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THE 2-IDEAL CLASS GROUPS OF $\mathbb{Q}(\zeta_l)$

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Abstract. For prime l we study the structure of the 2-part of the ideal class group Cl of $\mathbb{Q}(\zeta_l)$. We prove that $\mathrm{Cl} \otimes \mathbb{Z}_2$ is a cyclic Galois module for all $l < 10000$ with one exception and compute the explicit structure in several cases.

§1. Introduction

Let l be an odd prime number and let ζ_l be a primitive l-th root of unity. We denote by Cl the ideal class group of the field $\mathbb{Q}(\zeta_l)$. The aim of this paper is to study the structure of the 2-part of Cl as an abelian group. Let G be the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$. We have a natural decomposition $G = \Delta \times P$ where P is the 2-Sylow subgroup of G and Δ is the subgroup of G consisting of the elements of odd order. Let Cl^+ be the ideal class group of the field $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$; there exists a natural injective map $\mathrm{Cl}^+ \to \mathrm{Cl}$, and we denote by Cl^- its cokernel. In order to study the 2-part of class groups, it is useful to introduce 2-adic characters. Let $\chi : \Delta \to \overline{\mathbb{Q}}_2^*$ be a 2-adic character, and denote by $\mathcal{O}_\chi$ the ring $\mathbb{Z}_2(\chi)$. For any $\mathbb{Z}[G]$-module M, we define its χ-part $M(\chi)$ as $(M \otimes_{\mathbb{Z}} \mathbb{Z}_2) \otimes_{\mathbb{Z}_2[\Delta]} \mathcal{O}_\chi$. It is an $\mathcal{O}_\chi[P]$-module. In particular, the 2-part $M \otimes_{\mathbb{Z}} \mathbb{Z}_2$ of M is a direct sum of χ-parts. For more information on χ-parts, see [4]. In Section 2 we prove a cyclicity criterion, which is a version of théorème I.9 of [10] adapted to our situation:

Theorem 1. *Let $l \equiv 1 \pmod 4$. The group $\mathrm{Cl}(\chi)$ is a nontrivial cyclic $\mathcal{O}_\chi$-module if and only if $\#\mathrm{Cl}^-(\chi) = \#(\mathcal{O}_\chi/2)$.*

The above theorem allows one to determine $\mathrm{Cl}(\chi)$ in some cases. As an example, consider the field $\mathbb{Q}(\zeta_{9337})$. Let χ be the character of order 3. We have $\#\mathrm{Cl}^-(\chi) = \#(\mathcal{O}_\chi/2)$, and $\#\mathrm{Cl}^+(\chi) = \#(\mathcal{O}_\chi/8)$. Applying the theorem, we get $\mathrm{Cl}(\chi) \cong \mathcal{O}_\chi/16$.

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We then characterize the cohomological triviality of the $\mathcal{O}_\chi[P]$-module $\mathrm{Cl}(\chi)$. In Proposition 4 we show that $\mathrm{Cl}(\chi)$ is cohomologically trivial if and only if the χ-part of the units of $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$ have independent signs.

In Section 3 we study the cyclicity of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi[P]$-module. If $\mathrm{Cl}(\chi)$ is cyclic, then it is possible to compute explicitly the $\mathcal{O}_\chi[P]$-structure of $\mathrm{Cl}^+(\chi)$ and of $\mathrm{Cl}^-(\chi)$. The structure of $\mathrm{Cl}^-(\chi)$ is given by Proposition 2. The description of the structure of $\mathrm{Cl}^+(\chi)$ is more complicated. There exists an ideal $J^+(\chi)$ of $\mathcal{O}_\chi[P]$ such that $\mathrm{Cl}^+(\chi)$ and $\mathcal{O}_\chi[P]/J^+(\chi)$ have the same order. The definition of $J^+(\chi)$ can be found in Proposition 9 of [4]. The ideal $J^+(\chi)$ annihilates $\mathrm{Cl}^+(\chi)$ (Theorem 2.2 of [13]): this is proved using methods developed by F. Thaine. A more precise result is also given in [5]. Therefore, in case $\mathrm{Cl}^+(\chi)$ is cyclic over $\mathcal{O}_\chi[P]$, we have $\mathrm{Cl}^+(\chi) \cong \mathcal{O}_\chi[P]/J^+(\chi)$. The ideals $J^+(\chi)$ have been computed in [17] for all fields of prime conductor $l < 10000$. Cyclicity questions have been studied in [15], where it is proved that the minus class group Cl^- of $\mathbb{Q}(\zeta_l)$ is a cyclic Galois module for all primes $l \leq 509$. For class groups of real cyclic fields there are also some results in this direction [1, 7]. Numerical computations suggest that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module most of the times, and it is quite hard to find examples when this condition is not verified. We prove that $\mathrm{Cl}(\chi)$ is cyclic whenever $\mathrm{Cl}^+(\chi)$ is trivial (Propositions 5 and 6). Moreover, we prove the following:

Theorem 2. *If $l < 10000$ is a prime number not equal to 7687, then the group $\mathrm{Cl}(\chi)$ is a cyclic Galois module.*

In the case $l = 7687$ and χ a nontrivial cubic character, one could show by explicit computations that $\mathrm{Cl}(\chi)$ has actually two generators. In Section 4 we give several structure results on the $\mathcal{O}_\chi$-structure of $\mathrm{Cl}(\chi)$ in the case that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. In particular, we determine completely the $\mathcal{O}_\chi$-structure of $\mathrm{Cl}(\chi)$ when $l \equiv 3 \pmod 4$, $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module and $\#\mathrm{Cl}^+(\chi) \neq \#\mathrm{Cl}^-(\chi)$ (Propositions 7 and 8). These results allow us to determine in many cases the structure of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi$-module. The numerical results are presented in the table which is described in Section 5.

§2. Generalities on class groups

We maintain the same notations as in the introduction. Let G be the Galois group of the field $\mathbb{Q}(\zeta_l)$ over the rationals. The group G is a cyclic abelian group of order $l-1$. Let 2^e be the exact power of 2 dividing $l-1$.

Let $\chi : \Delta \to \overline{\mathbb{Q}}_2^*$ be a 2-adic character of Δ, and denote by $\mathcal{O}_\chi$ the discrete valuation ring $\mathbb{Z}_2(\chi)$. Let $\mathrm{Cl}(\chi)$ be the χ-part of the ideal class group of $\mathbb{Q}(\zeta_l)$. For any $\mathbb{Z}[G]$-module M, we define its χ-part

$$M(\chi) = (M \otimes_{\mathbb{Z}} \mathbb{Z}_2) \otimes_{\mathbb{Z}_2[\Delta]} \mathcal{O}_\chi.$$

We are interested in the structure of $\mathrm{Cl}(\chi)$ as a $\mathcal{O}_\chi[P]$-module. In order to proceed, we introduce some notation. Let d be the order of the character χ. We denote by K_e the subfield of $\mathbb{Q}(\zeta_l)$ fixed by $\mathrm{Ker}\,\chi$. It is a cyclic extension of $\mathbb{Q}$ of degree $d \cdot 2^e$. For all $0 \le i \le e$, we denote by K_i the unique subfield of K_e of degree $d \cdot 2^i$ over $\mathbb{Q}$. The fields K_i are totally real abelian fields for all $0 \le i \le e-1$. We denote by Cl_i and by Cl_i^∞ the ideal class group and the narrow ideal class group respectively of the field K_i. We also write Cl_e^+ for Cl_{e-1}. Observe that $\mathrm{Cl}_e \cong \mathrm{Cl}_e^\infty$.

