*Proceedings of the Royal Society of Edinburgh*, **143A**, 643–668, *2013*

# Global existence and non-existence for the degenerate and uniformly parabolic equations with gradient term

**Haifeng Shang**
School of Mathematics and Information Science,
Henan Polytechnic University, Jiaozuo 454000,
People's Republic of China
(hfshang@yahoo.cn)

(MS received 1 July 2011; accepted 27 March 2012)

We study the Cauchy problem for the degenerate and uniformly parabolic equations with gradient term. The local existence, global existence and non-existence of solutions are obtained. In the case of global solvability, we get the exact estimates of a solution. In particular, we obtain the global existence of solutions in the limiting case.

## 1. Introduction and statement of the main results

We will consider the non-negative solutions of the following Cauchy problem:

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = |Du^q|^\sigma \quad \text{in } S_T = \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N, \tag{1.2}$$

where $p \geqslant 2$, $q > 0$, $0 < \sigma < p$, $T > 0$, $N \geqslant 1$ and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is non-negative.

Concerning the case $p = 2$, the equation is typified as a viscous Hamilton–Jacobi equation. It has been proposed as an appropriate model for surface growth by ballistic deposition, and specifically for vapour deposition and the sputter deposition of thin films of aluminium and rare earth metals (see [24–27,31]). The problem (1.1), (1.2) has attracted much interest (see [1–4, 6–8, 10–16, 21, 23]) in recent years. In the earliest of these papers [6], the existence and uniqueness of a classical solution was proven under the assumption that $q = 1$, $\sigma = 1$ and $u_0 \in C^3_0(\mathbb{R}^N)$. In [4], the existence of a suitably defined weak solution, when $u_0$ is a Radon measure, was investigated. In [2], the existence of a unique classical solution was proven under the hypothesis that $q = 1$, $\sigma > 1$ and $u_0 \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)$. More recently, attention has focused on the questions of existence and uniqueness of suitably defined weak solutions when $q = 1$, $\sigma > 0$ and $u_0 \in L^p(\mathbb{R}^N)$, for some $1 \leqslant p < \infty$ or $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ or for when $u_0$ is a bounded measure (see [1, 9–12, 21, 22]).

For the case $p > 2$, with $q = 1$ and $1 \leqslant \sigma \leqslant p - 1$, the Cauchy problem (1.1), (1.2) has a local solution (see [28]) and the condition on initial data is that

$$\sup_{x \in \mathbb{R}^N} \int_{B_\rho(x)} u_0^h \, \mathrm{d}x < \infty,$$





https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

where $h \geqslant 1$ and $\rho$ is any fixed positive number. We remark that this optimal condition was first used in [5]. For $q\sigma \geqslant p-1$, the local existence of solutions was studied in [17, 30] with measures as initial data.

Here, for all $q > 0$ and $0 < \sigma < p$, we study the local existence, global existence and non-existence for the equation (1.1), with initial data in a suitable space $L^h_{\mathrm{loc}}(\mathbb{R}^N)(h \geqslant 1)$, motivated by the ideas in [4,5]. The results contained in this paper have the following new features. Firstly, we give the local existence of solutions.

(i) For $p = 2$, Andreucci [4] obtained the local existence of solutions of problem (1.1), (1.2) under optimal assumptions on the initial data for $q\sigma \geqslant 1$; here we generalize this result to the case $0 < q\sigma < 1$.

(ii) For $p > 2$, we generalize the local existence results in [28] to (1.1), (1.2).

Secondly, our interest is focused on the global existence and non-existence of solutions. Andreucci and Di Benedetto, in a fruitful paper [5], considered the Cauchy problem for the porous medium equation with strongly nonlinear source $u_t - \Delta u^m = u^\sigma$, where a Fujita-type result was obtained and the critical exponent was $\sigma_{\mathrm{c}} = m + 2/N$. Later, Zhao [32] (see also [28]) generalized the results in [5] to the evolution $p$-Laplacian equation $u_t - \mathrm{div}(|Du|^{p-2}Du) = u^\sigma$, where the critical exponent was $\sigma_{\mathrm{c}} = p - 1 + p/N$. Here, in the spirit of [5], we also obtain a Fujita-type result for (1.1), (1.2) and the critical exponent is

$$\sigma_{\mathrm{c}} = \frac{(N+1)p - N}{qN+1},$$

i.e.

$$\frac{N}{\kappa}\left(\frac{\sigma(pq-p+1)}{p-\sigma} - 1\right) = 1,$$

where the definition of $\kappa$ is as below. Moreover, the existence of local and global solutions in the limiting case

$$\frac{N}{\kappa_h}\left(\frac{\sigma(pq-p+1)}{p-\sigma} - 1\right) = 1$$

is also considered. The method here can be used to study the limiting cases $\sigma = m + 2h/N$ [5] and $\sigma = p - 1 + ph/N$ [32]. As far as we know, the results given in the present paper are new.

Definition 1.1. A non-negative measurable function $u(x,t)$ defined in $S_T$ is called a weak solution of (1.1), (1.2) if, for every bounded open set $\Omega$, with smooth boundary $\partial\Omega$,

$$u \in C_{\mathrm{loc}}(0,T; L^1(\Omega)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}(\Omega)) \cap L^\infty_{\mathrm{loc}}(S_T), \quad |Du^q|^\sigma \in L^1_{\mathrm{loc}}(S_T)$$

and

$$\int_\Omega u(x,t)\varphi(x,t)\,\mathrm{d}x + \int_s^t \int_\Omega [-u\varphi_\tau + |Du|^{p-2}DuD\varphi]\,\mathrm{d}x\,\mathrm{d}\tau = \int_\Omega u(x,s)\varphi(x,s)\,\mathrm{d}x + \int_s^t \int_\Omega |Du^q|^\sigma \varphi\,\mathrm{d}x\,\mathrm{d}\tau \quad (1.3)$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

for all $0 < s < t < T$ and all $\varphi \in W^{1,\infty}_{\mathrm{loc}}(0, T; L^\infty(\Omega)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_0(\Omega))$. Moreover,

$$\lim_{t\to 0} \int_{\mathcal{K}} |u(x,t) - u_0(x)| \, \mathrm{d}x = 0 \quad \forall \mathcal{K} \subset\subset \mathbb{R}^N. \tag{1.4}$$

Weak subsolutions (respectively supersolutions) are defined in the same way except that the '=' in (1.4) is replaced by '$\leqslant$' (respectively '$\geqslant$') and $\varphi$ is taken to be non-negative.

We make use of the norms in [5] and set

$$|||f|||_s \equiv \sup_{x\in\mathbb{R}^N} \|f\|_{s,B_\rho(x)},$$
$$\gamma^{-1} \sup_{x\in\mathbb{R}^N} \|f\|_{s,B_1(x)} \leqslant |||f|||_s \leqslant \gamma \sup_{x\in\mathbb{R}^N} \|f\|_{s,B_1(x)},$$

where $s \geqslant 1$, $f \in L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $\rho > 0$ and $\gamma = \gamma(N, s, \rho)$. We also define $\kappa_s = N(p-2) + ps$, $\kappa = \kappa_1 = N(p-2) + p$.

We use $\gamma(a_1, a_2, \dots, a_n)$ to denote positive constants depending only on specified quantities $a_1, a_2, \dots, a_n$.

First we state our main existence results as follows.

THEOREM 1.2. *Let* $|||u_0|||_1$ *be finite,* $0 < q\sigma < p - 1$ *and* $q > (p-1)/p$. *Then, there exists a solution to (1.1), (1.2) defined in* $\mathbb{R}^N \times (0, T_0)$, *where* $T_0 = T_0(N, p, q, \sigma)$, *such that for all* $0 < t < T_0$ *we have that*

$$|||u(\cdot,t)|||_1 \leqslant \gamma(|||u_0|||_1 + 1), \tag{1.5}$$

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \leqslant \gamma t^{-N/\kappa}(|||u_0|||_1^{p/\kappa} + 1), \tag{1.6}$$

*where* $\gamma = \gamma(N, p, q, \sigma)$, $\kappa = N(p-2) + p$.

THEOREM 1.3. *Let* $q\sigma \geqslant p - 1$ *and let* $|||u_0|||_h < \infty$ *with* $h \geqslant 1$ *satisfying*

$$\frac{N}{\kappa_h}\left(\frac{\sigma(pq - p + 1)}{p - \sigma} - 1\right) < 1, \tag{1.7}$$

*where* $\kappa_h = N(p-2) + ph$. *Then, there exists a constant* $\gamma = \gamma(N, p, q, \sigma, h)$ *and a positive time* $T_0' = T_0'(N, p, q, \sigma, h)$ *such that problem (1.1), (1.2) has a non-negative weak solution* $u$ *in the strip* $S_{T_0'}$, *satisfying*

$$|||u(\cdot,t)|||_h \leqslant \gamma |||u_0|||_h, \tag{1.8}$$

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \leqslant \gamma t^{-N/\kappa_h} |||u_0|||_h^{ph/\kappa_h} \tag{1.9}$$

*for all* $0 < t < T_0'$.

REMARK 1.4. The proofs in §§ 2 and 3 show that theorems 1.2 and 1.3 hold in the following two cases.

(1) $u_0$ can be of variable sign.

(2) If $h = 1$ is admissible in (1.7), $u_0$ can be replaced by a $\sigma$-finite Borel measure $\mu$ in $\mathbb{R}^N$, satisfying

$$|||\mu||| = \sup_{x\in\mathbb{R}^N} \mu(B_1(x)) < \infty.$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

REMARK 1.5. In [28], Lian *et al*. only considered the case of $p > 2$, $1 \leqslant \sigma \leqslant p-1$, and proved local existence of solutions with initial data in some $L^h_{\mathrm{loc}}(\mathbb{R}^N)$ $(h \geqslant 1)$. Here, we extend their results to the case of $q > 0$ and $0 < \sigma < p$.

In the limiting case, we have the following local existence result.

THEOREM 1.6. *Let*

$$\frac{N}{\kappa_h}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)=1, \qquad \frac{N}{\kappa}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)>1 \tag{1.10}$$

*and let*

$$\|u_0\|_{L^h(\mathbb{R}^N)} < \gamma_0, \tag{1.11}$$

*where $\gamma_0 = \gamma_0(N, p, q, \sigma, h)$ is sufficiently small. Then, there exists a positive time $T_0''$ such that the Cauchy problem (1.1), (1.2) has a solution $u$ in $S_{T_0''}$ and $u$ satisfies (1.8), (1.9) for all $0 < t < T_0''$.*

REMARK 1.7. The dependence of $T_0$, $T_0'$ and $T_0''$ on the quantities specified in the statement of theorems 1.2, 1.3 and 1.6 can be made explicit. We refer to the proofs of lemmata 2.3, 3.3 and theorem 1.6.

