COMPOSITIO MATHEMATICA

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Compositio Math. **157** (2021), 2377–2406.

doi:10.1112/S0010437X21007545

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 157 (2021) 2377–2406
doi:10.1112/S0010437X21007545

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Abstract

In this paper we give an interpretation, in terms of derived de Rham complexes, of Scholze's de Rham period sheaf and Tan and Tong's crystalline period sheaf.

Contents

1. Introduction

Fontaine's mysterious period rings are essential in formulating various p-adic comparison statements in p-adic Hodge theory. In the past decades there has been an effort to understand these period rings via other constructions related to differentials.

For instance, Colmez realized that one can put a topology on $\overline{\mathbb{Q}_p}$, related to Kähler differentials of $\overline{\mathbb{Z}_p}/\mathbb{Z}_p$, with respect to which the completion becomes the de Rham period ring B_{dR}^+; see [Fon94, Appendix] (which is polished and published in [Col12]).

Later on Beilinson [Bei12, § 1] gives another construction of B_{dR}^+ in terms of the derived de Rham cohomology (introduced by Illusie in [Ill72, Chapter VIII]) of $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$. In our notation, he shows that there is a filtered isomorphism

$$B_{\mathrm{dR}}^+ \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{\overline{\mathbb{Q}_p}/\mathbb{Q}_p};$$

see Construction 4.3 for the meaning of the right-hand side and Example 4.6.[1] In a similar vein, Bhatt [Bha12b, Proposition 9.9] exhibits a filtered isomorphism, realizing the crystalline period ring via the derived de Rham cohomology of $\overline{\mathbb{Z}_p}/\mathbb{Z}_p$,

$$A_{\mathrm{crys}} \cong \mathrm{dR}^{\mathrm{an}}_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p};$$

see Construction 3.1 and Example 3.5.

Received 24 August 2020, accepted in final form 29 April 2021, published online 6 October 2021.
2020 Mathematics Subject Classification 14F30, 14F40, 14G22, 14G45 (primary).
Keywords: period sheaf, derived de Rham cohomology, p-adic Hodge theory, crystalline cohomology.

[1] For the relation between these two constructions, see [Bei12, Proposition 1.6].

Fontaine's period rings admit various generalizations in geometric situations; see, for instance, [Fal89], [Bri08, §§5–6], [AI13, §2], [Sch13, §6] and [TT19, §2]. From now on let us focus on those introduced by Scholze. Recall that, in his proof of p-adic de Rham comparison for smooth proper rigid spaces over p-adic fields, Scholze introduces period sheaves $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ (see [Sch13, Definitions 6.1 and 6.8] and [Sch16]) on the pro-étale site of a smooth rigid space. Here by a p-adic field we mean a complete discrete-valued non-archimedean field extension of $\mathbb{Q}_p$ with perfect residue field, as in the setting of [Sch13, Definition 6.8]. However, the construction of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is somewhat complicated, and it takes one a fair amount of effort to understand $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. From this understanding Scholze deduces a long exact sequence [Sch13, Corollary 6.13],

$$0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{an}} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^{\dim X,\mathrm{an}} \to 0,$$

known as the p-adic analogue of the Poincaré sequence. Here ∇ is a connection which behaves like the classical Gauss–Manin connection (satisfying a certain Griffiths transversality and so on).

Following the theme, in this paper we explain how to understand Scholze's de Rham period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ in terms of suitable (analytic) derived de Rham sheaves.

Let k be a p-adic field. In this paper we introduce the (Hodge completed) analytic derived de Rham sheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ for the pro-étale site $X_{\text{proét}}$ relative to the analytic site X. Similarly, there is also a construction $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ for $X_{\text{proét}}$ relative to k. Our main result is the following theorem.

Theorem 1.1 (See Proposition 4.18 and Theorem 4.21 for the precise statement). *Let X be a smooth rigid space over k. We have natural filtered isomorphisms*

$$\mathbb{B}_{\mathrm{dR}}^+ \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \quad \textit{and} \quad \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}.$$

Moreover, from this viewpoint, one naturally gets the p-adic Poincaré sequence mentioned above. Indeed, in classical algebraic geometry, suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a triangle of smooth morphisms. Then one always has a sequence (see [KO68])

$$0 \to \Omega^*_{X/Z} \to \Omega^*_{X/Y} \xrightarrow{\nabla} \Omega^*_{X/Y} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^1_{Y/Z} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^*_{X/Y} \otimes_{f^{-1}\mathcal{O}_Y} \Omega^{\dim_{Y/Z}}_{Y/Z} \to 0,$$

whose totalization,[2] as well as the totalizations of the Hodge graded pieces (where $\Omega^i_{Y/Z}$ is given degree i), are all quasi-isomorphic to 0. In the framework of derived de Rham complexes, one has an intuitive base change formula for a triple of rings $A \to B \to C$,

$$\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B \cong \mathrm{dR}_{C/B},$$

which leads to a generalization of the above sequence (see §3.2). Applying this to the triangle $X_{\text{proét}} \to X \to k$ yields the following reinterpretation of the p-adic Poincaré sequence mentioned above.

Theorem 1.2 (See Theorem 4.20 for the precise statement). *Denote by $\nu\colon X_{\text{proét}} \to X$ the natural projection from the pro-étale site of X to the analytic site of X. The sequence*

$$0 \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\Omega_X^{\mathrm{an}} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\Omega_X^{\dim X,\mathrm{an}} \to 0$$

in $\widehat{\mathrm{DF}}(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k})$ is strict exact, where we give $\nu^{-1}\Omega_X^{i,\mathrm{an}}$ degree i.

Hence from this point of view, the connection ∇ defined by Scholze is indeed an incarnation of the Gauss–Manin connection.

[2] This is only heuristic, as totalizations do not make sense at the level of derived categories. See §§2.2 and 3.2.

The advantage of our perspective is that one can naturally generalize the above discussion to singular rigid spaces. Due to some technical issues, so far we have only worked out the case where the rigid space X is a local complete intersection over k (see Appendix A for a brief discussion of local complete intersections in rigid geometry). In this singular case, one no longer gets an ordinary sheaf but rather a sheaf in a derived ∞-category satisfying hyperdescent. In the local complete intersection case, the hypersheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$ is cohomologically bounded below by $-$ (embedded codimension of X). However, considering the zero-dimensional situation in § 4.5, we find that actually this hypersheaf always lives in cohomological degree 0 in that situation regardless of the input Artinian k-algebra. This leads to an interesting question that needs further exploration.

Question 1.3 (Same as Question 4.25; cf. [Bha12a]). In what generality should we expect $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$ to live in cohomological degree 0? And when that happens, can we reinterpret the underlying algebra via some construction similar to Scholze's $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ as in [Sch13, Sch16]?

Finally, we remark that we also have worked out a parallel story related to Tan and Tong's crystalline period sheaves [TT19, § 2]. We summarize the result in this direction as follows.

THEOREM 1.4 (See Theorem 3.21 and Corollary 3.19 for the precise statements). *Let k be an absolutely unramified p-adic field, with ring of integers $\mathcal{O}_k$, and let $\mathscr{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Denote by $w\colon X_{\mathrm{pro\acute{e}t}} \to \mathscr{X}$ the natural projection from the pro-étale site of the rigid generic fiber X of $\mathscr{X}$ to the Zariski site of $\mathscr{X}$. Then we have natural filtered isomorphisms*

$$\mathbb{A}_{\mathrm{crys}} \cong \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \quad \textit{and} \quad \mathcal{O}\mathbb{A}_{\mathrm{crys}} \cong \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}}.$$

Moreover, the sequence

$$0 \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathscr{X}}} w^{-1}\Omega^{1,\mathrm{an}}_{\mathscr{X}} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathscr{X}}} w^{-1}\Omega^{d,\mathrm{an}}_{\mathscr{X}} \to 0$$

in $\widehat{\mathrm{DF}}(\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k})$ is strict exact, where d is the relative dimension of $\mathscr{X}/\mathcal{O}_k$ and $w^{-1}\Omega^{i,\mathrm{an}}_{\mathscr{X}}$ is given degree i.

We want to mention that in our situation, we mostly care about the analytic derived de Rham complex for a map of adic spaces $X \to Y$, where X is a perfectoid space and Y is a rigid space (or their integral analogues). The analytic derived de Rham complex for a map of rigid spaces has been studied independently in [Ant20] and a forthcoming paper [Guo20] by the first named author.

Let us give a brief summary of the content of the following sections. In § 2 we explain the notation and conventions used in this paper, and we give a brief discussion of relevant facts about filtered derived ∞-categories and sheaves in them. In §§ 3 and 4 we work out, in a parallel way, the realizations of Scholze's and Tan and Tong's period sheaves. In both sections, we first introduce the relevant algebraic construction, then discuss the Poincaré sequence, and finally globalize (or sheafify) these constructions and show that they are (essentially) the same as the aforementioned period sheaves. In Appendix A we introduce the notion of local complete intersections in rigid geometry.

2. Notation and conventions

2.1 Notation

We fix k to be a complete discrete-valued p-adic field with a perfect residue field, and let $\mathcal{O}_k$ be its ring of integers. Denote by $\mathrm{Spa}(k)$ the adic spectrum $\mathrm{Spa}(k, \mathcal{O}_k)$.

Anything bearing the superscript $(-)^{\mathrm{an}}$ will mean a suitably p-completed version of the classical object $(-)$. The sense in which we are taking p-completion of these objects will be clear from the context.

The tensor products $\otimes$ appearing in this paper, if not otherwise specified, always denote derived tensor products. Similarly, completed tensor products always indicate derived completion of the derived tensor product (with respect to suitable filtrations to be specified in each case).

2.2 Filtrations

Many objects we are dealing with in this paper are viewed as objects either in the filtered derived ∞-category $\mathrm{DF}(R) \coloneqq \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathscr{D}(R))$ or in the full derived sub-∞-category $\widehat{\mathrm{DF}}(R) \subset \mathrm{DF}(R)$ consisting of objects that are derived complete with respect to the filtration, for some ring R which should be clear from the context. For a brief introduction to these, we refer readers to [BMS19, § 5.1].

We need a notion of *step sequence functor*, which is perhaps non-standard terminology. Given an integer $i \in \mathbb{N}$, we have a functor $\mathrm{Gr}^i \colon \mathrm{DF}(R) \to \mathscr{D}(R)$ sending a filtered object to its ith graded piece. This functor has a right adjoint which we call the *ith step sequence functor* and denote by $\mathrm{st}_i \colon \mathscr{D}(R) \to \mathrm{DF}(R)$. Concretely, the value of $\mathrm{st}_i(C)$ on j is given by

$$C_j = \begin{cases} C, & 0 \le j \le i, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $\mathcal{C}$ be a stable ∞-category; for example, $\mathcal{C}$ could be $\mathscr{D}(R)$, $\mathrm{DF}(R)$ or $\widehat{\mathrm{DF}}(R)$ for a discrete ring R. Consider a sequence of objects in $\mathcal{C}$,

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \cdots,$$

such that $d_{i+1} \circ d_i = 0$. If there exists an object L in the filtered ∞-category $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{C})$, satisfying the conditions

- $L(0) = A_0$,
- $L(i)/L(i+1) \cong A_{i+1}[-i]$,
- the natural map $L(0) \to L(0)/L(1)$ is identified with d_0,
- the natural connecting map of graded pieces $L(i)/L(i+1) \to L(i+1)/L(i+2)[1]$ is isomorphic to $d_{i+1}[-i]$,

then we say the sequence is *witnessed by the filtration L on A_0*. The notion is an ∞-analogue of a complex in the chain complex category.

When $\mathcal{C} = \mathrm{DF}(R)$, L can be regarded as an object $G(\bullet, \bullet) \in \mathrm{Fun}((\mathbb{N} \times \mathbb{N})^{\mathrm{op}}, \mathscr{D}(R))$, where we conventionally denote the first coordinate by i, the second coordinate by j, and $L(i) = G(i, 0)$. In this setting, we say the filtration $L(\bullet)$ on A_0 is *strict exact* if for any $j \in \mathbb{N}$, the object $G(0, j)$ is complete with respect to the filtration $G(i, j)$. Assume all of the $A_i = G(i-1, 0)/G(i, 0)[i-1]$ are cohomologically supported in degree 0 with filtrations (coming from the second coordinate) given by actual R-submodules. Then the sequence of A_i above can be thought of as a sequence of ordinary filtered R-modules, and our notion of strict exactness defined here agrees with the classical notion of strict exactness of a sequence of filtered R-modules.

https://doi.org/10.1112/S0010437X21007545 Published online by Cambridge University Press

2.3 Sheaves and hypersheaves

Here we give a quick review of sheaves valued in an ∞-category.

Let X be a site, and let $\mathscr{C}$ be a presentable ∞-category. The ∞-category of presheaves in $\mathscr{C}$, denoted by $\mathrm{PSh}(X, \mathscr{C})$, is defined to be the ∞-category $\mathrm{Fun}(X^{\mathrm{op}}, \mathscr{C})$ of contravariant functors from X to $\mathscr{C}$. The ∞-category $\mathrm{PSh}(X, \mathscr{C})$ admits a full sub-∞-category $\mathrm{Sh}(X, \mathscr{C})$ of *(infinity) sheaves in* $\mathscr{C}$, consisting of functors $\mathcal{F} : X^{\mathrm{op}} \to \mathscr{C}$ that send (finite) coproducts to products and satisfy the descent along Čech nerves: for any covering $U' \to U$ in X, the natural morphism to the limit below is required to be a weak equivalence

$$\mathcal{F}(U) \longrightarrow \lim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U'_n), \qquad (*)$$

where $U'_\bullet \to U$ is the Čech nerve associated with the covering $U' \to U$. Here we note that this is the ∞-categorical analogue of the classical sheaf condition in ordinary categories.