PROPOSITION 1. *Let the notations be as above. For $i \ge j$, we denote by $\sigma_{i,j}$ a generator of* $\mathrm{Gal}(K_i/K_j)$. *Then we have*

$$\mathrm{Cl}_i^\infty \cong \mathrm{Cl}_e/(\mathrm{Cl}_e)^{1-\sigma_{e,i}}, \forall\ 0 \le i \le e, \tag{1}$$

$$\mathrm{Cl}_i \cong \mathrm{Cl}_{e-1}/(\mathrm{Cl}_{e-1})^{1-\sigma_{e-1,i}}, \forall\ 0 \le i \le e-1. \tag{2}$$

Proof. The proof is the same as the one of Lemma 1 of [3]. We prove (1); the proof of (2) is analogous. The extension K_e/K_i is totally ramified at the unique prime ideal of K_i above l and unramified above all other finite primes. Class field theory implies that the norm map $N_{e,i} : \mathrm{Cl}_e \to \mathrm{Cl}_i^\infty$ is surjective. The group $(\mathrm{Cl}_e)^{1-\sigma_{e,i}}$ is clearly contained in the kernel. Applying the genus theory formula (see [11], Chapter 13, Lemma 4.1) and Hasse's principle, it follows that $\#\mathrm{Cl}_i^\infty = \#\mathrm{Cl}_e^{\mathrm{Gal}(K_e/K_i)} = \#\mathrm{Cl}_e/(\mathrm{Cl}_e)^{1-\sigma_{e,i}}$. Therefore the map $N_{e,i}$ induces an isomorphism, and our claim is proved.

We define the minus class group Cl_e^- to be the cokernel of the natural map $\mathrm{Cl}_{e-1} \to \mathrm{Cl}_e$.

Let $j \in G$ denote complex conjugation. Let $\zeta_{2^e} \in \overline{\mathbb{Q}}_2$ be a primitive 2^e-th root of unity. There exists an isomorphism $\mathrm{Cl}_e^-(\chi) \cong \mathrm{Cl}^-(\chi)$ induced by the norm map from $\mathbb{Q}(\zeta_l)$ to K_e. The group $\mathrm{Cl}^-(\chi)$ is an $\mathcal{O}_\chi[P]/(1+j) \cong \mathcal{O}_\chi[\zeta_{2^e}]$-module. Suppose that $\mathrm{Cl}^-(\chi)$ is a cyclic Galois module. Since $\mathcal{O}_\chi[\zeta_{2^e}]$ is a discrete valuation ring, there is a simple description of the structure of $\mathrm{Cl}^-(\chi)$. Let $2^f = \#(\mathcal{O}_\chi/2)$. The following is Proposition 3.4 of [15].

Proposition 2. *Suppose that A is a cyclic $\mathcal{O}_\chi[P]/(1+j)$-module. Moreover suppose that $\#A = 2^{ft}$. Then there is an isomorphism of $\mathcal{O}_\chi[\zeta_{2^e}]$-modules*

$$A \cong \mathcal{O}_\chi[\zeta_{2^e}]/(1-\zeta_{2^e})^t \tag{3}$$

and an isomorphism of $\mathcal{O}_\chi$-modules

$$A \cong (\mathcal{O}_\chi/2^r)^{(2^{e-1}-s)} \times (\mathcal{O}_\chi/2^{r+1})^s \tag{4}$$

where $r, s \in \mathbb{N}$ are determined by $t = r2^{e-1} + s$ and $0 \le s < 2^{e-1}$.

Proof. This follows because $\mathcal{O}_\chi[\zeta_{2^e}]$ is a discrete valuation ring with uniformizing element $1-\zeta_{2^e}$.

For any $\mathcal{O}_\chi$-module A, we denote by $\mathrm{rank}_{\mathcal{O}_\chi} A$ the dimension of the $\mathcal{O}_\chi/2$-vector space $A/2$.

Corollary 1. *Let $\#\mathrm{Cl}^-(\chi) = 2^{ft}$ and suppose that $\mathrm{Cl}^-(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. Then $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}^-(\chi) = \min(t, 2^{e-1})$.*

Proof. Since $1+j$ annihilates $\mathrm{Cl}^-(\chi)$, we can apply Proposition 2 with $A = \mathrm{Cl}^-(\chi)$. If $t < 2^{e-1}$, then $r = 0$, $s = t$ and (4) gives us $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}^-(\chi) = s = t$. If $t \ge 2^{e-1}$ then $r > 0$ and (4) gives us $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}^-(\chi) = 2^{e-1}$, as we wanted to show.

We are now ready to prove Theorem 1 of the introduction.

Proof of Theorem 1. Suppose that $\mathrm{Cl}(\chi) \cong \mathrm{Cl}_e(\chi)$ is nontrivial and cyclic over $\mathcal{O}_\chi$. This implies that $\mathrm{Cl}^-(\chi)$ is a nontrivial cyclic $\mathcal{O}_\chi$-module. The condition on l is equivalent to say that $e > 1$. Corollary 1 implies $t = 1$, therefore $\#\mathrm{Cl}^-(\chi) = 2^f = \#(\mathcal{O}_\chi/2)$. Suppose now that $\#\mathrm{Cl}^-(\chi) = \#(\mathcal{O}_\chi/2)$. In particular, $\mathrm{Cl}^-(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. We have $\mathrm{Cl}^-(\chi) \cong \mathrm{Cl}(\chi)/\mathrm{Cl}(\chi)^{1+j}$ and $1+j$ is in the maximal ideal of $\mathcal{O}_\chi[P]$. Nakayama's lemma implies that the group $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. We identify the ring $\mathcal{O}_\chi[P]$ with the ring $R = \mathcal{O}_\chi[T]/((1+T)^{2^e} - 1)$. Since $R/2 \cong \frac{\mathcal{O}_\chi}{2}[T]/(T^{2^e})$, we have $\mathrm{Cl}(\chi)/2 \cong \frac{\mathcal{O}_\chi}{2}[T]/(T^h)$ for some $h \le 2^e$. Since $1+j = 1+(1+T)^{2^{e-1}}$, we obtain $\mathrm{Cl}^-(\chi)/2 \cong \frac{\mathcal{O}_\chi}{2}[T]/(T^h, T^{2^{e-1}})$. Our assumption implies that $\min(h, 2^{e-1}) = 1$. Since $e > 1$, we must have $h = 1$: this means that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi$-module.

