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Maranda's theorem for pure-injective modules and duality

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Abstract. Let R be a discrete valuation domain with field of fractions Q and maximal ideal generated by π. Let Λ be an R-order such that $Q\Lambda$ is a separable Q-algebra. Maranda showed that there exists $k \in \mathbb{N}$ such that for all Λ-lattices L and M, if $L/L\pi^k \simeq M/M\pi^k$, then $L \simeq M$. Moreover, if R is complete and L is an indecomposable Λ-lattice, then $L/L\pi^k$ is also indecomposable. We extend Maranda's theorem to the class of R-reduced R-torsion-free pure-injective Λ-modules.

As an application of this extension, we show that if Λ is an order over a Dedekind domain R with field of fractions Q such that $Q\Lambda$ is separable, then the lattice of open subsets of the R-torsion-free part of the right Ziegler spectrum of Λ is isomorphic to the lattice of open subsets of the R-torsion-free part of the left Ziegler spectrum of Λ.

Furthermore, with k as in Maranda's theorem, we show that if M is R-torsion-free and $H(M)$ is the pure-injective hull of M, then $H(M)/H(M)\pi^k$ is the pure-injective hull of $M/M\pi^k$. We use this result to give a characterization of R-torsion-free pure-injective Λ-modules and describe the pure-injective hulls of certain R-torsion-free Λ-modules.

1 Introduction

Let R be a discrete valuation domain with maximal ideal generated by π and field of fractions Q. Let Λ be an order over R (i.e., an R-algebra that is finitely generated and projective as an R-module) such that $Q\Lambda$ is a separable Q-algebra. For example, $\Lambda = RG$, where G is a finite group and R is a discrete valuation domain whose field of fractions is characteristic zero. Maranda's theorem (see [13], [5, Theorem 30.14]) states that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0 + 1$ and Λ-lattices L, M, $L/L\pi^k \cong M/M\pi^k$ implies $L \cong M$ and if R is complete then L indecomposable implies $L/L\pi^k$ is indecomposable.

For any $M \in \text{Mod-}\Lambda$, $M/M\pi^k$ may be naturally viewed as a module over the $R/R\pi^k$-Artin algebra $\Lambda_k := \Lambda/\Lambda\pi^k$. In this paper, we study the functor from the category of R-torsion-free Λ-modules to the category of Λ_k-modules which sends M to $M/M\pi^k$ for k sufficiently large. In particular, in Section 3, we extend Maranda's theorem to a class of R-reduced R-torsion-free *pure-injective* Λ-modules and show that this functor preserves pure-injective hulls.

Pure-injective modules generalize injective modules, and they are "injective relative to pure embeddings." They correspond, via the tensor embedding, exactly

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to the injective objects in the category of additive functors from the category of finitely presented modules to abelian groups. Topologically, they are characterized as direct summands of compact Hausdorff modules. Pure-injective modules play a prominent role in the model theory of modules because every module is an elementary substructure of its pure-injective hull. Every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules, so the indecomposable pure-injective modules may be viewed as the building blocks of the module category up to elementary equivalence.

The set of isomorphism types of (right) indecomposable pure-injective modules over a ring S is equipped with the topology whose closed sets correspond to definable subcategories of Mod-S. The resulting space is called the (right) Ziegler spectrum Zg_S of S. This space captures the majority of model-theoretic information about Mod-S.

From the perspective of model theory of modules, the natural nonfinitely-presented generalization of a Λ-lattice is an R-torsion-free Λ-module. This is because the smallest definable subcategory of Mod-Λ containing all (right) Λ-lattices is exactly the category, Tf_Λ, of (right) R-torsion-free Λ-modules. We write ${}_\Lambda\mathrm{Tf}$ for the category of R-torsion-free left Λ-modules. Furthermore, the closed set of indecomposable pure-injective modules which are R-torsion-free is called the torsion-free part of the Ziegler spectrum of Λ and is denoted by Zg_Λ^{tf}. (This space is studied in [8, 14, 20].)

An alternative nonfinitely-presented version of a Λ-lattice, the generalized lattice, was introduced in [4] and further studied in [19, 23].

We must exclude the R-divisible R-torsion-free Λ-modules from our generalization of Maranda's theorem because if D is divisible then $D/D\pi^k = 0$. However, every R-torsion-free Λ-module decomposes as a direct sum $D \oplus N$ of an R-divisible module D and an R-reduced module N, i.e., $\bigcap_{i\in\mathbb{N}} N\pi^i = 0$. Thus, by restricting our generalization of Maranda's theorem further to the class of R-reduced R-torsion-free Λ-modules, we do not lose anything because the R-divisible R-torsion-free Λ-modules are just $Q\Lambda$-modules and, by assumption, $Q\Lambda$ is semisimple.

In Section 3, with k_0 as in the classical version of Maranda's theorem, we prove the following theorems.

Theorem 3.4 *Let M, N be R-torsion-free R-reduced pure-injective Λ-modules. If $M/M\pi^k \cong N/N\pi^k$ for some $k \geq k_0 + 1$, then $M \cong N$.*

Theorem 3.5 *Let $k \geq k_0 + 1$. If N is an indecomposable R-torsion-free R-reduced pure-injective Λ-module, then $N/N\pi^k$ is indecomposable.*

Unlike in the classical version of Theorem 3.5, we do not need to assume that R is complete. However, Λ-lattices are pure-injective if and only if R is complete. So this is not unexpected.

Using results from [8], which are applications of Maranda's theorem for Λ-lattices, we get the following.

Theorem 3.8 *Let $k \geq k_0 + 1$. Suppose that M is R-torsion-free and R-reduced. If $u : M \to H(M)$ is the pure-injective hull of M, then $\overline{u} : M/M\pi^k \to H(M)/H(M)\pi^k$ is the pure-injective hull of $M/M\pi^k$.*

Our proofs of these theorems and their applications rely on the fact that the functor taking $M \in \mathrm{Tf}_\Lambda$ to $M/M\pi^k \in \text{Mod-}\Lambda/\Lambda\pi^k$, which, for k sufficiently large, we will

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refer to as *Maranda's functor*, is an interpretation functor. The original definition (see Section 2) of an interpretation functor came out of the model-theoretic notion of an interpretation. However, from an algebraic perspective, interpretation functors are just additive functors which commute with direct limits and direct products.

Thanks to Maranda's theorem, in order to get information about the category of Λ-lattices, we may instead study a subcategory of the category of modules over the Artin algebra $\Lambda/\Lambda\pi^k$. The drawback of both the classical version of Maranda's theorem and our extended version is that mod-Λ_k, respectively, Mod-Λ_k, is almost always significantly more complicated than the category of Λ-lattices, respectively, Tf_Λ. For instance, the order $\mathbb{Z}_{(p)}C(p^2)$ is of finite lattice type (see [3]), but the category of $\mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)}$-free finitely generated $\mathbb{Z}_{(p)}/p^2\mathbb{Z}_{(p)}C(p^2)$-modules is wild [1].

Despite the above, we will see in Sections 4 and 5 that being able to move from Tf_Λ to a module category over an Artin algebra has useful applications.

We now describe the applications in Sections 4 and 5, which are largely independent of each other. Section 4 presents applications of Theorem 3.8 to pure-injectives and pure-injective hulls in Tf_Λ. We give the following characterization of pure-injective R-torsion-free Λ-modules.

