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UNIVERSAL FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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Abstract

It is proved that there exists a universal good inner function in the open unit polydisc U^n, that is its nonEuclidean translates are dense in the closed unit ball of $H^\infty(U^n)$ and that there exists a universal function in the open unit ball B_n of $\mathbf{C}^n$. These generalize Heins' result on universal Blaschke products.

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1. Introduction

Let Ω be a region in the space $\mathbf{C}^n$ of n complex variables, $\{\Phi\}$ a family of holomorphic automorphisms of Ω and $\mathscr{F}$ a family of holomorphic functions in Ω. A function $g \in \mathscr{F}$ is called a *universal function* of $\mathscr{F}$ relative to $\{\Phi\}$ if for any $f \in \mathscr{F}$, there exists a sequence $\{\Phi_k\}_1^\infty$ from $\{\Phi\}$ such that

$$g \circ \Phi_k \to f$$

uniformly on compact subsets of Ω.

G. D. Birkhoff (1929) was the first to obtain a universal function when he proved that in $\mathbf{C}$ there exists a universal entire function relative to translations of the form $z \to z+a$, $a \in \mathbf{R}$. This was later generalized by Seidel and Walsh (1941) to any simply-connected region instead of $\mathbf{C}$.

Heins (1955) considered universal functions in the family $\mathscr{B}$ of holomorphic functions in the open unit disc U bounded by 1 and obtained the following.

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HEINS' THEOREM. *There exist a Blaschke product b and a monotone increasing sequence $\{x_k\}$ with $x_0 = 0$ and* $\lim x_k = 1$ *such that*

(i) *the point $z = 1$ is the sole cluster point of the zeros of b,*

(ii) *to every $f \in \mathscr{B}$ corresponds a subsequence $\{x_{k(r)}\}$ for which*

$$b\left(\frac{z+x_{k(r)}}{1+x_{k(r)}z}\right) \to f(z)$$

uniformly on compact subsets of U.

He called b a universal Blaschke product.

For several complex variables, the results of Birkhoff and Seidel and Walsh hold true with little change in the proofs. The purpose of this paper is to consider analogues of Heins' theorem in the open unit polydisc U^n and in the open unit ball B_n of $\mathbf{C}^n$ respectively. We show that (Theorem 1) there exists a universal good inner function in U^n and that (Theorem 2) there exists a universal function in B_n. However, in the case of B_n, we were not able to show that the universal function can be chosen to be inner.

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2. Preliminaries

Points of $\mathbf{C}^n$ will be denoted by $z = (z_1, \ldots, z_n)$ where $z_j \in \mathbf{C}$, $1 \leqslant j \leqslant n$. The open unit polydisc U^n and the open unit ball B_n are defined by

$$U^n = \{z \in \mathbf{C}^n \colon |z_j| < 1,\ 1 \leqslant j \leqslant n\},$$

$$B_n = \{z \in \mathbf{C}^n \colon |z_1|^2 + \ldots + |z_n|^2 < 1\}.$$

The distinguished boundary of U^n is the n-dimensional torus

$$T^n = \{z \in \mathbf{C}^n \colon |z_j| = 1,\ 1 \leqslant j \leqslant n\}.$$

For an open subset Ω of $\mathbf{C}^n$, let $H(\Omega)$ denote the set of all holomorphic functions in Ω and $H^\infty(\Omega)$ the set of all bounded ones. Further, let

$$A(\Omega) = \{f \in H^\infty(\Omega) \colon f \text{ is continuous on } \overline{\Omega}\},$$

$$\mathscr{B}(\Omega) = \{f \in H^\infty(\Omega) \colon \|f\|_\infty \leqslant 1\}.$$

We let $H(\Omega)$ or its subsets have the compact-open topology so that convergence means uniform convergence on compact subsets of Ω.

