sup_{s \in I_j^\dagger} |w(s)|$; and hence for all l sufficiently large the hypothesis of our theorem is contradicted. This completes the proof of our theorem; and an argument similar to above also leads to the important conclusion that $\lim_{l\to\infty} z_{j,l}(s) = A_j D_{\nu_j}(s)$ uniformly in any compact subset of the s-axis, where $D_{\nu_j}(s)$ is the parabolic cylinder function and A_j is a nonzero constant (see also [2, pp. 136–137]).

Continuing with the argument of the above proof we wish now to construct a fundamental set of solutions for (2.1) in the interval $\delta_j \leq s \leq \mu_l^{1/32}$ for all sufficiently large values of l. Since $f_l(s) = O(\mu_l^{-5/32})$ as $l \to \infty$, uniformly in $0 \leq s \leq \mu_l^{1/32}$, and $\nu_j(\mu_l) - \nu_j = O(\mu_l^{-1/2})$ as $l \to \infty$ [3, Theorem 3.2], then with $F_l(s) = \nu_j - \nu_j(\mu_l) + f_l(s)$ we are led to write (2.1) in the form

$$(2.2)\qquad Y''(s) + \left(\nu_j + \frac{1}{2} - \frac{s^2}{4}\right) Y(s) = F_l(s) Y(s), \; s \in J_j^\dagger(\mu_l),$$

and for the interval $0 \leq s \leq \mu_l^{1/32}$ consider a solution of the form

$$(2.3)\qquad U_{1,l}(s) = D_{-\nu_j-1}(is)[1 + u_{1,l}(s)],$$

$u_{1,l}(0) = u'_{1,l}(0) = 0$. Substituting (2.3) into (2.2) we see [3, §2.1] that for each $l \geq 1$,

$$(2.4)\qquad u'_{1,l}(s) = D^{-2}_{-\nu_j-1}(is) \int_0^s F_l(t) D^2_{-\nu_j-1}(it)[1 + u_{1,l}(t)]\, dt, \qquad 0 \leq s \leq \mu_l^{1/32},$$

and $u_{1,l}(s)$ satisfies the Volterra integral equation

$$(2.5)\qquad u_{1,l}(s) = \int_0^s K_l(s,t)\, dt + \int_0^s K_l(s,t) u_{1,l}(t)\, dt, \qquad 0 \leq s \leq \mu_l^{1/32},$$

where

$$K_l(s,t) = ie^{i\pi\nu_j/2} F_l(t) \left[D_{-\nu_j-1}(it) D_{\nu_j}(t) - \frac{D_{\nu_j}(s)}{D_{-\nu_j-1}(is)} D^2_{-\nu_j-1}(it) \right].$$

Since $K_l(s, t)=O(\mu_l^{-5/32})$ as $l\to\infty$, uniformly in $0\le s, t\le\mu_l^{1/32}$ ([3, §2.1], [5, pp. 347–348]), we see that for each $l\ge 1$ equation (2.5) has a unique solution in $0\le s\le\mu_l^{1/32}$ which may be obtained by the usual method of successive approximations ([3, §2.1], [6, pp. 353–354]); and using the Gronwall lemma we see that $u_{1,l}(s)=O(\mu_l^{-1/8})$ as $l\to\infty$, uniformly in $0\le s\le\mu_l^{1/32}$. Also from (2.4) we see that $u'_{1,l}(s)=O(\mu_l^{-1/8})$ as $l\to\infty$, uniformly in $0\le s\le\mu_l^{1/32}$. Similarly for the interval $\delta_j\le s\le\mu_l^{1/32}$ we consider a second solution of (2.2) in the form

$$U_{2,l}(s) = D_{\nu_j}(s)[1+u_{2,l}(s)], \qquad \delta_j \le s \le \mu_l^{1/32}, \quad u_{2,l}(\mu_l^{1/32}) = u'_{2,l}(\mu_l^{1/32}) = 0;$$

and as above we have $u_{2,l}(s)=O(\mu_l^{-1/8})$, $u'_{2,l}(s)=O(\mu_l^{-1/8})$ as $l\to\infty$, uniformly in $\delta_j\le s\le\mu_l^{1/32}$. If W denotes the Wronskian, then we see that

$$\lim_{l\to\infty} W[U_{1,l}, U_{2,l}](\delta_j) = ie^{-i\pi\nu_j/2},$$

and therefore for all sufficiently large values of l, $U_{1,l}(s)$ and $U_{2,l}(s)$ form a fundamental set of solutions for (2.2) in $\delta_j\le s\le\mu_l^{1/32}$. Hence