There is a stronger descent condition which requires $(*)$ above to hold with respect to all *hypercovers* $U'_\bullet \to U$ in the site X. Sheaves satisfying such stronger condition are called *hypersheaves*. For example, given any bounded-below complex C of ordinary sheaves on a site X, the assignment $U \mapsto \mathrm{R}\Gamma(U, C)$ gives rise to a hypersheaf. The collection of hypersheaves in $\mathscr{C}$ forms a full sub-∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C})$ inside $\mathrm{Sh}(X, \mathscr{C})$.

Remark 2.1. Let $\mathscr{C} = \mathscr{D}(R)$ be the derived ∞-category of R-modules. Then the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C})$ of hypersheaves over X is in fact equivalent to the derived ∞-category $\mathscr{D}(X, R)$ of classical sheaves of R-modules over X, by [Lur18, Corollary 2.1.2.3]. Here the functor $\mathscr{D}(X, R) \to \mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C})$ associates a complex of ordinary sheaves C with the functor

$$U \mapsto R\Gamma(U, C), \quad \forall\, U \in X.$$

As an upshot, the underlying homotopy category of $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C})$ is the classical derived category of sheaves of R-modules over X. In particular, given a hypersheaf $\mathcal{F}$ of R-modules over X, we can always represent it by an actual complex of sheaves of R-modules.

2.4 Unfolding a hypersheaf

There is a way to define a hypersheaf on a site X via unfolding from a basis; cf. [BMS19, Proposition 4.31] and the discussion after it.

Let X be a site and let $\mathcal{B}$ be a basis of X, that is, $\mathcal{B}$ is a subcategory of X such that for each object U in X, there exists an object U' in $\mathcal{B}$ covering U. So any hypercover of an object in X can be refined to a hypercover with each term in $\mathcal{B}$. Let $\mathscr{C}$ be a presentable ∞-category.

Let $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathscr{C})$ be a hypersheaf on $\mathcal{B}$. We can then *unfold* the sheaf $\mathcal{F}$ into a hypersheaf $\mathcal{F}'$ on X, such that its evaluation at any $V \in X$ is given by

$$\mathcal{F}'(V) = \operatorname*{colim}_{U'_\bullet \to V} \varprojlim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U'_n),$$

where the colimit is indexed over all hypercovers $U'_\bullet \to V$ with $U'_n \in \mathcal{B}$ for all n. It can be shown that one hypercover suffices to compute the value of $\mathcal{F}'(V)$ in the above formula: actually, for a hypercover $U'_\bullet \to V$ with each U'_n in the basis $\mathcal{B}$, we have a natural weak equivalence

$$\lim_{[n]\in\Delta^{\mathrm{op}}} \mathcal{F}(U'_n) \longrightarrow \mathcal{F}'(V).$$

In particular, for any $U \in \mathcal{B}$, the natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$ is a weak equivalence.

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The above construction is functorial with respect to $\mathcal{F} \in \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathscr{C})$, and we get a natural unfolding functor

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathscr{C}) \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C}),$$

which is in fact an equivalence, with the inverse given by the restriction functor $\mathrm{Sh}^{\mathrm{hyp}}(X, \mathscr{C}) \to \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B}, \mathscr{C})$.

3. Integral theory

3.1 Affine construction

In this subsection we define the analytic cotangent complex and the analytic derived de Rham complex for a morphism of p-adic algebras. We refer readers to [Bha12b, §§ 2 and 3] for general background on the derived de Rham complex in a p-adic situation.

Construction 3.1 (Integral constructions). Let $A_0 \to B_0$ be a map of p-adically complete algebras over $\mathcal{O}_k$, and P be the standard polynomial resolution of B_0 over A_0.

We define the *analytic cotangent complex of* $A_0 \to B_0$, denoted by $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$, to be the derived p-completion of the complex $\Omega^1_{P/A_0} \otimes_P B_0$ of B_0-modules.

Next we denote by $(|\Omega^*_{P/A_0}|, \mathrm{Fil}^*)$ the direct sum totalization of the simplicial complex Ω^*_{P/A_0} together with its Hodge filtration, as an object in $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(A_0))$. As the de Rham complex of a simplicial ring admits a commutative differential graded algebra structure, we may regard $|\Omega^*_{P/A_0}|$ with its Hodge filtration as an object in $\mathrm{CAlg}(\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(A_0)))$. Then the *analytic derived de Rham complex of* B_0/A_0, denoted by $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$ in $\mathrm{CAlg}(\mathrm{DF}(A_0))$, is defined as the derived p-completion of the filtered E_∞ algebra $(|\Omega^*_{P/A_0}|, \mathrm{Fil}^*)$.

Remark 3.2. By construction, the graded pieces of the derived Hodge filtrations of $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$ are given by

$$\mathrm{Gr}^i(\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}) \cong (\mathrm{L}\wedge^i \mathbb{L}_{B_0/A_0})^{\mathrm{an}}[-i],$$

where $\mathrm{L}\wedge^i$ denotes the ith left derived wedge product; cf. [Bha12a, Construction 4.1].

Let us establish some properties of this construction before discussing an example.

Lemma 3.3. *Let $A \to B \to C$ be a triple of rings. Then we have a commutative diagram of filtered E_∞ algebras*

$$\begin{array}{ccc}
\mathrm{dR}_{B/A} & \longrightarrow & \mathrm{dR}_{C/A} \\
\downarrow & & \downarrow \\
B & \longrightarrow & \mathrm{dR}_{C/B}
\end{array}$$

where the left arrow is the projection to the zeroth graded piece of the derived Hodge filtration, and the other three arrows come from the functoriality of the construction of derived de Rham complex.

Proof. This follows from the left Kan extension of the case when B is a polynomial A-algebra and C is a polynomial B-algebra. □

The following theorem is the key ingredient in understanding the analytic derived de Rham complex in situations that are of interest to us.

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Theorem 3.4. *Let $A \to B \to C$ be ring homomorphisms of p-completely flat $\mathbb{Z}_p$-algebras, such that $A/p \to B/p$ is relatively perfect (see [Bha12b, Definition 3.6]). Then we have the following assertions.*

(1) $\mathbb{L}^{\mathrm{an}}_{B/A} = 0$, *and* $\mathrm{dR}^{\mathrm{an}}_{B/A} = B$.
(2) *The natural map* $\mathrm{dR}^{\mathrm{an}}_{C/A} \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ *is an isomorphism.*
(3) *We have a commutative diagram:*

$$\begin{array}{ccc} \mathrm{dR}^{\mathrm{an}}_{B/A} & \longrightarrow & \mathrm{dR}^{\mathrm{an}}_{C/A} \\ \downarrow \cong & & \downarrow \cong \\ B & \longrightarrow & \mathrm{dR}^{\mathrm{an}}_{C/B} \end{array}$$

(4) *Assume, furthermore, that $B \to C$ is surjective with kernel I and $B/p \to C/p$ is a local complete intersection. Then the natural map $B \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ exhibits the latter as $D_B(I)^{\mathrm{an}}$, the p-adic completion of the PD envelope of B along I. Moreover, the p-adic completion of the PD filtrations $\mathrm{Fil}^r = I^{[r],\mathrm{an}}$ is identified with the rth Hodge filtration.*

Note that, by [Bha12b, Lemma 3.38], $D_B(I)^{\mathrm{an}}$ is a p-complete flat $\mathbb{Z}_p$-algebra. Hence $I^{[r],\mathrm{an}}$, being submodules of a flat $\mathbb{Z}_p$-module, are also p-torsion-free for all r.

Proof. Statements (1) and (2) follow from the proof of [Bha12b, Corollary 3.8]: one immediately reduces modulo p and appeals to the conjugate filtration. Statement (3) follows from Lemma 3.3 by taking the derived p-completion.

As for (4), we first apply [Bha12b, Proposition 3.25] and [Ber74, Théorème V.2.3.2] to see that there is a natural filtered map $\mathscr{C}omp_{C/B}\colon \mathrm{dR}^{\mathrm{an}}_{C/B} \to D_B(I)^{\mathrm{an}}$ such that precomposing with $B \to \mathrm{dR}^{\mathrm{an}}_{C/B}$ gives the natural map $B = B^{\mathrm{an}} \to D_B(I)^{\mathrm{an}}$. By [Bha12b, Theorem 3.27] we see that $\mathscr{C}omp_{C/B}$ is an isomorphism for the underlying algebra. To show that the same holds for filtrations, it suffices to show that the induced map on graded pieces is an isomorphism as the map is compatible with filtrations. To that end, by a standard spread-out technique, we may reduce to the case where B is the p-adic completion of a finite type $\mathbb{Z}_p$ algebra, in particular it is Noetherian, in which case the identification of graded pieces via this natural map follows from a result of Illusie [Ill72, Corollaire VIII.2.2.8]. □

We are now ready to consider some examples. An inspiring arithmetic example is worked out by Bhatt.

Example 3.5 [Bha12b, Proposition 9.9]. There is a filtered isomorphism,

$$A_{\mathrm{crys}} \cong \mathrm{dR}^{\mathrm{an}}_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p}.$$

Let us work out a geometric example.

Example 3.6. Let n be a positive integer. Let

$$R = \mathbb{Z}_p\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle,$$
$$R_\infty = \mathbb{Z}_p\langle T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}\rangle = R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle/(T_i - S_i; 1 \leq i \leq n).$$

Applying (the derived p-completion of) the fundamental triangle of cotangent complexes to

$$\mathbb{Z}_p \to R \to R_\infty,$$

one obtains that $\mathbb{L}^{\mathrm{an}}_{R_\infty/R} = R_\infty \cdot \{dT_1, \ldots, dT_n\}[1]$.

On the other hand, the fundamental triangle associated with

$$R \to R\langle S_1^{1/p^\infty}, \dots, S_n^{1/p^\infty}\rangle \to R_\infty$$

gives us $\mathbb{L}^{\mathrm{an}}_{R_\infty/R} = R_\infty \cdot \{T_i - S_i; 1 \le i \le n\}[1]$.

The relation between these two presentations of $\mathbb{L}^{\mathrm{an}}_{R_\infty/R}$ is that

$$T_i - S_i = dT_i$$

in $\mathrm{H}_1(\mathbb{L}^{\mathrm{an}}_{R_\infty/R})$, as $(\partial/\partial T_i)(T_i - S_i) = 1$.[3]

Following the above notation, we describe $\mathrm{dR}^{\mathrm{an}}_{R_\infty/R}$.

Example 3.7. Applying Theorem 3.4 to $A = R$, $B = R\langle S_1^{1/p^\infty}, \dots, S_n^{1/p^\infty}\rangle$ and $I = (T_1 - S_1, \dots, T_n - S_n)$, we see that $\mathrm{dR}^{\mathrm{an}}_{R_\infty/R} = \big(\mathrm{D}_{\mathbb{Z}_p\langle T_1^{\pm 1},\dots,T_n^{\pm 1}, S_1^{1/p^\infty},\dots,S_n^{1/p^\infty}\rangle}(I)\big)^{\mathrm{an}}$ is the p-adic completion of the PD envelope of $R\langle S_1^{1/p^\infty}, \dots, S_n^{1/p^\infty}\rangle$ along I (notice that the PD envelope is p-torsion-free, hence derived completion agrees with classical completion), and the Hodge filtrations are (p-adically) generated by divided powers of $\{T_i - S_i\}$. Example 3.6 shows that the image of $(T_i - S_i)$ in $\mathrm{Gr}^1 = \mathbb{L}^{\mathrm{an}}_{R_\infty/R}[-1] = R_\infty \otimes_R \Omega^{1,\mathrm{an}}_{R/\mathbb{Z}_p}$ is identified with $1 \otimes dT_i$. This precise identification will be used later (see Example 4.7 and the proof of Theorem 4.21) when we compare certain rational version of the analytic derived de Rham complex with Scholze's period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$.

3.2 Derived de Rham complex for a triple

Given a pair of smooth morphisms $A \to B \to C$, there is a natural Gauss–Manin connection $\mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^1_{B/A}$, such that $\mathrm{dR}_{C/A}$ is naturally identified with the 'totalization' of the following sequence:

$$\mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^1_{B/A} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \Omega^{\dim_{B/A}}_{B/A}.$$

Katz and Oda [KO68] observed that this can be explained by a filtration on $\mathrm{dR}_{C/A}$. In this subsection we shall show how to generalize this to the context of the derived de Rham complex for a pair of arbitrary morphisms $A \to B \to C$.

We first need to introduce a way to put filtration on a tensor product of filtered modules over a filtered E_∞-algebra. The following fact about Bar resolution is well known, and we thank Bhargav Bhatt for teaching us in this generality.

Lemma 3.8. *Let A be an ordinary ring, let R be an E_∞-algebra over A, and let M and N be two objects in $\mathscr{D}(R)$. Then the following augmented simplicial object in $\mathscr{D}(A)$ displays $M \otimes_R N$ as the colimit of the simplicial objects in $\mathscr{D}(A)$:*

$$\Big(\cdots \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} M \otimes_A R \otimes_A R \otimes_A N \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} M \otimes_A R \otimes_A N \rightrightarrows M \otimes_A N\Big) \longrightarrow M \otimes_R N.$$

Here the arrows are given by 'multiplying two factors together'.