A function $f \in H^\infty(U^n)$ is an *inner function* in U^n if $\lim_{r \to 1} f(rz) = f^*(z)$ has absolute value 1 for almost all $z \in T^n$ (relative to the Haar measure m_n on T^n). An inner function f in U^n is said to be *good* if

$$\lim_{r\to 1}\int_{T^n} \log|f(rw)|\,dm_n(w)=0.$$

When $n=1$, the good inner functions are precisely the Blaschke products. (See Rudin (1969).)

A subset E of Ω is a *uniqueness set* (or D-set) for a family $\mathscr{F}\subseteq H(\Omega)$ if $f\in\mathscr{F}$ and $f=0$ on E imply $f=0$ in Ω.

The following version of Vitali's theorem will be useful.

VITALI'S THEOREM. *Let the sequence $\{f_k\}$ form a normal family in a region Ω in $\mathbf{C}^n$, E a uniqueness set for $\overline{\mathscr{F}}$, the closure of $\mathscr{F}$ in the compact-open topology. If f_k converges pointwise on E, then f_k converges uniformly on compact subsets of Ω.*

For a proof in one variable, see for example Markushevich (1965) Section 87. The same proof works for several variables.

3. Universal inner function in U^n

Let $x_k=(x_{k1},\dots,x_{kn})\in\mathbf{R}^n$, $0\leqslant x_{kj}<1$, $1\leqslant j\leqslant n$. Let

$$\varphi_{kj}(z_j)=\frac{z_j+x_{kj}}{1+x_{kj}z_j},\quad 1\leqslant j\leqslant n$$

and

$$\Phi_k(z)=(\varphi_{k1}(z_1),\dots,\varphi_{kn}(z_n)),\quad z\in U^n. \tag{1}$$

Then each Φ_k is a holomorphic automorphism of U^n, and is continuous on $\overline{U}^n$. In this section, we show that there exists a universal good inner function in U^n relative to the automorphisms $\{\Phi_k\}$.

THEOREM 1. *There exist a good inner function g in U^n and a sequence $\{x_k\}$ in $\mathbf{R}^n$ with $0\leqslant x_{kj}<1$ and $\lim_{k\to\infty}x_{kj}=1$, $1\leqslant j\leqslant n$, with the property that to every $f\in\mathscr{B}=\mathscr{B}(U^n)$ corresponds a subsequence $\{x_{k(r)}\}$ such that*

$$g\circ\Phi_{k(r)}\to f$$

uniformly on compact subsets of U^n, where Φ^k is obtained from x_k as given by (1).

The following result of Rudin (1969), Theorem 5.5.1, is needed for the proof.

LEMMA 2. *There exists a countable set $\{A_k\}_1^\infty$ which is dense in $\mathscr{B}$ in the compact-open topology, where $A_k\in A(U^n)$, $|A_k(z)|=1$ for all $z\in T^n$.*

For our purposes, we need to modify the functions A_k in Lemma 2 so that they have the value 1 at the points $\pm(1, \ldots, 1)$. To that end, let $c \in U$, $\eta_1, \eta_2 \in T$ be given. Fix an nth root of each of these and denote them by $c^{1/n}$, $\eta_1^{1/n}$, $\eta_2^{1/n}$ respectively. It was shown in Heins (1955) that there exists $\varphi\colon \bar{U} \to \bar{U}$, which is holomorphic in U and

$$\varphi(0) = c^{1/n}, \quad \varphi(1) = \eta_1^{1/n}, \quad \varphi(-1) = \eta_2^{1/n}.$$

For $z \in \bar{U}^n$, define $\Psi(z) = \varphi(z_1) \ldots \varphi(z_n)$. Then

$$\Psi(0) = c, \quad \Psi(1, \ldots, 1) = \eta_1, \quad \Psi(-1, \ldots, -1) = \eta_2.$$

With A_k as in Lemma 2, we define $G_k = A_k \Psi_k$, where Ψ_k is of the kind just described, with $\Psi_k(0) = 1 - 2^{-k}$,

$$\Psi_k(1, \ldots, 1) = \frac{1}{A_k(1, \ldots, 1)}, \quad \Psi_k(-1, \ldots, -1) = \frac{1}{A_k(-1, \ldots, -1)}.$$