Theorem 4.6 *Let $M \in Tf_\Lambda$. Then M is pure-injective if and only if*

(1) *$M/M\pi^k$ is pure-injective for all $k \in \mathbb{N}$ and*
(2) *M is pure-injective as an R-module.*

We also give information about the pure-injective hull of an R-reduced R-torsion-free module M in terms of pure-injective hulls of $M/M\pi^k$ for all $k \geq k_0 + 1$. In particular, when M is reduced, R-torsion-free, and $M/M\pi^k$ is pure-injective for all $k \in \mathbb{N}$, we show (Theorem 4.6) that the pure-injective hull of M is the inverse limit of the Λ-modules $M/M\pi^k$ along the canonical projections.

We use these results to answer the questions at the end of [20]. In particular, we describe the pure-injective hulls of the Prüfer-like modules, denoted T in [20]. We show that these pure-injective hulls are indecomposable and hence are points of the $\widehat{\mathbb{Z}_{(2)}}$-torsion-free part of the Ziegler spectrum of the $\widehat{\mathbb{Z}_{(2)}}$-order $\widehat{\mathbb{Z}_{(2)}}C_2 \times C_2$. As far as we are aware, until now, the only points of Zg_Λ^{tf}, for any order Λ, which have been explicitly described as modules are $\widehat{\Lambda}$-lattices, where $\widehat{R}$ is the completion of R and $\widehat{\Lambda} := \widehat{R} \otimes \Lambda$, and the R-divisible modules, which are just the indecomposable $Q\Lambda$-modules.

The theme of Section 5 is connections between Tf_Λ and ${}_\Lambda\mathrm{Tf}$. Here, we extend our setting to include the case where R is a Dedekind domain with field of fractions Q and Λ is an R-order such that $Q\Lambda$ is a separable Q-algebra. We write ${}_S\mathrm{Zg}$ for the left Ziegler spectrum of S and ${}_\Lambda\mathrm{Zg}^{tf}$ for the torsion-free part of the left Ziegler spectrum of Λ.

Herzog [9] showed that for any ring S, the lattice of open subsets of Zg_S and the lattice of open subsets of ${}_S\mathrm{Zg}$ are isomorphic. Applying Herzog's result directly to Zg_Λ shows that the lattice of open subsets of Zg_Λ^{tf} is isomorphic to the lattice of open subsets of the closed subset of R-divisible modules in ${}_\Lambda\mathrm{Zg}$. Despite this, we are able to show (Theorem 5.2) that the lattice of open subsets of Zg_Λ^{tf} is also isomorphic, in a natural way, to the lattice of open subsets of ${}_\Lambda\mathrm{Zg}^{tf}$. This is the main result of Section 5.

We finish Section 5 by showing (Corollary 5.19) that the m-dimension of the lattice of (right) pp formulas of Λ with respect to the theory of Tf_Λ is equal to the m-

dimension of the lattice of (left) pp formulas of Λ with respect to the theory of ${}_\Lambda\mathrm{Tf}$. As a consequence, we show (Corollary 5.20) that the Krull–Gabriel dimension of $(\mathrm{Latt}_\Lambda, \mathrm{Ab})^{fp}$ is equal to the Krull–Gabriel dimension of $({}_\Lambda\mathrm{Latt}, \mathrm{Ab})^{fp}$, where Latt_Λ is the category of right Λ-lattices and ${}_\Lambda\mathrm{Latt}$ is the category of left Λ-lattices.

Before starting the main body of the paper, the reader should be *warned* that the word *lattice* has two meanings in this paper; the first, a particular type of Λ-module and the second a partially ordered set with meets and joins. Since these objects are so different in character, it should not cause confusion.

2 Preliminaries

We start by introducing some notation and basic definitions relating to orders. For a general introduction to orders and their categories of lattices, we suggest [5].

Let R be a Dedekind domain. We assume throughout that R is not a field. An R-order Λ is an R-algebra which is finitely generated and R-torsion-free as an R-module. A Λ-lattice is a finitely generated Λ-module which is R-torsion-free. We will write Latt_Λ (respectively, ${}_\Lambda\mathrm{Latt}$) for the category of right (respectively, left) Λ-lattices and Tf_Λ (respectively, ${}_\Lambda\mathrm{Tf}$) for the category of right (respectively, left) R-torsion-free modules.

Let $\mathrm{Max}R$ denote the set of nonzero prime ideals of R. If $P \in \mathrm{Max}R$, then Λ_P, the localization of Λ at the multiplicative set $R\backslash P$, is an R_P-order. Let $\widehat{R_P}$ and $\widehat{\Lambda_P}$ denote the P-adic completions of R_P and Λ_P, respectively. Note that $\widehat{\Lambda_P}$ is an $\widehat{R_P}$-order. If $L \in \mathrm{Latt}_\Lambda$ and $P \in \mathrm{Max}R$, then L_P will denote $R_P \otimes_R L$. If $L \in \mathrm{Latt}_\Lambda$, then $\widehat{L_P}$ will denote the P-adic completion of L. Note that if $L \in \mathrm{Latt}_\Lambda$, then L_P is a Λ_P-lattice and $\widehat{L_P}$ is a $\widehat{\Lambda_P}$-lattice.

We will assume that $Q\Lambda$ is a separable Q-algebra. This is used in two principal ways: first, it is an assumption of Maranda's theorem for lattices over orders (see [5, 30.12]), and second, it implies that for all nonzero prime ideals $P \lhd R$, $\widehat{Q}\widehat{\Lambda_P} = \widehat{Q}\Lambda$ is a semisimple $\widehat{Q}$-algebra, where $\widehat{Q}$ denotes the field of fractions of $\widehat{R_P}$.

We now give a summary of the notions from model theory of modules that will be used in this paper. For a more detailed introduction, the reader is referred to [15, 17].

We will write $\mathbf{x}$ for tuples of variables and likewise $\mathbf{m}$ for tuples of elements in a module.

Let S be a ring. A (right) *pp-n-formula* is a formula in the language of S-modules of the form

$$\exists \mathbf{y}\ (\mathbf{y}, \mathbf{x})A = 0,$$

where A is an $(l+n) \times m$ matrix with entries from S, $\mathbf{y}$ is an l-tuple of variables, $\mathbf{x}$ is an n-tuple of variables, and l, n, m are natural numbers.

If $M \in \text{Mod-}S$ and ϕ is a pp-n-formula, then we write $\phi(M)$ for the *solution set* of ϕ in M. For any pp-n-formula ϕ and S-module M, $\phi(M)$ is an $\mathrm{End}(M)$-submodule of M^n under the diagonal action of $\mathrm{End}(M)$ on M^n.

After identifying (right) pp-n-formulas ϕ, ψ such that $\phi(M) = \psi(M)$ for all $M \in \text{Mod-}S$, the set of pp-n-formulas becomes a lattice under inclusion of solution sets, i.e., $\psi \le \phi$ if $\psi(M) \subseteq \psi(M)$ for all $M \in \text{Mod-}S$. We denote this lattice by pp_S^n and the left module version by ${}_S\mathrm{pp}^n$. If X is a collection of (right) S-modules, then we write

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$\mathrm{pp}_S^n X$ for the quotient of pp_S^n under the equivalence relation $\phi \sim_X \psi$ if $\phi(M) = \psi(M)$ for all $M \in X$.