Proof. Since the ∞-category $\mathscr{D}(R)$ is generated by shifts of R [Lur17, 7.1.2.1], commuting tensor with colimit, we may assume that both M and N are just R. In this case, the statement holds for merely E_1-algebras, as we have a null homotopy $R^{\otimes_A n} \to R^{\otimes_A (n+1)}$ given by tensoring $R^{\otimes_A n}$ with the natural map $A \to R$. □

[3] Here we follow the sign conventions in the Stacks Project; see [Sta20, Tag 07MC footnote 1]

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Construction 3.9. Let A be an ordinary ring, let R be a filtered E_∞-algebra over A, and let M and N be two filtered R-modules with filtrations compatible with that on R. Then we regard $M \otimes_R N$ as an object in $\mathrm{DF}(A)$ via the Bar resolution in Lemma 3.8, with

$$\mathrm{Fil}^i(M \otimes_R N) := \mathrm{colim}_{\Delta^{\mathrm{op}}}\big(\cdots \Rrightarrow \mathrm{Fil}^i(M \otimes_A R \otimes_A R \otimes_A N) \Rrightarrow \mathrm{Fil}^i(M \otimes_A R \otimes_A N) \rightrightarrows \mathrm{Fil}^i(M \otimes_A N)\big),$$

where the filtrations on $M \otimes_A R \otimes_A \cdots \otimes_A R \otimes_A N$ are given by the usual Day involution.

LEMMA 3.10. *Let A, R, M, N be as in Construction 3.9. Then we have*

$$\mathrm{Gr}^*(M \otimes_R N) \cong \mathrm{Gr}^*(M) \otimes_{\mathrm{Gr}^*(R)} \mathrm{Gr}^*(N).$$

Proof. We have

$$\mathrm{Gr}^*(M \otimes_R N) \cong \mathrm{colim}_{\Delta^{\mathrm{op}}}\big(\cdots \Rrightarrow \mathrm{Gr}^*(M \otimes_A R \otimes_A R \otimes_A N) \Rrightarrow \mathrm{Gr}^*(M \otimes_A R \otimes_A N) \rightrightarrows \mathrm{Gr}^*(M \otimes_A N)\big)$$

$$\cong \mathrm{colim}_{\Delta^{\mathrm{op}}}\big(\cdots \Rrightarrow \mathrm{Gr}^*(M) \otimes_A \mathrm{Gr}^*(R) \otimes_A \mathrm{Gr}^*(N) \rightrightarrows \mathrm{Gr}^*(M) \otimes_A \mathrm{Gr}^*(N)\big) \cong \mathrm{Gr}^*(M) \otimes_{\mathrm{Gr}^*(R)} \mathrm{Gr}^*(N).$$

□

PROPOSITION 3.11. *Let $A \to B \to C$ be a triple of rings. Then the diagram of filtered E_∞-algebras in Lemma 3.3 induces a filtered isomorphism of filtered E_∞-algebras over B:*

$$\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B \cong \mathrm{dR}_{C/B}.$$

Here the left-hand side is equipped with the filtration in Construction 3.9 with the Hodge filtrations on $\mathrm{dR}_{C/A}$ and $\mathrm{dR}_{B/A}$, and $\mathrm{Fil}^i(B) = 0$ for $i \geq 1$. The right-hand side is equipped with the Hodge filtration. Denote by $\Omega^*_{B/A} := \oplus_i \mathrm{st}_i(\mathrm{L} \wedge^i \mathbb{L}_{B/A})[-i]$ the graded algebra associated with the Hodge filtration.

Proof. After cofibrant replacing B by a simplicial polynomial A-algebra and C by a simplicial polynomial B-algebra, we reduce the statement to the case where B is a polynomial A-algebra and C is a polynomial B-algebra. One verifies directly that in this case we have

$$\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B \cong \mathrm{dR}_{C/B} \quad \text{and} \quad \Omega^*_{C/A} \otimes_{\Omega^*_{B/A}} B \cong \Omega^*_{C/B}.$$

We conclude the proof by recalling that a filtered morphism with isomorphic underlying object is a filtered isomorphism if and only if the induced morphisms of graded pieces are isomorphisms.

□

Construction 3.12. Let $A \to B \to C$ be a triple of rings. Then we put a filtration on $\mathrm{dR}_{C/A}$ given by $L(i) = \mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} \mathrm{Fil}^i_{\mathrm{H}}(\mathrm{dR}_{B/A})$, viewed as a commutative algebra object in $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{DF}(A)) = \mathrm{Fun}((\mathbb{N} \times \mathbb{N})^{\mathrm{op}}, \mathscr{D}(A))$, where the filtration on $L(i)$ is as in Construction 3.9 with each factor being equipped with its own Hodge filtrations. We have $L(0) \cong \mathrm{dR}_{C/A}$, and we call $L(i)$ *the ith Katz–Oda filtration on* $\mathrm{dR}_{C/A}$ and denote it by $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A})$.

We caution readers that each $\mathrm{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A})$ is equipped with yet another filtration, which we shall still call the Hodge filtration; the index is often denoted by j. The graded pieces of the Katz–Oda filtration when both arrows in $A \to B \to C$ are smooth were studied by Katz and Oda [KO68], although in a different language, hence the name.

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Lemma 3.13. *Let $A \to B \to C$ be a triple of rings. Then the following statements hold.*

(1) *We have a filtered isomorphism*

$$\mathrm{Gr}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}) \cong \mathrm{dR}_{C/B} \otimes_B \mathrm{st}_i((\mathrm{L} \wedge^i \mathbb{L}_{B/A})[-i]).$$

(2) *Under the above filtered isomorphism, the Katz–Oda filtration on $\mathrm{dR}_{C/A}$ witnesses the following sequence:*

$$\mathrm{dR}_{C/A} \to \mathrm{dR}_{C/B} \xrightarrow{\nabla} \mathrm{dR}_{C/B} \otimes_B \mathrm{st}_1(\mathbb{L}_{B/A}) \xrightarrow{\nabla} \cdots.$$

Here ∇ denotes connecting homomorphisms, which is $\mathrm{dR}_{C/A}$-linear and satisfies Newton–Leibniz rule.

(3) *The induced Katz–Oda filtration on $\mathrm{Gr}^j_{\mathrm{H}}(\mathrm{dR}_{C/A})$ is complete. In fact $\mathrm{Fil}^i_{\mathrm{KO}} \mathrm{Gr}^j_{\mathrm{H}}(\mathrm{dR}_{C/A}) = 0$ whenever $i > j$.*

(4) *If $A \to B$ is smooth of equidimension d, then $\mathrm{Fil}^i_{\mathrm{KO}} \mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}_{C/A}) \cong 0$ for any $i > d$. In particular, combining with the previous point, we get that in this situation the Katz–Oda filtration is strict exact in the sense of § 2.2.*

Proof. For (1) we have

$$\mathrm{Gr}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}) \cong \mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} \mathrm{st}_i(\mathrm{L} \wedge^i \mathbb{L}_{B/A})[-i] \cong (\mathrm{dR}_{C/A} \otimes_{\mathrm{dR}_{B/A}} B) \otimes_B \mathrm{st}_i(\mathrm{L} \wedge^i \mathbb{L}_{B/A})[-i],$$

and by Proposition 3.11 the right-hand side can be identified with $\mathrm{dR}_{C/B} \otimes_B \mathrm{st}_i(\mathrm{L} \wedge^i \mathbb{L}_{B/A})[-i]$.

For (2) we just need to show the properties of these ∇s. With any multiplicative filtration on an E_∞-algebra R, we get a natural filtered map $\mathrm{Fil}^i \otimes_R \mathrm{Fil}^j \to \mathrm{Fil}^{i+j}(R)$ where the left-hand side is equipped with the Day convolution filtration (over the underlying algebra R). Now we look at the following commutative diagram:

$$\begin{array}{ccccc}
(\mathrm{Gr}^i \otimes_R \mathrm{Gr}^{j+1}) \oplus (\mathrm{Gr}^{i+1} \otimes_R \mathrm{Gr}^j) & \longrightarrow & \mathrm{Fil}^{i+j} / \mathrm{Fil}^{i+j+2}(\mathrm{Fil}^i \otimes_R \mathrm{Fil}^j) & \longrightarrow & \mathrm{Gr}^i \otimes_R \mathrm{Gr}^j \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}^{i+j+1} & \longrightarrow & \mathrm{Fil}^{i+j} / \mathrm{Fil}^{i+j+2}(R) & \longrightarrow & \mathrm{Gr}^{i+j} \xrightarrow{+1}
\end{array}$$

to conclude that the connecting morphisms are R-linear and satisfy the Newton–Leibniz rule. Since $\mathrm{Fil}^i_{\mathrm{KO}}$ is a multiplicative filtration on $\mathrm{dR}_{C/A}$, we get the desired properties of ∇.

Statement (3) follows from the distinguished triangle of cotangent complexes and their exterior powers.

Statement (4) follows from the definition of the Katz–Oda filtration in Construction 3.12 and the fact that $\mathrm{Fil}^i_{\mathrm{H}}(\mathrm{dR}_{B/A}) = 0$ whenever $i > d$. □

We do not need the following construction in this paper, but mention it for the sake of completeness of our discussion.

Construction 3.14. We denote the graded algebra associated with the Hodge filtration on the derived de Rham complex by $\mathrm{L}\Omega^*_{-/-}$.[4] Let $A \to B \to C$ be a triple of rings. Note that $\mathrm{L}\Omega^*_{C/A} \cong \mathrm{L} \wedge^*_C (\mathrm{st}_1(\mathbb{L}_{C/A}))[-*]$, and we have a functorial filtration $\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A}$ with quotient $\mathbb{L}_{C/B}$. Hence there is a functorial multiplicative exhaustive increasing filtration on $\mathrm{L}\Omega^*_{C/A}$, called the *vertical filtration* and denoted by Fil^v_i, consisting of graded-$\mathrm{L}\Omega^*_{B/A}$-submodules with graded pieces given by $\mathrm{Gr}^v_i = \mathrm{L}\Omega^*_{B/A} \otimes_B \mathrm{st}_i(\mathrm{L} \wedge^i \mathbb{L}_{C/B})[-i]$.

[4] We warn readers that this is not standard notation; elsewhere the symbol LΩ is often used to denote the derived de Rham complex.

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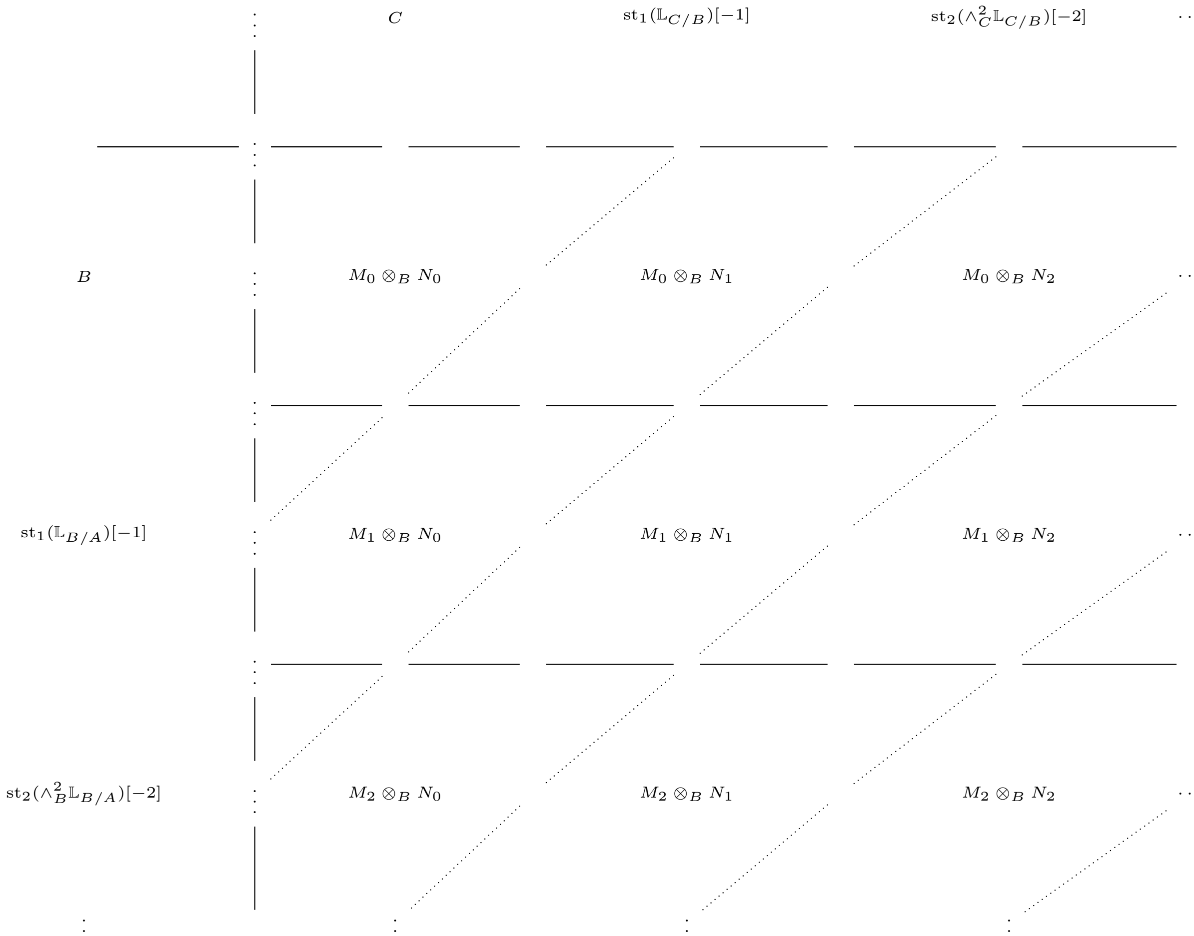
Let us summarize the picture of (the graded pieces of) these filtrations in the following diagram:



In this diagram, $M_i = \mathrm{st}_i(\wedge^i_B \mathbb{L}_{B/A})[-i]$, and $N_j = \mathrm{st}_j(\wedge^j_C \mathbb{L}_{C/B})[-j]$, for $i, j \in \mathbb{N}$. Let us explain the diagram: it describes graded pieces of filtrations on $\mathrm{dR}_{C/A}$. Here the rows represent graded pieces of the Katz–Oda filtration, and the dotted lines indicate the Hodge filtration (given by things below the dotted line). Once we take graded pieces with respect to the Hodge filtration, then the vertical filtration is literally induced by vertical columns, starting from left to right, hence the name.