Lastly, we state the global existence and non-existence results.

THEOREM 1.8. *Let*

$$\frac{N}{\kappa}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)>1 \tag{1.12}$$

*and let*

$$\|u_0\|_{h,\mathbb{R}^N} + \|u_0\|_{1,\mathbb{R}^N} < \gamma_0, \tag{1.13}$$

*where $h > 1$ satisfies (1.7) and $\gamma_0 = \gamma_0(N, p, q, \sigma, h)$ is sufficiently small. Then, the Cauchy problem (1.1), (1.2) has a solution in $S_\infty = \mathbb{R}^N \times (0, \infty)$, and*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \leqslant \gamma t^{-N/\kappa}\left(\sup_{0<\tau<t}\int_{\mathbb{R}^N} u(x,\tau)\,\mathrm{d}x\right)^{p/\kappa}, \tag{1.14}$$

$$\sup_{0<\tau<t}\int_{\mathbb{R}^N} u(x,\tau)\,\mathrm{d}x \leqslant \gamma\int_{\mathbb{R}^N} u_0\,\mathrm{d}x \tag{1.15}$$

*for all $t \in (0, \infty)$.*

THEOREM 1.9. *Let*

$$\frac{N}{\kappa_h}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)=1, \qquad \frac{N}{\kappa}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)>1 \tag{1.16}$$

*and let*

$$\|u_0\|_{h,\mathbb{R}^N} + \|u_0\|_{1,\mathbb{R}^N} < \gamma_0, \tag{1.17}$$

*where $\gamma_0 = \gamma_0(N, p, q, \sigma, h)$ is sufficiently small. Then, the Cauchy problem (1.1), (1.2) has a solution in $S_\infty = \mathbb{R}^N \times (0, \infty)$.*

THEOREM 1.10. *Let*

$$\frac{N}{\kappa}\left(\frac{\sigma(pq-p+1)}{p-\sigma}-1\right)<1. \tag{1.18}$$

*Then there cannot exist a global non-trivial non-negative solution to the Cauchy problem (1.1), (1.2).*

This paper is structured as follows. In § 2, we give the proof of theorem 1.2. In § 3, we finish the proof of theorem 1.3. In § 4, we prove theorem 1.6. In § 5, we give the proofs of theorems 1.8 and 1.9 and in § 6, we give the proof of theorem 1.10.

## 2. *A priori* estimates and proof of theorem 1.2

In this section, we let $0<q\sigma<p-1$ and $q>p-1/p$.

Set

$$T^*=\min\{T,1\}.$$

First, we give the $L^\infty$ estimate.

LEMMA 2.1. *Let $u$ be a non-negative continuous weak subsolution of (1.1) in $S_{T^*}$. Assume that there exists a time $0<T'<T^*$ such that*

$$t\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{p-2}\leqslant 1 \quad \forall 0<t<T'. \tag{2.1}$$

*Then we have that*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}\leqslant \gamma t^{-N/\kappa}|||u(\cdot,t)|||_1^{p/\kappa}+t \quad \forall 0<t<T', \tag{2.2}$$

*where $\gamma=\gamma(N,p,q,\sigma)$, $\kappa=N(p-2)+p$.*

*Proof.* Let $\epsilon\in(0,\frac{1}{2})$ and $x_0\in\mathbb{R}^N$ be fixed. For $n=0,1,2,\dots$, set

$$B_n=B_{\rho_n}(x_0),\quad \rho_n=\frac{1}{2}+\frac{\epsilon}{2^{n+1}},\qquad k_n=k-\frac{k}{2^{n+1}},$$
$$Q_n=B_n\times(t_n,t),\quad t_n=\frac{t}{2}-\left(\frac{\epsilon}{2^{n+1}}\right)^p t,\quad 0<t_n<t\leqslant T',$$

where $k\geqslant t$ is to be chosen. Let $\zeta_n(x,\tau)$ be a smooth cut-off function in $Q_n$ with $0\leqslant\zeta_n(x,\tau)\leqslant 1$, such that

$$\zeta_n\equiv 1\quad\text{in } Q_{n+1},\quad 0\leqslant\frac{\partial\zeta_n}{\partial\tau}\leqslant\gamma\frac{2^{(n+2)p}}{\epsilon^p t},\quad |D\zeta_n|\leqslant\gamma\frac{2^{n+1}}{\epsilon}.$$

Taking $\varphi=(u-k_{n+1})_+\zeta_n^p$ as a test function in (1.3), we get that

$$\begin{aligned}
&\frac{1}{2}\int_{B_n}(u-k_{n+1})_+^2(x,t')\zeta_n^p\,\mathrm{d}x+\int_{t_n}^{t'}\int_{B_n}|D(u-k_{n+1})_+|^p\zeta_n^p\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+p\int_{t_n}^{t'}\int_{B_n}(u-k_{n+1})_+\zeta_n^{p-1}|Du|^{p-1}DuD\zeta_n\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad=\frac{p}{2}\int_{t_n}^{t'}\int_{B_n}(u-k_{n+1})_+^2\zeta_n^{p-1}\zeta_{n\tau}\,\mathrm{d}x\,\mathrm{d}\tau+\int_{t_n}^{t'}\int_{B_n}|Du^q|^\sigma(u-k_{n+1})_+\zeta_n^p\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned} \tag{2.3}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

where $t_n < t' < t$. By Young's inequality, we obtain

$$\left|p\int_{t_n}^{t'}\int_{B_n}(u-k_{n+1})_+\zeta_n^{p-1}|Du|^{p-1}DuD\zeta_n\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant \frac{1}{4}\int_{t_n}^{t'}\int_{B_n}|D(u-k_{n+1})_+|^p\zeta_n^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_{t_n}^{t'}\int_{B_n}(u-k_{n+1})_+^p|D\zeta_n|^p\,\mathrm{d}x\,\mathrm{d}\tau, \tag{2.4}$$

$$\int_{t_n}^{t'}\int_{B_n}|Du^q|^\sigma(u-k_{n+1})_+\zeta_n^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \frac{1}{4}\int_{t_n}^{t'}\int_{B_n}|D(u-k_{n+1})_+|^p\zeta_n^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_{t_n}^{t'}\int_{B_n}u^{\sigma(pq-p+1)/(p-\sigma)}(u-k_{n+1})_+\,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.5}$$

If $u > 2k_n$, then

$$(u-k_n)_+^2 \geqslant \tfrac{1}{2}u(u-k_{n+1})_+.$$

If $k_{n+1} \leqslant u \leqslant 2k_n$, then we have that

$$(u-k_n)_+^2 \geqslant (u-k_n)_+(k_{n+1}-k_n) \geqslant 2^{-n-3}u(u-k_{n+1})_+.$$

Hence,

$$\int_{t_n}^{t'}\int_{B_n}u^{\sigma(pq-p+1)/(p-\sigma)}(u-k_{n+1})_+\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 2^n\gamma\int_{t_n}^{t'}\int_{B_n}u^{\sigma(pq-p+1)/(p-\sigma)-1}(u-k_n)_+^2\,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.6}$$

Since $q\sigma < p-1$ and $q > (p-1)/p$, we have that

$$\frac{\sigma(pq-p+1)}{p-\sigma} < p-1.$$

If

$$0 \leqslant \frac{\sigma(pq-p+1)}{p-\sigma} - 1 < p-2,$$

by virtue of (2.1), we get that

$$u^{\sigma(pq-p+1)/(p-\sigma)-1}(x,\tau) \leqslant \left(\frac{1}{\tau}\right)^{(\sigma(pq-p+1)/(p-\sigma)-1)/(p-2)} \leqslant \frac{1}{\tau} < \frac{4}{t}, \tag{2.7}$$

since $0 < \tau < 1$ and $\tau > t_n \geqslant \frac{1}{4}t$.

If

$$\frac{\sigma(pq-p+1)}{p-\sigma} - 1 < 0,$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

then we have that

$$u^{\sigma(pq-p+1)/(p-\sigma)-1}(x,\tau) \leqslant \left(\frac{1}{k_{n+1}}\right)^{1-\sigma(pq-p+1)/(p-\sigma)} \leqslant \frac{2}{t}, \tag{2.8}$$

since $q > (p-1)/p$ and $k \geqslant t$.

Combining (2.4)–(2.8) with (2.3), we obtain

$$\sup_{t_n<\tau<t} \int_{B_n} (u-k_{n+1})_+^2 \zeta_n^p(x,\tau)\,\mathrm{d}x + \iint_{Q_n} |D((u-k_{n+1})_+\zeta_n)|^p\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma \frac{2^{2n}}{\epsilon^p t} \iint_{Q_n} (u-k_n)_+^2\,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.9}$$

Here we also use (2.1). Then, by the recursive inequality (2.9), following the iterative process in [30, lemma (2.2)], we can obtain

$$\|u(\cdot,t)\|_{\infty,B_{1/2}} \leqslant \gamma t^{-(N+p)/\kappa} \left(\int_0^t \int_{B_1} u\,\mathrm{d}x\,\mathrm{d}\tau\right)^{p/\kappa} + t. \tag{2.10}$$

The above inequality implies (2.2). □

Now we give the estimate of gradient term $|Du^q|^\sigma$.