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We now study the cohomology of the groups $\mathrm{Cl}(\chi)$. We say that $\mathrm{Cl}(\chi)$ is cohomologically trivial if the Tate cohomology groups $\widehat{H}^i(P, \mathrm{Cl}(\chi))$ are trivial for all $i \in \mathbb{Z}$. Since P is a cyclic group and $\mathrm{Cl}(\chi)$ has finite order, Tate cohomology has period 2 and the Herbrand quotient is 1. Therefore saying that $\mathrm{Cl}(\chi)$ is cohomologically trivial is equivalent to say that there exists an i such that $\widehat{H}^i(P, \mathrm{Cl}(\chi))$ is trivial.

We need to recall some notations and results. For any field E, we denote by $\mathcal{O}_E^*$ the unit group of its ring of integers. If E is a totally real field, we also denote by E_+ the set of totally positive elements of E, and by $\mathcal{O}_{E,+}^*$ the group of totally positive units in $\mathcal{O}_E^*$. Combining Theorem 1 of [4] and Proposition 7 (iii) of [4] we get that

$$\widehat{H}^0(P, \mathrm{Cl}(\chi)) \cong (\mathcal{O}_{K_0,+}^*/N_{K_0}^{K_e}\mathcal{O}_{K_e}^*)(\chi) \tag{5}$$

where $N_{K_0}^{K_e}$ is the norm map from K_e to K_0. We need to recall another result.

Proposition 3. *Let K be a totally real number field and let K/E be a quadratic extension. Suppose that $(\mathcal{O}_E^*)^2 = \mathcal{O}_{E,+}^*$. We then have a natural isomorphism*

$$K^*/(K_+^*\mathcal{O}_K^*) \cong \widehat{H}^0(\mathrm{Gal}(K/E), \mathcal{O}_K^*).$$

For a proof see [2], Theorem 12.11, page 61. We now give a criterion for the cohomological triviality of $\mathrm{Cl}(\chi)$ in terms of the signature of the units.

Proposition 4. *The cohomology group $\widehat{H}^0(P, \mathrm{Cl}(\chi))$ is trivial if and only if $(\mathcal{O}_{K_{e-1},+}^*/(\mathcal{O}_{K_{e-1}}^*)^2)(\chi) \cong 0$.*

Proof. Since $\mathcal{O}_{K_e}^* = \mu(K_e)\mathcal{O}_{K_{e-1}}^*$, where $\mu(K_e)$ are the roots of unity in K_e (the Hasse index is 1 in our situation), $N_{K_0}^{K_e}\mathcal{O}_{K_e}^* = (N_{K_0}^{K_{e-1}}\mathcal{O}_{K_{e-1}}^*)^2$. We have

$$(N_{K_0}^{K_{e-1}}\mathcal{O}_{K_{e-1}}^*)^2 \subset (\mathcal{O}_{K_0}^*)^2 \subset \mathcal{O}_{K_0,+}^*.$$

Therefore, by (5), $\widehat{H}^0(P, \mathrm{Cl}(\chi)) \cong 0$ is equivalent to

$$(N_{K_0}^{K_{e-1}}\mathcal{O}_{K_{e-1}}^*)(\chi) = \mathcal{O}_{K_0}^*(\chi) \text{ and } (\mathcal{O}_{K_0}^*)^2(\chi) = \mathcal{O}_{K_0,+}^*(\chi). \tag{6}$$

If $e = 1$, then $K_{e-1} = K_0$ and we are done. From now on, we suppose that $e > 1$. Since the isomorphism in Proposition 3 is natural, it remains true if we take χ-parts. Suppose that (6) holds. The first condition, which can be stated as [1]

$$\widehat{H}^0(\mathrm{Gal}(K_{e-1}/K_0), \mathcal{O}^*_{K_{e-1}}(\chi)) \cong 0 \tag{7}$$

implies that

$$\widehat{H}^0(\mathrm{Gal}(K_{i+1}/K_i), \mathcal{O}^*_{K_{i+1}}(\chi)) \cong 0, \ \forall\, 0 \le i \le e-2.$$

We apply inductively Proposition 3 to the extensions K_{i+1}/K_i, until $i = e-2$. At each step we get $(K_{i+1})^*/((K_{i+1})^*_+\mathcal{O}^*_{K_{i+1}})(\chi) \cong 0$, which is equivalent to $(\mathcal{O}^*_{K_{i+1},+}/(\mathcal{O}^*_{K_{i+1}})^2)(\chi) \cong 0$. The last step gives our claim. Vice versa, suppose that the group $(\mathcal{O}^*_{K_{e-1},+}/(\mathcal{O}^*_{K_{e-1}})^2)(\chi)$ is trivial. Since the extension K_{e-1}/K_0 is totally ramified above $l \neq 2$, we have that $(\mathcal{O}^*_{K_i,+}/(\mathcal{O}^*_{K_i})^2)(\chi)$ is trivial for all $i = 1, \dots, e-1$. Therefore using Proposition 3 again, we have that at each step the cohomology group $\widehat{H}^0(K_{i+1}/K_i, \mathcal{O}^*_{K_{i+1}}(\chi))$ is trivial. This implies that $\mathcal{O}^*_{K_0}(\chi) \cong N^{K_{e-1}}_{K_0}\mathcal{O}^*_{K_{e-1}}(\chi)$, thus (6) is satisfied.

§3. Cyclicity of $\mathrm{Cl}(\chi)$ as a Galois module

In this section we study the case when $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. Numerical computations suggest that this is almost always the case, and in this situation we have more information on the structure of $\mathrm{Cl}(\chi)$. We maintain the notations from the previous sections. The norm map $\mathrm{Cl} \to \mathrm{Cl}^+$ is surjective by class field theory, and the natural map $\mathrm{Cl}^+ \to \mathrm{Cl}$ is injective [11, Chap. 3, Th. 4.2]. The composition of these maps is multiplication by $1+j$; this allows us to identify Cl^+ with Cl^{1+j}.

We first state a criterion of cyclicity of $\mathrm{Cl}_0(\chi)$ as a $\mathcal{O}_\chi$-module.

THEOREM 3. (T. Berthier) *Suppose that χ is not the trivial character. If $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_0(\chi) = 1$ then there exists a prime number $r \equiv 3 \pmod 4$ which is split in $K_0/\mathbb{Q}$, such that the χ-part*[2] *$\mathrm{Cl}_{\chi,r}$ of the ideal class group of the field $K_0(\sqrt{-r})$ has order $\#\mathrm{Cl}_0(\chi)\#(\mathcal{O}_\chi/2)$. On the other hand, if there exists a prime r as above, then $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_0(\chi) \le 1$.*

[1] We remark that if χ is not the trivial character, then it is true that (7) is equivalent to say that $\mathcal{O}^*_{K_{e-1}}(\chi)$ is a free one dimensional $\mathcal{O}_\chi[\mathrm{Gal}(K_{e-1}/K_0)]$-module, but we do not need this.