THEOREM 2.3. *For all l sufficiently large,*

$$z_{j,l}(s) = d_1(\mu_l)D_{-\nu_j-1}(is)[1+u_{1,l}(s)]+d_2(\mu_l)D_{\nu_j}(s)[1+u_{2,l}(s)]$$

in $\delta_j\le s\le\mu_l^{1/32}$, where $u_{k,l}(s)=O(\mu_l^{-1/8})$, $u'_{k,l}(s)=O(\mu_l^{-1/8})$ as $l\to\infty$, uniformly in $\delta_j\le s\le\mu_l^{-1/32}$, $k=1, 2$, and $\lim_{l\to\infty} d_1(\mu_l)=0$, $\lim_{l\to\infty} d_2(\mu_l)=A_j$. We also have as $l\to\infty$,

$$d_1(\mu_l) = O(\mu_l^{(\nu_j+1)/32}e^{-1/4}\mu_l^{1/16}) \quad \textit{and} \quad z_{j,l}(\mu_l^{1/64}) = O(\mu_l^{\nu_j/64}e^{-1/4}\mu_l^{1/32}).$$

Similarly for all l sufficiently large

$$z_{j,l}(s) = d_3(\mu_l)D_{-\nu_j-1}(is)[1+u_{3,l}(s)]+d_4(\mu_l)D_{\nu_j}(s)[1+u_{4,l}(s)]$$

in $-\mu_l^{1/32}\le s\le -\delta_j$, where $u_{k,l}(s)=O(\mu_l^{-1/8})$, $u'_{k,l}(s)=O(\mu_l^{-1/8})$ as $l\to\infty$, uniformly in $-\mu_l^{1/32}\le s\le -\delta_j$, $k=3, 4$, and $\lim_{l\to\infty} d_3(\mu_l)=0$, $\lim_{l\to\infty} d_4(\mu_l)=A_j$. We also have as $l\to\infty$,

$$d_3(\mu_l)=O(\mu_l^{(\nu_j+1)/32}e^{-1/4}\mu_l^{1/16}) \quad \textit{and} \quad z_{j,l}(-\mu_l^{1/64}) = O(\mu_l^{\nu_j/64}e^{-1/4}\mu_l^{1/32}).$$

Proof. For the interval $\delta_j\le s\le\mu_l^{1/32}$ the proof is completed by observing that for all l sufficiently large, $|z_{j,l}(\mu_l^{1/32})|<|z_{j,l}(s_l)|$, Theorem 2.1, and hence

$$|z_{j,l}(\mu_l^{1/32})|\le 1+\sup_{t\in I_j^{\dagger}} |A_jD_{\nu_j}(s)|;$$

then bounds for $d_1(\mu_l)$ and $z_{j,l}(\mu_l^{1/64})$ follow directly from [5, pp. 347–348]. The results for the interval $-\mu_l^{1/32}\le s\le -\delta_j$ can be shown in the same manner as above.

Carrying on, let us now denote by E the set of integers $\{i\}_{i=1}^{p}$; and if $p>1$ then

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for each i, $1 \le i \le p$, $i \ne j$, we again proceed as in Theorem 2.2 and introduce the substitution $s=(4\mu a_i)^{1/4}(x-c_i)$, $x \in J_i$, and put

$$z_{i,l}(s) = v(x, \mu_l) = \frac{y_m(x, \mu_l)}{y_m(x_l, \mu_l)}, \quad x \in J_i,$$

and where we see that for $l=1, 2, \ldots, z_{i,l}(s)$ satisfies (2.1) with j replaced by i. Assuming a further selection of a sequence of $\{\mu_l\}_{l=1}^{\infty}$ if necessary and relabelling suitably, we introduce a partition of E into the two subsets E_1 and E_2; i.e., $E=E_1 \cup E_2$, $E_1 \cap E_2=\varnothing$, and where $E_2=\varnothing$ if $p=1$. An element $i \in E$ will be a member of E_1 if

(a) ν_i is a nonnegative integer,
(b) $\lim_{l\to\infty} z_{i,l}(s)=A_i D_{\nu_i}(s)$ uniformly in every compact subset of the s-axis,
(c) and where A_i is a constant,

$$\max\{|z_{i,l}(\mu_l^{1/32})|, |z_{i,l}(-\mu_l^{1/32})|\} \le \sup_{s\in I_i^\dagger} |z_{i,l}(s)|,$$

for all l sufficiently large.