Specializing to the p-adic setting, we get the following lemma.

Lemma 3.15. *Let $A \to B \to C$ be a triangle of p-complete flat $\mathbb{Z}_p$-algebras. Suppose B/p is smooth over A/p of relative equidimension n. Then we have a p-adic Katz–Oda filtration on $\mathrm{dR}_{C/A}$ which is strict exact and witnesses the following sequence:*

$$0 \to \mathrm{dR}^{\mathrm{an}}_{C/A} \to \mathrm{dR}^{\mathrm{an}}_{C/B} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B} \otimes_B \mathrm{st}_1(\Omega^{1,\mathrm{an}}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B} \otimes_B \mathrm{st}_n(\Omega^{n,\mathrm{an}}_{B/A}) \to 0.$$

Recall that the superscript $(-)^{\mathrm{an}}$ denotes the derived p-completion of the corresponding objects. Note that since $\Omega^{i,\mathrm{an}}_{B/A}$ are all finite flat B-modules by assumption and $\mathrm{dR}^{\mathrm{an}}_{C/B}$ is p-complete, the tensor products appearing above are already p-complete.

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Proof. Taking the derived p-completion of the Katz–Oda filtration on $\mathrm{dR}_{C/A}$, we get such a strict exact filtration by Lemma 3.13. □

3.3 Integral de Rham sheaves

For the rest of this section we focus on the situation spelled out as follows. Let κ be a perfect field in characteristic $p > 0$, and let $k = W(\kappa)[1/p]$ be the absolutely unramified discrete-valued p-adic field with the ring of integers $\mathcal{O}_k = W(\kappa)$. Fix a separated formally smooth p-adic formal scheme $\mathscr{X}$ over $\mathcal{O}_k$. Denote by X its generic fiber, viewed as an adic space over the Huber pair $(k, \mathcal{O}_k)$.

In this situation, there is a natural map of ringed sites

$$w\colon (X_{\text{proét}}, \widehat{\mathcal{O}}_X^+) \longrightarrow (\mathscr{X}, \mathcal{O}_{\mathscr{X}})$$

which sends an open subset $\mathscr{U} \subset \mathscr{X}$ to the open subset $U \in X_{\text{proét}}$, where U is the generic fiber of $\mathscr{U}$. This allows us to define the inverse image $w^{-1}\mathcal{O}_{\mathscr{X}}$ of the integral structure sheaf $\mathcal{O}_{\mathscr{X}}$, as a sheaf on the pro-étale site $X_{\text{proét}}$.

On the pro-étale site of X, we have a morphism of sheaves of p-complete $\mathcal{O}_k$-algebras:

$$\mathcal{O}_k \longrightarrow w^{-1}\mathcal{O}_{\mathscr{X}} \longrightarrow \widehat{\mathcal{O}}_X^+. \tag{$\boxdot$}$$

We refer readers to [Sch13, §§ 3 and 4] for a detailed discussion around the pro-étale site of a rigid space and structure sheaves on it. There is a subcategory $X^{\omega}_{\text{proét}/\mathscr{X}} \subset X_{\text{proét}}$ consisting of affinoid perfectoid objects $U = \mathrm{Spa}(B, B^+) \in X_{\text{proét}}$ whose image in X is contained in $w^{-1}(\mathrm{Spf}(A_0))$, the generic fiber of an affine open $\mathrm{Spf}(A_0) \subset \mathscr{X}$. The class of such objects forms a basis for the pro-étale topology by (the proof of) [Sch13, Proposition 4.8]. We first study the behavior of the derived de Rham complex for the triangle equation ($\boxdot$) on $X^{\omega}_{\text{proét}/\mathscr{X}}$.

Proposition 3.16. *Let $U = \mathrm{Spa}(B, B^+) \in X_{\text{proét}}$ be an object in $X^{\omega}_{\text{proét}/\mathscr{X}}$, and choose $\mathrm{Spf}(A_0) \subset \mathscr{X}$ such that the image of U in X is contained in $w^{-1}(\mathrm{Spf}(A_0))$. Then:*

(1) *the natural surjection $\theta\colon A_{inf}(B^+) \twoheadrightarrow B^+$ exhibits $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k} = A_{\mathrm{crys}}(B^+)$, the p-completion of the divided envelope of $A_{inf}(B^+)$ along $\ker(\theta)$;*
(2) *the natural surjection $w^\sharp \otimes \theta\colon A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+) \twoheadrightarrow B^+$ exhibits $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}$ as the p-completion of the divided envelope of $A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+)$ along $\ker(w^\sharp \otimes \theta)$;*
(3) *in both cases, the Hodge filtrations are identified as the p-completion of PD filtrations;*
(4) *the filtered algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}$ is independent of the choice of A_0, and we denote it by $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathscr{X}}$.*

Remark 3.17. In particular, (1) and (2) tells us that these derived de Rham complexes are actually quasi-isomorphic to an honest algebra viewed as a complex supported on cohomological degree 0; (4) tells us that sending $U = \mathrm{Spa}(B, B^+) \in X^{\omega}_{\text{proét}/\mathscr{X}}$ to $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathscr{X}}$ gives a well-defined presheaf on $X^{\omega}_{\text{proét}/\mathscr{X}}$.

Proof of Proposition 3.16. Applying Theorem 3.4(4) to the triangles

$$\mathcal{O}_k \to A_{inf}(B^+) \to B^+ \quad \text{and} \quad A_0 \to A_0 \hat{\otimes}_{\mathcal{O}_k} A_{inf}(B^+) \to B^+$$

proves (1) and (2) respectively and (3).[5] As for (4), using the separatedness of $\mathscr{X}$, we reduce to the situation where the image of U in X is in a smaller open $w^{-1}(\mathrm{Spf}(A_1)) \subset w^{-1}(\mathrm{Spf}(A_0))$. It suffices to show that the natural map $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0} \to \mathrm{dR}^{\mathrm{an}}_{B^+/A_1}$ is a filtered isomorphism, which follows from Lemma 3.15 as $A_0/p \to A_1/p$ is étale. □

[5] Here we use the unramifiedness of $\mathcal{O}_k$ to verify the relative perfectness assumption in Theorem 3.4.

Recall that the subcategory $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathscr{X}} \subset X_{\mathrm{pro\acute{e}t}}$ gives a basis for the topology on $X_{\mathrm{pro\acute{e}t}}$. Hence any presheaf on $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathscr{X}}$ can be sheafified to a sheaf on $X_{\mathrm{pro\acute{e}t}}$.

We define the analytic de Rham sheaf for $\widehat{\mathcal{O}}^+_X$ over $\mathcal{O}_k$ and $w^{-1}\mathcal{O}_{\mathscr{X}}$ as follows.

Construction 3.18 ($\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$ and $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}}$). The *analytic de Rham sheaf of* $\widehat{\mathcal{O}}^+_X/\mathcal{O}_k$, denoted by $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$, is the p-adic completion of the unfolding of the presheaf on $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathscr{X}}$ which assigns each $U = \mathrm{Spa}(B, B^+)$ the algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k}$. We equip it with the decreasing Hodge filtration $\mathit{Fil}^r_{\mathrm{H}}$ given by the image of p-completion of the unfolding of the presheaf assigning each $U = \mathrm{Spa}(B, B^+)$ the rth Hodge filtration in $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathcal{O}_k}$.

The *analytic de Rham sheaf of* $\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}$, denoted by $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}}$, is the p-adic completion of the unfolding of the presheaf on $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathscr{X}}$ which assigns each $U = \mathrm{Spa}(B, B^+)$ the filtered algebra $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathscr{X}}$. Similarly, we equip it with the decreasing Hodge filtration Fil^r_H given by the image of p-completion of the unfolding of the presheaf whose value on each $U = \mathrm{Spa}(B, B^+)$ is the rth Hodge filtration in $\mathrm{dR}^{\mathrm{an}}_{B^+/\mathscr{X}}$.

The fact that these definitions/constructions make sense follows from Proposition 3.16 and Remark 3.17.

One may also define the corresponding mod p^n version of these sheaves. Since sheafifying commutes with arbitrary colimit, the p-adic completion of the sheafification of a presheaf F is the same as the inverse limit over n of the sheafification of presheaves F/p^n. Therefore we have that $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}/p^n$ is the same as the sheafification of the presheaf $\mathrm{dR}_{B^+/\mathcal{O}_k}/p^n$. Its rth Hodge filtration agrees with the sheafification of the presheaf $\mathrm{Fil}^r_H(\mathrm{dR}_{B^+/\mathcal{O}_k}/p^n)$, as sheafifying is an exact functor. Similar statements can be made for the mod p^n version of $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}}$ and its Hodge filtrations.

Now the strict exact Katz–Oda filtration obtained in the Lemma 3.15 gives us the following corollary.

COROLLARY 3.19 (Crystalline Poincaré lemma). *There is a functorial* $\mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k}$*-linear strict exact sequence of filtered sheaves on* $X_{\mathrm{pro\acute{e}t}}$,

$$0 \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \to \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathscr{X}}} \mathrm{st}_1(w^{-1}\Omega^{1,\mathrm{an}}_{\mathscr{X}}) \xrightarrow{\nabla} \cdots$$

$$\cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \otimes_{w^{-1}\mathcal{O}_{\mathscr{X}}} \mathrm{st}_d(w^{-1}\Omega^{d,\mathrm{an}}_{\mathscr{X}}) \to 0,$$

where d is the relative dimension of $\mathscr{X}/\mathcal{O}_k$.

Proof. Using the discussion before this corollary, we reduce to checking it at the level of presheaves on $X^{\omega}_{\mathrm{pro\acute{e}t}/\mathscr{X}}$. Since now everything in sight is supported cohomologically in degree 0 with filtrations given by submodules because of Proposition 3.16, the strict exact Katz–Oda filtration in Lemma 3.15 implies what we want. □

Remark 3.20. We can drop the separatedness assumption on $\mathscr{X}$ as follows. Since any formal scheme is covered by affine ones, and affine formal schemes are automatically separated, we may define all these de Rham sheaves on each slice subcategory of the pro-étale site of the rigid generic fiber of affine opens of $\mathscr{X}$. Similar to the proof of Proposition 3.16(4), we can show that these de Rham sheaves satisfy the base change formula with respect to maps of affine opens of $\mathscr{X}$ (by appealing to Lemma 3.15 again), hence these sheaves on the slice subcategories glue to

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a global one. The crystalline Poincaré lemma obtained above holds verbatim as exactness of a sequence of sheaves may be checked locally.

3.4 Comparing with Tan and Tong's crystalline period sheaves

Finally, we shall identify the two de Rham sheaves defined above with two period sheaves that show up in the work of Tan–Tong [TT19]. We refer readers to [TT19, Definitions 2.1. and 2.9] for the meaning of the period sheaves $\mathbb{A}_{\mathrm{crys}}$ and $\mathcal{O}\mathbb{A}_{\mathrm{crys}}$ and their PD filtrations.

We look at the triangle of sheaves of rings:

$$\mathcal{O}_k \to w^{-1}(\mathcal{O}_{\mathscr{X}})\hat{\otimes}_{\mathcal{O}_k}\mathbb{A}_{inf} \xrightarrow{w^\sharp\hat{\otimes}\theta} \widehat{\mathcal{O}}^+_X.$$

Theorem 3.21. *The triangle above induces a filtered isomorphism of sheaves:* $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \cong \mathbb{A}_{\mathrm{crys}}$ *and* $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathscr{X}}} \cong \mathcal{O}\mathbb{A}_{\mathrm{crys}}$. *Moreover, under this identification, the crystalline Poincaré sequence in Corollary 3.19 agrees with the one obtained in [TT19, Corollary 2.17].*

Proof. We check these isomorphisms modulo p^n for any n. In both cases, the de Rham sheaf and the crystalline period sheaf are both unfoldings of the same PD envelope presheaf (with its PD filtrations) on $X^{\omega}_{\mathrm{proét}/\mathscr{X}}$: for the de Rham sheaves this statement follows from Proposition 3.16 and the base change formula of the PD envelope (note that taking the PD envelope is a left adjoint functor, hence commutes with the colimit; in particular, it commutes with modulo p^n for any n), and for the crystalline period sheaf it follows from the definition (note that although the $\mathcal{O}\mathbb{A}_{inf}$ defined in Tan and Tong's work uses an uncompleted tensor of $w^{-1}(\mathcal{O}_{\mathscr{X}})$ and $\mathbb{A}_{inf}$ instead of the completed tensors we use here, the difference vanishes when we modulo any power of p and restrict to the basis of affinoid perfectoid objects).

Therefore, in both cases, we have natural isomorphisms modulo p^n for any n, and taking the inverse limit gives the result we want as all sheaves are p-adic completions of their modulo p^n versions.

The claim about matching Poincaré sequences follows by unwinding definitions. Indeed, we need to check that the maps ∇ defined in these two sequences agree, but since ∇ is linear over $\mathrm{dR}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_k} \cong \mathbb{A}_{\mathrm{crys}}$, it suffices to check that the ∇ agree on u_i which is the image of $T_i - S_i$ (notation from [TT19] and Example 3.7, respectively) by functoriality of the Poincaré sequence. One checks that in both cases their image under ∇ is $1 \otimes dT_i$. □

4. Rational theory

For the rest of this paper, we shall study a rational version of the previously derived de Rham complex. Let us spell out the setup by recalling the following notation: k is a p-adic field with ring of integers denoted by $\mathcal{O}_k$ and X is a separated[6] rigid space over k which we view as an adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$.

4.1 Affinoid construction

In this subsection we recall the construction of the analytic cotangent complex and give the construction of the analytic derived de Rham complex, for a map of Huber rings over a p-adic field k. For a detailed discussion of the analytic cotangent complex (for topological finite type algebras), we refer readers to [GR03, §§ 7.1–7.3].