LEMMA 2.2. *Let $u$ be a non-negative continuous weak subsolution of (1.1) in $S_{T^*}$. If $p > 2$, we have that*

$$\int_0^t \int_{B_{1/2}(x_0)} |Du|^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma |||u(\cdot,t)|||_1 (t^{1/\kappa} |||u(\cdot,t)|||_1^{(p-2)/\kappa} + t^{(p-1)/p}) + \gamma t |||u(\cdot,t)|||_1^{1/p}. \tag{2.11}$$

*If $p = 2$, we have that*

$$\int_0^t \int_{B_{\rho/2}(x_0)} |Du|\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma (t^{1/2-Nh_0/2} |||u(\cdot,t)|||_1 + t^{1/2+h_0/2} |||u(\cdot,t)|||_1^{1-h_0/2} + t |||u(\cdot,t)|||_1^{(1-h_0)/2}), \tag{2.12}$$

*where $\gamma = \gamma(N,p,q,\sigma)$ and $h_0 > 0$ is a constant such that the exponents in (2.12) are positive.*

*Moreover, for all $p \geqslant 2$, the following two statements hold.*

(1) *Let $\sigma(pq-p+1)/(p-\sigma) \leqslant 1$ and let (2.1) hold, then we have that*

$$\int_0^t \int_{B_{1/2}(x_0)} |Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma (t^{1-\sigma/p-N\sigma r_0/p\kappa} |||u(\cdot,t)|||_1^{\sigma r_0/\kappa} + t^{r_0})(|||u(\cdot,t)|||_1 + 1), \tag{2.13}$$

*where $0 < t < T'$ ($T'$ is as in lemma 2.1), $\gamma = \gamma(N, p, q, \sigma)$ and $r_0 = r_0(N, p, q, \sigma)$ such that the exponents in (2.13) are positive.*

(2) *Let $\sigma(pq - p + 1)/(p - \sigma) > 1$, assume that a time $0 < T'' < T^*$ is given such that*

$$t\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{p-2} + t\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1} \leqslant 1 \quad \forall 0 < t < T''. \tag{2.14}$$

*Then, we have that*

$$\begin{aligned}\int_0^t \int_{B_{1/2}(x_0)} |Du^q|^\sigma \,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \gamma |||u(\cdot,t)|||_1 \{t^{1-\sigma/p-(N/\kappa)((\sigma(pq-p+1)-p+\sigma)/p)} |||u(\cdot,t)|||_1^{(\sigma(pq-p+1)-p+\sigma)/\kappa} \\ &\quad + t^{1-\sigma/p+\sigma r_1/p-(N/\kappa)((\sigma(pq-p+1)-p+\sigma-\sigma r_1)/p)} \\ &\quad \times |||u(\cdot,t)|||_1^{(\sigma(pq-p+1)-p+\sigma-\sigma r_1)/\kappa} \\ &\quad + t^{\sigma(pq-p+1)/p-(\sigma r_1/p)(N/\kappa+1)} |||u(\cdot,t)|||_1^{\sigma r_1/\kappa} + t^{\sigma(pq-p+1)/p}\},\end{aligned} \tag{2.15}$$

*where $0 < t < T''$, $\gamma = \gamma(N, p, q, \sigma)$ and $r_1 = r_1(N, p, q, \sigma)$ such that the exponents in (2.15) are positive.*

*Proof.* Here, we only prove (2.11)–(2.13). The proof of (2.15) is similar to (2.13) and we omit the details.

Firstly, we prove (2.11). Set $B_1 = B_1(x_0)$. Take

$$\varphi = t^{p\beta/(p-1)} u^{1-1/(p-1)} \zeta^p, \qquad \frac{N(p-2)}{\kappa p} < \beta < \frac{1}{p},$$

as a test function in (1.3), where $\zeta$ is a piecewise smooth cut-off function in $B_1$, such that

$$0 \leqslant \zeta \leqslant 1 \quad \text{in } B_1, \qquad \zeta = 1 \quad \text{in } B_{1/2}, \qquad |D\zeta| \leqslant \gamma.$$

Thus, we obtain

$$\begin{aligned}\left(1 - \frac{1}{p-1}\right) \int_0^t \int_{B_1} \tau^{p\beta/(p-1)} u^{-1/(p-1)} |Du|^p \zeta^p \,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \gamma \int_0^t \int_{B_1} \tau^{p\beta/(p-1)-1} u^{2-1/(p-1)} \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + p \int_0^t \int_{B_1} \tau^{p\beta/(p-1)} \zeta^{p-1} u^{1-1/(p-1)} |Du|^{p-1} |D\zeta| \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \int_0^t \int_{B_1} \tau^{p\beta/(p-1)} u^{1-1/(p-1)} |Du^q|^\sigma \zeta^p \,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{2.16}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Young's inequality implies that

$$p\left|\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}\zeta^{p-1}u^{1-1/(p-1)}|Du|^{p-1}|D\zeta|\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant \frac{1}{4}\left(1-\frac{1}{p-1}\right)\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{-1/(p-1)}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{p-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}\tau, \tag{2.17}$$

$$\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{1-1/(p-1)}|Du^q|^\sigma\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \frac{1}{4}\left(1-\frac{1}{p-1}\right)\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{-1/(p-1)}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{p/(p-\sigma)(q\sigma-\sigma+1)-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.18}$$

Since $q\sigma < p-1$ and

$$\frac{p(q\sigma-\sigma+1)}{p-\sigma}-\frac{1}{p-1}>0,$$

then we have that

$$u^{(p/(p-\sigma))(q\sigma-\sigma+1)-1/(p-1)} \leqslant u^{p-1/(p-1)}+1. \tag{2.19}$$

Combining (2.17)–(2.19) with (2.16), together with (2.1), (2.2), we get that

$$\begin{aligned}\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}\zeta^p u^{-1/(p-1)}|Du|^p\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \gamma\int_0^t\int_{B_1}\tau^{p\beta/(p-1)-1}u^{2-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}\tau+\gamma t^{p\beta/(p-1)+1}\\ &\leqslant \gamma|||u(\cdot,t)|||_1\int_0^t\tau^{p\beta/(p-1)-1}\|u(\cdot,\tau)\|_{\infty,B_1}^{1-1/(p-1)}\,\mathrm{d}\tau+\gamma t^{p\beta/(p-1)+1}\\ &\leqslant \gamma|||u(\cdot,t)|||_1(t^{p\beta/(p-1)-N(p-2)/\kappa(p-1)}|||u(\cdot,t)|||_1^{p(p-2)/\kappa(p-1)}\\ &\qquad + t^{p\beta/(p-1)+(p-2)/(p-1)})+\gamma t^{p\beta/(p-1)+1}.\end{aligned} \tag{2.20}$$

Next, we estimate, by the Hölder inequality, that

$$\int_0^t\int_{B_1}|Du|^{p-1}\zeta^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \left(\int_0^t\int_{B_1}\tau^{p\beta/(p-1)}u^{-1/(p-1)}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(p-1)/p}\left(\int_0^t\int_{B_1}\tau^{-p\beta}u\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/p}.$$

Combining this inequality with (2.20), we obtain (2.11).

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Secondly, we prove (2.12) and (2.13). Take $\varphi = t^\beta u^r \zeta^p$ as a test function in (1.3), where $\zeta$ is as above and $\beta > 0$, $r > 0$ are to be chosen. Thus, we obtain

$$\begin{aligned} r\int_0^t\int_{B_1} \tau^\beta u^{r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \frac{\beta}{1+r}\int_0^t\int_{B_1}\tau^{\beta-1}u^{1+r}\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + p\int_0^t\int_{B_1}\tau^\beta u^r\zeta^{p-1}|Du|^{p-1}|D\zeta|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \int_0^t\int_{B_1}\tau^\beta u^r\zeta^p|Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{2.21}$$

Young's inequality implies that

$$\begin{aligned} p\int_0^t\int_{B_1}\tau^\beta u^r\zeta^{p-1}|Du|^{p-1}|D\zeta|\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \frac{r}{4}\int_0^t\int_{B_1}\tau^\beta u^{r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^\beta u^{p-1+r}\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \int_0^t\int_{B_1}\tau^\beta u^r\zeta^p|Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \frac{r}{4}\int_0^t\int_{B_1}\tau^\beta u^{r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^\beta u^{\sigma(pq-p+1)/(p-\sigma)+r}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \frac{r}{4}\int_0^t\int_{B_1}\tau^\beta u^{r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^\beta u^{1+r}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \gamma\int_0^t\int_{B_1}\tau^\beta\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{2.23}$$

since $\sigma(pq-p+1)/(p-\sigma) \leqslant 1$.

Substituting (2.22) and (2.23) into (2.21), recalling (2.1), (2.2), we obtain

$$\begin{aligned} \int_0^t\int_{B_1}\tau^\beta u^{r-1}|Du|^p\zeta^p\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \gamma\int_0^t\int_{B_1}\tau^{\beta-1}u^{1+r}\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^\beta\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \gamma|||u(\cdot,t)|||_1\int_0^t\tau^{\beta-1}\|u(\cdot,\tau)\|_{\infty,B_1}^r\,\mathrm{d}\tau + \gamma\int_0^t\int_{B_1}\tau^\beta\|u(\cdot,\tau)\|_{\infty,B_1}^r\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \gamma(t^{\beta-Nr/\kappa}|||u(\cdot,\tau)|||_1^{pr/\kappa} + t^{\beta+r})(|||u(\cdot,t)|||_1 + 1), \end{aligned} \tag{2.24}$$

provided that

$$\beta - \frac{Nr}{\kappa} > 0. \tag{2.25}$$

Now, we prove (2.12). For $0 < \beta < 1$, we choose $0 < r < \min\{1, 2\beta/N\}$. By (2.24) and (2.2), we obtain

$$\begin{aligned}\int_0^t \int_{B_1} \tau^{\beta} u^{r-1} |Du|^p \zeta^2 \, dx \, d\tau &\leqslant \gamma \biggl( \int_0^t \int_{B_\rho} \tau^{\beta-1} u^{r+1} \, dx \, d\tau + t^{\beta+1} \biggr) \\ &\leqslant \gamma |||u(\cdot,t)|||_1 \biggl( \int_0^t \tau^{\beta-1-Nr/2} \, d\tau |||u(\cdot,t)|||_1^r + \int_0^t \tau^{\beta-1+r} \, d\tau \biggr) + t^{\beta+1} \\ &= \gamma |||u(\cdot,t)|||_1 (t^{\beta-Nr/2} |||u(\cdot,t)|||_1^r + t^{\beta+r}) + t^{\beta+1}. \end{aligned} \tag{2.26}$$

Next, by the Hölder inequality,

$$\int_0^t \int_{B_1} |Du| \zeta \, dx \, d\tau \leqslant \biggl( \int_0^t \int_{B_1} \tau^{\beta} u^{r-1} |Du|^2 \zeta^2 \, dx \, d\tau \biggr)^{1/2} \biggl( \int_0^t \int_{B_1} \tau^{-\beta} u^{1-r} \, dx \, d\tau \biggr)^{1/2}. \tag{2.27}$$

Also, estimate the last integral by the Hölder inequality,

$$\int_0^t \int_{B_1} \tau^{-\beta} u^{1-r} \, dx \, d\tau \leqslant \int_0^t \tau^{-\beta} \biggl( \int_{B_1} u(x,\tau) \, dx \biggr)^{1-r} d\tau \leqslant \gamma t^{1-\beta} |||u(\cdot,t)|||_1^{1-r}. \tag{2.28}$$

Then, collecting (2.26)–(2.28), we prove (2.12).

