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An Alternative Proof of the Linearity of the Size-Ramsey Number of Paths

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The size-Ramsey number $\hat{r}(F)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that every colouring of the edges of G with two colours yields a monochromatic copy of F. In 1983, Beck provided a beautiful argument that shows that $\hat{r}(P_n)$ is linear, solving a problem of Erdős. In this note, we provide another proof of this fact that actually gives a better bound, namely, $\hat{r}(P_n) < 137n$ for n sufficiently large.

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1. Introduction

Given two finite graphs F and G, we write $G \to F$ if *every* colouring of the edges of G with two colours (say blue and red) contains a monochromatic copy of F (that is, a copy that is either blue or red). The *size-Ramsey number* of a graph F, introduced by Erdős, Faudree, Rousseau and Schelp [7] in 1978, is defined as follows:

$$\hat{r}(F) = \min\{|E(G)| : G \to F\}.$$

In this note, we consider the size-Ramsey number of the path P_n on n vertices. It is obvious that $\hat{r}(P_n) = \Omega(n)$ and that $\hat{r}(P_n) = O(n^2)$ (for example, $K_{2n} \to P_n$), but the exact behaviour of $\hat{r}(P_n)$ was not known for a long time. In fact, Erdős [6] offered \$100 for a

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proof or disproof that

$$\hat{r}(P_n)/n \to \infty \quad \text{and} \quad \hat{r}(P_n)/n^2 \to 0.$$

This problem was solved by Beck [2] in 1983 who, quite surprisingly, showed that $\hat{r}(P_n) < 900n$ for sufficiently large n. A variant of his proof was provided by Bollobás [5]; it gives $\hat{r}(P_n) < 720n$ for sufficiently large n. These bounds are not given by explicit constructions; later Alon and Chung [1] gave an explicit construction of graphs G on $O(n)$ vertices with $G \to P_n$.

Here we provide an alternative and elementary proof of the linearity of the size-Ramsey number of paths that gives a better bound. The proof relies on a simple observation, Lemma 2.1, which may be applicable elsewhere.

Theorem 1.1. *For n sufficiently large, $\hat{r}(P_n) < 137n$.*

In order to show the result, similarly to Beck and Bollobás, we are going to use binomial random graphs. The *binomial random graph* $G(n,p)$ is the random graph G with vertex set $[n]$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may, and usually does, tend to zero as n tends to infinity. Throughout, all asymptotics are as $n \to \infty$. We say that a sequence of events $\mathcal{E}_n$ in a probability space holds *with high probability* (or w.h.p.) if the probability that $\mathcal{E}_n$ holds tends to 1 as $n \to \infty$. For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptotic calculations we will make.

2. Proof of Theorem 1.1

We start with the following elementary observation.[1]

Lemma 2.1. *Let $c > 1$ be a real number and let $G = (V,E)$ be a graph on cn vertices. Suppose that every edge of G is coloured blue or red and that there is no monochromatic P_n. Then there exist disjoint sets $U, W \subseteq V$ of size $n(c-1)/2$ such that there is no blue edge between U and W.*

Proof. We perform the following algorithm on G to construct a blue path P. Let v_1 be an arbitrary vertex of G, let $P = (v_1)$, $U = V \setminus \{v_1\}$, and $W = \emptyset$. We investigate all edges from v_1 to U searching for a blue edge. If such an edge is found (say from v_1 to v_2), we extend the blue path as $P = (v_1, v_2)$ and remove v_2 from U. We continue extending the blue path P this way for as long as possible. Since there is no monochromatic P_n, we must reach a point of the process in which P cannot be extended, that is, there is a blue path from v_1 to v_k ($k < n$) and there is no blue edge from v_k to U. This time, v_k is moved to W and we try to continue extending the path from v_{k-1}, reaching another critical point in which another vertex will be moved to W, etc. If P is reduced to a single vertex v_1

[1] A similar result was obtained independently by Pokrovskiy [10].

and no blue edge to U is found, we move v_1 to W and simply restart the process from another vertex from U, again arbitrarily chosen.

An obvious but important observation is that during this algorithm there is never a blue edge between U and W. Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. Finally, since there is no monochromatic P_n, the number of vertices of the blue path P is always smaller than n. Hence, at some point of the process both U and W must have size at least $n(c-1)/2$. The result follows by removing some vertices from U or W, if needed, so that both sets have size precisely $n(c-1)/2$. $\square$

Now, we prove the following straightforward properties of random graphs. For disjoint sets S and T, $e(S,T)$ denotes the number of edges between S and T.

Lemma 2.2. *Let $c = 7.29$ and $d = 5.14$, and consider $G = (V,E) \in G(cn, d/n)$. Then, the following two properties hold w.h.p.:*

(i) $|E(G)| = (1+o(1))nc^2d/2 < 137n$,

(ii) *for every two disjoint sets of vertices S and T such that $|S| = |T| = n(c-3)/4$, we have* $e(S,T) > 0$.