[6] Just as Remark 3.20 suggests, we can remove the separatedness assumption in the end.

Let $f\colon (A, A^+) \to (B, B^+)$ be a map of complete Huber rings over k. Denote by $\mathcal{C}_{B/A}$ the filtered category of pairs (A_0, B_0), where A_0 and B_0 are rings of definition of (A, A^+) and (B, B^+) separately, such that $f(A_0) \subset B_0$.

Construction 4.1 (Analytic cotangent complex, affinoid). For each $(A_0, B_0) \in \mathcal{C}_{B/A}$, denote by $\mathbb{L}^{\mathrm{an}}_{B_0/A_0}$ the integral analytic cotangent complex of $A_0 \to B_0$ as in Construction 3.1. The *analytic cotangent complex of* $f\colon (A, A^+) \to (B, B^+)$, denoted by $\mathbb{L}^{\mathrm{an}}_{B/A}$, is defined as the filtered colimit

$$\mathbb{L}^{\mathrm{an}}_{B/A} \coloneqq \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right].$$

For the convenience of readers, let us list a few properties of the analytic cotangent complex for a morphism of rigid affinoid algebras obtained by Gabber and Romero.

Theorem 4.2. *Let $A \to B$ be a morphism of k-affinoid algebras. Then we have the following assertions.*

1. *$\mathbb{L}^{\mathrm{an}}_{B/A}$ is in $\mathscr{D}^{\le 0}(B)$ and is pseudo-coherent over B; see [GR03, Theorem 7.1.33(i)].*
2. *The zeroth cohomology of the analytic cotangent complex is given by the analytic relative differential, $\mathrm{H}_0(\mathbb{L}^{\mathrm{an}}_{B/A}) \simeq \Omega^{\mathrm{an}}_{B/A}$; see [GR03, Lemma 7.1.27(iii) and Equation (7.2.36)];*
3. *If $A \to B$ is smooth, then $\mathbb{L}^{\mathrm{an}}_{B/A} \simeq \Omega^{\mathrm{an}}_{B/A}[0]$; see [GR03, Theorem 7.2.42(ii)].*
4. *If $A \to B$ is surjective, then the analytic cotangent complex agrees with the classical cotangent complex, $\mathbb{L}_{B/A} \simeq \mathbb{L}^{\mathrm{an}}_{B/A}$; see [GR03, Lemma 7.2.46(ii)].*

Construction 4.3 (Analytic derived de Rham complex, affinoid). Let $f\colon (A, A^+) \to (B, B+)$ be a map of complete Huber rings over k. For each $(A_0, B_0) \in \mathcal{C}_{B/A}$, by Construction 3.1 we could define the integral analytic derived de Rham complex $\mathrm{dR}^{\mathrm{an}}_{B_0/A_0}$, as an object in $\mathrm{CAlg}(\mathrm{DF}(A_0))$. Then the *analytic derived de Rham complex* $\mathrm{dR}^{\mathrm{an}}_{B/A}$ of (B, B^+) over (A, A^+), as an object in $\mathrm{CAlg}(\mathrm{DF}(A))$, is defined to be the filtered colimit

$$\mathrm{dR}^{\mathrm{an}}_{B/A} \coloneqq \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathrm{dR}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right].$$

Moreover, the *(Hodge) completed analytic derived de Rham complex* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ of (B, B^+) over (A, A^+), as an object in $\mathrm{CAlg}(\widehat{\mathrm{DF}}(A))$, is defined as the derived filtered completion of $\mathrm{dR}^{\mathrm{an}}_{B/A}$.

By the construction, the graded pieces of the filtered complete A-complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ are given by

$$\begin{aligned}\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) &\cong \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \mathrm{Gr}^i(G(A_0,B_0)) \\ &\cong \operatorname*{colim}_{(A_0,B_0)\in\mathcal{C}_{B/A}} \left(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B_0/A_0}\left[\frac{1}{p}\right]\right)[-i] \\ &\cong (\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i], \end{aligned} \qquad (⚁)$$

due to the fact that the functor Gr^i preserves filtered colimits.

Remark 4.4 (Complexity of the construction). The two rational constructions above involve colimits among all rings of definitions and seem to be very complicated. A naive attempt would be to take the usual cotangent/derived de Rham complex of $A^+ \to B^+$, apply the derived p-adic completion and invert p (and do the filtered completion, in the derived de Rham complex

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case) directly. This would *not* give us the expected answer in general, which is essentially due to the possible existence of nilpotent elements in (A, A^+) and (B, B^+).

Take the map $(k, \mathcal{O}_k) \to (B, B^+)$ for $B = k\langle\epsilon\rangle/(\epsilon^2)$ as an example. Then a ring of definition B_0 of B could be $\mathcal{O}_k\langle\epsilon\rangle/(\epsilon^2)$, while there is only one open integral subring of B that contains $\mathcal{O}_k$, namely $\mathcal{O}_k \oplus k\cdot\epsilon$. In this case, it is easy to see that the derived p-completions of cotangent complexes $\mathbb{L}_{B^+/\mathcal{O}_k}$ and $\mathbb{L}_{B_0/\mathcal{O}_k}$ are different, and remain so after inverting p.

Remark 4.5 (Simplified construction for uniform Huber pairs). Assume both of the Huber pairs $(A, A^+) \to (B, B^+)$ are *uniform*; that is, the subrings of power bounded elements A° and B° are bounded in A and B separately. Then both A^+ and B^+ are rings of definition of A and B separately. In particular, Construction 4.1 and Construction 4.3 can be simplified as follows:

$$\begin{aligned}\mathbb{L}^{\mathrm{an}}_{B/A} &= \mathbb{L}^{\mathrm{an}}_{B^+/A^+}\left[\frac{1}{p}\right],\\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} &= \text{filtered completion of }\left((\text{derived } p\text{-completion of } \mathrm{dR}_{B^+/A^+})\left[\frac{1}{p}\right]\right),\end{aligned}$$

where we recall that $\mathbb{L}^{\mathrm{an}}_{B^+/A^+}$ is the derived p-completion of the classical cotangent complex $\mathbb{L}_{B^+/A^+}$, and dR_{B^+/A^+} is the classical derived de Rham complex of B^+/A^+, as in [BMS19, Examples 5.11–5.12].

Examples of uniform Huber pairs include reduced affinoid algebras over discrete-valued or algebraically closed non-Archimedean fields [FvdP04, Theorem 3.5.6], and perfectoid affinoid algebras [Sch12, Theorem 6.3].

An arithmetic example of the Hodge completed analytic derived de Rham complex has been worked out by Beilinson.

Example 4.6 [Bei12, Proposition 1.5]. We have a filtered isomorphism

$$B^+_{\mathrm{dR}} \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{\overline{\mathbb{Q}_p}/\mathbb{Q}_p}.$$

Next we work out a geometric example. Let us compute the Hodge completed analytic derived de Rham complex of a perfectoid torus over a rigid analytic torus. Following the notation in Example 3.6, let $R = \mathbb{Z}_p\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle$ and $R_\infty = \mathbb{Z}_p\langle T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}\rangle = R\langle S_1^{1/p^\infty}, \ldots, S_n^{1/p^\infty}\rangle/(T_i - S_i; 1 \le i \le n)$.

Example 4.7. Continue with Example 3.7. After inverting p and completing along Hodge filtrations, we see that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]}$ is given by the completion of $\mathbb{Q}_p\langle T_i^{\pm 1}, S_i^{1/p^\infty}\rangle$ along $\{T_i - S_i; 1 \le i \le n\}$. Here we use Remark 4.5 to relate $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty/R}$ and $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]}$. A more explicit presentation is

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{R_\infty[1/p]/R[1/p]} = \mathbb{Q}_p\langle S_1^{\pm 1/p^\infty}, \ldots, S_n^{\pm 1/p^\infty}\rangle[\![X_1, \ldots, X_n]\!]$$

via the change of variables $T_i = X_i + S_i$ (hence $T_i^{-1} = S_i^{-1}\cdot(1 + S_i^{-1}X_i)^{-1}$); cf. the notation before [Sch13, Proposition 6.10].

We need to understand the output of these constructions for general perfectoid affinoid algebras relative to affinoid algebras. The following tells us that in this situation, the Hodge completed analytic derived de Rham complex can be computed with any ring of definition inside the affinoid algebra.

LEMMA 4.8. *Let (A, A^+) be a topologically finite type complete Tate ring over $(k, \mathcal{O}_k)$, with $A_0 \subset A^+$ being a ring of definition. Let (B, B^+) be a perfectoid algebra over (A, A^+). Then we have the following assertions.*

(1) *The analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A} \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p]$.*
(2) *The Hodge completed analytic derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \cong \widehat{\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}}[1/p]$, where the latter is the Hodge completion of $\mathrm{dR}^{\mathrm{an}}_{B^+/A_0}[1/p]$.*

In the proof below we will show a stronger statement: the transition morphisms of the colimit process computing the left-hand side in Constructions 4.1 and 4.3 are all isomorphisms.

Proof. Let $A_0' \subset A^+$ be another ring of definition containing A_0. It suffices to show that $\mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p] \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}[1/p]$ and similarly for their Hodge completed analytic derived de Rham complexes. Since the Hodge completed analytic derived de Rham complexes of both sides are derived complete with respect to the Hodge filtration, whose graded pieces, by Equation (⚀), are derived wedge products of relevant analytic cotangent complexes, we see that the statement about Hodge completed analytic derived de Rham complexes follows from the statement about analytic cotangent complex.

To show $\mathbb{L}^{\mathrm{an}}_{B^+/A_0}[1/p] \cong \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}[1/p]$, we appeal to the fundamental triangle of (analytic) cotangent complexes:

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0} \otimes_{A_0'} B^+ \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}.$$

Here the tensor product does not need an extra p-completion as $\mathbb{L}_{A_0'/A_0}$ is pseudo-coherent; see [GR03, Theorem 7.1.33]. By [GR03, Theorem 7.2.42], the p-complete cotangent complex $\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}$ satisfies

$$\mathbb{L}^{\mathrm{an}}_{A_0'/A_0}\left[\frac{1}{p}\right] = \Omega^{1,\mathrm{an}}_{A_0'[1/p]/A_0[1/p]},$$

which vanishes as $A_0'[1/p]$ and $A_0[1/p]$ are both equal to A. Therefore the natural map

$$\mathbb{L}^{\mathrm{an}}_{B^+/A_0}\left[\frac{1}{p}\right] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0'}\left[\frac{1}{p}\right]$$

induced by $A_0 \to A_0'$ is a quasi-isomorphism. □

We can understand the associated graded algebra of analytic de Rham complexes of perfectoid affinoid algebras over affinoid algebras via the following Theorem 4.9. Let K be a perfectoid field extension of k that contains p^n-roots of unity for all $n \in \mathbb{N}$.

THEOREM 4.9. *Let (A, A^+) be a topologically finite type complete Tate ring over $(k, \mathcal{O}_k)$. Assume (B, B^+) is a perfectoid algebra containing both $(K, \mathcal{O}_K)$ and (A, A^+). Then the graded algebra $\mathrm{Gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ admits a natural graded quasi-isomorphism to the derived divided power algebra $\mathrm{L}\Gamma^*_B(\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}))$, where the first graded piece fits into a distinguished triangle:*

$$B(1) \longrightarrow \mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathbb{L}^{\mathrm{an}}_{B/A}[-1] \longrightarrow B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k},$$

which is functorial in $(B, B^+)/(A, A^+)$. In particular, the graded pieces are B-pseudo-coherent.

Here $B(1)$ denote $\ker(\theta)/\ker(\theta)^2$, where $\theta\colon A_{inf}(B^+)[1/p] \twoheadrightarrow B$ is Fontaine's θ map. Our assumption of (B, B^+) containing $(K, \mathcal{O}_K)$ ensures that this is (non-canonically) isomorphic to B itself; see [Sch13, Lemma 6.3]. After sheafifying everything, it corresponds to a suitable Tate twist of B.

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Proof. The identification $\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathbb{L}^{\mathrm{an}}_{B/A}[-1]$ is already spelled out by (⊡).

Let us fix a single choice of pair of rings of definition (A_0, B^+) in $\mathcal{C}_{B/A}$. Here A_0 is topologically finitely presented over $\mathcal{O}_k$, and B^+ contains $\mathcal{O}_K$ for K a perfectoid field containing all p^nth roots of unity.

Consider the triple $\mathcal{O}_k \longrightarrow A_0 \longrightarrow B^+$, which induces the triangle

$$\mathbb{L}^{\mathrm{an}}_{A_0/\mathcal{O}_k} \otimes_{A_0} B^+ \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k} \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/A_0}.$$

Here again we have used the pseudo-coherence [GR03, Theorem 7.1.33] of $\mathbb{L}^{\mathrm{an}}_{A_0/\mathcal{O}_k}$. We need to show $\mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k}[1/p] \cong B(1)[1]$. To that end, let W be the Witt ring of the residue field of $\mathcal{O}_k$. By looking at the triple $W \to \mathcal{O}_k \to B^+$, we get another sequence

$$\mathbb{L}^{\mathrm{an}}_{B^+/W} \cong B^+(1)[1] \longrightarrow \mathbb{L}^{\mathrm{an}}_{B^+/\mathcal{O}_k} \longrightarrow \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W} \otimes_{\mathcal{O}_k} B^+[1],$$

where the first identification follows from Proposition 3.16, and the tensor product does not need an extra completion again by coherence of $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W}$. Since $k/W[1/p]$ is finite étale, we conclude that $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_k/W}[1/p] = 0$ by [GR03, Theorem 7.2.42]. This ends the proof of the structure of $\mathbb{L}^{\mathrm{an}}_{B/A}$.