We now prove (2.13). By the Hölder inequality, we have that

$$\begin{aligned}\int_0^t \int_{B_1} |Du^q|^{\sigma} \zeta^{\sigma} \, dx \, d\tau &\leqslant \biggl( \int_0^t \int_{B_1} \tau^{\beta} u^{r-1} |Du|^p \zeta^p \, dx \, d\tau \biggr)^{\sigma/p} \\ &\quad \times \biggl( \int_0^t \int_{B_1} \tau^{-\beta\sigma/(p-\sigma)} u^{\sigma(pq-p+1-r)/(p-\sigma)} \, dx \, d\tau \biggr)^{(p-\sigma)/p}. \end{aligned} \tag{2.29}$$

Assume that

$$pq - p + 1 - r > 0, \tag{2.30}$$

since

$$\frac{\sigma(pq-p+1-r)}{p-\sigma} < \frac{\sigma(pq-p+1)}{p-\sigma} \leqslant 1,$$

then Young's inequality implies that

$$\begin{aligned}\int_0^t \int_{B_1} \tau^{-\beta\sigma/(p-\sigma)} u^{\sigma(pq-p+1-r)/(p-\sigma)} \, dx \, d\tau &\leqslant \int_0^t \int_{B_1} \tau^{-\beta\sigma/(p-\sigma)} (u+1) \, dx \, d\tau \\ &\leqslant t^{1-\beta\sigma/(p-\sigma)} (|||u(\cdot,t)|||_1 + 1), \end{aligned} \tag{2.31}$$

provided that

$$1 - \frac{\beta\sigma}{p-\sigma} > 0. \tag{2.32}$$

Substituting (2.24) and (2.31) into (2.29), we obtain (2.13). It is left to prove that $\beta$ and $r$ can be chosen such that (2.25), (2.30) and (2.32) hold. We first fix $\beta$ such that (2.32) holds. Then we can fix $r = r_0$ such that $0 < r_0 < \min\{pq-p+1, \beta\kappa/N\}$. The proof is complete. □

Finally, we give an *a priori* bound of solutions to (1.1) in terms of the initial data.

LEMMA 2.3. *Let $u \geqslant 0$ be a bounded and uniformly continuous solution to (1.1), (1.2) in $S_{T^*}$. Then the following statements hold.*

(1) *If $\sigma(pq-p+1)/(p-\sigma) \leqslant 1$, then there exists $T_0 = T_0(N,p,q,\sigma) < T^*$ such that*

$$|||u(\cdot,t)|||_1 \leqslant \gamma(|||u_0|||_1 + 1) \quad \forall 0 < t < T_0 \tag{2.33}$$

*and (2.1), (2.2) hold for all $0 < t < T_0$, where $\gamma = \gamma(N,p,q,\sigma)$.*

(2) *If $\sigma(pq-p+1)/(p-\sigma) > 1$, then there exists $T_{01} < T^*$ such that (2.33), (2.14) and (2.2) hold for all $0 < t < T_{01}$.*

*Proof.* Here, we only prove the case $\sigma(pq-p+1)/(p-\sigma) \leqslant 1$ and $p > 2$; the proof of other cases are similar. Define

$$t_0 = \sup\{0 < T' < T^* \mid (2.1) \text{ holds}\}.$$

Choose $0 < t < t_0$ and let $B_1 = B_1(x_0)$. Take $\zeta$ as a test function in (1.3), where $\zeta$ is as in lemma 2.2. Direct calculation shows that

$$\begin{aligned}
&\int_{B_{1/2}} u(x,t)\,\mathrm{d}x \\
&\quad\leqslant \int_{B_1} u_0\,\mathrm{d}x + \gamma\int_0^t\int_{B_1} |Du|^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{B_1} |Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqslant \int_{B_1} u_0\,\mathrm{d}x + \gamma|||u(\cdot,t)|||_1(t^{1/\kappa}|||u(\cdot,t)|||_1^{(p-2)/\kappa} + t^{(p-1)/p}) + \gamma t|||u(\cdot,t)|||_1^{1/p} \\
&\qquad\quad + \gamma(t^{1-\sigma/p-N\sigma r_0/p\kappa}|||u(\cdot,t)|||_1^{\sigma r_0/\kappa} + t^{r_0})(|||u(\cdot,t)|||_1 + 1)
\end{aligned} \tag{2.34}$$

for all $0 < t < t_0$; here we use (2.11) and (2.13). Since $x_0 \in \mathbb{R}^N$ is arbitrarily chosen, we have that

$$|||u(\cdot,t)|||_1 \leqslant |||u_0|||_1 + \gamma M_1(t)|||u(\cdot,t)|||_1 + \gamma t|||u(\cdot,t)|||_1^{1/p} + \gamma M_2(t)(|||u(\cdot,t)|||_1 + 1), \tag{2.35}$$

where the meanings of $M_1(t)$ and $M_2(t)$ are obvious. Set

$$t_1 = \sup\{0 < t < T^* \mid t^{p/\kappa}|||u(\cdot,t)|||_1^{p(p-2)/\kappa} + M_1(t) + M_2(t) + t|||u(\cdot,t)|||_1^{1/p} < \delta\}, \tag{2.36}$$

where $\delta > 0$ (small) is to be chosen. Note that $t_1$ is well defined because the stipulated assumptions ensure that $|||u(\cdot,t)|||_1$ is continuous in $[0,T^*]$, and the exponents of $t$ in (2.36) are positive. By lemma 2.1 and (2.36), it is easily seen that $t_1 < t_0$, by a suitable choice of $\delta$. Then, if we choose $\delta < 1/4\gamma$, we obtain

$$|||u(\cdot,t)|||_1 \leqslant \gamma(|||u_0|||_1 + 1) \quad \forall 0 < t < t_1. \tag{2.37}$$

Therefore, all the claims hold. The number $t_1$ is still only qualitatively known. A quantitative lower bound $T_0$ can be found by substituting (2.37) into the definition, (2.36), of $t_1$. □

*Proof of theorem 1.2.* Consider the approximating problems

$$\left.\begin{aligned} u_{nt} - \operatorname{div}(|Du_n|^{p-2}Du_n) &= \min\{|Du_n^q|^\sigma, n\} && \text{in } B_n \times (0,\infty), \\ u_n(x,t) &= 0 && \text{in } \partial B_n \times (0,\infty), \\ u_n(x,0) &= u_{0n}(x) && \text{on } B_n, \end{aligned}\right\} \tag{2.38}$$

where $B_n = \{x \in \mathbb{R}^N \mid |x| < n\}$ and $u_{0n} \in C_0^\infty(\mathbb{R}^N)$ is non-negative and has compact support in $B_n$, which satisfies

$$\lim_{n\to\infty} \int_{\mathcal{K}} |u_{0n} - u_0| \,\mathrm{d}x = 0 \quad \forall \mathcal{K} \subset\subset \mathbb{R}^N$$

and

$$|||u_{0n}|||_1 \leqslant \gamma |||u_0|||_1.$$

The existence and Hölder continuity of solution $u_n$ for (2.38) follow from [19,27,29]. Then, theorem 1.2 can be proved by lemmata 2.1–2.3, following the methods in [4,5] (see also [30]). □

## 3. Proof of theorem 1.3

In this section, we let $q\sigma \geqslant p-1$. This implies that $\sigma(pq-p+1)/(p-\sigma) > 1$. To prove theorem 1.3, we firstly prove several lemmata.

The following supremum estimate will play an important role in proving the existence result.

LEMMA 3.1. *Let $u$ be a non-negative continuous weak subsolution of (1.1) in $S_T$. Also, assume that a time $0 < T'' < T$ is given, such that*

$$t\rho^{-p}\|u(\cdot,t)\|_{\infty,B_{2\rho}(x_0)}^{p-2} + t\|u(\cdot,t)\|_{\infty,B_{2\rho}(x_0)}^{\sigma(pq-p+1)/(p-\sigma)-1} \leqslant 1 \quad \forall 0 < t < T''. \tag{3.1}$$

*There exists a constant $\gamma = \gamma(N,p,q,\sigma,h)$ such that, for every ball $B_{2\rho}(x_0)$,*

$$\|u(\cdot,t)\|_{\infty,B_\rho(x_0)} \leqslant \gamma t^{-(N+p)/\kappa_h} \left( \int_0^t \int_{B_{2\rho}(x_0)} u^h \,\mathrm{d}x\,\mathrm{d}\tau \right)^{p/\kappa_h} \quad \forall 0 < t < T'', \tag{3.2}$$

*where $\kappa_h = N(p-2) + ph$, $h \geqslant 1$ and $x_0 \in \mathbb{R}^N$ is fixed.*

*Proof.* Let $\rho > 0$, $\epsilon \in (0, \frac{1}{2})$ and $k > 0$ is to be chosen. For $n = 0,1,2,\dots$, set

$$B_n = B_{\rho_n}(x_0), \quad \rho_n = \rho + \frac{\epsilon}{2^{n+1}}\rho, \qquad k_n = k - \frac{k}{2^{n+1}},$$
$$Q_n = B_n \times (t_n, t), \quad 0 < t_n < t \leqslant T'', \quad t_n = \frac{t}{2} - \left(\frac{\epsilon}{2^{n+1}}\right)^p t.$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Let $\zeta_n(x,\tau)$ be a smooth cut-off function in $Q_n$ with $0 \leqslant \zeta_n(x,\tau) \leqslant 1$, such that

$$\zeta_n \equiv 1 \quad \text{in } Q_{n+1}, \qquad 0 \leqslant \frac{\partial \zeta_n}{\partial \tau} \leqslant \gamma \frac{2^{(n+2)p}}{\epsilon^p t}, \qquad |D\zeta_n| \leqslant \gamma \frac{2^{n+2}}{\epsilon \rho}.$$