It is clear that if $i \in E_1$ then the results of Theorem 2.3 hold with j replaced by i, and that $E_1 \ne \varnothing$ since $j \in E_1$. The remaining elements of E, if any, are members of E_2; we observe that if $E_2 \ne \varnothing$ and $i \in E_2$, then for all sufficiently large values of l, there is an $x_{i,l} \in J_i - I_i(\mu_l)$ such that $|v(x_{i,l}, \mu_l)| > \sup_{x\in I_i(\mu_l)} |v(x, \mu_l)|$.

THEOREM 2.4. *If*

$$X_l = [0, 1] - \bigcup_{i\in E_1} [c_i-(4a_i)^{-1/4}\mu_l^{-15/64}, c_i+(4a_i)^{-1/4}\mu_l^{-15/64}],$$

then for all sufficiently large values of l,

$$\sup_{x\in X_l} |v(x, \mu_l)| \le v_l = \max_{i\in E_1} \max\{|v(c_i-(4a_i)^{-1/4}\mu_l^{-15/64})|, |v(c_i+(4a_i)^{-1/4}\mu_l^{-15/64})|\}.$$

Proof. From the proof of Theorem 2.1, we see our statement is true if $E_2=\varnothing$. Suppose then that $E_2 \ne \varnothing$ and the theorem is false; then there is an l and an $x_l \in X_l$ such that $|v(x_l, \mu_l)| > v_l$, and where l can be assumed sufficiently large to satisfy all statements of the preceding paragraph and proof of Theorem 2.1.

Let us first consider the case $x_l \in I_i(\mu_l)$, $i \in E_2$; then there is an $x_{i,l} \in J_i - I_i(\mu_i)$ such that

$$|v(x_{i,l}, \mu_l)| > \sup_{x\in I_i(\mu_l)} |v(x, \mu_l)| > v_l.$$

Assuming $x_{i,l} > x_l$, we see that if $i=p$ or $(i+1) \in E_1$ we have a contradiction; and if $i<p$ and $(i+1) \in E_2$ we can proceed on to find an $x_{i+k-1,l} \in X_l - \bigcup_{n\in E_2} I_n(\mu_l)$, $k \ge 2$, $x_{i+k-1,l} > c_{i+k-1,l} + \delta(\mu_l)$, such that

$$|v(x_{i+k-1,l}, \mu_l)| > \sup_{x\in I_{i+k-1}(\mu_l)} |v(x, \mu_l)| > v_l$$

and with the property that either $(i+k) \in E_1$ or $(i+k-1)=p$, and which again leads to a contradiction. A similar argument for the cases $x_{i,l} < x_l \in I_i(\mu_l)$ and $x_l \in X_l$

$-\bigcup_{i\in E_2} I_i(\mu_l)$ lead again to contradictions. This completes the proof of our theorem.

Continuing on, let us now denote by $\phi_m(x, \mu)$ the eigenfunction of (1.1), (1.2) corresponding to $\lambda_m(\mu)$ such that $\int_0^1 \phi_m^2(x, \mu)\, dx = 1$. Then we have

$$\phi_m(x, \mu_l) = k(\mu_l)v(x, \mu_l), \qquad k^2(\mu_l) \int_{c_j-(4a_j)^{-1/4}\mu_l^{-15/64}}^{c_j+(4a_j)^{-1/4}\mu_l^{-15/64}} v^2(x, \mu_l)\, dx < 1,$$

and

$$(4\mu_l a_j)^{-1/4} k^2(\mu_l)\left[\int_{-\mu_l^{1/64}}^{-\delta_j} z_{j,l}^2(s)\, ds + \int_{\delta_j}^{\mu_l^{1/64}} z_{j,l}^2(s)\, ds\right] < 1;$$

hence from Theorem 2.3 ([2, p. 137], [5, p. 350]), and the fact that $A_j \neq 0$, we see that as $l \to \infty$, $k(\mu_l) = O(\mu_l^{1/8})$.