Now we turn to the higher graded piece. The ith graded piece $\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ is quasi-isomorphic to $(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i]$, which by rewriting in terms of the first graded piece is

$$(\mathrm{L}\wedge^i (\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})[1]))[-i].$$

So by the relation between the derived wedge product and the derived divided power functor (with bounded above input; see [Ill71, V.4.3.5]), we get

$$\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathrm{L}\Gamma^i_B(\mathrm{Gr}^1(\widehat{\mathrm{dR}}_{B/A})),$$

and we get the divided power algebra structure of the graded algebra $\mathrm{Gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$. □

Consequently we get cohomological bounds for perfectoid affinoid algebras over various types of affinoid algebras. The notion of local complete intersection and embedded codimension (in the situation that we are working with) is discussed in Appendix A.

Corollary 4.10. *Let $(B, B^+)/(A, A^+)$ be as in the statement of Theorem 4.9. Then we have:*

(1) $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathscr{D}^{\leq 0}(A)$;
(2) *if A/k is smooth, then* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathscr{D}^{[0,0]}(A)$;
(3) *if A/k is local complete intersection with embedded codimension c, then* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \in \mathscr{D}^{[-c,0]}(A)$.

Proof. Since the output of $\widehat{\mathrm{dR}}^{\mathrm{an}}$ is always derived complete with respect to its Hodge filtration, it suffices to show these statements for the graded pieces of Hodge filtration.

For (1), this follows from the fact that $\mathbb{L}^{\mathrm{an}}_{B/A} \in \mathscr{D}^{\leq 0}(B)$. Statement (2) follows from (3) as a smooth affinoid algebra has embedded codimension 0.

As for (3), we check that the graded pieces of Hodge filtration in this case are in $\mathscr{D}^{[-c,0]}$. In fact, we shall show that the graded pieces, as objects in $\mathscr{D}(B)$, have Tor amplitude $[-c, 0]$. Since B contains $\mathbb{Q}$, we have

$$\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}) \cong \mathrm{L}\Gamma^i_B(\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})) \cong \mathrm{LSym}^i_B(\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})).$$

Using the triangle in Theorem 4.9, it suffices to show that $\mathrm{LSym}^j_B(B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k})$ have Tor amplitude $[-c, 0]$ for all j. Since $\mathrm{LSym}^j_B(B \otimes_A \mathbb{L}^{\mathrm{an}}_{A/k}) \cong B \otimes_A \mathrm{LSym}^j_A(\mathbb{L}^{\mathrm{an}}_{A/k})$, we are done by Proposition A.7. □

4.2 Poincaré sequence

In this subsection we explain the Poincaré sequence for Hodge completed de Rham complexes.

LEMMA 4.11. *Let $B \to C$ be an A-algebra morphism. Then for every $j \in \mathbb{N}$, the Katz–Oda filtration on $\mathrm{dR}_{C/A}$ induces a functorial strict exact filtration on $\mathrm{dR}_{C/A}/\operatorname{Fil}^j_{\mathrm{H}}$, witnessing the following sequence:*

$$\mathrm{dR}_{C/A}/\operatorname{Fil}^j \to \mathrm{dR}_{C/B}/\operatorname{Fil}^j \xrightarrow{\nabla} \mathrm{dR}_{C/B}/\operatorname{Fil}^{j-1} \otimes_B \mathrm{st}_1(\mathbb{L}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}_{C/B}/\operatorname{Fil}^1 \otimes_B \mathrm{st}_{j-1}(\mathrm{L}\wedge^{j-1}\mathbb{L}_{B/A}).$$

Here $\mathrm{dR}_{C/A}$ and $\mathrm{dR}_{C/B}$ are equipped with Hodge filtrations.

Moreover, $\operatorname{Fil}^i_{\mathrm{KO}}(\mathrm{dR}_{C/A}/\operatorname{Fil}^j_{\mathrm{H}}) = 0$ whenever $i > j$.

Proof. We consider the induced Katz–Oda filtration on $\mathrm{dR}_{C/A}/\operatorname{Fil}^i_{\mathrm{H}}$. Since we have modded out Hodge filtration, Lemma 3.13(3) implies the desired vanishing of the $\operatorname{Fil}^i_{\mathrm{KO}}$ when $i > j$, and this in turn implies the strict exactness of these filtrations. □

Specializing to the p-adic situation, we get the following lemma.

LEMMA 4.12. *Let $(A, A^+) \to (B, B^+) \to (C, C^+)$ be a triangle of complete Huber rings over k. Then for each $j \in \mathbb{N}$, we have a functorial strict exact filtration on $\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j$, still denoted by $\operatorname{Fil}^i_{\mathrm{KO}}$, witnessing the following sequence:*

$$\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j \to \mathrm{dR}^{\mathrm{an}}_{C/B}/\operatorname{Fil}^j \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B}/\operatorname{Fil}^{j-1} \widehat{\otimes}_B \mathrm{st}_1(\mathbb{L}^{\mathrm{an}}_{B/A}) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{C/B}/\operatorname{Fil}^1 \widehat{\otimes}_B \mathrm{st}_{j-1}(\mathrm{L}\wedge^{j-1}\mathbb{L}^{\mathrm{an}}_{B/A}).$$

Here $\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j$ and $\mathrm{dR}^{\mathrm{an}}_{C/B}/\operatorname{Fil}^j$ are equipped with Hodge filtrations.

Moreover, $\operatorname{Fil}^i_{\mathrm{KO}}(\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j) = 0$ whenever $i > j$.

Proof. For any triangle of rings of definition $A_0 \to B_0 \to C_0$, we p-complete the filtration from Lemma 4.11 and invert p, then we take the colimit over all triangles of such triples of rings of definition to get the required filtration. Since all the operations involved are (derived) exact, the resulting filtration still vanishes : $\operatorname{Fil}^i_{\mathrm{KO}} = 0$ whenever $i > j$, and this again implies strict exactness. □

In the setting of the above lemma, after taking the limit as j goes to ∞, we get the following corollary.

COROLLARY 4.13 (Poincaré lemma). *Let $(A, A^+) \to (B, B^+) \to (C, C^+)$ be a triangle of complete Huber rings over k. Then there is a functorial strict exact filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}$ witnessing the sequence*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B} \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B} \hat{\otimes}_B \mathrm{st}_1(\mathbb{L}^{\mathrm{an}}_{B/A}) \to \cdots . \qquad (⚁)$$

The maps ∇ are $\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}$-linear and satisfy the Newton–Leibniz rule.

Proof. Taking the limit in j of the Katz–Oda filtrations on $\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j$ in Lemma 4.12 gives the desired filtration. Indeed, the inverse limit of complete filtrations is again complete. Moreover, we have

$$\operatorname{Gr}^i_{\mathrm{KO}}(\widehat{\mathrm{dR}}^{\mathrm{an}}_{C/A}) \cong \lim_j \operatorname{Gr}^i_{\mathrm{KO}}(\mathrm{dR}^{\mathrm{an}}_{C/A}/\operatorname{Fil}^j) \cong \lim_j \left(\mathrm{dR}^{\mathrm{an}}_{C/B}/\operatorname{Fil}^{j-i} \widehat{\otimes}_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i]\right) \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{C/B} \hat{\otimes}_B \mathrm{st}_i(\mathrm{L}\wedge^i \mathbb{L}^{\mathrm{an}}_{B/A})[-i],$$

so we get the statement about the sequence that this filtration witnesses.

Finally, the statement about ∇ is the consequence of a general statement about multiplicative filtrations on E_∞-algebras; see the proof of Lemma 3.13(2). □

Remark 4.14. In fact, the discussion of the Poincaré sequence above could be obtained via a product formula,

$$\widehat{\mathrm{dR}}_{C/A}\widehat{\otimes}_{\widehat{\mathrm{dR}}_{B/A}} B \cong \widehat{\mathrm{dR}}_{C/B},$$

similar to the discussion in § 3.2. Here the formula can be obtained via a filtered completion, by p-completing the formula in Proposition 3.11 and inverting p.

We mention that this formula could also be proved by applying the symmetric monoidal functor Gr^* and checking the graded pieces, where the claim is reduced to the distinguished triangle of analytic cotangent complexes for a triple of Huber pairs.

4.3 Rational de Rham sheaves

In this subsection we apply the construction of the (Hodge completed) analytic derived de Rham complexes to the triangle of sheaves of Huber rings $(k, \mathcal{O}_k) \to (\nu^{-1}\mathcal{O}_X, \nu^{-1}\mathcal{O}_X^+) \to (\widehat{\mathcal{O}}_X, \widehat{\mathcal{O}}_X^+)$ on the pro-étale site, where $\nu\colon X_{\text{proét}} \to X$ is the standard map of sites. The procedure is similar to what we did in § 3.3, except now we allow X to be a locally complete intersection[7] over k, and we shall use the unfolding as discussed in § 2.4.

Let K be a perfectoid field extension of k that contains p^n-roots of unity for all $n \in \mathbb{N}$. There is a subcategory $X^{\omega}_{\text{proét}} \subset X_{\text{proét}}$ consisting of affinoid perfectoid objects $U = \mathrm{Spa}(B, B^+) \in X_{K,\text{proét}}$ whose image in X is contained in an affinoid open $\mathrm{Spa}(A, A^+) \subset X$. The class of such objects form a basis for the pro-étale topology by (the proof of) [Sch13, Proposition 4.8].

Proposition 4.15. *Let $U = \mathrm{Spa}(B, B^+) \in X^{\omega}_{\text{proét}}$, and choose $\mathrm{Spa}(A, A^+) \subset X$ such that the image of U in X is contained in $\mathrm{Spa}(A, A^+)$. Then:*

1. *the natural surjection $\theta\colon A_{inf}(B^+)[1/p] \twoheadrightarrow B$ exhibits $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} = B^+_{\mathrm{dR}}(B)$, and the Hodge filtrations are identified with the $\ker(\theta)$-adic filtrations;*
2. *the presheaf defined by sending U to $\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k})$ is a hypersheaf;*
3. *the assignment sending U to $\mathrm{dR}^{\mathrm{an}}_{B/A}/\,\mathrm{Fil}^n$ is independent of the choice of $\mathrm{Spa}(A, A^+)$, hence so is the assignment sending U to $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$, which we denote by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$;*
4. *assuming X/k is a local complete intersection, the presheaf assigning U to $\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X})$ is a hypersheaf.*

Proof. Statements (1) and (3) follow from the same proof of Proposition 3.16(1) and (4), respectively.

Now we prove (2). The ith graded piece of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$ is isomorphic to $B(i)$ by Theorem 4.9 (with (A, A^+) there being $(k, \mathcal{O}_k)$). These are hypersheaves as they are supported in cohomological degree 0 and satisfy higher acyclicity by [Sch13, Lemma 4.10].

Finally, we turn to (4). The graded pieces of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$, by (2), are the same as those of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ for any choice of A. Notice that, by Theorem 4.9, the $\mathrm{Gr}^i(\mathrm{dR}^{\mathrm{an}}_{B/A})$ have a finite step filtration with graded pieces given by $(\mathrm{L}\wedge^j \mathbb{L}^{\mathrm{an}}_{A/k}) \otimes_A B(i-j)$. Since the hypersheaf property satisfies the two-out-of-three principle in a triangle, it suffices to show that the assignment sending

$$\mathrm{Spa}(B, B^+) = U \mapsto \big(\mathrm{L}\wedge^j \mathbb{L}^{\mathrm{an}}_{A/k}\big) \otimes_A B(i-j)$$

is a hypersheaf. This follows from the fact that $\mathbb{L}^{\mathrm{an}}_{A/k}$ is a perfect complex (as X is assumed to be a local complete intersection over k) and, again, that sending U to $B(m)$ is a hypersheaf for any $m \in \mathbb{Z}$. □

[7] See Appendix A for the notion of local complete intersection that we are using here.

In particular, Proposition 4.15 tells us that the presheaves given by

$$\mathrm{Spa}(B,B^+) = U \in X^{\omega}_{\text{proét}} \mapsto \begin{cases} \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}/\,\mathrm{Fil}^n \ \text{ or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \ \text{or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}/\,\mathrm{Fil}^n \ \text{ or} \\ \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X} \end{cases}$$

are all hypersheaves on $X^{\omega}_{\text{proét}}$ (assuming X/k is a local complete intersection for the latter two), using the fact that the hypersheaf property is preserved under taking limits, so we may unfold them to get a hypersheaf on $X_{\text{proét}}$.

The authors believe that the conclusion of Proposition 4.15(4) (or a variant) should still hold for general rigid spaces instead of only the local complete intersection ones. Hence we pose the following question.

Question 4.16. Given any rigid space X/k, is it true that the presheaf assigning U to $\mathrm{Gr}^i(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X})$ is always a hypersheaf?

The subtlety is that a pro-étale map of affinoid perfectoid algebras need not be flat.

We are now ready to define the hypersheaf version of the relative de Rham cohomology.

DEFINITION 4.17. The *Hodge completed analytic derived de Rham complex* of $X_{\text{proét}}$ over k, denoted by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$, is defined to be the unfolding of the hypersheaf on $X^{\omega}_{\text{proét}}$ whose value at $U = \mathrm{Spa}(B,B^+) \in X^{\omega}_{\text{proét}}$ is $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$.

Similarly, we define a filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ by unfolding the Hodge filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}$. Since values of unfolding are computed by derived limits, we see immediately that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is derived complete with respect to the filtration.

This construction is related to Scholze's period sheaf $\mathbb{B}^+_{\mathrm{dR}}$ (see [Sch13, Definition 6.1(ii)]) by the following proposition.