[2] Here we view χ as a character of $\mathrm{Gal}(K_0/\mathbb{Q})$ and extend it to $\mathrm{Gal}(K_0(\sqrt{-r})/\mathbb{Q}(\sqrt{-r}))$.

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This result is a special case of [1, Th. 2.4.3]. The proof of the first part is difficult. Here we sketch the proof of the second part. The conditions imposed on r imply that the places which ramify in $K_0(\sqrt{-r})/K_0$ are precisely the infinite ones and the ones above r. If we apply the χ-ambiguous class number formula of genus theory[3], we get that $\mathrm{Cl}_{\chi,r}^{\mathrm{Gal}(K_0(\sqrt{-r})/K_0)}$ has order at least $\#\mathrm{Cl}_0(\chi)\#(\mathcal{O}_\chi/2)$. Therefore our hypothesis force $\mathrm{Cl}_{\chi,r}$ to be $\mathrm{Gal}(K_0(\sqrt{-r})/K_0)$-invariant. Since the field $K_0(\sqrt{-r})$ is totally imaginary, the field K_0 is totally real, and χ is not the trivial character, it is not hard to show that the natural map $\mathrm{Cl}_0(\chi) \to \mathrm{Cl}_{\chi,r}$ is injective. Moreover, since the extension $K_0(\sqrt{-r})/K_0$ is ramified, the norm map $\mathrm{Cl}_{\chi,r} \to \mathrm{Cl}_0(\chi)$ is surjective. Let σ be a generator of $\mathrm{Gal}(K_0(\sqrt{-r})/K_0)$. We have $\mathrm{Cl}_{\chi,r}^2 = \mathrm{Cl}_{\chi,r}^{1+\sigma} \cong \mathrm{Cl}_0(\chi)$. Therefore

$$\#(\mathrm{Cl}_{\chi,r}/\mathrm{Cl}_{\chi,r}^2) = \#(\mathcal{O}_\chi/2)$$

and we get that $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_{\chi,r} = 1$. Since $\mathrm{Cl}_0(\chi)$ is an epimorphic image of $\mathrm{Cl}_{\chi,r}$, we get $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_0(\chi) \leq 1$.

COROLLARY 2. *If* $\mathrm{Cl}_0(\chi)$ *is a cyclic* $\mathcal{O}_\chi$*-module, then* $\mathrm{Cl}_0^\infty(\chi)$ *is a cyclic* $\mathcal{O}_\chi$*-module as well.*

Proof. We can assume that $\mathrm{Cl}_0(\chi)$ is nontrivial, otherwise $\mathrm{Cl}_0^\infty(\chi)$ is either trivial or isomorphic to $\mathcal{O}_\chi/2$, hence cyclic. In this situation χ is not the trivial character and we can apply Theorem 3. Therefore there exists a quadratic totally imaginary extension $E = K_0(\sqrt{-r})$ of K_0 such that the χ-part $\mathrm{Cl}_E(\chi)$ of the ideal class group of E has $\mathcal{O}_\chi$-rank equal to 1 (see the proof of Theorem 3). Moreover the extension E/K_0 is ramified at the finite primes above r. Therefore the norm map $\mathrm{Cl}_E(\chi) \to \mathrm{Cl}_0^\infty(\chi)$ between narrow ideal class groups is surjective. The group $\mathrm{Cl}_0^\infty(\chi)$ is then a surjective image of $\mathrm{Cl}_E(\chi)$ (they are actually isomorphic). Therefore $\mathrm{Cl}_0^\infty(\chi)$ is a cyclic $\mathcal{O}_\chi$-module.

Observe that in the case -1 is a power of 2 modulo the order of χ, B. Oriat already proved the equality $\mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_0(\chi) = \mathrm{rank}_{\mathcal{O}_\chi}\mathrm{Cl}_0^\infty(\chi)$ using the *Spiegelungssatz* [12, Cor. 2 c].

Remark. In [15, Th. 3.3] a sufficient condition for $\mathrm{Cl}^-(\chi)$ to be a cyclic $\mathcal{O}_\chi[P]$-module is given.

[3] This is the χ-version of Lemma 4.1, Chapter 13 of [11]. See also [6].

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Proposition 5. *The following assertions are equivalent:*

1. $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module;*
2. $\mathrm{Cl}^-(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module;*
3. $\mathrm{Cl}^+(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module;*
4. $\mathrm{Cl}_0^\infty(\chi)$ *is a cyclic* $\mathcal{O}_\chi$*-module;*
5. $\mathrm{Cl}_0(\chi)$ *is a cyclic* $\mathcal{O}_\chi$*-module.*

Proof. The ring $\mathcal{O}_\chi[P]$ is a local ring with maximal ideal $(2, 1-\sigma)$, where σ is a generator of P. By definition

$$\mathrm{Cl}^-(\chi) \cong \mathrm{Cl}(\chi)/\mathrm{Cl}^+(\chi) \cong \mathrm{Cl}(\chi)/\mathrm{Cl}(\chi)^{1+j}.$$

Nakayama's lemma gives the equivalence of 1 and 2. The equivalence of 1 and 4 follows again by Nakayama's lemma and Equation (1) of Proposition 1. Similarly, 3 and 5 are equivalent by Nakayama's lemma and Equation (2) of Proposition 1. Since $\mathrm{Cl}_0(\chi)$ is a surjective image of $\mathrm{Cl}_0^\infty(\chi)$, condition 4 implies 5. Moreover 5 implies 4 by Corollary 2.

Sometimes it is easy to show that $\mathrm{Cl}(\chi)$ is a cyclic Galois module.

Proposition 6. *Suppose that* $\mathrm{Cl}^+(\chi) \cong 0$*. Then* $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module.*

Proof. This is immediate from Proposition 5, but we give a direct proof independent of Corollary 2. In the notation of the previous section, the group $\mathrm{Cl}_0(\chi)$ is trivial, because it is a surjective image of $\mathrm{Cl}_{e-1}(\chi) = \mathrm{Cl}^+(\chi)$ under the norm map. This implies that $\mathrm{Cl}_0^\infty(\chi)$ is either trivial, or isomorphic to $\mathcal{O}_\chi/2$. Therefore in both cases $\mathrm{Cl}_0^\infty(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. By Proposition 1 we have $\mathrm{Cl}_0^\infty(\chi) \cong \mathrm{Cl}(\chi)/\mathrm{Cl}(\chi)^{1-\sigma}$. By Nakayama's lemma, we get that $\mathrm{Cl}(\chi)$ is cyclic, as we had to show.