Since $\| \Psi_k \|_\infty \leqslant 1$ for all k, $\{\Psi_k\}$ is a normal family in U^n. Since $\Psi_k(0) = 1 - 2^k$, the maximum modulus theorem implies that any convergent subsequence of $\{\Psi_k\}$ has the constant limit 1 in U^n. Since $\{A_k\}$ is dense in $\mathscr{B}$, any $f \in \mathscr{B}$ is the limit of a convergent subsequence $\{A_{k(r)}\}$. Since the corresponding sequence $\{\Psi_{k(r)}\}$ is a normal family, there exists a convergent subsequence $\{\Psi_{k(r_j)}\}$, with limit 1 in U^n. Then the sequence $\{G_{k(r_j)}\}$ converges to f. Hence $\{G_k\}$ is dense in $\mathscr{B}$. Further, $G_k(1, \ldots, 1) = G_k(-1, \ldots, -1) = 1$.

Now let

$$x_k \in \mathbf{R}^n, \quad 0 \leqslant x_{kj} < 1, \quad 1 \leqslant j \leqslant n,$$

$$I_k = \{(y_1, \ldots, y_n) \in \mathbf{R}^n\colon -1 \leqslant y_j \leqslant x_{kj}, \quad 1 \leqslant j \leqslant n\}$$

and

$$V_k = \{z \in U\colon |1 - z| < 2^{-k}\}.$$

Let Φ_k be the automorphism associated with x_k as described by (1). Let $g_k = G_k \circ \Phi_k^{-1}$. Then the real numbers x_{kj} can be chosen so that $x_{kj} \to 1$ and

(i) $g_{k+1}(I_k) \subseteq V_{k+1}$,

(ii) $g_r(x_{k+1}) \in V_{k+1}$, $r = 1, 2, \ldots, k$.

This is a consequence of the fact that $G_k(1, \ldots, 1) = G_k(-1, \ldots, -1) = 1$ and that each G_k is continuous on $\bar{U}^n$.

With g_k chosen as above, we define $g(z) = \prod_1^\infty g_k(z)$. Then g satisfies the requirements of Theorem 1. The infinite product converges on $I_1 = \bigcap_1^\infty I_k$ by (i). Since I_1 is a uniqueness set for $H^\infty(U^n)$, the product converges uniformly on compact subsets of U^n, by Vitali's theorem. Since each factor g_k is a good inner function in U^n, so is g (see Rudin (1969), p. 118).

It remains to show that $\{g \circ \Phi_k\}$ is dense in $\mathscr{B}$. Since $\{G_k\}$ is dense in $\mathscr{B}$, it is enough to show that

(2) $$\lim_{k \to \infty} (G_k - g \circ \Phi_k) = 0.$$

But $G_k - g\circ\Phi_k = G_k(1-\Pi_{j\neq k} g_j\circ\Phi_k)$. Since the product is in $\mathscr{B}$, (2) is a consequence of

$$\lim_{k\to\infty}\prod_{j\neq k}(g_j\circ\Phi_k)(0)=1.$$

Since $\Phi_k(0)=x_k$, it suffices to show that

$$\lim_{k\to\infty}\prod_{j\neq k} g_j(x_k)=1.$$

Let $g_j(x_k)=1+u_j(k)$. Then by (i) and (ii),

$$|u_j(k)|<2^{-k}\quad\text{if } j<k\quad\text{and}\quad |u_j(k)|<2^{-j}\quad\text{if } j>k.$$

Hence

$$\begin{aligned}\Big|\prod_{j\neq k} g_j(x_k)-1\Big| &\leqslant \prod_{j\neq k}(1+|u_j|)-1\\ &\leqslant (1+2^{-k})^{k-1}\prod_{j=k+1}^{\infty}(1+2^{-j})-1\\ &\to 0\quad\text{as } k\to\infty.\end{aligned}$$