THEOREM 2.5. *It is the case that*

$$\mu \int_0^1 (b^\dagger(x)\phi_m(x, \mu))^2\, dx = O(1) \quad \text{as } \mu \to \infty.$$

Proof. Suppose that the theorem is false; then there exists a sequence of values of μ, $\{\mu_l\}_{l=1}^\infty$, $\mu_1 < \mu_2 < \cdots$, $\lim_{l\to\infty} \mu_l = \infty$, such that

$$\lim_{l\to\infty} \left\{\mu_l \int_0^1 (b^\dagger(x)\phi_m(x, \mu_l))^2\, dx\right\} = \infty.$$

By a selection of a subsequence of $\{\mu_l\}_{l=1}^\infty$ and relabelling suitably we may assume the results of Theorems 2.3 and 2.4 as well as the results immediately preceding the statement of this theorem are valid. Hence

$$\mu_l \int_0^1 (b^\dagger(x)\phi_m(x, \mu_l))^2\, dx = \mu_l k^2(\mu_l) \int_{X_l} (b^\dagger(x)v(x, \mu_l))^2\, dx$$
$$+ \sum_{i\in E_1} \int_{-\mu_l^{1/64}}^{\mu_l^{1/64}} s^4 R_{i,l}(s) z_{i,l}^2(s)\, ds = S_{1,l} + S_{2,l},$$

and where $|R_{i,l}(s)| \leq R$, and R is a positive constant independent of i, l and s. From Theorems 2.3 and 2.4 we see that $S_{1,l} = o(1)$ as $l \to \infty$; and from Theorem 2.3 and [5, p. 347] we have $S_{2,l} = O(1)$ as $l \to \infty$. Hence we are led to a contradiction.

3. **Final Results.**

THEOREM 3.1. *For* $n = 0, 1, \ldots,$ *we have as* $t \to 0$,

(i) $\lambda_n(\mu, t) - \lambda_n(\mu) = O(t)$, ($\mu$ fixed), and

(ii) $\Lambda_n(t) - \Lambda_n = O(t)$; (*and not uniformly with respect to* n). *In fact if we fix a* $\mu^\dagger > 0$, *we have for* $\mu \geq \mu^\dagger$ *and as* $t \to 0$,

(iii) (a) $\Lambda_0(\mu, t) - \Lambda_0(\mu) = O(t)$;

(b) *if* $n > 0$ *and* $\Lambda_n > \Lambda_{n-1}$, *then* $\Lambda_n(\mu, t) - \Lambda_n(\mu) = O(t)$;

(c) *if* $\Lambda_n = \Lambda_{n+1} = \cdots = \Lambda_{n+l}$, $l \geq 1$, *and* $\Lambda_{n-1} < \Lambda_n$ *if* $n > 0$, *and* $\Lambda_{n+l} < \Lambda_{n+l+1}$, *then* $\Lambda_{n+l}(\mu, t) - \Lambda_{n+l}(\mu) = O(t)$, *and if* $l > 1$ *and* $1 \leq m \leq l-1$ *then*

$$\min_{m \leq k \leq l} |\Lambda_{n+m}(\mu, t) - \Lambda_{n+k}(\mu)| = O(t);$$

and where the results are uniform in μ *for each n, but not uniform with respect to n.*

Proof. We shall only prove parts (ii), (iii); part (i) can be proved using similar arguments (see also [6, pp. 231–232]). We shall also assume for the remainder of this proof that $\mu \geq \mu^\dagger$, where $\mu^\dagger$ is given above. Then for $n = 0, 1, 2, \ldots$, let $\phi_n(x, \mu)$ denote the eigenfunction of (1.1), (1.2) corresponding to $\lambda_n(\mu)$ such that $\int_0^1 \phi_n^2(x, \mu)\, dx = 1$ and let $\phi_n(x, \mu, t)$ denote the eigenfunction of (1.3), (1.2) corresponding to $\lambda_n(\mu, t)$ such that $\int_0^1 \phi_n^2(x, \mu, t) = 1$, and put

$$F_{m,n}(\mu, t) = \int_0^1 \phi_m(x, \mu, t)\phi_n(x, \mu)\, dx,$$

$$G_{m,n}(\mu, t) = \int_0^1 b^\dagger(x)\phi_m(x, \mu, t)\phi_n(x, \mu)\, dx;$$

and for $n = 0, 1, \ldots$, let us introduce the positive constants B_n and C_n, where $B_n^2 = \sup_{\mu^\dagger \leq \mu < \infty} \{\mu \int_0^1 (b^\dagger(x)\phi_n(x, \mu))^2\, dx\}$, (see Theorem 2.5), and $C_n^2 = \sum_{r=0}^n B_r^2$. Now putting $L \equiv d^2/dx^2 - \mu(B - b(x)) + q(x)$, we have from equations (1.3) and (1.1),

$$L[\phi_m(x, \mu, t)] + \mu^{1/2}\Lambda_m(u, t)\phi_m(x, \mu, t) = t\mu b^\dagger(x)\phi_m(x, \mu, t), \quad 0 \leq x \leq 1,$$

$$L[\phi_n(x, \mu)] + \mu^{1/2}\Lambda_n(\mu)\phi_n(x, \mu) = 0, \quad 0 \leq x \leq 1.$$