Proof of Theorem 2. Because of Proposition 6, we can suppose that $\mathrm{Cl}^+(\chi)$ is not trivial. In [4] we determined all 2-adic characters χ of conductor $l < 10000$ such that $\mathrm{Cl}^+(\chi)$ is not trivial. They also appear in the table at the end of this paper. By Proposition 5 we can rule out all cases with either $\#\mathrm{Cl}^+(\chi) \leq \#(\mathcal{O}_\chi/2)$ or $\#\mathrm{Cl}^-(\chi) \leq \#(\mathcal{O}_\chi/2)$. Only few cases remain; they are precisely the characters of order 3 and conductors $l = 349$,

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709, 1777, 4261, 4297, 4357, 4561, 6247, 7687, 9109. For these characters, it is enough to check whether condition 5 of Proposition 5 holds. Looking at the tables in [8], one sees that for $l = 349$, 709, 4261, 4357, 4561, 9109, the group $\mathrm{Cl}_0(\chi)$ has order $4 = \#(\mathcal{O}_\chi/2)$, hence it is $\mathcal{O}_\chi$-cyclic. Since we exclude $l = 7687$, to complete the proof we are left with the three cases $l = 1777$, 4297, 6247. To prove the cyclicity of $\mathrm{Cl}_0(\chi)$ it is enough to find in each case an auxiliary prime r satisfying the conditions of Theorem 3. This has been done in [1]: if $l = 1777$ or $l = 4297$ one can take $r = 7$, for $l = 6247$ one takes $r = 11$. The proof of the theorem is now complete.

§4. Structure of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi$-module

Suppose we are given a prime number l, a character χ as in the previous sections, and we know that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. In several cases it is possible to determine the structure of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi$-module from the knowledge of $h_\chi^+ = \#\mathrm{Cl}^+(\chi)$, $h_\chi^- = \#\mathrm{Cl}^-(\chi)$ and the order of the cohomology groups. In this section we give several criteria in this direction.

4.1. The case $l \equiv 3 \pmod 4$

The case $l \equiv 3 \pmod 4$ is simpler because P is cyclic of order 2, generated by complex conjugation j. Suppose that $\mathrm{Cl}(\chi)$ is a cyclic Galois module. In this situation both $\mathrm{Cl}^+(\chi)$ and $\mathrm{Cl}^-(\chi)$ are cyclic $\mathcal{O}_\chi$-modules. Let f_χ be the dimension of $\mathcal{O}_\chi/2$, as a $\mathbb{Z}/2\mathbb{Z}$-vector space. We denote by a_χ^+ and by a_χ^- respectively the integers $\sqrt[f_\chi]{h_\chi^+}$ and $\sqrt[f_\chi]{h_\chi^-}$. They are defined in such a way that

$$\#(\mathcal{O}_\chi/a_\chi^+) = h_\chi^+ \text{ and } \#(\mathcal{O}_\chi/a_\chi^-) = h_\chi^-.$$

Let $a_\chi^{\max} = \max(a_\chi^+, a_\chi^-)$ and $a_\chi^{\min} = \min(a_\chi^+, a_\chi^-)$.

LEMMA 1. *Assume that $l \equiv 3 \pmod 4$. Then $\mathrm{Cl}(\chi)$ is annihilated by $2a_\chi^{\max}$.*

Proof. Let $x \in \mathrm{Cl}(\chi)$. We have

$$x^2 = x^{1+j}x^{1-j}. \tag{8}$$

We have $\mathrm{Cl}(\chi)^{a_\chi^+(1+j)} = 1$. Since

$$\mathrm{Cl}(\chi)^{1-j} \subset \mathrm{Ker}(1 + j : \mathrm{Cl}(\chi) \to \mathrm{Cl}(\chi)^{1+j})$$

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we get that

$$\#\mathrm{Cl}(\chi)^{1-j} \leq h_\chi^-$$

therefore $\mathrm{Cl}(\chi)^{a_\chi^-(1-j)} = 1$. If we apply $a_\chi^{\max}$ to (8) we get $x^{2a_\chi^{\max}} = 1$, as we wanted to show.

Corollary 3. *Let $l \equiv 3 \pmod 4$. Suppose that $\mathrm{Cl}(\chi)$ is a nontrivial cyclic $\mathcal{O}_\chi[P]$-module. Then, either*

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/a_\chi^+) \times (\mathcal{O}_\chi/a_\chi^-),$$

or

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/2a_\chi^{\max}) \times (\mathcal{O}_\chi/(a_\chi^{\min}/2))$$

as $\mathcal{O}_\chi$-modules.

Proof. The hypothesis implies that both $\mathrm{Cl}^+(\chi)$ and $\mathrm{Cl}^-(\chi)$ are cyclic $\mathcal{O}_\chi$-modules. Therefore $\mathrm{Cl}(\chi)$ has $\mathcal{O}_\chi$-rank at most 2. The result now follows from Lemma 1.

We now show that if $l \equiv 3 \pmod 4$, $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module, and $h_\chi^+ \neq h_\chi^-$, then the structure of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi$-module, can be determined.

Proposition 7. *Let $l \equiv 3 \pmod 4$, $h_\chi^+ > h_\chi^-$ and suppose that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. Then there is an isomorphism of $\mathcal{O}_\chi$-modules:*

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/2a_\chi^+) \times (\mathcal{O}_\chi/(a_\chi^-/2)).$$

Proof. Let x be a generator of $\mathrm{Cl}(\chi)$. We consider

$$x^2 = x^{1+j}x^{1-j}.$$

Multiplying by $a_\chi^+/2$, we get

$$x^{a_\chi^+} = x^{(a_\chi^+/2)(1+j)}$$

since $a_\chi^+/2 \geq a_\chi^-$ kills $\mathrm{Cl}(\chi)^{1-j}$. Since $\mathrm{Cl}(\chi)^{1+j} \cong \mathrm{Cl}^+(\chi)$ is cyclic and has exponent a_χ^+, the right hand side is not trivial. Thus x has order $2a_\chi^+$, and we are in the second case of Corollary 3.

Proposition 8. *Let $l \equiv 3 \pmod 4$, $h_\chi^+ < h_\chi^-$ and suppose that $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. Then we have, as $\mathcal{O}_\chi$-modules:*

1. $\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/a_\chi^-) \times (\mathcal{O}_\chi/a_\chi^+)$ *if* $\mathrm{Cl}(\chi)$ *is not cohomologically trivial*;
2. $\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/2a_\chi^-) \times (\mathcal{O}_\chi/(a_\chi^+/2))$ *if* $\mathrm{Cl}(\chi)$ *is cohomologically trivial.*

Proof. Because of Corollary 3, it is enough to prove that the first condition is verified if and only if $\mathrm{Cl}(\chi)$ is not cohomologically trivial. Suppose that

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/a_\chi^-) \times (\mathcal{O}_\chi/a_\chi^+)$$

as $\mathcal{O}_\chi$-modules. The module $\mathrm{Cl}(\chi)$ is killed by a_χ^- and by $a_\chi^+(1+j)$. Since $\mathrm{Cl}(\chi)$ is $\mathcal{O}_\chi[P]$-cyclic, counting orders we must have an isomorphism of $\mathcal{O}_\chi[P]$-modules

$$\mathrm{Cl}(\chi) \cong \frac{\mathcal{O}_\chi[P]}{(a_\chi^-, a_\chi^+(1+j))}.$$