Take $\varphi = (u-k_{n+1})_+^h \zeta_n^p$ as a test function in (1.3). We get that

$$\begin{aligned}
&\frac{1}{h+1}\int_{B_n} (u-k_{n+1})_+^{h+1}(x,t')\zeta_n^p \,\mathrm{d}x \\
&\quad + h\int_{t_n}^{t'}\int_{B_n} (u-k_{n+1})_+^{h-1}|D(u-k_{n+1})_+|^p \zeta_n^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + p\int_{t_n}^{t'}\int_{B_n} \zeta_n^{p-1}(u-k_{n+1})_+^h |Du|^{p-2} Du D\zeta_n \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\qquad = \frac{p}{h+1}\int_{t_n}^{t'}\int_{B_n} (u-k_{n+1})_+^{h+1}\zeta_n^{p-1}\zeta_{n\tau}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\qquad\quad + \int_{t_n}^{t'}\int_{B_n} |Du^q|^\sigma (u-k_{n+1})_+^h \zeta_n^p \,\mathrm{d}x\,\mathrm{d}\tau, \qquad (3.3)
\end{aligned}$$

where $t_n < t' < t$. By Young's inequality, we obtain

$$\begin{aligned}
&p\left|\int_{t_n}^{t'}\int_{B_n} \zeta_n^{p-1}(u-k_{n+1})_+^h |Du|^{p-2} Du D\zeta_n \,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\qquad \leqslant \frac{h}{4}\int_{t_n}^{t'}\int_{B_n} (u-k_{n+1})_+^{h-1}|D(u-k_{n+1})_+|^p \zeta_n^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\quad + \gamma\int_{t_n}^{t'}\int_{B_n} (u-k_{n+1})_+^{p-1+h}|D\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}\tau, \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
&\int_{t_n}^{t'}\int_{B_n} |Du^q|^\sigma (u-k_{n+1})_+^h \zeta_n^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \frac{h}{4}\int_{t_n}^{t'}\int_{B_n} (u-k_{n+1})_+^{h-1}|D(u-k_{n+1})_+|^p \zeta_n^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\quad + \gamma\int_{t_n}^{t'}\int_{B_n} u^{p\sigma(q-1)/(p-\sigma)}(u-k_{n+1})_+^{h+\sigma/(p-\sigma)} \,\mathrm{d}x\,\mathrm{d}\tau. \qquad (3.5)
\end{aligned}$$

If $u > 2k_n$, then

$$(u-k_n)_+^{h+1} \geqslant \tfrac{1}{2}u(u-k_{n+1})_+^h.$$

If $k_{n+1} \leqslant u \leqslant 2k_n$, then

$$(u-k_n)_+^{h+1} \geqslant (u-k_n)_+^h(k_{n+1}-k_n) \geqslant 2^{-n-3}u(u-k_{n+1})_+^h.$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Hence,

$$\int_{t_n}^{t'}\int_{B_n} u^{p\sigma(q-1)/(p-\sigma)}(u-k_{n+1})_+^{h+\sigma/(p-\sigma)}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 2^n\gamma\int_{t_n}^{t'}\int_{B_n} u^{\sigma(pq-p+1)/(p-\sigma)-1}(u-k_n)_+^{h+1}\,\mathrm{d}x\,\mathrm{d}\tau. \quad (3.6)$$

Substituting (3.4)–(3.6) into (3.3), we obtain

$$\sup_{t_n<\tau<t}\int_{B_n}(u-k_{n+1})_+^{h+1}\zeta_n^p(x,\tau)\,\mathrm{d}x + \iint_{Q_n}|D((u-k_{n+1})_+^{(p-1+h)/p}\zeta_n)|^p\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma\frac{2^{np}}{\epsilon^p t}(1+M)\iint_{Q_n}(u-k_n)_+^{h+1}\,\mathrm{d}x\,\mathrm{d}\tau, \quad (3.7)$$

where

$$M=\sup_{0<\tau<t}(\tau\rho^{-p}\|u(\cdot,\tau)\|_{\infty,B_{2\rho}(x_0)}^{p-2}+\tau\|u(\cdot,\tau)\|_{\infty,B_{2\rho}(x_0)}^{\sigma(pq-p+1)/(p-\sigma)-1}).$$

By an iteration process similar to [30, lemma (2.1)], we obtain

$$\|u(\cdot,t)\|_{\infty,B_{\rho/2}}\leqslant\gamma\biggl(\frac{1+M}{t}\biggr)^{(N+p)/\kappa_h}\biggl(\iint_{Q_0}u^h\,\mathrm{d}x\,\mathrm{d}\tau\biggr)^{p/\kappa_h}. \quad (3.8)$$

This implies (3.2), since (3.1) holds. □

We also need the estimates of $|Du^q|^\sigma$, which are as follows.

LEMMA 3.2. *Let the assumptions of lemma 2.1 and (1.7) hold. Then, for every $B_{2\rho}(x_0)\subset\mathbb{R}^N$, $0<t<T''$. If $p>2$, $\sigma\geqslant 1$ or $p=2$, $\sigma>1$, then the following statements hold.*

(i) *If $h=1$, then*

$$\int_0^t\int_{B_\rho}|Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau \leqslant\gamma t^{(p-\sigma)/p-(N(\sigma(pq-p+1)-p+\sigma))/p\kappa}G^{1+(\sigma(pq-p+1)-p+\sigma)/\kappa}(t), \quad (3.9)$$

*where*

$$G(t)=\sup_{0<\tau<t}\int_{B_{2\rho(x_0)}}u(x,\tau)\,\mathrm{d}x,\qquad \gamma=\gamma(N,p,q,\sigma).$$

(ii) *If $h>1$, then*

$$\int_0^t\int_{B_\rho}|Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau \leqslant\gamma t^{(p-\sigma)/p-N(\sigma(pq-p+1)-p+\sigma)/p\kappa_h}\rho^{N(h-1)/h}\phi^{1/h+(\sigma(pq-p+1)-p+\sigma)/\kappa_h}(t), \quad (3.10)$$

*where*

$$\phi(t) = \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u^h(x,\tau)\,\mathrm{d}x, \qquad \gamma = \gamma(N,p,q,\sigma,h).$$

*If $p=2$, $\sigma=1$, we have that*

$$\int_0^t \int_{B_{\rho/2}(x_0)} |Du|\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \gamma t^{1/2-Nh_1/2} \rho^{Nh_1/2} G(t), \tag{3.11}$$

*where $\gamma = \gamma(N,p,q,\sigma)$ and $h_1 > 0$ is a constant such that the exponents in (3.11) are positive.*

*Proof.* Set $B_{2\rho} = B_{2\rho}(x_0)$. We only prove (3.9) and (3.10); (3.11) can be proved similarly to (2.12) and we omit the details. Take $\varphi = t^\beta u^r \zeta^p$ as a test function in (1.3), where $\zeta$ is a piecewise smooth cut-off function in $B_{2\rho}$, such that

$$0 \leqslant \zeta \leqslant 1 \quad \text{in } B_{2\rho}, \qquad \zeta = 1 \quad \text{in } B_\rho, \qquad |D\zeta| \leqslant \frac{\gamma}{\rho}$$

and $\beta > 0$, $r > 0$ are to be chosen. As in the proof of (2.24), together with (3.1) and (3.2), we obtain

$$\begin{aligned}
&\int_0^t \int_{B_{2\rho}} \tau^\beta u^{r-1} |Du|^p \zeta^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \gamma \int_0^t \int_{B_{2\rho}} \tau^{\beta-1} u^{1+r} (\tau u^{p-2} + \tau u^{\sigma(pq-p+1)/(p-\sigma)-1})\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \gamma \int_0^t \int_{B_{2\rho}} \tau^{\beta-1} u^{1+r}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \gamma G(t) \int_0^t \int_{B_{2\rho}} \tau^{\beta-1} \|u(\cdot,\tau)\|^r_{\infty,B_{2\rho}}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \gamma G(t) t^{\beta - Nr/\kappa_h} \phi^{pr/\kappa_h}(t),
\end{aligned} \tag{3.12}$$

where

$$\phi(t) = \sup_{0<\tau<t} \int_{B_{2\rho}(x_0)} u^h(x,\tau)\,\mathrm{d}x,$$

provided that

$$\beta - \frac{Nr}{\kappa_h} > 0. \tag{3.13}$$

By the Hölder inequality, we have that

$$\begin{aligned}
&\int_0^t \int_{B_{2\rho}} |Du^q|^\sigma \zeta^\sigma \,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad \leqslant \biggl( \int_0^t \int_{B_{2\rho}} \tau^\beta u^{r-1} |Du|^p \zeta^p \,\mathrm{d}x\,\mathrm{d}\tau \biggr)^{\sigma/p} \\
&\qquad\quad \times \biggl( \int_0^t \int_{B_{2\rho}} \tau^{-\beta\sigma/(p-\sigma)} u^{\sigma(pq-p+1-r)/(p-\sigma)} \,\mathrm{d}x\,\mathrm{d}\tau \biggr)^{(p-\sigma)/p}.
\end{aligned} \tag{3.14}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Then, by Young's inequality and (3.2), we have that

$$\begin{aligned}\int_0^t \int_{B_{2\rho}} &\tau^{-\beta\sigma/(p-\sigma)} u^{\sigma(pq-p+1-r)/(p-\sigma)}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant G(t)\int_0^t \int_{B_{2\rho}} \tau^{-\beta\sigma/(p-\sigma)} \|u(\cdot,\tau)\|_{\infty,B_{2\rho}}^{\sigma(pq-p+1-r)/(p-\sigma)-1}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \gamma G(t) t^{1-\beta\sigma/(p-\sigma)-(N/\kappa_h)(\sigma(pq-p+1-r)/(p-\sigma)-1)} \phi^{(p/\kappa_h)(\sigma(pq-p+1-r)/(p-\sigma)-1)}(t),\end{aligned} \tag{3.15}$$

provided that

$$\frac{\sigma(pq-p+1-r)}{p-\sigma} - 1 > 0, \tag{3.16}$$

$$1 - \frac{\beta\sigma}{p-\sigma} - \frac{N}{\kappa_h}\left(\frac{\sigma(pq-p+1-r)}{p-\sigma} - 1\right) > 0. \tag{3.17}$$

Substituting (3.12) and (3.15) into (3.14), we obtain

$$\begin{aligned}\int_0^t \int_{B_\rho} &|Du^q|^\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant \gamma G(t) t^{(p-\sigma)/p - N(\sigma(pq-p+1)-p+\sigma)/p\kappa_h} \phi^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h}(t).\end{aligned} \tag{3.18}$$