For $\phi, \psi \in \mathrm{pp}_S^n$, we will write $\phi + \psi$ for the join (least upper bound) of ϕ and ψ in pp_S^n and $\phi \wedge \psi$ for the meet (greatest lower bound) of ϕ and ψ in pp_S^n. Note that, for all $M \in \text{Mod-}S$, $(\phi + \psi)(M) = \phi(M) + \psi(M)$ and $(\phi \wedge \psi)(M) = \phi(M) \cap \psi(M)$.

A *pp- n-pair*, written ϕ / ψ, is a pair of pp-n-formulas ϕ, ψ such that $\phi(M) \supseteq \psi(M)$ for all S-modules M. If ϕ / ψ is a pp-n-pair, then we write $[\psi, \phi]$ for the interval in pp_S^n, that is, the set of $\sigma \in \mathrm{pp}_S^n$ such that $\psi \leq \sigma \leq \phi$. If X is a collection of (right) S-modules, we will write $[\psi, \phi]_X$ for the corresponding interval in $\mathrm{pp}_S^n X$.

If $\mathbf{m}$ is an n-tuple of elements from a module M, then the *pp-type* of $\mathbf{m}$ is the set of pp-n-formulas ϕ such that $\mathbf{m} \in \phi(M)$. If $M \in \text{mod-}S$ and $\mathbf{m}$ is an n-tuple of elements from M, then [17, Lemma 1.2.6] there exists $\phi \in \mathrm{pp}_S^n$ such that ψ is in the pp-type of $\mathbf{m}$ if and only if $\psi \geq \phi$. In this case, we say that ϕ *generates* the pp-type of $\mathbf{m}$.

For each $n \in \mathbb{N}$, Prest defined a lattice anti-isomorphism $D : \mathrm{pp}_S^n \to {}_S\mathrm{pp}^n$ (see [15, Theorem 8.21], [17, Section 1.3.1]). As is standard, we denote its inverse ${}_S\mathrm{pp}^n \to \mathrm{pp}_S^n$ also by D. Apart from the fact that for $a \in S$, $D(xa = 0)$ is $a|x$ and $D(a|x)$ is $ax = 0$, we will not need to explicitly take the dual of a pp formula here, so we will not give its definition.

An embedding $f : M \to N$ is a *pure-embedding* if for all $\phi \in \mathrm{pp}_S^1$, $\phi(N) \cap f(M) = f(\phi(M))$. Equivalently, for all $L \in S\text{-mod}$, $f \otimes - : M \otimes L \to N \otimes L$ is an embedding. We say N is *pure-injective* if every pure-embedding $g : N \to M$ is a split embedding. Equivalently, N is pure-injective if and only if it is *algebraically compact* [17, Theorem 4.3.11]. That is, for all $n \in \mathbb{N}$, if for each $i \in \mathcal{I}$, $\mathbf{a_i} \in N$ is an n-tuple and ϕ_i is a pp-n-formula, then $\bigcap_{i \in \mathcal{I}} \mathbf{a_i} + \phi_i(N) = \varnothing$ implies there is some finite subset $\mathcal{I}'$ of $\mathcal{I}$ with $\bigcap_{i \in \mathcal{I}'} \mathbf{a_i} + \phi_i(N) = \varnothing$.

We will write pinj_S (respectively, ${}_S\mathrm{pinj}$) for the set of (isomorphism types of) indecomposable pure-injective right (respectively, left) S-modules.

We say a pure-embedding $i : M \to N$ with N pure-injective is a *pure-injective hull* of M if for every other pure-embedding $g : M \to K$ where K is pure-injective, there is a pure-embedding $h : N \to K$ such that $hi = g$. The pure-injective hull of M is unique up to isomorphism over M, and we will write $H(M)$ for any module N such that the inclusion of M in N is a pure-injective hull of M.

The following lemma will be used in Section 5. Its proof is exactly as in [14, Lemma 3.1].

Lemma 2.1 *Let M be a Λ-lattice. The pure-injective hull of M is isomorphic to $\prod_{P \in MaxR} \widehat{M_P}$.*

A full subcategory of a module category Mod-S is a *definable subcategory* if it satisfies the equivalent conditions in the following theorem.

Theorem 2.2 [17, Theorem 3.4.7] *The following statements are equivalent for $\mathcal{X}$ a full subcategory of Mod-S.*

(1) *There exists a set of pp-pairs $\{\phi_i/\psi_i \mid i \in I\}$ such that $M \in \mathcal{X}$ if and only if $\phi_i(M) = \psi_i(M)$ for all $i \in I$.*
(2) *$\mathcal{X}$ is closed under direct products, direct limits, and pure submodules.*

(3) *$\mathcal{X}$ is closed under direct products, reduced products, and pure submodules.*
(4) *$\mathcal{X}$ is closed under direct products, ultrapowers, and pure submodules.*

For an R-order Λ, a particularly important definable subcategory is, Tf_Λ, the class of all R-torsion-free Λ-modules. It is the class of Λ-modules such that for all nonzero $r \in R$, the solution set of $xr = 0$ in M is equal to the solution set of $x = 0$ in M.

Given a class of modules $\mathcal{C}$, let $\langle \mathcal{C} \rangle$ denote the smallest definable subcategory containing $\mathcal{C}$. Since all modules in Tf_Λ are direct unions of their finitely generated submodules and a finitely generated R-torsion-free module is a Λ-lattice, $\langle \text{Latt}_\Lambda \rangle = \text{Tf}_\Lambda$.

If $\mathcal{C} \subseteq \text{Mod-}S$, then we will write $\text{pinj}(\mathcal{C})$ for the set of (isomorphism types of) indecomposable pure-injective S-modules contained in $\mathcal{C}$. By [17, Corollary 5.1.4], definable subcategories of Mod-S are determined by the indecomposable pure-injective S-modules they contain, i.e., $\mathcal{C} = \langle \text{pinj}(\mathcal{C}) \rangle$.

The (right) *Ziegler spectrum* of a ring S, denoted Zg_S, is a topological space whose points are isomorphism classes of indecomposable pure-injective (right) S-modules and which has a basis of open sets given by

$$(\phi / \psi) = \{M \in \text{pinj}_S \mid \phi(M) \supsetneq \psi(M)\phi(M)\},$$

where φ, ψ range over (right) pp-1-formulas. We write ${}_S\text{Zg}$ for the left Ziegler spectrum of S.

The sets (ϕ / ψ) are compact, in particular, Zg_S is compact.

From (i) of Theorem 2.2, it is clear that if $\mathcal{X}$ is a definable subcategory of Mod-S, then $\mathcal{X} \cap \text{pinj}_S$ is a closed subset of Zg_S and that all closed subsets of Zg_S arise in this way. Since definable subcategories are determined by the indecomposable pure-injective modules they contain, if $\mathcal{X}, \mathcal{Y}$ definable subcategories of Mod-S, then $\mathcal{X} \cap \text{Zg}_S = \mathcal{Y} \cap \text{Zg}_S$ if and only if $\mathcal{X} = \mathcal{Y}$. Thus, there is an inclusion preserving correspondence between the closed subsets of Zg_S and the definable subcategories of Mod-S. If $\mathcal{X}$ is a definable subcategory of Mod-S, then we will write $\text{Zg}(\mathcal{X})$ for the Ziegler spectrum of $\mathcal{X}$, that is, $\mathcal{X} \cap \text{Zg}_S$ with the topology inherited from Zg_S. When Λ is an R-order, we will write Zg^{tf}_Λ (respectively, ${}_\Lambda\text{Zg}^{tf}$) for $\text{Zg}(\text{Tf}_\Lambda)$ (respectively, $\text{Zg}({}_\Lambda\text{Tf})$).