4. Universal functions in B_n

The open unit ball B_n has the following family of automorphisms. For each positive integer k, let x_k and y_k be nonnegative real numbers satisfying $x_k^2+y_k^2=1$. Let

(3) $$\Phi_k(z)=(\varphi_{k1}(z),\varphi_{k2}(z),\dots,\varphi_{kn}(z)),\quad z\in B_n,$$

where

$$\varphi_{k1}(z)=\frac{z_1+x_k}{1+x_kz_1},$$

$$\varphi_{kj}(z)=\frac{y_kz_j}{1+x_kz_1},\quad 2\leqslant j\leqslant n.$$

Each Φ_k is an automorphism of $\bar{B}_n$, with the inverse obtained from the above by replacing x_k by $-x_k$.

THEOREM 3. *There exists a universal function in $\mathscr{B}(B_n)$ relative to the automorphisms $\{\Phi_k\}$ gjven by* (3).

The proof of Theorem 3 is exactly parallel to that of Theorem 1, the main difficulty being an analogue of Lemma 2. We give only the proof of Lemma 4 below and omit the rest of the proof of Theorem 3.

Lemma 4. *There exists a sequence $\{A_k\}$ which is dense in $\mathscr{B}=\mathscr{B}(B_n)$, where $A_k \in A(B_n)$, $A_k \in \mathscr{B}$, and*

$$|A_k(\pm 1, 0, \ldots, 0)| = 1.$$

Proof. We use a method discussed in Curtis (1969). As in the proof of Lemma 2, it is sufficient to show that given a polynomial P with $|P(z)| \leqslant r < 1$ for all $z \in \bar{B}_n$, a compact subset K of B_n and an $\varepsilon > 0$, there exists an $f \in A(B_n) \cap \mathscr{B}$ with $|f(\pm 1, 0, \ldots, 0)| = 1$ and $|P - f| < \varepsilon$ on K. Fix such a compact set K.

Let

$$M = \frac{4}{1-r}, \quad \eta = \frac{1+r}{2} \quad \text{and} \quad S = \{z \in \bar{B}_n \colon |z_1| = 1\}.$$

For any positive integer m, let

$$g_m(z) = \frac{P(z) + z_1^{2m}}{1 + z_1^{2m} Q(z)}, \quad z \in \bar{B}_n,$$

where $\overline{Q(z)} = P(1/\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$. Then g_m is in $A(B_n)$ for sufficiently large m, and for all m,

$$|g_m(z)| = 1 \quad \text{for all } z \in S \tag{4}$$

since $z \in S$ implies $z = (z_1, 0, \ldots, 0)$ with $|z_1| = 1$. In particular, $g_m(\pm 1, 0, \ldots, 0)$ is independent of m and has absolute value 1. Note that since $|z_1| < 1$ on K,

$$\| g_m - P \|_K \to 0 \quad \text{as } m \to \infty.$$

Furthermore, there exists an $m_0 > 0$ such that

$$|g_m| \leqslant M \quad \text{on } \bar{B}_n \quad \text{for all } m \geqslant m_0,$$

and given any compact subset F of $\bar{B}_n$ with $F \cap S = \varnothing$, there exists an m_F such that $|g_m| \leqslant \eta$ on F, for all $m \geqslant m_F$.

Let $0 < r_1 < r_2 < \ldots < 1$, $r_k \to 1$. Let

$$V_k = \{z \in \bar{B}_n \colon |z_1| > r_k\}.$$

Choose r_1 such that $V_1 \cap K = \varnothing$. Choose a such that

$$0 < a < 1 \quad \text{and} \quad aM + (1-a)\eta < 1.$$

Let

$$\varepsilon_k = (1-a)^k (1 - aM - (1-a)\eta).$$

Choose $m_1 > m_0$ such that $\| g_{m_1} - P \|_K < \varepsilon$, and

$$|g_{m_1}(z)| \leqslant \eta, \quad z \in F_0,$$

where F_0 is a compact set containing K with $F_0 \cap S = \varnothing$. Let $f_1 = g_{m_1}$. Define

$$W_1 = V_1 \cap \{z \in \bar{B}_n \colon \| a f_1(z) | - a | < \varepsilon_1\}$$