PROPOSITION 4.18. *On $X^{\omega}_{\text{proét}}$ we have a filtered isomorphism $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$ of hypersheaves. Consequently, the zeroth cohomology sheaf of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is identified with the sheaf $\mathbb{B}^+_{\mathrm{dR}}$ as filtered sheaves on $X_{\text{proét}}$.*

Before the proof, we want to mention that under the equivalence $\mathscr{D}(X,k) \cong \mathrm{Sh}^{\mathrm{hyp}}(X,k)$ and its filtered version (cf. Remark 2.1), this proposition implies that the derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ is represented by the ordinary sheaf $\mathbb{B}^+_{\mathrm{dR}}$. Here the induced filtration on $\mathscr{H}^0(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k})$ is given by $\mathscr{H}^0(\mathrm{Fil}^* \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k})$.

Proof. The first sentence follows from Proposition 4.15(1).

Given a hypersheaf F supported in cohomological degree 0 on a basis of a site S, it also defines an ordinary sheaf on S (by taking the zeroth cohomology). The unfolding of F is a hypersheaf in $\mathscr{D}^{\geq 0}$, and its zeroth cohomological sheaf is the associated ordinary sheaf.

In our situation, we have the basis $X^{\omega}_{\text{proét}}$ of the site $X_{\text{proét}}$, and Scholze's $\mathbb{B}^+_{\mathrm{dR}}$ (and its filtrations) are defined as the ordinary sheaf obtained from $\mathbb{B}^+_{\mathrm{dR}}(\hat{\mathcal{O}}^+_X)$ (and its $\ker(\theta)$-adic filtrations). Now the previous paragraph and the first statement give us the second statement. $\square$

DEFINITION 4.19. Let X be a local complete intersection rigid space over k. Then the *Hodge completed analytic derived de Rham complex* of $X_{\text{proét}}$ over X, denoted by $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$, is defined

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to be the unfolding of the hypersheaf on $X^{\omega}_{\text{proét}}$ whose value at $U = \operatorname{Spa}(B, B^+) \in X^{\omega}_{\text{proét}}$ is $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$.

Similarly, we define a filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ by unfolding the Hodge filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/X}$. So $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ is also derived complete with respect to the filtration.

If X is a local complete intersection rigid space over k with embedded codimension c, then by Corollary 4.10(3) we see that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ lives in $\mathrm{Sh}^{\mathrm{hyp}}(X_{\text{proét}}, \mathscr{D}^{\geq -c}(k))$.

The Poincaré lemma obtained in the previous subsection now immediately yields the following theorem.

Theorem 4.20. *Let X be a local complete intersection rigid space over k. Then there is a functorial strict exact filtration on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$ witnessing the following sequence:*

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_1(\nu^{-1}(\mathbb{L}^{\mathrm{an}}_{X/k})) \xrightarrow{\nabla} \cdots.$$

If X is further assumed to be smooth over k of equidimension d, then the $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k}$-linear sequence

$$0 \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \xrightarrow{\nabla} \mathrm{dR}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_1(\nu^{-1}(\mathbb{L}^{\mathrm{an}}_{X/k})) \xrightarrow{\nabla} \cdots$$
$$\cdots \xrightarrow{\nabla} \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \otimes_{\nu^{-1}\mathcal{O}_X} \mathrm{st}_d(\nu^{-1}(\mathrm{L} \wedge^d \mathbb{L}^{\mathrm{an}}_{X/k})) \to 0$$

is strict exact.

Note that as X/k is assumed to be a local complete intersection, these wedge powers of the analytic cotangent complex are (locally) perfect complexes, hence the completed tensor is the same as just a tensor.

Proof. Since both unfolding and taking Gr^i commute with taking limits, the above follows from unfolding Corollary 4.13, and the fact that the completed tensor in Corollary 4.13 is the same as the tensor for local complete intersections X/k.

When X is smooth over k, everything in sight (on the basis of affinoid perfectoids in $X^{\omega}_{\text{proét}}$) is supported cohomologically in degree 0 with filtrations given by submodules because of Theorem 4.9, Corollary 4.10, and Proposition 4.15, and the strict exact Katz–Oda filtration gives what we want. ☐

4.4 Comparison with Scholze's de Rham period sheaf

In this subsection we show that when X is smooth, the de Rham sheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ defined above is related to Scholze's de Rham period sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$. We refer readers to [Sch16, part (3)] for the its definition. Following the notation of [Sch16] , let $\operatorname{Spa}(R_i, R_i^+)$ be an affinoid perfectoid in $X_{\text{proét}}$ with $\operatorname{Spa}(R_0, R^+_))$ an affinoid open in X. Then for any i, we have maps

$$R_i^+ \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+} \quad \text{and} \quad \mathbb{A}_{inf}(R^+) = \mathrm{dR}^{\mathrm{an}}_{R^+/W(\kappa)} \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+},$$

which is compatible with maps to R^+; here κ denotes the residue field of k. The equality above is deduced from Theorem 3.4(1). Therefore we get an induced map

$$R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{inf}(R^+) \to \mathrm{dR}^{\mathrm{an}}_{R^+/R_i^+} \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i}.$$

Taking the composition map above, inverting p and completing along the kernel of the surjection onto R (note that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i}$ lives in cohomological degree 0 by Corollary 4.10(2) and is already

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complete with respect to this filtration), we get a natural arrow

$$\big((R_i^+\hat{\otimes}_{W(\kappa)}\mathbb{A}_{inf}(R^+))[1/p]\big)^{\wedge} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_i} \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{R/R_0};$$

here we apply Corollary 4.13 to $(R_0, R_0^+) \to (R_i, R_i^+) \to (R, R^+)$ to see the filtered isomorphism above. This arrow is compatible with index i, hence after taking the colimit, we get the following map of sheaves on $X^{\omega}_{\mathrm{pro\acute{e}t}}$ (see the discussion before Proposition 4.15 for the meaning of $X^{\omega}_{\mathrm{pro\acute{e}t}}$):

$$f\colon \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}\,|_{X^{\omega}_{\mathrm{pro\acute{e}t}}} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X},$$

which is compatible with maps to $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$ and maps from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$.

THEOREM 4.21. *The map f above induces a filtered isomorphism of sheaves on $X^{\omega}_{\mathrm{pro\acute{e}t}}$. Hence we get that $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ is the zeroth cohomology sheaf of the hypersheaf $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$ on $X_{\mathrm{pro\acute{e}t}}$.*

Similar to Proposition 4.18, under the equivalence $\mathscr{D}(X,k) \cong \mathrm{Sh}^{\mathrm{hyp}}(X,k)$ and its filtered version (cf. Remark 2.1), this theorem implies that the derived de Rham complex $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$ is represented by the ordinary sheaf $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$.

Proof. The second sentence follows from the first sentence, due to the same argument in the proof of the second statement of Proposition 4.18. So it suffices to show the first sentence.

On both sheaves there are natural filtrations: on $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ we have the $\ker(\theta)$-adic filtration where $\theta\colon \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \to \hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$, and on $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$ we have the Hodge filtration with the first Hodge filtration being the kernel of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X} \twoheadrightarrow \hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$. Since f is compatible with maps to $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$ and the Hodge filtration is multiplicative, it suffices to show that f induces an isomorphism on their graded pieces. Now locally on $X^{\omega}_{\mathrm{pro\acute{e}t}}$, we have that $\mathrm{Gr}^*(\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}) \cong \mathrm{Sym}^*_{\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}}(\mathrm{Gr}^1\,\mathcal{O}\mathbb{B}^+_{\mathrm{dR}})$ by [Sch13, Proposition 6.10], and similarly $\mathrm{Gr}^*(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}) \cong \mathrm{Sym}^*_{\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}}(\mathrm{Gr}^1\,\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X})$ by Theorem 4.9 (note that in characteristic 0 divided powers are the same as symmetric powers). Therefore we are reduced to showing that f induces an isomorphism on the first graded pieces. Their first graded pieces admit a common submodule given by the first graded pieces of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$, which are $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}(1)$.

Now we get the diagram

$$\begin{array}{ccccc}
\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}(1) & \longrightarrow & \mathrm{Gr}^1\,\mathcal{O}\mathbb{B}^+_{\mathrm{dR}} & \longrightarrow & \hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega^{\mathrm{an}}_X \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{Gr}^1 f} & & \downarrow{\scriptstyle g} \\
\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}(1) & \longrightarrow & \mathrm{Gr}^1\,\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X} & \longrightarrow & \hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega^{\mathrm{an}}_X
\end{array}$$

with both rows being short exact (by [Sch13, Corollary 6.14] and Theorem 4.9, respectively) and the left vertical arrow being an isomorphism as f is compatible with the maps from $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$, which is why we get the induced arrow g. Moreover, f is linear over $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/k} \simeq \mathbb{B}^+_{\mathrm{dR}}$, which implies that g is linear over $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$. Therefore it suffices to show that g induces an isomorphism.

As the statement is étale local, we may assume that $X = \mathbb{T}^n = \mathrm{Spa}(k\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle, \mathcal{O}_k\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle)$. Denote by $\mathbb{T}^n_{\infty}$ the pro-finite-étale tower above $\mathbb{T}^n$ given by adjoining p-power roots of the coordinates T_i. We have the following diagram:

$$\begin{array}{ccc} \mathbb{Z}_p\langle T_i^{\pm 1}, S_i^{1/p^\infty}\rangle = \mathbb{Z}_p\langle T_i^{\pm 1}\rangle \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p\langle S_i^{1/p^\infty}\rangle & \xrightarrow{\alpha} & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ |_{\mathbb{T}^n_\infty} \\ \downarrow \beta & & \downarrow f \\ \mathbb{Q}_p\langle S_i^{\pm 1/p^\infty}\rangle[\![X_i]\!] & \xrightarrow{\gamma} & \widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X} |_{\mathbb{T}^n_\infty} \end{array}$$

Here the arrow β is given by sending T_i to $X_i + S_i$, and S_i is sent to $1 \otimes [T_i^\flat]$ under α. The element $\alpha(T_i - S_i)$ is $u_i \in \mathrm{Fil}^1\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ whose image in $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{an}}$ is $1 \otimes dT_i$; see the discussion before [Sch13, Proposition 6.10]. On the other hand, the element $\beta(T_i - S_i)$ is X_i, and the image of $\gamma(X_i)$ in $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{an}}$ is also $1 \otimes dT_i$ by Examples 4.7, 3.6 and 3.7. Therefore we get that $g(1 \otimes dT_i) = 1 \otimes dT_i$, since g is linear over $\hat{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}}$ and Ω_X^{an} is generated by the dT_i, we see that g is an isomorphism, and the proof is complete. □

Remark 4.22. In the process of the proof above, we also see that under the identification in Proposition 4.18 and Theorem 4.21, the Poincaré sequence obtained in Theorem 4.20 matches the one in Scholze's paper [Sch13, Corollary 6.13]; cf. proof of the second statement of Theorem 3.21.

Also the Faltings extension (see [Sch13, Corollary 6.14] and Theorem 4.9), being the first graded pieces of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \cong \mathscr{H}^0(\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X})$, is matched. In some sense, our proof above reduces to identifying the Faltings extension, and this is a well-known fact to experts. In fact, this project was initiated after Bhargav Bhatt explained to us how to get the Faltings extension from the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_{\mathrm{pro\acute{e}t}}/X}$.

4.5 An example

In this complementary subsection, we would like to compute the Hodge completed analytic derived de Rham complex of a perfectoid algebra over a zero-dimensional k-affinoid algebra. Surprisingly, the underlying algebra (forgetting its filtration) one gets always lives in cohomological degree 0, which leads us to Question 4.25 below.

Without loss of generality, let (K, K^+) be a perfectoid field over k, containing all p-power roots of unity, and let A be an Artinian local finite k-algebra with residue field being k as well. Let (B, B^+) be a perfectoid affinoid algebra containing (K, K^+) and let $A \to B$ be a morphism of k-algebras. Since perfectoid affinoid algebras are reduced, we get a sequence of maps $k \to A \to k \to B$.

By the above sequence, we get natural filtered k-linear maps

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \quad \text{and} \quad \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

This induces a filtered map,

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \otimes_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A},$$

where the filtration on the source comes from the symmetric monoidal structure on $\mathrm{DF}(k)$. Since this map is compatible with the filtration and the target is complete with respect to its filtration, we get an induced map

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

Proposition 4.23. *The map* $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \hat{\otimes}_k \widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}$ *above is a filtered isomorphism.*

Proof. Since both are complete with respect to their filtrations, it suffices to show that the map induces an isomorphism on the graded pieces. The graded algebras of both sides are

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the symmetric algebras (over B) on their first graded pieces, hence it suffices to check that $\mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}\hat{\otimes}_k\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}) \longrightarrow \mathrm{Gr}^1(\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A})$ is an isomorphism. This follows from the decomposition of analytic cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{B/A} \cong \mathbb{L}^{\mathrm{an}}_{B/k} \oplus (\mathbb{L}^{\mathrm{an}}_{A/k} \otimes_A B),$$

which is deduced from contemplating the sequence $k \to A \to k \to B$. □

We know that $\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k} \cong \mathbb{B}^+_{\mathrm{dR}}(B)$, and a result of Bhatt tells us the underlying algebra of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong A$, explained in below. Since $A \to k$ is a surjection, the analytic cotangent complex agrees with the classical cotangent complex, hence we have a filtered isomorphism

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \longrightarrow \widehat{\mathrm{dR}}_{k/A}.$$

Now [Bha12a, Theorem 4.10] implies that the underlying algebra $\widehat{\mathrm{dR}}_{k/A}$ is isomorphic to the completion of A along the surjection $A \to k$. Since A is an Artinian local ring, this completion is simply A itself. Therefore we get a map of the underlying algebras:

$$\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}\hat{\otimes}_k\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}.$$

PROPOSITION 4.24. *The map $\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \longrightarrow \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}\hat{\otimes}_k\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A}$ above is an isomorphism. Consequently we have an isomorphism*

$$\mathbb{B}^+_{\mathrm{dR}}(B) \otimes_k A \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{B/A}.$$

Proof. By definition, we have

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}\hat{\otimes}_k\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \lim_n \lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m,$$

where we have used the (filtered) identification $\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \widehat{\mathrm{dR}}_{k/A}$ spelled out before this proposition.