We finish this section by introducing interpretation functors and proving a result about them which we will need in Section 5.

Let $\mathcal{C} \subseteq \text{Mod-}S$ and $\mathcal{D} \subseteq \text{Mod-}T$ be definable subcategories. Let ϕ/ψ be a pp-m-pair over S and for each $t \in T$, let $\rho_t(\overline{x}, \overline{y})$ be a pp-$2m$-formula such that for each $M \in \mathcal{C}$, the solution set $\rho_t(M, M) \subseteq M^m \times M^m$ defines an endomorphism ρ_t^M of the abelian group $\phi(M)/\psi(M)$ and such that $\phi(M)/\psi(M)$ is a T-module in $\mathcal{D}$ when for all $t \in T$, the action of t on $\phi(M)/\psi(M)$ is given by ρ_t^M. In this situation, $(\phi/\psi; (\rho_t)_{t \in T})$ defines an additive functor $I : \mathcal{C} \to \mathcal{D}$. Following [16], we call any functor equivalent to one defined in this way an *interpretation functor*.

From the definition, it is clear that for $k \in \mathbb{N}$, the functor $I : \text{Tf}_\Lambda \to \text{Mod-}\Lambda/\pi^k\Lambda$ which sends $M \in \text{Tf}_\Lambda$ to $M/M\pi^k$ is an interpretation functor. We will consider another interpretation functor, Butler's functor, at the end of Section 4.

The following theorem, due to Prest in full generality and Krause in a special case, gives a completely algebraic characterization of interpretation functors.

Theorem 2.3 [11, Theorem 7.2], [18, Corollary 25.3] *An additive functor $I : \mathcal{C} \to \mathcal{D}$ is an interpretation functor if and only if it commutes with direct products and direct limits.*

There are many ways to see that interpretation functors preserve pure-injectivity. Working with pp formulas, it is easiest to show that interpretation functors preserve algebraic compactness by translating systems of cosets of solution sets of pp formulas for IN into a system of cosets of solution sets of pp formulas for N via I. For the more categorically minded, the most direct route is to use the fact [17, Theorem 4.3.6] that a module M is pure-injective if and only if for any cardinal κ, the summation map $\Sigma_M : M^{(\kappa)} \to M$ factors through the canonical embedding of $M^{(\kappa)}$ into M^κ. Note that since interpretation functors are additive and commute with direct limits, they commute with infinite direct sums. One sees that $I\Sigma_M$ is the summation map $\Sigma_{IM} : IM^{(\kappa)} \to IM$ because it is the unique map which is the identity when composed with the component maps IM into $IM^{(\kappa)}$.

Define $\ker I$ to be the definable subcategory of objects $L \in \mathcal{C}$ such that $IL = 0$. For $\mathcal{D}'$ a definable subcategory of $\mathcal{D}$, let $I^{-1}\mathcal{D}'$ be the definable subcategory of objects $L \in \mathcal{C}$ such that $IL \in \mathcal{D}'$.

The following lemma is used in various places in the literature. It follows easily from (3) of Theorem 2.2.

Lemma 2.4 *Let $I : \mathcal{C} \to \mathcal{D}$ be an interpretation functor and $\mathcal{C}'$ a definable subcategory of $\mathcal{C}$. Then the closure of $I\mathcal{C}'$ under pure-subobjects is a definable subcategory of $\mathcal{D}$.*

Lemma 2.5 *Let $I : \mathcal{C} \to \mathcal{D}$ be an interpretation functor such that for all $N \in pinj(\mathcal{C})$, $IN = 0$, or $IN \in pinj(\mathcal{D})$ and if $N, M \in pinj(\mathcal{C})$, $IN, IM \neq 0$, and $IN \cong IM$, then $N \cong M$.*

(i) *If $\mathcal{C}'$ is a definable subcategory of $\mathcal{C}$ containing* $\ker I$, *then $I^{-1}\langle I\mathcal{C}'\rangle = \mathcal{C}'$.*
(ii) *If $\mathcal{D}'$ is a definable subcategory of $\langle I\mathcal{C}\rangle$, then $\langle I(I^{-1}\mathcal{D}')\rangle = \mathcal{D}'$.*

Proof (i) Suppose $M \in \mathcal{C}'$. Then $IM \in \langle I\mathcal{C}'\rangle$. So $M \in I^{-1}\langle I\mathcal{C}'\rangle$.

Suppose $N \in \text{pinj}(\mathcal{C})$ and $N \in I^{-1}\langle I\mathcal{C}'\rangle$. If $IN = 0$, then $N \in \mathcal{C}'$, since $\ker I \subseteq \mathcal{C}'$. So we may assume that $IN \neq 0$ and IN is a pure-subobject of IL for some $L \in \mathcal{C}'$ by Lemma 2.4. Since N is pure-injective, so is IN. Hence, IN is a direct summand of IL. By the hypotheses on I, IN is indecomposable. So by [17, Proposition 18.2.24], there exists $L' \in \text{pinj}(\mathcal{C}')$ such that IN is a direct summand of IL'. By the hypothesis on I, IL' is indecomposable and hence $IN \cong IL'$. By the other hypothesis on I, $L' \cong N$. Thus, $N \in \mathcal{C}'$, as required.

Since definable subcategories are determined by the indecomposable pure-injective modules they contain, $I^{-1}\langle I\mathcal{C}'\rangle \subseteq \mathcal{C}'$.

(ii) Suppose $\mathcal{D}'$ is a definable subcategory of $\langle I\mathcal{C}\rangle$. Since $\mathcal{D}'$ is a definable subcategory, $\langle I(I^{-1}\mathcal{D}')\rangle \subseteq \mathcal{D}'$ if and only if $I(I^{-1}\mathcal{D}') \subseteq \mathcal{D}'$. Take $M \in I^{-1}\mathcal{D}'$. By definition, $IM \in \mathcal{D}'$. So $I(I^{-1}\mathcal{D}') \subseteq \mathcal{D}'$.

We now show that $\mathcal{D}' \subseteq \langle I(I^{-1}\mathcal{D}')\rangle$. Suppose $N \in \text{pinj}(\mathcal{D}')$. Since $\mathcal{D}' \subseteq \langle I\mathcal{C}\rangle$, by Lemma 2.4, there exists $L \in \mathcal{C}$ such that N is pure-subobject of IL. Thus, N is a direct summand of IL. By [17, Proposition 18.2.24], we may assume L is also indecomposable pure-injective. Thus, $N \cong IL$. So $L \in I^{-1}\mathcal{D}'$ and $N \cong IL \in I(I^{-1}\mathcal{D}')$, as required. ∎

Corollary 2.6 *Let $I : \mathcal{C} \to \mathcal{D}$ be an interpretation functor such that for all $N \in pinj(\mathcal{C})$, $IN = 0$, or $IN \in pinj(\mathcal{D})$ and if $N, M \in pinj(\mathcal{C})$, $IN, IM \neq 0$, and $IN \cong IM$, then $N \cong$*

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M. The maps

$$\ker I \subseteq \mathcal{C}' \subseteq \mathcal{C} \mapsto \langle I\mathcal{C}' \rangle$$

and

$$\mathcal{D}' \subseteq \langle I\mathcal{C} \rangle \mapsto I^{-1}\mathcal{D}'$$

give a inclusion preserving bijective correspondence between definable subcategories in $\langle I\mathcal{C} \rangle$ *and definable subcategories of* $\mathcal{C}$ *containing* $\ker I$.