It is immediately verified that this is not cohomologically trivial. Now suppose that $\mathrm{Cl}(\chi)$ is not cohomologically trivial. Let x an element of $\mathrm{Cl}(\chi)$. We have

$$x^2 = x^{1+j}x^{1-j}. \tag{9}$$

Since $\widehat{H}^1(P, \mathrm{Cl}(\chi))$ is not trivial, we get

$$\#\mathrm{Cl}(\chi)^{1-j} < \#\mathrm{Ker}(1+j) = \#\mathrm{Cl}^-(\chi) = \#(\mathcal{O}_\chi/a_\chi^-).$$

This implies that $a_\chi^-/2$ kills $\mathrm{Cl}(\chi)^{1-j}$. But, since $h_\chi^+ < h_\chi^-$, the number $a_\chi^-/2$ kills also $\mathrm{Cl}(\chi)^{1+j}$. If we multiply (9) by $a_\chi^-/2$, we get

$$x^{a_\chi^-} = 1.$$

This implies that $\mathrm{Cl}(\chi)$ has exponent a_χ^-. Therefore

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/a_\chi^-) \times (\mathcal{O}_\chi/a_\chi^+)$$

as $\mathcal{O}_\chi$-modules. This completes the proof.

If $h_\chi^+ = h_\chi^-$, then we do not have a criterion to determine the structure of $\mathrm{Cl}(\chi)$ as an $\mathcal{O}_\chi$-module in general. Anyway, the following is true:

PROPOSITION 9. *Let* $l \equiv 3 \pmod 4$, $h_\chi^+ = h_\chi^-$ *and suppose that* $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module. If* $\mathrm{Cl}(\chi)$ *is not cohomologically trivial, then there is an isomorphism of* $\mathcal{O}_\chi$*-modules*:

$$\mathrm{Cl}(\chi) \cong (\mathcal{O}_\chi/2a_\chi^+) \times (\mathcal{O}_\chi/(a_\chi^+/2)).$$

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Proof. In our situation we have that $\widehat{H}^i(P, \mathrm{Cl}(\chi))$ is isomorphic to $\mathcal{O}_\chi/2$, for all $i \in \mathbb{Z}$. By the properties of Tate cohomology groups

$$\widehat{H}^1(P, \mathrm{Cl}(\chi)) \cong {}_N\mathrm{Cl}(\chi)/\mathrm{Cl}(\chi)^{1-j}$$

where ${}_N\mathrm{Cl}(\chi)$ denotes the kernel of the norm map

$$\mathrm{Cl}(\chi) \to \mathrm{Cl}(\chi) : x \to x^{1+j}.$$

We have that $\#({}_N\mathrm{Cl}(\chi)) = h_\chi^-$. From this we get easily that

$$\#\mathrm{Cl}(\chi)^{1-j} = \#(\mathcal{O}_\chi/(a_\chi^-/2)).$$

In particular, $a_\chi^-/2$ kills $\mathrm{Cl}(\chi)^{1-j}$. On the other hand, since $\mathrm{Cl}^+(\chi) \cong \mathrm{Cl}(\chi)^{1+j}$ is a cyclic $\mathcal{O}_\chi$-module, $\mathrm{Cl}(\chi)^{1+j} \cong \mathcal{O}_\chi/a_\chi^+$. Therefore a_χ^+ is the exponent of $\mathrm{Cl}(\chi)^{1+j}$. Now let x be a generator of $\mathrm{Cl}(\chi)$. We have

$$x^2 = x^{1+j}x^{1-j}.$$

It is now easy to see that x has order $2a_\chi^+$. Thus we are in the second case of Corollary 3.

4.2. The general case

In this subsection we give some results which are a generalization of the ones in the previous subsection. If A is any finite abelian group, we denote by $\mathrm{Exp}(A)$ its exponent.

PROPOSITION 10. *Suppose that* $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module. Let* Q *be the cyclic group of order* 2 *generated by* j*. Suppose that*

$$\#\widehat{H}^0(Q, \mathrm{Cl}(\chi)) = h_\chi^-.$$

Then

$$\#(\mathrm{Cl}(\chi)/\mathrm{Cl}(\chi)^2) = h_\chi^-$$

and $\mathrm{Exp}(\mathrm{Cl}(\chi)) = 2\mathrm{Exp}(\mathrm{Cl}(\chi)^{1+j})$.

Proof. We have an isomorphism

$$\widehat{H}^0(Q, \mathrm{Cl}(\chi)) \cong \mathrm{Cl}(\chi)^Q/\mathrm{Cl}(\chi)^{1+j}.$$

This implies that our assumption on the order of the cohomology group is equivalent to the equality $\mathrm{Cl}(\chi)^Q = \mathrm{Cl}(\chi)$. Since in this case j acts as identity on $\mathrm{Cl}(\chi)$, we get $\mathrm{Cl}(\chi)^2 = \mathrm{Cl}(\chi)^{1+j}$. The proof of the first part of the proposition follows substituting these relations in our hypothesis. Let now x be a generator of $\mathrm{Cl}(\chi)$. The element $x^2 = x^{1+j}$ is a generator of $\mathrm{Cl}(\chi)^{1+j} \cong \mathrm{Cl}^+(\chi)$. Thus $2\mathrm{Exp}(\mathrm{Cl}(\chi)^{1+j}) = \mathrm{Exp}(\mathrm{Cl}(\chi))$, as we wanted to prove.

PROPOSITION 11. *Suppose that* $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module. Suppose that* $\mathrm{Exp}(\mathrm{Cl}^+(\chi)) > \mathrm{Exp}(\mathrm{Cl}^-(\chi))$. *Then* $\mathrm{Exp}(\mathrm{Cl}(\chi)) = 2\mathrm{Exp}(\mathrm{Cl}^+(\chi))$.

Proof. We first want to show that $\#\mathrm{Cl}(\chi)^{1-j} \leq h_\chi^-$. This is true because

$$\mathrm{Cl}(\chi)^{1-j} \subset \mathrm{Ker}(1 + j : \mathrm{Cl}(\chi) \to \mathrm{Cl}(\chi)^{1+j})$$

and the right hand side has order h_χ^-. Both $\mathrm{Cl}(\chi)^{1-j}$ and $\mathrm{Cl}^-(\chi)$ are cyclic modules over the discrete valuation ring $\mathcal{O}_\chi[P]/(1 + j)$. Since $\mathrm{Cl}(\chi)^{1-j}$ has order less or equal than $\#\mathrm{Cl}^-(\chi) = h_\chi^-$, it follows that $\mathrm{Cl}(\chi)^{1-j}$ is isomorphic to a quotient of $\mathrm{Cl}^-(\chi)$. This implies that $\mathrm{Exp}(\mathrm{Cl}(\chi)^{1-j}) \leq \mathrm{Exp}(\mathrm{Cl}^-(\chi)) < \mathrm{Exp}(\mathrm{Cl}^+(\chi))$. Let x be a generator of $\mathrm{Cl}(\chi)$. We consider the identity

$$x^2 = x^{1+j}x^{1-j}.$$

It is easy to see that the order of the right hand side is $\mathrm{Exp}(\mathrm{Cl}(\chi)^{1+j}) = \mathrm{Exp}(\mathrm{Cl}^+(\chi))$. Looking at the left hand side, we get that the order of x is $2\mathrm{Exp}(\mathrm{Cl}^+(\chi))$. This completes the proof.