We claim that for any given n, we have an isomorphism

$$\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A \cong \lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m.$$

Indeed for each $i \in \mathbb{Z}$, we have the following short exact sequence:

$$0 \longrightarrow \mathrm{R}^1\lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\right) \longrightarrow$$
$$\mathrm{H}^i(\lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m\right)) \longrightarrow \lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^i(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\right) \longrightarrow 0.$$

Since for each m and i, the vector space $\mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)$ is finite-dimensional over k, we see that the inverse system $\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{H}^{i-1}(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)$ satisfies the Mittag-Leffler condition, hence the $\mathrm{R}^1\lim$ term vanishes. By [Bha12a, Theorem 4.10], we have that the inverse system $\{\mathrm{H}^i(\mathrm{dR}_{k/A}/\mathrm{Fil}^m)\}_m$ is pro-isomorphic to 0 if $i \neq 0$ and is pro-isomorphic to A (since A is finite-dimensional over k) if $i = 0$, therefore the above short exact sequence becomes

$$\mathrm{H}^i\big(\lim_m \left(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\mathrm{Fil}^m\right)\big) \cong \begin{cases} 0, & i \neq 0, \\ \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A, & i = 0. \end{cases}$$

This gives us the claim above.

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Now we have

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{B/k}\hat{\otimes}_k\widehat{\mathrm{dR}}^{\mathrm{an}}_{k/A} \cong \lim_n(\lim_m \mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k \mathrm{dR}_{k/A}/\operatorname{Fil}^m) \cong \lim_n(\mathbb{B}^+_{\mathrm{dR}}(B)/(\xi)^n \otimes_k A) \cong \mathbb{B}^+_{\mathrm{dR}}(B)\otimes_k A$$

as desired, where the last identification follows from the fact that A is finite over k. □

If one contemplates the example $A = k[\epsilon]/(\epsilon^2)$, one sees that $\mathrm{dR}^{\mathrm{an}}_{B/A}/\operatorname{Fil}^i$ does not live in cohomological degree 0 alone for any $i \geq 2$.

As a consequence of the above proposition, for the $X = \operatorname{Spa}(A)$ we have an equality of presheaves on $X^{\omega}_{\text{proét}}$,

$$\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X} \cong \mathbb{B}^+_{\mathrm{dR}} \otimes_k \nu^{-1}\mathcal{O}_X;$$

in particular, the underlying algebra of $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}$ pro-étale locally lives in cohomological degree 0. Motivated by this computation and results in [Bha12a], we end this paper by posing the following question.

Question 4.25. In what generality should we expect $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X_{\text{proét}}/X}|_{X^{\omega}_{\text{proét}}}$ to live in cohomological degree 0? And when that happens, can we reinterpret the underlying algebra via some construction similar to Scholze's $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ as in [Sch13, Sch16]?

Acknowledgements

We are grateful to Bhargav Bhatt for suggesting this project to us and for many discussions related to it. We thank David Hansen heartily for listening to our project, drawing our attention to Tan and Tong's period sheaves, and sharing excitement with the second named author. The second named author would also like to thank Johan de Jong for discussions around the notion of local complete intersections in rigid geometry and suggesting the proof of a lemma. The first named author is partially funded by Department of Mathematics, University of Michigan, and by NSF grant DMS 1801689 through Bhargav Bhatt.

Appendix A. Local complete intersections in rigid geometry

In this appendix we discuss the notion of local complete intersection morphisms in rigid geometry. We remark that the results recorded here hold verbatim with k being a general complete non-Archimedean field.

In order to talk about local complete intersections, we need to understand how being of finite Tor dimension[8] behaves under base change in rigid geometry.

Lemma A.1. *Let A and B be two affinoid k-algebras, and $A \to B$ a morphism of Tor dimension m. Let $P := A\langle T_1, \ldots, T_n\rangle \twoheadrightarrow B$ be a surjection. Then we have*

$$\operatorname{Tor}\dim_P(B) \leq m + n.$$

The following proof was suggested to us by Johan de Jong.

Proof. Choose a resolution of B by finite free P-modules

$$\cdots \xrightarrow{d_i} M_i \xrightarrow{d_{i-1}} M_{i-1} \cdots \xrightarrow{d_0} M_0 \twoheadrightarrow B.$$

Since P is flat over A, we see that $M := \operatorname{Coker}(d_m)$ is flat over A as $A \to B$ is assumed to be of Tor dimension m [Sta20, Tag 0653]. Moreover, M is finitely generated over P since P

[8] In classical literature such as [Avr99] this corresponds to the notion of having finite flat dimension.

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is Noetherian. Now we use [Li20, Lemma 6.3] to see that M admits a projective resolution over P of length n. Therefore we get that B has a projective resolution over P of length $m+n$. □

LEMMA A.2. *Let A and B be two affinoid k-algebras, and $A \to B$ a morphism of finite Tor dimension. Let C be any affinoid A-algebra. Then the base change (in the realm of rigid geometry) $C \to B\hat{\otimes}_A C$ is also of finite Tor dimension.*

Proof. Choose a surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$, which again is of finite Tor dimension by Lemma A.1. Then we have a factorization

$$C \to C\langle T_1, \dots, T_n\rangle \to B \otimes_{A\langle T_1,\dots,T_n\rangle} C\langle T_1, \dots, T_n\rangle \cong B\hat{\otimes}_A C.$$

Since the first arrow is flat and the second arrow, being the base change of an arrow of finite Tor dimension, is of finite Tor dimension, we conclude that the composition is of finite Tor dimension [Sta20, Tag 066J]. □

PROPOSITION A.3. *Let $A \to B$ a morphism of k-affinoid algebras. Then the following statements are equivalent.*

(1) *Any surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ is a local complete intersection.*
(2) *There exists a surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ which is a local complete intersection.*
(3) *$A \to B$ is of finite Tor dimension and the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a perfect B-complex.*

Moreover, any of these three equivalent conditions implies that $\mathbb{L}^{\mathrm{an}}_{B/A}$ is a perfect complex with Tor amplitude in $[-1, 0]$.

Proof. It is easy to see that (1) implies (2).

To see that (2) implies (3), first of all $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ being a local complete intersection implies that it is of finite Tor dimension. Since $A \to A\langle T_1, \dots, T_n\rangle$ is flat, we see that $A \to B$ is also of finite Tor dimension by [Sta20, Tag 0653]. Next we look at the triangle $A \to A\langle T_1, \dots, T_n\rangle \to B$, which gives rise to a triangle of analytic cotangent complexes:

$$\mathbb{L}^{\mathrm{an}}_{A\langle T_1,\dots,T_n\rangle/A} \otimes_A B \to \mathbb{L}^{\mathrm{an}}_{B/A} \to \mathbb{L}^{\mathrm{an}}_{B/A\langle T_1,\dots,T_n\rangle}.$$

Now Theorem 4.2(3) gives that the first term is a perfect complex with Tor amplitude in $[0, 0]$, while condition (2) and Theorem 4.2(4) imply that the third term is a perfect complex with Tor amplitude in $[-1, -1]$, hence we see that (2) implies (3) and gives the last sentence as well.

Finally, we need to show that (3) implies (1). To that end we apply Avramov's solution of Quillen's conjecture [Avr99]. As $A \to B$ is of finite Tor dimension, we see that any surjection $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ has finite Tor dimension by Lemma A.1. The previous paragraph shows that $\mathbb{L}^{\mathrm{an}}_{B/A}$ being a perfect complex is equivalent to the classical cotangent complex $\mathbb{L}_{B/A\langle T_1,\dots,T_n\rangle}$ being a perfect complex. Now we use Avramov's result [Avr99, Theorem 1.4] to conclude that $A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B$ is a local complete intersection. □

DEFINITION A.4. Let $A \to B$ be a morphism of k-affinoid algebras. The morphism $A \to B$ of k-affinoid algebras is called *a local complete intersection* if one of the three equivalent conditions in Proposition A.3 is satisfied.

Let $Y \to X$ be a morphism of rigid spaces over k. Then this morphism is called *a local complete intersection* if for any pair of affinoid domains U and V in X and Y, such that the image of V is contained in U, the induced map of k-affinoid algebras is a local complete intersection.

We leave it as an exercise (using Theorem 4.2) that a morphism being a local complete intersection may be checked locally on the source and target. We caution readers that there is

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a notion of local complete intersection morphism between Noetherian rings, while the notion we define here should (clearly) only be considered in the situation of rigid geometry. These two notions agree when the morphism considered is a surjection. We hope this slight abuse of this notion will not cause any confusion. But as a sanity check, let us show here that this notion matches the corresponding notion in classical algebraic geometry under rigid analytification. The following proposition was suggested to us by David Hansen.

Proposition A.5. *Let $f\colon X \to Y$ be a morphism of schemes locally of finite type over a k-affinoid algebra A with rigid analytification $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Then f is a local complete intersection (in the classical sense) if and only if f^{an} is a local complete intersection (in the sense of Definition A.4).*

Proof. We first reduce to the case where both of X and Y are affine. Then we may check this after taking the fiber product Y with an affine space so that f is a closed embedding. In this situation, we have an identification of ringed sites $X^{\mathrm{an}} \cong X \times_Y Y^{\mathrm{an}}$ and an identification of cotangent complexes:

$$\iota^*\mathbb{L}_{X/Y} \simeq \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}},$$

where $\iota\colon X^{\mathrm{an}} \to X$ is the natural map of ringed sites.

Now we use the fact that classical Tate points on X^{an} are in bijection with closed points on X, and for any such point x, the map $\iota^\sharp\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ of local rings is faithfully flat. Therefore we can check $\mathbb{L}_{X/Y}$ is perfect by pulling back along ι, hence $\mathbb{L}_{X/Y}$ is perfect if and only if $\mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}/Y^{\mathrm{an}}}$ is perfect, and this finishes the proof. □

Next we address the localization of analytic cotangent complexes for a local complete intersection morphism.

Let us introduce some notions.

Definition A.6. Let $A \to B$ be a morphism of k-affinoid algebras. Let $\mathfrak{m} \subset B$ be a maximal ideal. The *embedded dimension of B/A at $\mathfrak{m}$* is defined to be

$$\dim_{B/A,\mathfrak{m}} := \dim_{\kappa(\mathfrak{m})}(\Omega^{\mathrm{an}}_{B/A} \otimes_B B/\mathfrak{m}).$$

Let $\mathfrak{n}$ be the preimage of $\mathfrak{m}$ in A (which is also a maximal ideal). We define the *embedded codimension of B/A at $\mathfrak{m}$* to be

$$\dim_{B/A,\mathfrak{m}} + \dim(A_{\mathfrak{n}}) - \dim(B_{\mathfrak{m}}).$$

The *embedded codimension of B/A* is the supremum of that at all maximal ideals $\mathfrak{m} \subset B$.

Proposition A.7. *Let $A \to B$ be a local complete intersection morphism of k-affinoid algebras. Then at any maximal ideal $\mathfrak{m} \subset B$, there is a presentation of the analytic cotangent complex*

$$\mathbb{L}^{\mathrm{an}}_{B/A} \otimes_B B_{\mathfrak{m}} \simeq \left[B_{\mathfrak{m}}^{\oplus c(\mathfrak{m})} \to B_{\mathfrak{m}}^{\oplus d(\mathfrak{m})}\right],$$

where $c(\mathfrak{m})$ is the embedded codimension of B/A at $\mathfrak{m}$ and $d(\mathfrak{m})$ is the embedded dimension of B/A at $\mathfrak{m}$. Here $B_{\mathfrak{m}}^{\oplus d(\mathfrak{m})}$ is put in degree 0.

In particular, the Tor amplitude of $\mathrm{LSym}^i\mathbb{L}^{\mathrm{an}}_{B/A}$ is always in $[-\min\{c, i\}, 0]$ where c is the embedded codimension of B/A.

Proof. We may always replace B by a rational domain containing the point $\mathfrak{m}$ (viewed as a classical Tate point on the associated adic space), so we can assume there are power bounded elements $f_1, \ldots, f_{d(\mathfrak{m})}$ whose differentials generate the stalk of $\Omega^{\mathrm{an}}_{B/A}$ at $\mathfrak{m}$. Thus we have a map

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$A' := A\langle T_1, \dots, T_{d(\mathfrak{m})}\rangle \to B$ which is unramified at $\mathfrak{m}$; see [Hub96, § 1.6]. By [Hub96, Proposition 1.6.8] we can factorize the map $A' \to B$ as $A' \xrightarrow{h} C \xrightarrow{g} B$ where h is étale and g is surjective.

One checks that the étaleness of h guarantees that the surjection $C \xrightarrow{g} B$ has finite Tor dimension. Moreover, Theorem 4.2 implies that $\mathbb{L}_{B/C}$ is a perfect complex because of the triangle

$$\mathbb{L}^{\mathrm{an}}_{C/A} \otimes_C B \to \mathbb{L}^{\mathrm{an}}_{B/A} \to \mathbb{L}_{B/C}.$$

Hence $C \to B$ is a surjective local complete intersection. Hence the kernel of $C \to B$ around $\mathfrak{m}$ is generated by a regular sequence of length $c(\mathfrak{m})$. This in turn implies that $\mathbb{L}_{B/C} \otimes_B B_{\mathfrak{m}} \simeq B_{\mathfrak{m}}^{\oplus c(\mathfrak{m})}[1]$, which together with the triangle above gives the local presentation we want in the statement.

The statement concerning Tor amplitude can be checked at every maximal ideal which, by our presentation, follows from the formula $\mathrm{LSym}^i(C[1]) \simeq \mathrm{L} \wedge^i (C)[i]$; see [Ill71, V.4.3.4]. □

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