Proof We have shown that if $\mathcal{C}'$ is a definable subcategory of $\mathcal{C}$ containing $\ker I$, then $I^{-1}\langle I\mathcal{C}' \rangle = \mathcal{C}'$, and if $\mathcal{D}'$ is a definable subcategory of $\langle I\mathcal{C}' \rangle$, then $\langle I(I^{-1}\mathcal{D}') \rangle = \mathcal{D}'$.

That this correspondence is inclusion preserving follows directly from its definition. ■

The following is very close to [11, Theorem 7.8], [16, Theorem 3.19], and [17, Corollary 18.2.26], but our hypotheses are slightly different. This statement will be needed in Section 5.

Proposition 2.7 *Let* $I : \mathcal{C} \to \mathcal{D}$ *be an interpretation functor such that for all* $N \in \mathrm{pinj}(\mathcal{C})$, $IN = 0$, *or* $IN \in \mathrm{pinj}(\mathcal{D})$ *and if* $N, M \in \mathrm{pinj}(\mathcal{C})$, $IN, IM \neq 0$, *and* $IN \cong IM$, *then* $N \cong M$. *The assignment* $N \mapsto IN$ *induces a homeomorphism between* $\mathrm{Zg}(\mathcal{C}) \setminus \ker I$ *and its image in* $\mathrm{Zg}(\mathcal{D})$ *which is closed.*

Proof Suppose $L \in \langle I\mathcal{C} \rangle \cap \mathrm{Zg}(\mathcal{D})$. Then L is a pure-subobject of some IN for some $N \in \mathrm{Zg}(\mathcal{C})$. By hypothesis on I, IN is indecomposable. So $L \cong IN$. Thus, the closed set $\langle I\mathcal{C} \rangle \cap \mathrm{Zg}(\mathcal{D})$ is the image of $\mathrm{Zg}(\mathcal{C}) \setminus \ker I$ under I.

Suppose X is a closed subset of $\mathrm{Zg}(\mathcal{D})$ contained in $I\mathrm{Zg}(\mathcal{C})$. Let $\mathcal{X}$ be the definable subcategory of $\mathcal{D}$ generated by X. Let $\mathcal{Y} := I^{-1}\mathcal{X}$ and $Y := \mathcal{Y} \cap \mathrm{Zg}(\mathcal{C})$. Since $\mathcal{X} \subseteq \langle I\mathcal{C} \rangle$, $IL \in \mathcal{X}$ if and only if $L \in \mathcal{Y}$ by Lemma 2.5. So $N \in Y$ if and only if $IN \in X$. Thus, $N \mapsto IN$ is continuous.

Suppose Y is a closed subset of $\mathrm{Zg}(\mathcal{C})$. We may replace Y by the closed subset $Y \cup (\ker I \cap \mathrm{Zg}(\mathcal{C}))$ without changing its intersection with $\mathrm{Zg}(\mathcal{C}) \setminus \ker I$. Let $\mathcal{Y}$ be the definable subcategory of $\mathcal{C}$ generated by Y, and let $X = \langle I\mathcal{Y} \rangle \cap \mathrm{Zg}(\mathcal{D})$. Now, $N \in \mathcal{Y}$ if and only $N \in I^{-1}\langle I\mathcal{Y} \rangle$ by Lemma 2.5. So $N \in Y$ if and only if $IN \in X$. Thus, the inverse of $N \mapsto IN$ is continuous. ■

3 Maranda's functor

Throughout this section, R will be a discrete valuation domain with field of fractions Q and maximal ideal generated by π, and Λ will be an R-order such that $Q\Lambda$ is a separable Q-algebra.

The basis of Maranda's theorem is the existence[1] of a nonnegative integer l such that for all Λ-lattices L and M,

$$\pi^l \operatorname{Ext}^1(L, M) = 0.$$

Throughout this section, let k_0 be the smallest such nonnegative integer. We will call this natural number *Maranda's constant* (for Λ as an R-order).

[1] The existence of such a nonnegative integer is implied by the fact that $Q\Lambda$ is separable (see [5, Corollary 29.5] and the discussion just after [5, 30.12]).

Note that since Λ is Noetherian, $\text{Ext}^1(L,-)$ is finitely presented as a functor in $(\text{mod-}\Lambda, \text{Ab})$ (see [17, Theorem 10.2.35]). Hence, $\pi^{k_0}\text{Ext}^1(L,-)$ is also finitely presented. Since Tf_Λ is the smallest definable subcategory containing Latt_Λ, $\pi^{k_0}\text{Ext}^1(L,N) = 0$ for all $L \in \text{Latt}_\Lambda$ and $N \in \text{Tf}_\Lambda$.

Throughout this section, when $k \in \mathbb{N}$ is clear from the context, for $M \in \text{Mod-}\Lambda$ and $m \in M$, we will often write $\overline{M}$ for $M/M\pi^k$ and $\overline{m}$ for $m + M\pi^k$. If $f : M \to N \in \text{Mod-}\Lambda$, then we will write $\overline{f}$ for the induced homomorphism from $M/M\pi^k$ to $N/N\pi^k$. This is to allow us to use subscripts on modules as indices and to ease readability. We will write Λ_k for the ring $\Lambda/\pi^k\Lambda$.

The proof of the next lemma can easily be extracted from the proof of [5, Theorem 30.14].

Lemma 3.1 *Let $L \in \text{Latt}_\Lambda$ and $M \in \text{Tf}_\Lambda$. If $k \geq k_0 + 1$, then for all $g \in \text{Hom}_{\Lambda_k}(L/L\pi^k, M/M\pi^k)$, there exists $h \in \text{Hom}_\Lambda(L,M)$ such that for all $m \in L$, $\pi^{k-k_0} + \Lambda\pi^k | \overline{h(m)} - g(\overline{m})$.*

The following proposition is key to proving both parts of our extension of Maranda's theorem.

Proposition 3.2 *Let M, N be R-torsion-free Λ-modules with N pure-injective. If $k \geq k_0 + 1$, then for all $g \in \text{Hom}_{\Lambda_k}(M/M\pi^k, N/N\pi^k)$, there exists $h \in \text{Hom}_\Lambda(M,N)$ such that for all $m \in M$, $\pi^{k-k_0} + \Lambda\pi^k | \overline{h(m)} - g(\overline{m})$.*

Proof Since $M \in \text{Tf}_\Lambda$, there exists a directed system of Λ-lattices L_i for $i \in I$ and $\sigma_{ij} : L_i \to L_j$ for $i \leq j \in I$ such that M is the direct limit of this directed system. Let $f_i : L_i \to M$ be the component maps.