The following proposition deals with a very ad hoc situation. It will enable us to determine the exponent of $\mathrm{Cl}(\chi)$ in the cases $l = 397$ and $l = 9421$.

PROPOSITION 12. *Let* $l \equiv 5 \pmod 8$. *Suppose that* $\#\mathrm{Cl}^-(\chi) = \#\mathcal{O}_\chi/8$, *and* $\#\mathrm{Cl}^+(\chi) = \#\mathcal{O}_\chi/2$. *If* $\mathrm{Cl}(\chi)$ *is a cyclic* $\mathcal{O}_\chi[P]$*-module, then it has exponent equal to* 4.

Proof. Combining Theorem 1 of [4] and Proposition 7 (iii) of [4] we get that

$$\widehat{H}^0(Q, \mathrm{Cl}(\chi)) \cong (\mathcal{O}_{F,+}^*/(\mathcal{O}_F^*)^2)(\chi)$$

where $F = \mathbb{Q}(\zeta_l + \zeta_l^{-1})$ and Q is the subgroup of P of order 2 generated by j. We have $e = 2$ and $\mathrm{Cl}^+(\chi) = \mathrm{Cl}_1(\chi)$. By contradiction, suppose that $\#\mathrm{Cl}_1^\infty(\chi) = \#\mathrm{Cl}_1(\chi) = \#(\mathcal{O}_\chi/2)$. Since by Proposition 5 the group $\mathrm{Cl}_0^\infty(\chi)$ is not trivial, we get that the surjective map $\mathrm{Cl}_1^\infty(\chi) \to \mathrm{Cl}_0^\infty(\chi)$ induced by the norm is actually an isomorphism. Let σ be a generator of P. From Equation (1) of Proposition 1 we obtain

$$\mathrm{Cl}(\chi)^{1-\sigma} = \mathrm{Cl}(\chi)^{1-\sigma^2} = (\mathrm{Cl}(\chi)^{1-\sigma})^{1+\sigma}.$$

Since the element $1+\sigma$ is contained in the maximal ideal of the local ring $\mathcal{O}_\chi[P]$, Nakayama's lemma gives $\mathrm{Cl}(\chi)^{1-\sigma} \cong 0$. Proposition 1 then implies that $\mathrm{Cl}(\chi) \cong \mathrm{Cl}_1(\chi) \cong \mathrm{Cl}^+(\chi)$, which is absurd, because $\mathrm{Cl}^-(\chi)$ is not trivial. Therefore $\#\mathrm{Cl}_1^\infty(\chi) > \#\mathrm{Cl}_1(\chi)$ and $(\mathcal{O}^*_{F,+}/(\mathcal{O}^*_F)^2)(\chi)$ is not trivial. This implies that the Tate cohomology group $\widehat{H}^1(Q, \mathrm{Cl}(\chi))$ is not trivial. Therefore $\mathrm{Cl}(\chi)^{1-j}$ has order strictly smaller than h_χ^-, hence $\#\mathrm{Cl}(\chi)^{1-j} \leq \#(\mathcal{O}_\chi/4)$. The group $\mathrm{Cl}(\chi)^{1-j}$ is a cyclic module over $\mathcal{O}_\chi[P]/(1+j)$. By Proposition 2 we get that $\mathrm{Cl}(\chi)^{1-j}$ can have at most exponent equal to 2. Let x be a generator of $\mathrm{Cl}(\chi)$. From the usual identity

$$x^2 = x^{1+j}x^{1-j}$$

we see that $\mathrm{Cl}(\chi)$ has at most exponent 4. If we show that $\mathrm{Cl}^-(\chi)$ has exponent 4, then the proof is complete. But again $\mathrm{Cl}^-(\chi)$ is a cyclic $\mathcal{O}_\chi[P]/(1+j)$-module of order $\#\mathcal{O}_\chi/8$. By Proposition 2 such a module has exponent 4.

§5. Tables

Let l be a prime number and let χ be a 2-adic character as in the previous sections. The theory developed allows us to determine much information about $\mathrm{Cl}(\chi)$, and sometimes the whole structure as an $\mathcal{O}_\chi$-module. In this section we present a table containing our numerical results. If $\mathrm{Cl}^+(\chi)$ is trivial then $\mathrm{Cl}(\chi) \cong \mathrm{Cl}^-(\chi)$. In this case, using Propositions 6 and 2 it is easy to determine the whole $\mathcal{O}_\chi[P]$-structure of $\mathrm{Cl}^-(\chi)$ from the knowledge of its order. The table has an entry for each prime number $l < 10000$ such that $\mathrm{Cl}^+(\chi)$ is not trivial. For each l we determine various quantities. The number d denotes the degree of the field $K_e \subset \mathbb{Q}(\zeta_l)$ fixed by $\mathrm{Ker}(\chi)$. The numbers h_χ^+ and h_χ^- denote the order of $\mathrm{Cl}^+(\chi)$ and of $\mathrm{Cl}^-(\chi)$ respectively. They have been computed in [4]. The column $\#\widehat{H}^0$ contains the order of the Tate cohomology group $\widehat{H}^0(Q, \mathrm{Cl}(\chi))$, where Q is the group of order 2 generated by complex conjugation. This quantity can be easily computed using the table and Theorem 1 of [4]. The other entries contain the structure of the groups $\mathrm{Cl}^+(\chi)$, $\mathrm{Cl}^-(\chi)$ and $\mathrm{Cl}(\chi)$ as $\mathcal{O}_\chi$-modules. In the table we write n for $\mathcal{O}_\chi/n$. Observe that as an abelian group we have

$$\mathcal{O}_\chi/2^k \cong (\mathbb{Z}/2^k\mathbb{Z})^{f_\chi}$$

where $f_\chi = [\mathbb{Z}_2(\chi) : \mathbb{Z}_2]$ is the multiplicative order of 2 in $(\mathbb{Z}/\mathrm{ord}(\chi))^*$. We are not able to determine these structures in all cases. Therefore some entries are left blank. The structure of $\mathrm{Cl}^+(\chi)$ and of $\mathrm{Cl}^-(\chi)$ are computed in all cases when $\mathrm{Cl}(\chi)$ is a cyclic $\mathcal{O}_\chi[P]$-module. By Theorem 2, this happens when $l \neq 7687$. The groups $\mathrm{Cl}^+(\chi)$ have been computed from the ideals $J^+(\chi)$ which can be found in the table of [17] (see the remarks in the introduction). The groups $\mathrm{Cl}^-(\chi)$ can be computed using Proposition 2. The groups $\mathrm{Cl}(\chi)$ have been computed in some cases, using the results mentioned in the column labelled as "notes". It turns out that for each $l < 10000$ there is at most one character χ such that $\mathrm{Cl}^+(\chi)$ is not trivial, except for $l = 7841$. For $l = 7841$ there are exactly two characters with this property; in this case the table has two entries corresponding to each character. In the two cases $l = 397$ and $l = 9421$, we are able to determine only the exponent of the class group $\mathrm{Cl}(\chi)$.