Hence, if $h = 1$, (3.18) implies (3.9). If $h > 1$, applying the Hölder inequality in (3.18), we obtain (3.10). It is left to prove that $\beta$ and $r$ can be chosen such that (3.13), (3.16) and (3.17) hold. Fix

$$0 < \beta < \frac{p-\sigma}{\sigma}. \tag{3.19}$$

Then, by (1.7), we easily find $r$ such that (3.13), (3.16) and (3.17) hold. □

LEMMA 3.3. *Let $u \geqslant 0$ be a bounded and uniformly continuous solution to (1.1), (1.2) in $S_T$ and let (1.7) hold, then there exists $T_0' = T_0'(N,p,q,\sigma) < T$ such that*

$$|||u(\cdot,t)|||_h \leqslant \gamma |||u_0|||_h \quad \forall 0 < t < T_0' \tag{3.20}$$

*and (3.1), (3.2) hold for all $0 < t < T_0'$, where $\gamma = \gamma(N,p,q,\sigma)$.*

*Proof.* Let $\rho \geqslant 1$ be fixed and let $t_0$ be the largest time satisfying

$$t\rho^{-p}\|u(\cdot,t)\|_{\infty,B_{2\rho}(x_0)}^{p-2} + t\|u(\cdot,t)\|_{\infty,B_{2\rho}(x_0)}^{\sigma(pq-p+1)/(p-\sigma)-1} \leqslant 1 \tag{3.21}$$

for all $0 < t < t_0$. By lemma 3.1, we have that

$$u(x,t) \leqslant \gamma t^{-N/\kappa_h} \phi(t)^{p/\kappa_h}. \tag{3.22}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Let $\zeta \geqslant 0$ be a piecewise smooth cut-off function in $B_{2\rho}(x_0)$, with $\zeta = 1$ on $B_\rho(x)$ and $|D\zeta| \leqslant \gamma/\rho$. We take $u^{h-1}\zeta^p$ as a test function. If $h > 1$, we obtain

$$\begin{aligned}\int_{B_{2\rho}(x_0)(x)} u^h(y,t)\zeta^p \,\mathrm{d}y + h(h-1)\int_0^t \int_{B_{2\rho}(x_0)} u^{h-2}|Du|^p\zeta^p \,\mathrm{d}y\,\mathrm{d}\tau \\ \leqslant \int_{B_{2\rho}(x_0)} u_0^h(y)\zeta^p \,\mathrm{d}y + ph\int_0^t \int_{B_{2\rho}(x_0)} u^{h-1}|Du|^{p-1}\zeta^{p-1}|D\zeta| \,\mathrm{d}y\,\mathrm{d}\tau \\ + h\int_0^t \int_{B_{2\rho}(x_0)} |Du^q|^\sigma u^{h-1}\zeta^p \,\mathrm{d}y\,\mathrm{d}\tau. \qquad (3.23)\end{aligned}$$

Applying Young's inequality in (3.23), and using (3.22), we have that

$$\begin{aligned}&\int_{B_\rho(x_0)} u^h(y,t)\,\mathrm{d}y \\ &\leqslant \int_{B_{2\rho}(x_0)} u_0^h(y)\,\mathrm{d}y + \gamma\int_0^t\int_{B_{2\rho}(x_0)} u^{p-2+h}\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \gamma\int_0^t\int_{B_{2\rho}(x_0)} u^{\sigma(pq-p+1)/(p-\sigma)-1+h}\,\mathrm{d}y\,\mathrm{d}\tau \\ &\leqslant |||u_0|||_h^h \\ &\quad + \gamma\phi(t)\biggl(\int_0^t \tau^{-N(p-2)/\kappa_h}\phi^{p(p-2)/\kappa_h}(\tau)\,\mathrm{d}\tau \\ &\qquad + \int_0^t \tau^{-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\phi^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}(\tau)\,\mathrm{d}\tau\biggr) \\ &\leqslant |||u_0|||_h^h + \gamma M_1(t)\phi(t), \qquad (3.24)\end{aligned}$$

where

$$\begin{aligned}M_1(t) = t^{ph/\kappa_h}\phi^{p(p-2)/\kappa_h}(t) \\ + t^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\phi^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}(t).\end{aligned}$$

If $h = 1$, by lemma 3.2 and (3.22), we get that

$$\begin{aligned}\int_{B_\rho(x_0)} u(y,t)\,\mathrm{d}y &\leqslant \int_{B_{2\rho}(x_0)} u_0(y)\,\mathrm{d}y + \frac{\gamma}{\rho}\int_0^t\int_{B_{2\rho}(x_0)} |Du|^{p-1}\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \int_0^t\int_{B_{2\rho}(x_0)} |Du^q|^\sigma\,\mathrm{d}y\,\mathrm{d}\tau \\ &\leqslant \int_{B_{2\rho}(x_0)} u_0(y)\,\mathrm{d}y + \gamma t^{1/\kappa}G(t)^{1+(p-2/\kappa)} \\ &\quad + \gamma t^{(p-\sigma)/p-N(\sigma(pq-p+1)-p+\sigma)/p\kappa}G^{1+(\sigma(pq-p+1)-p+\sigma)/\kappa}(t) \\ &\leqslant |||u_0|||_1 + \gamma M_2(t)\phi(t), \qquad (3.25)\end{aligned}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

where

$$M_2(t) = t^{1/\kappa}\phi^{(p-2)/\kappa}(t) + t^{(p-\sigma)/p-N(\sigma(pq-p+1)-p+\sigma)/p\kappa}\phi^{(\sigma(pq-p+1)-p+\sigma)/\kappa}(t).$$

Set

$$t_1 = \sup\{t > 0 \mid M_1(t) + M_2(t) < \delta\}, \tag{3.26}$$

where $\delta > 0$ is to be chosen. Note that for $0 < t < t_1$, by (3.22), we have that

$$\begin{aligned} t\|u(\cdot,t)\|^{p-2}_{\infty,B_{2\rho}(x_0)} + t\|u(\cdot,t)\|^{\sigma(pq-p+1)/(p-\sigma)-1}_{\infty,B_{2\rho}(x_0)} &\leqslant \gamma(t^{ph/\kappa_h}\psi^{p(p-2)/\kappa_h}(t) \\ &\quad + t^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\phi^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}(t)) \\ &\leqslant \gamma\delta \\ &< \tfrac{1}{2}, \end{aligned}$$

where $\delta$ is chosen sufficiently small. By the definition of $t_0$ and $t_1$, noting that $\rho \geqslant 1$, the above inequality implies that $t_1 < t_0$. Also, possibly by choosing $\delta$ even smaller, noting that $x_0 \in \mathbb{R}^N$ is arbitrarily chosen, by (3.24) and (3.25), we have that

$$|||u|||^h_h \leqslant \gamma|||u_0|||^h_h \quad \forall 0 < t < t_1. \tag{3.27}$$

A quantitative lower bound $T_0'$ of $t_1$ can be found by substituting (3.27) into the definition of $t_1$ in (3.26). Therefore, all the claims made in the lemma will follow, using the supremum estimate (3.27). □

Now, we use lemmata 3.1–3.3 to prove theorem 1.3.

*Proof of theorem 1.3.* Considering the same approximating problems (2.38), the only difference is the approximation process to the initial data. Here $u_{0n} \in C_0^\infty(\mathbb{R}^N)$ is non-negative and has compact support in $B_n$, which satisfies

$$\lim_{n\to\infty}\int_{\mathcal{K}} |u_{0n} - u_0|^h \, \mathrm{d}x = 0 \quad \forall \mathcal{K} \subset\subset \mathbb{R}^N$$

and

$$|||u_{0n}|||_h \leqslant \gamma|||u_0|||_h.$$

The existence and Hölder continuity of solution $u_n$ for (2.38) follow from [18, 19, 27, 29]. Then, by lemmata 3.1–3.3 we can prove theorem 1.3, following the methods in [4, 5] (see also [30]). □

## 4. Proof of theorem 1.6

We again consider the approximating problems in the proof of theorem 1.3. In fact, if we can only prove the uniform estimates of (1.8), (1.9) to $u_n$, then theorem 1.6 can be proved in a similar way to the proof of theorem 1.3. Let $\rho \geqslant 1$ be fixed and let $t_0$ be as in (3.21). Note that (1.10) implies $h > 1$. Then by lemma 3.1 we also have (3.22).

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

Let $u_n$ be the solution of the approximating problem in the proof of theorem 1.3. Take $u_n^{h-1}$ as a test function. Then by direct calculations we get that

$$\frac{1}{h}\int_{B_n(x_0)} u_n^h(y,t)\,\mathrm{d}y + \frac{h-1}{2}\left(\frac{p}{p+h-2}\right)^p \int_s^t \int_{B_n(x_0)} |Du_n^{(p+h-2)/p}|^p\,\mathrm{d}y\,\mathrm{d}\tau \leqslant \frac{1}{h}\int_{B_n(x_0)} u_n^h(y,s)\,\mathrm{d}y + \gamma_1 \int_s^t \int_{B_n(x_0)} u_n^{\sigma(pq-p+1)/(p-\sigma)-1+h}\,\mathrm{d}y\,\mathrm{d}\tau, \tag{4.1}$$

where $0<s<t<T$, $\gamma_1=\gamma_1(N,p,q,\sigma,h)$. By (1.10), we get that

$$\frac{\sigma(pq-p+1)}{p-\sigma} > p-1.$$

By the Gagliardo–Nirenberg inequality in [27], we have that

$$\int_{B_n(x_0)} u_n^{\sigma(pq-p+1)/(p-\sigma)-1+h}\,\mathrm{d}y \leqslant \gamma_2 \|u_n\|_{L^h(B_n(x_0))}^{\sigma(pq-p+1)/(p-\sigma)-p+1} \|Du_n^{(p+h-2)/p}\|_{L^p(B_n(x_0))}^p, \tag{4.2}$$

where $\gamma_2=\gamma_2(N,p,q,\sigma,h)$. Hence, (4.1) and (4.2) imply that

$$\frac{1}{h}\int_{B_n(x_0)} u_n^h(y,t)\,\mathrm{d}y + \int_s^t \left\{\frac{h-1}{2}\left(\frac{p}{p+h-2}\right)^p - \gamma_3 \|u_n(\tau)\|_{L^h(B_n(x_0))}^{\sigma(pq-p+1)/(p-\sigma)-p+1}\right\} \times \|Du_n^{(p+h-2)/p}(\tau)\|_{L^p(B_n(x_0))}^p\,\mathrm{d}\tau \leqslant \frac{1}{h}\int_{B_n(x_0)} u_n^h(y,s)\,\mathrm{d}y, \tag{4.3}$$

where $\gamma_3=\gamma_1\gamma_2$. Choose $\gamma_0$ in (1.11) sufficiently small, such that

$$\begin{aligned}\|u_{0n}\|_{L^h(B_n(x_0))}^h &\leqslant \gamma \|u_0\|_{L^h(B_n(x_0))}^h \\ &< \gamma\gamma_0 \\ &\leqslant \left(\frac{h-1}{2\gamma_3}\right)^{h/(\sigma(pq-p+1)/(p-\sigma)-p+1)} \left(\frac{p}{p+h-2}\right)^{ph/(\sigma(pq-p+1)/(p-\sigma)-p+1)}.\end{aligned} \tag{4.4}$$