Proof. Part (i) is obvious. The expected number of edges in G is

$$\binom{cn}{2}\frac{d}{n} = (1+o(1))nc^2\frac{d}{2},$$

and concentration around the expectation follows immediately from Chernoff's bound.

For part (ii), let X be the number of pairs of disjoint sets S and T of the desired size such that $e(S,T) = 0$. Setting $\alpha = \alpha(c) = (c-3)/4$, we have

$$\begin{aligned}\mathbb{E}[X] &= \binom{cn}{\alpha n}\binom{(c-\alpha)n}{\alpha n}\left(1-\frac{d}{n}\right)^{\alpha n\cdot\alpha n}\\ &\leqslant \frac{(cn)!}{(\alpha n)!(\alpha n)!((c-2\alpha)n)!}\exp\bigl(-d\alpha^2 n\bigr).\end{aligned}$$

Using Stirling's formula ($x! \sim \sqrt{2\pi x}(x/e)^x$), we see that $\mathbb{E}[X] \leqslant \exp(f(c,d)n)$, where

$$f(c,d) = c\ln c - 2\alpha\ln\alpha - (c-2\alpha)\ln(c-2\alpha) - d\alpha^2.$$

For $c = 7.29$ and $d = 5.14$, we have $f(c,d) < -0.008$, and so $\mathbb{E}[X] \to 0$ as $n \to \infty$. (The values of c and d were chosen so as to minimize $c^2d/2$ under the condition $f(c,d) < 0$.) Now part (ii) follows by Markov's inequality. $\square$

Now, we are ready to prove the main result.

Proof of Theorem 1.1. Let $c = 7.29$ and $d = 5.14$, and consider $G = (V,E) \in G(cn, d/n)$. We show that w.h.p. $G \to P_n$, which will finish the proof by Lemma 2.2(i).

Suppose that $G \not\to P_n$. Thus, there is a blue–red colouring of E with no monochromatic P_n. It follows (deterministically) from Lemma 2.1 that V can be partitioned into

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three sets P, U, W such that $|P| = n$, $|U| = |W| = n(c-1)/2$, and there is no blue edge between U and W. Similarly, V can be partitioned into three sets P', U', W' such that $|P'| = n$, $|U'| = |W'| = n(c-1)/2$, and there is no red edge between U' and W'.

Now, consider $X = U \cap U', Y = U \cap W', X' = W \cap U', Y' = W \cap W'$ and let $x = |X|$, $y = |Y|$, $x' = |X'|$, $y' = |Y'|$. Observe that

$$x + y = |U \cap (U' \cup W')| = |U \setminus P'| \geqslant |U| - |P'| = n(c-3)/2. \tag{2.1}$$

Similarly, one can show that $x' + y' \geqslant n(c-3)/2$, $x + x' \geqslant n(c-3)/2$, and that $y + y' \geqslant n(c-3)/2$. We say that a set is *large* if its size is at least $n(c-3)/4$; otherwise, it is *small*.

Claim 2.3. Either both X and Y' are large or both Y and X' are large.

Proof of the claim. Suppose that at least one of X, Y' is small *and* at least one of Y, X' is small, say, X and Y are small. Then $x + y < n(c-3)/4 + n(c-3)/4 = n(c-3)/2$, which contradicts (2.1). The remaining three cases are symmetric, and so the claim holds. □

Now, let us return to the proof. Without loss of generality, we may assume that $X = U \cap U'$ and $Y' = W \cap W'$ are large. Since $X \subseteq U$ and $Y' \subseteq W$, there is no blue edge between X and Y'. Similarly, since $X \subseteq U'$ and $Y' \subseteq W'$, there is no red edge between X and Y', and so $e(X, Y') = 0$, and therefore G does not have the property in Lemma 2.2(ii). Since, by Lemma 2.2, G does have this property w.h.p., we deduce that w.h.p. $G \to P_n$, as required. □

3. Remarks

In this note we have shown that $\hat{r}(P_n) < 137n$. The best known lower bound,

$$\hat{r}(P_n) \geqslant (1 + \sqrt{2})n - 2,$$

was given by Bollobás [4], who improved the previous result of Beck [3] that $\hat{r}(P_n) \geqslant \frac{9}{4}n$. Decreasing the gap between the lower and upper bounds might be of some interest. One approach to improving the upper bound could be to deal with non-symmetric cases in our claim or to use random d-regular graphs instead of binomial graphs.

Another related problem deals with longest monochromatic paths in $G(n, p)$. Observe that it follows from the proof of Theorem 1.1 that, for every $\omega = \omega(n)$ tending to infinity as $n \to \infty$, we have that w.h.p. any 2-colouring of the edges of $G(n, \omega/n)$ yields a monochromatic path of length $((1-\varepsilon)/3)n$ for an arbitrarily small $\varepsilon > 0$. On the other hand, a simple construction of Gerencsér and Gyárfás [8] shows that such a path cannot be longer than $\frac{2}{3}n$. We conjecture that actually $(1 + o(1))\frac{2}{3}n$ is the right answer for random graphs with average degree tending to infinity.[2]

[2] The conjecture was recently proved by Letzter [9].

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