l	d	h_χ^+	h_χ^-	$\#\widehat{H}^0$	$\mathrm{Cl}^+(\chi)$	$\mathrm{Cl}^-(\chi)$	$\mathrm{Cl}(\chi)$	notes
163	6	2^2	2^2	1	2	2		
277	12	2^2	2^4	2^2	2	$2,2$		
349	12	2^4	2^4	1	$2,2$	$2,2$		
397	12	2^2	2^6	2^2	2	$2,4$	Exp=4	prop. 12
491	14	2^3	2^3	1	2	2		
547	6	2^2	2^2	1	2	2		
607	6	2^2	2^4	1	2	4	8	prop. 8
709	12	2^4	2^4	1	$2,2$	$2,2$		
827	14	2^3	2^3	1	2	2		
853	12	2^2	2^2	2^2	2	2	4	th. 1
937	24	2^4	2^2	2^2	4	2	8	th. 1
941	20	2^4	2^8	2^4	2	$2,2$		
1009	48	2^2	2^2	2^2	2	2	4	th. 1
1399	6	2^2	2^4	1	2	4	8	prop. 8
1699	6	2^2	2^2	2^2	2	2	4	prop. 9
1777	48	2^4	2^4	2^4	4	$2,2$	$2,8$	prop. 11
1789	12	2^2	2^4	2^2	2	$2,2$		
1879	6	2^2	2^2	1	2	2		
1951	6	2^2	2^2	1	2	2		

l	d	h_χ^+	h_χ^-	$\#\widehat{H}^0$	$\mathrm{Cl}^+(\chi)$	$\mathrm{Cl}^-(\chi)$	$\mathrm{Cl}(\chi)$	notes
2131	6	2^2	2^2	1	2	2		
2161	80	2^4	2^4	2^4	2	2	4	th. 1
2311	6	2^2	2^2	1	2	2		
2689	384	2^2	2^2	2^2	2	2	4	th. 1
2797	12	2^2	2^4	2^2	2	$2, 2$		
2803	6	2^2	2^2	1	2	2		
2927	14	2^3	2^3	1	2	2		
3037	12	2^2	2^4	2^2	2	$2, 2$		
3271	6	2^2	2^2	1	2	2		
3301	20	2^4	2^4	2^4	2	2	4	th. 1
3517	12	2^2	2^2	2^2	2	2	4	th. 1
3727	6	2^2	2^2	1	2	2		
3931	10	2^8	2^4	1	4	2	8	prop. 7
4099	6	2^2	2^2	1	2	2		
4219	6	2^2	2^2	1	2	2		
4261	12	2^4	2^4	1	$2, 2$	$2, 2$		
4297	24	2^8	2^4	2^4	$2, 8$	$2, 2$	$4, 16$	prop. 10
4327	14	2^3	2^3	1	2	2		
4357	12	2^4	2^4	2^2	4	$2, 2$	$2, 8$	prop. 11
4561	48	2^4	2^4	2^4	$2, 2$	$2, 2$	$4, 4$	prop. 10
4567	6	2^2	2^2	1	2	2		
4639	6	2^2	2^2	1	2	2		
4789	12	2^2	2^4	2^2	2	$2, 2$		
4801	192	2^2	2^4	2^4	2	$2, 2$	$2, 4$	prop. 10
5197	12	2^2	2^4	2^2	2	$2, 2$		
5479	6	2^2	2^2	1	2	2		
5531	14	2^3	2^3	1	2	2		
5659	6	2^2	2^2	1	2	2		
5779	6	2^2	2^2	1	2	2		
5953	192	2^2	2^2	2^2	2	2	4	th. 1
6037	12	2^2	2^2	2^2	2	2	4	th. 1
6079	6	2^2	2^2	1	2	2		
6163	6	2^2	2^6	1	2	8	16	prop. 8
6247	6	2^4	2^6	1	4	8	$2, 16$	prop. 8
6301	28	2^3	2^3	2^3	2	2	4	th. 1
6553	24	2^2	2^2	2^2	2	2	4	th. 1

l	d	h_χ^+	h_χ^-	$\#\widehat{H}^0$	$\mathrm{Cl}^+(\chi)$	$\mathrm{Cl}^-(\chi)$	$\mathrm{Cl}(\chi)$	notes
6637	12	2^2	2^2	2^2	2	2	4	th. 1
6709	12	2^2	2^2	2^2	2	2	4	th. 1
6833	112	2^3	2^3	2^3	2	2	4	th. 1
7027	6	2^2	2^2	2^2	2	2	4	prop. 9
7297	384	2^2	2^4	2^4	2	$2, 2$	$2, 4$	prop. 10
7489	192	2^6	2^2	2^2	8	2	16	th. 1
7589	28	2^3	2^3	2^3	2	2	4	th. 1
7639	6	2^2	2^4	2^2	2	4	$2, 4$	prop. 8
7687	6	2^4	2^4	1				
7841	224	2^3	2^3	2^3	2	2	4	th. 1
	224	2^3	2^3	2^3	2	2	4	th. 1
7867	6	2^2	2^2	1	2	2		
7879	6	2^2	2^2	1	2	2		
8011	6	2^2	2^2	1	2	2		
8191	6	2^2	2^4	2^2	2	4	$2, 4$	prop. 8
8209	48	2^2	2^2	2^2	2	2	4	th. 1
8629	12	2^2	2^2	2^2	2	2	4	th. 1
8647	6	2^2	2^2	1	2	2		
8731	6	2^2	2^2	1	2	2		
8831	10	2^4	2^4	1	2	2		
8887	6	2^2	2^2	1	2	2		
9109	12	2^4	2^4	2^2	4	$2, 2$	$2, 8$	prop. 11
9283	6	2^2	2^2	2^2	2	2	4	prop. 9
9319	6	2^2	2^2	1	2	2		
9337	24	2^6	2^2	1	8	2	16	th. 1
9391	6	2^2	2^2	1	2	2		
9421	12	2^2	2^6	2^2	2	$2, 4$	Exp=4	prop. 12
9601	384	2^4	2^2	2^2	4	2	8	th. 1
9649	48	2^2	2^4	2^4	2	$2, 2$	$2, 4$	prop. 10
9721	24	2^2	2^2	2^2	2	2	4	th. 1

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