Since $\|u_n(t)\|_{L^h(B_n(x_0))}^h$ is continuous in $[0,T]$, (4.3) and (4.4) thus imply that $\|u_n(t)\|_{L^h(B_n(x_0))}^h$ is non-increasing in $0\leqslant t<T$, i.e.

$$\|u_n(t)\|_{L^h(B_n(x_0))}^h \leqslant \|u_{0n}\|_{L^h(B_n(x_0))}^h \leqslant \gamma \|u_0\|_{L^h(\mathbb{R}^N)}^h \tag{4.5}$$

for all $0<t<T$. Since, for any $\rho\geqslant 1$, there exists a natural number $n_0$, such that, for all $n\geqslant n_0$, we have $B_{2\rho}(x_0)\subset B_n(x_0)$, (4.5) implies that

$$\|u_n(t)\|_{L^h(B_{2\rho}(x_0))}^h \leqslant \gamma \|u_0\|_{L^h(\mathbb{R}^N)}^h \tag{4.6}$$

for all $0<t<T$. Hence (4.6) and (3.22) imply (1.8) and (1.9).

Therefore, we are left with the task of estimating below $t_0 \geqslant T_0''$. This can be accomplished at once by substituting (4.6) and (3.22) into (3.21), the definition of $t_0$.

## 5. Proof of theorems 1.8 and 1.9

*Proof of theorem 1.8.* In what follows we drop the index $n$, denoting $u_n$, $u_{0n}$ as $u$, $u_0$, respectively. If we prove *a priori* $L^\infty$-estimates for all $t > 0$, the existence of global solutions will follow at once by reasoning as in § 3. Set

$$G(t) = \sup_{0<\tau<t} \int_{\mathbb{R}^N} u(x,\tau)\,\mathrm{d}x,$$

$$G(0) = \int_{\mathbb{R}^N} u_0\,\mathrm{d}x,$$

$$T = \sup\{t\colon t\|u(\cdot,t)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1} \leqslant 1\},$$

$$T_G = \sup\{G(t) \leqslant 2G(0)\}.$$

Let $\rho = \infty$ in lemma 3.1. Then, for all $s \geqslant 1$, we obtain

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \leqslant \gamma t^{-N/\kappa_s} \sup_{0<\tau<t} \|u(\cdot,\tau)\|_{s,\mathbb{R}^N}^{p/\kappa_s} \quad \forall 0 < t < T, \tag{5.1}$$

where $\gamma = \gamma(N,p,q,\sigma,s)$.

Thus, to obtain the uniform $L^\infty$-estimates for all $t > 0$, it will suffice to show that $T_G \geqslant T$ and $T = \infty$. We divide the proof into three steps.

Step 1. We claim that $\min\{T_G, T\} > 1$.

We prove this assertion by contradiction. We set

$$T_\phi = \sup\{t\colon \phi(t) < 2\phi(0)\},$$

where

$$\phi(t) = \sup_{0<\tau<t} \int_{\mathbb{R}^N} u^h(x,\tau)\,\mathrm{d}x, \qquad \phi(0) = \int_{\mathbb{R}^N} u_0^h\,\mathrm{d}x,$$

and divide the proof into three subcases.

Firstly, in (i) and (ii), we prove that $\min\{T, T_\phi\} > 1$.

(i) Case $T_\phi < T$. Take $u^{h-1}\zeta^p$ as a test function, where $\zeta$ is as in the proof of lemma 3.2. Then, we get that

$$\begin{aligned}\frac{1}{h}\int_{B_{2\rho}} u^h(x,t)\zeta^p\,\mathrm{d}x + (h-1)\int_0^t\int_{B_{2\rho}} u^{h-2}|Du|^p\zeta^2\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant \frac{1}{h}\int_{B_{2\rho}} u_0^h(x)\zeta^p\,\mathrm{d}x + p\int_0^t\int_{B_{2\rho}} u^{h-1}|Du|^{p-1}\zeta^{p-1}|D\zeta|\,\mathrm{d}x\,\mathrm{d}\tau \\ + \int_0^t\int_{B_{2\rho}} u^{h-1}|Du^q|^\sigma\zeta^2\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{5.2}$$

https://doi.org/10.1017/S030821051100103X Published online by Cambridge University Press

for all $0 < t < T_\phi$. Applying Young's inequality in (5.2), together with (5.1), we have that

$$\begin{aligned}
\sup_{0<t<T_\phi} \int_{B_\rho} u^h(x,t)\,\mathrm{d}x
&\leqslant \int_{B_{2\rho}} u_0^h(x)\,\mathrm{d}x + \gamma\rho^{-p}\int_0^{T_\phi}\int_{B_{2\rho}} u^{p-2+h}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \gamma\int_0^{T_\phi}\int_{B_{2\rho}} u^{\sigma(pq-p+1)/(p-\sigma)-1+h}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \phi(0) + \gamma\phi(T_\phi)\int_0^{T_\phi}(\rho^{-p}\|u\|_{\infty,B_{2\rho}}^{p-2} + \|u\|_{\infty,B_{2\rho}}^{\sigma(pq-p+1)/(p-\sigma)-1})\,\mathrm{d}\tau \\
&\leqslant \phi(0) + \gamma\phi(T_\phi) \\
&\quad\times\int_0^{T_\phi}(\rho^{-p}\tau^{-N(p-2)/\kappa_h}\phi(\tau)^{p(p-2)/\kappa_h} \\
&\qquad + \tau^{-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\phi(\tau)^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h})\,\mathrm{d}\tau \\
&\leqslant \phi(0) + \gamma\phi(T_\phi)(\rho^{-p}T_\phi^{p/\kappa_h}\gamma_0^{p(p-2)/\kappa_h} \\
&\qquad + T_\phi^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}),
\end{aligned}$$

since (1.13) holds. Letting $\rho \to \infty$ in the above inequality, we have that

$$\phi(T_\phi) \leqslant \phi(0) + \gamma\phi(T_\phi)T_\phi^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}. \tag{5.3}$$

Let $T_\phi \leqslant 1$ and choose $\gamma_0$ sufficiently small that

$$\gamma\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h} \leqslant \tfrac{1}{4}.$$

Hence,

$$\gamma T_\phi^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h} \leqslant \tfrac{1}{4}$$

holds by virtue of (1.7). Then, (5.3) implies $\phi(T_\phi) \leqslant \frac{3}{4}$, in contradiction with the definition of $T_\phi$.

(ii) Case $T_\phi \geqslant T$.

Let us suppose $T < \infty$. Hence, by the definition of $T$, we get that

$$\begin{aligned}
1 &= \sup_{0<\tau<T}\{\tau\|u(\cdot,\tau)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1}\} \\
&\leqslant \gamma T^{1-N(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h}.
\end{aligned}\tag{5.4}$$

If $T \leqslant 1$, then, choosing $\gamma_0$ sufficiently small, such that

$$\gamma\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h} \leqslant \tfrac{1}{2},$$

we get a contradiction. Thus, from (i) and (ii) we obtain $\min\{T, T_\phi\} > 1$.

(iii) If $T_G \geqslant \min\{T, T_\phi\}$, then the claim is obvious. Assume $T_G < \min\{T, T_\phi\}$ and take $u/(u+\epsilon)\zeta^p$ ($\epsilon > 0$) as a test function. Then we have that

$$\begin{aligned}\int_{B_{2\rho}} \int_0^{u(t)} \frac{z}{z+\epsilon}\zeta^p \,\mathrm{d}z \,\mathrm{d}x + \int_0^{T_G} \int_{B_{2\rho}} \frac{\epsilon}{(u+\epsilon)^2}|Du|^p \zeta^p \,\mathrm{d}x \,\mathrm{d}\tau \\ \leqslant \int_{B_{2\rho}} \int_0^{u_0} \frac{z}{z+\epsilon} \,\mathrm{d}z \,\mathrm{d}x + p\int_0^t \int_{B_{2\rho}} \frac{u}{u+\epsilon}|Du|^{p-1}\zeta^{p-1}|D\zeta| \,\mathrm{d}x \,\mathrm{d}\tau \\ + \int_0^t \int_{B_{2\rho}} \frac{u}{u+\epsilon}|Du^q|^\sigma \zeta^p \,\mathrm{d}x \,\mathrm{d}\tau \qquad (5.5)\end{aligned}$$

for all $0 < t < T_G$. Let $\epsilon \to 0$ in (5.5), together with (3.18). Then we have

$$\begin{aligned}&\sup_{0<t<T_G} \int_{B_\rho} u(x,t) \,\mathrm{d}x \\ &\leqslant \int_{B_{2\rho}} u_0 \,\mathrm{d}x \\ &\quad + \gamma G(T_G)\biggl(\frac{1}{\rho} T_G^{h/\kappa_h} \phi^{(p-2)/\kappa_h}(T_G) \\ &\qquad + T_G^{(p-\sigma)/p - N(\sigma(pq-p+1)-p+\sigma)/p\kappa_h} \phi^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h}(T_G)\biggr) \\ &\leqslant G(0) + \gamma G(T_G)\biggl(\frac{1}{\rho} T_G^{h/\kappa_h} \gamma_0^{(p-2)/\kappa_h} \\ &\qquad + T_G^{(p-\sigma)/p - N(\sigma(pq-p+1)-p+\sigma)/p\kappa_h} \gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h}\biggr),\end{aligned}$$

since (1.13) holds. Let $\rho \to \infty$ in the above inequality. Then we obtain

$$G(T_G) \leqslant G(0) + \gamma G(T_G) T_G^{(p-\sigma)/p - N(\sigma(pq-p+1)-p+\sigma)/p\kappa_h} \gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h}. \qquad (5.6)$$

If $T_G \leqslant 1$, then, choosing $\gamma_0$ sufficiently small such that

$$\gamma\gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h} \leqslant \tfrac{1}{4},$$

it follows from (5.6) that $G(T_G) \leqslant \frac{3}{4}G(0)$, in contradiction with the definition of $T_G$. Hence the claim holds.