Our aim is to find $h_i : L_i \to N$ for all $i \in I$ such that $h_i = h_j\sigma_{ij}$ and for all $a \in L_i$, $\pi^{k-k_0} + \Lambda\pi^k | \overline{h_i(a)} - g(\overline{f_i(a)})$.

If we can do this, then there exists $h : M \to N$ such that $h_i = hf_i$ for all $i \in I$. This homomorphism is then as required by the statement of the proposition for the following reasons. For all $m \in M$, there exist $i \in I$ and $a \in L_i$ such that $f_i(a) = m$. So

$$\overline{h(m)} - g(\overline{m}) = \overline{hf_i(a)} - g(\overline{f_i(a)}) = \overline{h_i(a)} - g(\overline{f_i(a)})$$

is divisible by $\pi^{k-k_0} + \Lambda\pi^k$.

For each $i \in I$, let $\varepsilon_i : L_i \to N$ be such that for all $a \in L_i$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{\varepsilon_i(a)} - g(\overline{f_i(a)})$. Such an ε_i exists by Lemma 3.1 since L_i is a Λ-lattice.

Let $\mathbf{c_i} := (c_{i1}, \dots, c_{il_i})$ generate L_i as an R-module, and let ϕ_i generate the pp-type of $\mathbf{c_i}$. Note that $\mathbf{m} \in \phi_i(N)$ if and only if there exists a $q : L_i \to N$ such that $q(\mathbf{c_i}) = \mathbf{m}$.

Let

$$\chi_i(x_1, \dots, x_{l_i}) := \phi_i(x_1, \dots, x_{l_i}) \wedge \bigwedge_{j=1}^{l_i} \pi^{k-k_0} | x_j.$$

We now show that $\mathbf{m} - \varepsilon_i(\mathbf{c_i}) \in \chi_i(N)$ if and only if there exists a homomorphism $q \in \text{Hom}(L_i, N)$ such that $q(\mathbf{c_i}) = \mathbf{m}$ and for all $a \in L_i$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{q(a)} - g(\overline{f_i(a)})$.

Suppose $\mathbf{m} - \varepsilon_i(\mathbf{c}_i) \in \chi_i(N)$. Since $\varepsilon_i(\mathbf{c_i}) \in \phi_i(N)$, $\mathbf{m} \in \phi_i(N)$, and hence there exists $q \in \text{Hom}(L_i, N)$ such that $q(\mathbf{c_i}) = \mathbf{m}$. For each $1 \leq j \leq l_i$, π^{k-k_0} divides $q(c_{ij})$ −

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$\varepsilon_i(c_{ij}) = m_j - \varepsilon_i(c_{ij})$. By definition of ε_i, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{\varepsilon_i(c_{ij})} - g(\overline{f_i(c_{ij})})$. So $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{q(c_{ij})} - g(\overline{f_i(c_{ij})})$ for $1 \leq j \leq l_i$. Since $\mathbf{c_i}$ generates L_i, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{q(a)} - g(\overline{f_i(a)})$ for all $a \in L_i$.

Now, suppose that $q \in \mathrm{Hom}(L_i, N)$ is such that $q(\mathbf{c_i}) = \mathbf{m}$ and that for all $a \in L_i$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{q(a)} - g(\overline{f_i(a)})$. Then $\mathbf{m} - \varepsilon_i(\mathbf{c_i}) = (q - \varepsilon_i)(\mathbf{c_i}) \in \phi_i(N)$. By definition of ε_i, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{\varepsilon_i(a)} - g(\overline{f_i(a)})$ for all $a \in L_i$. So $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{q(a)} - \overline{\varepsilon_i(a)}$ for all $a \in L_i$. Since $k \geq k - k_0$, π^{k-k_0} divides $q(a) - \varepsilon_i(a)$ for all $a \in L_i$. So, in particular, π^{k-k_0} divides $q(c_{ij}) - \varepsilon_i(c_{ij}) = m_j - \varepsilon_i(c_{ij})$ for all $1 \leq j \leq l_i$. Thus, $\mathbf{m} - \varepsilon_i(c_i) \in \chi_i(N)$, as required.

For $i \leq j \in I$, let $\mathbf{t_{ij}} \in R^{l_j \times l_i}$ be such that $\sigma_{ij}(\mathbf{c_i}) = \mathbf{c_j} \cdot \mathbf{t_{ij}}$.

Consider the system of linear equations and cosets of pp-definable subsets

$$(1)_i \qquad \mathbf{x_i} \in \varepsilon_i(\mathbf{c_i}) + \chi_i(N)$$

for $i \in I$ and

$$(2)_{ij} \qquad \mathbf{x_i} = \mathbf{x_j} \cdot \mathbf{t_{ij}}$$

for $i \leq j \in I$.

Let $I_0 \subseteq I$ be a finite subset of I. Since I is directed, by adding an element to I_0 if necessary, we may assume that there is a $p \in I_0$ such that $i \leq p$ for all $i \in I_0$.

Let $\mathbf{m_p} = \varepsilon_p(\mathbf{c_p})$ and for $i \in I_0$, and let $\mathbf{m_i} = \mathbf{m_p} \cdot \mathbf{t_{ip}}$. Then

$$\mathbf{m_i} = \varepsilon_p(\mathbf{c_p}) \cdot \mathbf{t_{ip}} = \varepsilon_p(\mathbf{c_p} \cdot \mathbf{t_{ip}}) = \varepsilon_p(\sigma_{ip}(\mathbf{c_i})),$$

for all $i \in I_0$.

Suppose that $i \leq j \in I_0$. Then $\sigma_{ip} = \sigma_{jp} \circ \sigma_{ij}$. So

$$\mathbf{m_i} = \varepsilon_p(\sigma_{jp} \circ \sigma_{ij}(\mathbf{c_i})) = \varepsilon_p(\sigma_{jp}(\mathbf{c_j} \cdot \mathbf{t_{ij}})) = \varepsilon_p(\sigma_{jp}(\mathbf{c_j})) \cdot \mathbf{t_{ij}} = \mathbf{m_j} \cdot \mathbf{t_{ij}}.$$

Thus, $(\mathbf{m_i})_{i \in I_0}$ satisfies $(2)_{ij}$ for all $i \leq j \in I_0$.

We now need to show that for all $i \in I_0$, $\mathbf{m_i} - \varepsilon_i(\mathbf{c_i}) \in \chi_i(N)$. Let $q := \varepsilon_p \circ \sigma_{ip}$. Then $q(\mathbf{c_i}) = \varepsilon_p(\sigma_{ip}(\mathbf{c_i})) = \mathbf{m_i}$; furthermore, by definition of ε_p, for all $a \in L_i$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{\varepsilon_p(\sigma_{ip}(a))} - g(\overline{f_p(\sigma_{ip}(a))}) = \overline{q(a)} - g(\overline{f_i(a)})$. Thus, using the characterization of the solution set of $\chi_i(N)$ proved earlier, $\mathbf{m_i} - \varepsilon_i(\mathbf{c_i}) \in \chi_i(N)$.