Hence from Green's formula and equation (1.2) we have

$$(3.1) \qquad (\Lambda_m(\mu, t) - \Lambda_n(\mu))F_{m,n}(\mu, t) = t\mu^{1/2}G_{m,n}(\mu, t).$$

Since $\Lambda_n(\mu, t) - \Lambda_n(\mu) \geq 0$, $n = 0, 1, \ldots$, [6, pp. 87–90], we see from equation (3.1) and the Parseval theorem [4, p. 199] that

$$(\Lambda_0(\mu, t) - \Lambda_0(\mu))^2 \leq \sum_{m=0}^{\infty} (\Lambda_m(\mu, t) - \Lambda_0(\mu))^2 F_{m,0}^2(\mu, t)$$
$$= t^2\mu \sum_{m=0}^{\infty} G_{m,0}^2(\mu, t) \leq t^2 B_0^2,$$

and our results follow for $n = 0$.

Similarly, if $\Lambda_1 > \Lambda_0$, then there is a $\Delta > 0$ such that $\Lambda_1(\mu) - \Lambda_0(\mu) \geq \Delta$ for $\mu \geq \mu^\dagger$; hence if $t \leq \Delta/4C_1$ and $\delta(\mu, t) = \min\{\Lambda_1(\mu, t) - \Lambda_1(\mu), \Lambda_1(\mu) - \Lambda_0(\mu, t)\}$, then

$$(\delta(\mu, t))^2 \leq \sum_{m=0}^{\infty} (\Lambda_m(\mu, t) - \Lambda_1(\mu))^2 F_{m,1}^2(\mu, t)$$
$$= t^2\mu \sum_{m=0}^{\infty} G_{m,1}^2(\mu, t) \leq t^2 B_1^2,$$

and our results follow for $n = 1$ for this case.

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If $\Lambda_0=\Lambda_1=\cdots=\Lambda_l$, $l\geq 1$, and $\Lambda_{l+1}>\Lambda_l$, then there is a $\Delta^*>0$ such that $\Lambda_{l+1}(\mu)-\Lambda_l(\mu)\geq\Delta^*$ for $\mu\geq\mu^\dagger$. Hence if $0\leq n\leq l$, then

$$(\Delta^*)^2 \sum_{=l+1}^{\infty} F_{m,n}^2(\mu, t) \leq \sum_{m=l+1}^{\infty} (\Lambda_m(\mu, t)-\Lambda_n(\mu))^2 F_{m,n}^2(\mu, t) \leq t^2 B_n^2,$$

and

$$\sum_{m=0}^{l} F_{m,n}^2(\mu, t) \geq 1-t^2(B_n/\Delta^*)^2,$$

$$\sum_{n=0}^{l}\sum_{m=0}^{l} F_{m,n}^2(\mu, t) \geq (l+1)-t^2(C_l/\Delta^*)^2$$

and

$$\sum_{m=0}^{l}\sum_{n=l+1}^{\infty} F_{m,n}^2(\mu, t) \leq t^2(C_l/\Delta^*)^2.$$

Therefore for $0\leq m\leq l$,

$$\sum_{n=l+1}^{\infty} F_{m,n}^2(\mu, t) \leq t^2(C_l/\Delta^*)^2 \quad\text{and}\quad \sum_{n=0}^{l} F_{m,n}^2(\mu, t) \geq 1-t^2(C_l/\Delta^*)^2.$$

Hence for $0\leq m\leq l$,

$$\min_{m\leq k\leq l}\{(\Lambda_m(\mu, t)-\Lambda_k(\mu))^2\}\sum_{n=0}^{l} F_{m,n}^2(\mu, t) \leq \sum_{n=0}^{l}(\Lambda_m(\mu, t)-\Lambda_n(\mu))^2 F_{m,n}^2(\mu, t)$$
$$= t^2\mu\sum_{n=0}^{l} G_{m,n}^2(\mu, t) \leq t^2 C_l^2;$$

so

$$\min_{m\leq k\leq l}(\Lambda_m(\mu, t)-\Lambda_k(\mu))^2 \leq t^2(4C_l^2/3) \quad \text{if } t\leq(\Delta^*/2C_l),$$

and again our results follow.

The proof of parts (ii) and (iii) of our theorem is then completed by arguing in the same way for all values of n.

In conclusion we would like to state that similar results also hold under suitable conditions if $c_1=0$ or $c_p=1$.

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