Step 2. We claim that $T_G \geqslant T$.

Assume that $T_G < T$ and the claim is obvious otherwise. Take $u/(u+\epsilon)\zeta^p$ as a test function again and, using a similar proof as in (iii), we have that

$$\begin{aligned}G(T_G) &\leqslant G(0) + \gamma G(T_G)(\gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h} \\ &\qquad + T_G^{(p-\sigma)/p - N(\sigma(pq-p+1)-p+\sigma)/p\kappa} \gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa}) \\ &\leqslant G(0) + \gamma G(T_G)(\gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h} + \gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa}), \qquad (5.7)\end{aligned}$$

where we use (3.18), (1.7) and (1.12). Hence, if $\gamma_0$ is chosen small enough, such that

$$\gamma(\gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa_h} + \gamma_0^{(\sigma(pq-p+1)-p+\sigma)/\kappa}) \leqslant \tfrac{1}{4},$$

then (5.7) implies a contradiction with the definition of $T_G$. Thus the claim holds.

Step 3. We claim that $T = \infty$.

Assuming, on the contrary, that $T < \infty$, then we have that

$$\begin{aligned}1 &= \sup_{0<\tau<T}\{\tau\|u(\cdot,\tau)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1}\} \\ &\leqslant \sup_{0<\tau<1}\{\tau\|u(\cdot,\tau)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1}\} + \sup_{1<\tau<T}\{\tau\|u(\cdot,\tau)\|_{\infty,\mathbb{R}^N}^{\sigma(pq-p+1)/(p-\sigma)-1}\} \\ &\leqslant \gamma(\gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa_h} + \gamma_0^{p(\sigma(pq-p+1)/(p-\sigma)-1)/\kappa}), \end{aligned} \tag{5.8}$$

since (5.1), (1.7) and (1.12), (1.13) hold. If we choose $\gamma_0$ sufficiently small, then (5.8) implies a contradiction with the definition of $T$. The proof of theorem 1.8 is complete. □

*Proof of theorem 1.9.* Theorem 1.9 can be proved similarly to the proof of theorem 1.8 and we omit the details. □

## 6. Proof of theorem 1.10

We proceed by contradiction and consider a solution $u \not\equiv 0$ in $\mathbb{R}^N \times (0, \infty)$. Then, by [20, theorem 3.1] (see also [30]), we get that

$$u(x,t) \geqslant \gamma_0 t^{-N/\kappa}, \qquad |x| < \gamma_0 t^{1/\kappa}, \quad t > 1, \tag{6.1}$$

where $\gamma_0 = \gamma_0(N, p, u)$. Using a similar proof to that of [17, theorem 1.2], which was strictly based on [3, lemma 4.1], we can obtain

$$\int_{B_\rho} u^\epsilon(x,t)\,\mathrm{d}x \leqslant \gamma_1 \max\{\rho^{\epsilon\sigma/(q\sigma-1)+N} t^{-\epsilon/(q\sigma-1)}, \rho^{-\epsilon(p-\sigma)/(q\sigma-p+1)+N}\}, \tag{6.2}$$

where $0 < \epsilon < 1$, $\rho = \gamma_2 t^{1/\kappa}$. Then (6.1), (6.2) imply that

$$\gamma_0^\epsilon t^{-N\epsilon/\kappa} \rho^N \leqslant \gamma_1 \max\{\rho^{\epsilon\sigma/(q\sigma-1)+N} t^{-\epsilon/(q\sigma-1)}, \rho^{-\epsilon(p-\sigma)/(q\sigma-p+1)+N}\}, \tag{6.3}$$

i.e.

$$\begin{aligned}\gamma_0 \leqslant \gamma_2 \max\{&t^{-(1-N/\kappa(\sigma(q\sigma-p+1)/(p-\sigma)-1))/\kappa(q\sigma-1)}, \\ &t^{-(1-N/\kappa(\sigma(q\sigma-p+1)/(p-\sigma)-1))/\kappa(q\sigma-1)}\}.\end{aligned} \tag{6.4}$$

Due to (1.18), this clearly leads to an inconsistency as $t \to \infty$, and theorem 1.10 is proved.

## Acknowledgements

This work was supported by the National Natural Science Foundation through the Tianyuan Fund for Mathematics (Grant no. 11126119) and Youth of China (Grant no. 11201124). H.S. thanks the referee for several helpful suggestions which improved the presentation of the manuscript.

## References


1 N. Alaa. Solutions faibles d'équations paraboliques quasi-linéaires avec données initiales mesures. *Ann. Math. Blaise Pascal* **3** (1996), 1–15.

2 L. Amour and M. Ben-Artzi. Global existence and decay for viscous Hamilton–Jacobi equations. *Nonlin. Analysis TMA* **31** (1998), 621–628.

3 D. Andreucci. New results on the Cauchy problem for parabolic systems and equations with strongly nonlinear sources. *Manuscr. Math.* **77** (1992), 127–159.

4 D. Andreucci. Degenerate parabolic equations with initial data measures. *Trans. Am. Math. Soc.* **349** (1997), 3911–3923.

5 D. Andreucci and E. Di Benedetto. On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources. *Annali Scuola Norm. Sup. Pisa IV* **18** (1991), 393–441.

6 M. Ben-Artzi. Global existence and decay for a nonlinear parabolic equation. *Nonlin. Analysis* **19** (1992), 763–768.

7 M. Ben-Artzi. Global properties of some nonlinear parabolic equations. In *Nonlinear partial differential equations and their applications: Collège de France seminar, vol. XIX* (ed. D. Cioranescu and J.-L. Lions) Studies in Mathematics and Its Applications, vol. 31, pp. 57–67 (Elsevier, 2002).

8 M. Ben-Artzi and H. Koch. Decay of mass for a semilinear parabolic equation. *Commun. PDEs* **24** (1999), 869–881.

9 M. Ben-Artzi, P. Souplet and F. B. Weissler. Sur la non-existence et la non-unicite des solutions du problème de Cauchy pour une equation parabolique semi-lineaire. *C. R. Acad. Sci. Paris Sér. I* **329** (1999), 371–376.

10 M. Ben-Artzi, J. Goodman and A. Levy. Remarks on a nonlinear parabolic equation. *Trans. Am. Math. Soc.* **352** (2000), 731–751.

11 M. Ben-Artzi, P. Souplet and F. B. Weissler. The local theory for viscous Hamilton–Jacobi equations in Lebesgue spaces. *J. Math. Pures Appl.* **81** (2002), 343–378.

12 S. Benachour and P. Laurençot. Global solutions to viscous Hamilton–Jacobi equations with irregular initial data. *Commun. PDEs* **24** (1999), 1999–2021.

13 S. Benachour and P. Laurençot. Very singular solutions to a nonlinear parabolic equation with absorption. I. Existence. *Proc. R. Soc. Edinb.* A **131** (2001), 27–44.

14 S. Benachour, P. Laurençot and D. Schmitt. Extinction and decay estimates for viscous Hamilton–Jacobi equations in $\mathbb{R}^N$. *Proc. Am. Math. Soc.* **130** (2001), 1103–1111.

15 S. Benachour, P. Laurençot, D. Schmitt and P. Souplet. Extinction and non-extinction estimates for viscous Hamilton–Jacobi equations in $\mathbb{R}^N$. *Asymp. Analysis* **31** (2002), 229–246.

16 P. Biler, M. Guedda and G. Karch. Asymptotic properties of solutions of the viscous Hamilton–Jacobi equation. *J. Evol. Eqns* **4** (2004), 75–97.

17 M. Y. Chen and J. N. Zhao. On the Cauchy problem of evolution $p$-Laplacian equation with nonlinear gradient term. *Chin. Annals Math.* B **30** (2009), 1–16.

18 E. Di Benedetto. *Degenerate parabolic equations* (Springer, 1993).

19 E. Di Benedetto and A. Friedman. Hölder estimates for nonlinear degenerate parabolic systems. *J. Reine Angew. Math.* **357** (1985), 1–22.

20 E. Di Benedetto and M. A. Herrero. On the Cauchy problem and initial traces for a degenerate parabolic equation. *Trans. Am. Math. Soc.* **134** (1989), 187–224.

21 B. H. Gilding. The Cauchy problem for $u_t = \Delta u + |Du|^q$, large-time behaviour. *J. Math. Pures Appl.* **84** (2005), 753–785.

22 B. H. Gilding, M. Guedda and R. Kersner. The Cauchy problem for $u_t = \Delta u + |Du|^q$. *J. Math. Analysis Applic.* **284** (2003), 733–755.

23 T. Halpin-Healy and Y. C. Zhang. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. *Phys. Rep.* **254** (1995), 215–414.

24 M. Kardar, G. Parisi and Y. C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** (1986), 889–892.

25 J. Krug and H. Spohn. Universality classes for deterministic surface growth. *Phys. Rev.* A **38** (1988), 4271–4283.

26 J. Krug and H. Spohn. Kinetic roughening of growing surfaces. In *Solids far from equilibrium* (ed. C. Godréche), pp. 479–582 (Cambridge University Press, 1991).

27 O. A. Ladyzhenskaja, V. A. Solonikov and N. N. Ural′cema. *Linear and quasilinear equations of parabolic type* (Providence, RI: American Mathematical Society, 1968).

28 S. Z. Lian, H. J. Yuan, C. L. Cao, W. J. Gao and X. J. Xu. On the Cauchy problem for the evolution $p$-Laplacian equations with gradient term and source. *J. Diff. Eqns* **235** (2007), 544–585.

29 J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites nonlinéaires* (Paris: Dunod, 1969).

30 H. F. Shang and F. Q. Li. On the Cauchy problem for the evolution $p$-Laplacian equations with gradient term and source and measures as initial data. *Nonlin. Analysis* **72** (2010), 3396–3411.

31 T. Vicsek. *Fractal growth phenomena*, 2nd edn (Singapore: World Scientific, 1992).

32 J. N. Zhao. On the Cauchy problem and initial traces for the evolution $p$-Laplacian equations with strongly nonlinear sources. *J. Diff. Eqns* **121** (1995), 329–383.

(*Issued 7 June 2013*)