Since the system of equations $(1)_i$ and $(2)_{ij}$ is finitely solvable and N is pure-injective, there exists $(\mathbf{m_i})_{i \in I}$ with $\mathbf{m_i} \in N$ satisfying $(1)_i$ and $(2)_{ij}$ for all $i \leq j \in I$. For each $i \in I$, let $h_i : L_i \to N$ be the homomorphism which sends $\mathbf{c_i}$ to $\mathbf{m_i}$. Condition $(2)_{ij}$ ensures that for all $i \leq j \in I$, $h_i = h_j \circ \sigma_{ij}$. This is because $h_j(\sigma_{ij}(\mathbf{c_i})) = h_j(\mathbf{c_j} \cdot \mathbf{t_{ij}}) = h_j(\mathbf{c_j}) \cdot \mathbf{t_{ij}} = \mathbf{m_j} \cdot \mathbf{t_{ij}} = \mathbf{m_i}$. Condition $(1)_i$ ensures that $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{h_i(a)} - g(\overline{f_i(a)})$ for all $a \in L_i$. ∎

Lemma 3.3 *Let $N \in$ Mod-Λ_k, and let $g, \sigma \in \mathrm{End}N$. Suppose that for all $m \in N$, $\pi + \Lambda\pi^k | \sigma(m)$. Then $g - \sigma$ is an isomorphism if and only if g is an isomorphism.*

Proof Suppose that g is an isomorphism. Then $(g - \sigma)g^{-1} = \mathrm{Id}_N - \sigma g^{-1}$. Let $h := \sigma g^{-1}$ and $f := \mathrm{Id}_N + h + \cdots h^{k-1}$. Since $\pi + \Lambda\pi^k | \sigma(m)$ for all $m \in N$, $h^k = 0$. Thus, $(\mathrm{Id}_N - h) \circ f = f \circ (\mathrm{Id}_N - h) = \mathrm{Id}_N$. So $(g - \sigma)g^{-1}f = \mathrm{Id}_N$ and $g^{-1}f(g - \sigma) = g^{-1}f(g - \sigma)g^{-1}g = \mathrm{Id}_N$. Therefore, $g - \sigma$ is an isomorphism. For the converse,

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note that for all $m \in N$, $\pi + \Lambda\pi^k | - \sigma(m)$. Thus, the implication we have just proved also shows that if $g - \sigma$ is an isomorphism, then $g = (g - \sigma) - (-\sigma)$ is an isomorphism. ■

Theorem 3.4 *Let $M, N \in Tf_\Lambda$ be R-reduced and pure-injective. If $M/M\pi^k \cong N/N\pi^k$ for some $k \geq k_0 + 1$, then $M \cong N$.*

Proof We first show that if $f : M \to N$ is such that $\overline{f} : \overline{M} \to \overline{N}$ is an isomorphism, then f is an isomorphism.

Suppose $\overline{f}$ is an isomorphism and $f(m) = 0$. If $m \neq 0$, then since M is reduced, there exists $n \in M$ and l a nonnegative integer such that $m = n\pi^l$ where π does not divide n. Since N is R-torsion-free, $f(m) = f(n)\pi^l = 0$ implies $f(n) = 0$. So $\overline{f}(\overline{n}) = 0$. Therefore, $\overline{n} = 0$. This implies π divides n, contradicting our assumption. So $m = 0$. Therefore, f is injective.

We now show that f is surjective. Since $\overline{f}$ is surjective, for all $n \in N$, there exists $m \in M$ such that $n - f(m) \in N\pi^k$. Suppose m_l is such that $n - f(m_l) \in N\pi^{lk}$. Let $a\pi^{lk} = n - f(m_l)$. There exists $b \in M$ such that $a - f(b) \in N\pi^k$. Thus, $a\pi^{lk} - f(b)\pi^{lk} \in N\pi^{(l+1)k}$. So $n - f(b\pi^{lk} + m_l) \in N\pi^{(l+1)k}$ and $(b\pi^{lk} + m_l) - m_l \in M\pi^{lk}$. So there exists a sequence $(m_l)_{l\in\mathbb{N}}$ in M such that for all $l \in \mathbb{N}$, $n - f(m_l) \in N\pi^{lk}$ and $m_{l+1} - m_l \in M\pi^{lk}$. Since M is pure-injective, there exists an $m \in M$ such that $m - m_l \in M\pi^{kl}$ for all $l \in \mathbb{N}$. Thus, $f(m) - n = f(m - m_l) - (n - f(m_l)) \in N\pi^{kl}$ for all $l \in \mathbb{N}$. Since N is reduced, $f(m) = n$.

Suppose that $g : \overline{M} \to \overline{N}$ is an isomorphism with inverse $h : \overline{N} \to \overline{M}$. There exists $e \in \mathrm{Hom}_\Lambda(M, N)$ such that for all $m \in M$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{e(m)} - g(\overline{m})$ and $f \in \mathrm{Hom}_\Lambda(N, M)$ such that for all $m \in N$, $\pi^{k-k_0} + \Lambda\pi^k$ divides $\overline{f(m)} - h(\overline{m})$. Since $\overline{f} \circ \overline{e} = (\overline{f} - h) \circ (\overline{e} - g) + (\overline{f} - h) \circ g + h \circ (\overline{e} - g) + h \circ g$, Lemma 3.3 implies that $\overline{f} \circ \overline{e}$ is an isomorphism. Similarly, we can show that $\overline{e} \circ \overline{f}$ is an isomorphism. Thus, $\overline{e}$ and $\overline{f}$ are both isomorphisms. So the above arguments imply that e and f are both isomorphisms. ■

Theorem 3.5 *Let $k \geq k_0 + 1$. If N is an indecomposable R-torsion-free R-reduced pure-injective Λ-module, then $N/N\pi^k$ is indecomposable.*

Proof We will show that for all $f \in \mathrm{End}\overline{N}$, either f is an isomorphism, or $1 - f$ is an isomorphism. Hence, $\mathrm{End}\overline{N}$ is local.

Proposition 3.2 implies that the homomorphism sending $f \in \mathrm{End}N$ to $\overline{f} \in \mathrm{End}\overline{N}$ induces a surjective ring homomorphism from $\mathrm{End}N$ to $\mathrm{End}\overline{N}/\{g \in \mathrm{End}\overline{N} \mid g(n) \in \overline{N}\pi \text{ for all } n \in \overline{N}\}$.

Suppose $f \in \mathrm{End}\overline{N}$ is not an isomorphism. There exist $g \in \mathrm{End}N$ and $\sigma \in \mathrm{End}\overline{N}$ such that $f = \overline{g} + \sigma$ and $\sigma(n) \in \overline{N}\pi$ for all $n \in \overline{N}$. By Lemma 3.3, $\overline{g}$ is not an isomorphism, and hence neither is g. Since $\mathrm{End}N$ is local, $\mathrm{Id}_N - g$ is an isomorphism. Thus, $\mathrm{Id}_{\overline{N}} - \overline{g}$ is an isomorphism. So, by Lemma 3.3, $\mathrm{Id}_{\overline{N}} - f = \mathrm{Id}_{\overline{N}} - (\overline{g} + \sigma)$ is an isomorphism, as required. ■

We now show that Maranda's functor preserves pure-injective hulls. The proof uses somewhat different techniques to those used so far and relies on [8, Proposition 4.6]. In order to avoid introducing various definitions that will not be used in the rest of this paper, we state only the part of that proposition which we need.