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and $F_1 = \bar{B}_n - W_1$. By (4), $S \subseteq W_1$ and hence $F_1 \cap S = \varnothing$. Now choose m_2 such that with $f_2 = g_{m_2}$, we have

$$|f_2(z)| \leqslant \eta, \quad z \in F_1,$$

and $\|f_2 - P\|_K < \varepsilon$. Then $z \in W_1$ implies

$$a|f_1(z)| + a(1-a)|f_2(z)| < a + \varepsilon_1 + a(1-a)M = 1 - (1-a)^2 \eta.$$

Define

$$W_2 = V_2 \cap \left\{ z \in W_1 : \left| \sum_{k=1}^{2} a(1-a)^{k-1} \left| f_k(z) \right| - \sum_{k=1}^{2} a(1-a)^{k-1} \right| < \varepsilon_2 \right\},$$

and $F_2 = \bar{B}_n - W_2$. Then $F_2 \cap S = \varnothing$.

Continuing inductively, we assume that open sets

$$\bar{B}_n = W_0 \supset W_1 \supset W_2 \supset \ldots \supset W_{k-1}$$

and functions $f_1, \ldots, f_k \in A(B_n)$ have been chosen such that

$$|f_j(z)| \leqslant M \quad \text{for } z \in \bar{B}_n; \quad |f_j(z)| = 1 \quad \text{for } z \in S;$$

$$\|f_j - P\|_K < \varepsilon; \quad |f_j(z)| \leqslant \eta \quad \text{for } z \in F_{j-1} = \bar{B}_n - W_{j-1}, \quad 1 \leqslant j \leqslant k,$$

and $z \in W_{j-1}$ implies $\sum_{i=1}^{j} a(1-a)^{i-1} |f_i(z)| \leqslant 1 - (1-a)^j \eta$. Now define

$$W_k = V_k \cap \left\{ z \in W_{k-1} : \left| \sum_{i=1}^{k} a(1-a)^{i-1} \left| f_i(z) \right| - \sum_{i=1}^{k} a(1-a)^{i-1} \right| < \varepsilon_k \right\}$$

and $F_k = \bar{B}_n - W_k$. Then $F_k \cap S = \varnothing$.

Choose m_{k+1} such that with $f_{k+1} = g_{m_{k+1}}$, we have

$$|f_{k+1}(z)| \leqslant M, \quad z \in \bar{B}_n; \quad |f_{k+1}(z)| = 1, \quad z \in S;$$

$$|f_{k+1}(z)| \leqslant \eta, \quad z \in F_k$$

and

$$\|f_{k+1} - P\|_K < \varepsilon.$$

Then $z \in W_k$ implies

$$\sum_{i=1}^{k+1} a(1-a)^{i-1} |f_i(z)| < \sum_{i=1}^{k} a(1-a)^{i-1} + \varepsilon_k + a(1-a)^k M$$
$$= 1 - (1-a)^{k+1} \eta.$$

With the sequence $\{f_k\}$ chosen as above, we define

$$f = \sum_{k=1}^{\infty} a(1-a)^{k-1} f_k.$$

Then

$$f \in A(B_n), \quad \|f - P\|_K \leqslant \sum_{k=1}^{\infty} a(1-a)^{k-1} \|f_k - P\|_K < \varepsilon,$$

and

$$|f(\pm 1, 0, \ldots, 0)| = 1.$$

We claim that $f \in \mathscr{B}$. Let $z \in B_n$. Then there exists k such that $z \in W_{k-1} - W_k$. So,

$$\begin{aligned} |f(z)| &\leqslant \sum_{i=1}^{k} a(1-a)^{i-1} |f_i(z)| + \sum_{i=k+1}^{\infty} a(1-a)^{i-1} |f_i(z)| \\ &< 1-(1-a)^k \eta + (1-a)^k \eta = 1. \end{aligned}$$

This completes the proof of Lemma 4.

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