Proposition 3.6 *Let $k \geq k_0 + 1$. For all $\psi \in [\pi^{k-k_0} | \mathbf{x}, \mathbf{x} = \mathbf{x}] \subseteq pp^n_\Lambda$, there exists $\widehat{\psi} \in [\pi^{k-k_0} + \Lambda\pi^k | \mathbf{x}, \mathbf{x} = \mathbf{x}] \subseteq pp^n_{\Lambda_k}$ such that for all $M \in Tf_\Lambda$ and $\mathbf{m} \in M$, $\mathbf{m} \in \psi(M)$ if and only if $\mathbf{m} + M\pi^k \in \widehat{\psi}(M/M\pi^k)$.*

The following useful lemma was communicated to me by Prest.

Lemma 3.7 *Let $M \in Mod\text{-}S$, $H(M)$ be its pure-injective hull, and let $\mathbf{b} \in H(M)$ be an n-tuple. Suppose that $\mathbf{b} \in \phi(H(M)) \setminus \bigcup_{i=1}^{l} \psi_i(H(M))$, where $\phi, \psi_1, \ldots, \psi_n$ are pp-n-formulas. There exist an n-tuple $\mathbf{b}' \in M$ and a pp-n-formula θ such that $\theta(\mathbf{b}' - \mathbf{b})$ holds and*

$$H(M) \models \theta(\mathbf{b}' - \mathbf{y}) \to \phi(\mathbf{y}) \wedge \bigwedge_{i=1}^{n} \neg\psi_i(\mathbf{y}).$$

Proof Let $\mathbf{b} \in H(M)$. Suppose that $\mathbf{b} \in \phi(H(M))$ and $\mathbf{b} \notin \bigcup_{i=1}^{l} \psi_i(H(M))$.

By [15, Lemma 4.1] and [15, Theorem 4.10(c)], there exist $\mathbf{a} \in M$ and a pp formula $\chi(\mathbf{x}, \mathbf{y})$ such that $\chi(\mathbf{a}, \mathbf{b})$ holds in $H(M)$ and

$$H(M) \models \chi(\mathbf{a}, \mathbf{y}) \to \phi(\mathbf{y}) \wedge \bigwedge_{i=1}^{n} \neg\psi_i(\mathbf{y}).$$

Since $H(M)$ is an elementary extension of M, there exists $\mathbf{b}' \in M$ such that $\chi(\mathbf{a}, \mathbf{b}')$ holds in M and hence in $H(M)$. Thus, $\chi(\mathbf{0}, \mathbf{b}' - \mathbf{b})$ holds in $H(M)$. Set $\theta(\mathbf{z}) := \chi(\mathbf{0}, \mathbf{z})$. So $\theta(\mathbf{b}' - \mathbf{b})$ holds in $H(M)$.

Suppose $\mathbf{c} \in H(M)$ and $\theta(\mathbf{b}' - \mathbf{c})$ holds in $H(M)$. Then $\chi(\mathbf{a}, \mathbf{c})$ holds in $H(M)$. Thus, $\phi(\mathbf{c}) \wedge \bigwedge_{i=1}^{l} \neg\psi_i(\mathbf{c})$ holds in $H(M)$. So $\theta(\mathbf{b}' - \mathbf{b})$ holds and

$$H(M) \models \theta(\mathbf{b}' - \mathbf{y}) \to \phi(\mathbf{y}) \wedge \bigwedge_{i=1}^{l} \neg\psi_i(\mathbf{y}). \qquad \blacksquare$$

The following theorem is motivated by [16, Lemma 3.16].

Theorem 3.8 *Let $k \geq k_0 + 1$ and $M \in Tf_\Lambda$. If $u : M \to H(M)$ is a pure-injective hull of M, then the induced map $\overline{u} : M/M\pi^k \to H(M)/H(M)\pi^k$ is a pure-injective hull for $M/M\pi^k$.*

Proof We identify M with its image in $H(M)$. Our aim is to show that for all $b \in H(M)$ with $\overline{b} \neq 0$, there exists $a \in M$ and $\chi(x, y) \in \text{pp}^2_{\Lambda_k}$ such that $\chi(\overline{a}, \overline{b})$ holds in $H(M)/H(M)\pi^k$ and $\chi(\overline{a}, \overline{0})$ does not hold in $H(M)/H(M)\pi^k$.

Suppose that π does not divide $b \in H(M)$. Since $H(M)$ is the pure-injective hull of M, by Lemma 3.7, there exist $a \in M$ and a pp formula $\theta(x) \in \text{pp}^1_\Lambda$ such that $\theta(a - b)$ holds in $H(M)$ and $\theta(a - x) \to \neg\pi|x$. Let $\Delta(x) := \theta(x) + \pi|x$. Then $\Delta(a - b)$ holds in $H(M)$, and for all $c \in H(M)$, $\Delta(a - c\pi)$ does not hold. Let $\widehat{\Delta}$ be as in Proposition 3.6. So $\widehat{\Delta}(\overline{a} - \overline{b})$ holds in $H(M)/H(M)\pi^k$.

Now, suppose that $e \in H(M) \setminus H(M)\pi^k$, $e = b\pi^n$, and π does not divide b. Note that this implies $n < k$. Let Δ and $a \in M$ be as in the previous paragraph, i.e., $\Delta \geq \pi|x$, $\Delta(a - b)$ holds in $H(M)$, and for all $c \in H(M)$, $\Delta(a - c\pi)$ does not hold. Let $\chi(x, y) := \exists z\, \widehat{\Delta}(x - z) \wedge y = z\pi^n \in \text{pp}^2_{\Lambda_k}$. Suppose that $\chi(\overline{a}, \overline{0})$ holds. Then there exists $d \in H(M)$ such that $\overline{d}\pi^n = \overline{0}$ and $\widehat{\Delta}(\overline{a} - \overline{d})$ holds. But then $d\pi^n \in H(M)\pi^k$. Since M and hence $H(M)$ and R-torsion-free, $d \in H(M)\pi^{k-n}$. This contradicts the definition of Δ. Thus, $\chi(\overline{a}, \overline{e})$ holds and $\chi(\overline{a}, \overline{0})$ does not hold in $H(M)/H(M)\pi^k$.

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Suppose that $H(M)/H(M)\pi^k = N \oplus N'$ and $M/M\pi^k \subseteq N$. If $\overline{c} \in H(M)/H(M)\pi^k$ is nonzero, then we have shown that there exist $\overline{a} \in \overline{M}$ and $\chi(x,y) \in \mathrm{pp}^2_{\Lambda_k}$ such that $\chi(\overline{a},\overline{c})$ holds and $\chi(\overline{a},\overline{0})$ does not hold. Since the solution sets of pp formulas commute with direct sums, this implies that if $\overline{c} \in N'$, then $\overline{c} = \overline{0}$. Thus, N' is the zero module, and $H(M)/H(M)\pi^k$ is the pure-injective hull of $M/M\pi^k$. ∎

4 Pure-injectives and pure-injective hulls

As in the previous section, R will be a discrete valuation domain with field of fractions Q and maximal ideal generated by π, and Λ will be an R-order such that $Q\Lambda$ is a separable Q-algebra.

We start this section by showing that the pure-injective hull of an R-reduced R-torsion-free Λ-module is R-reduced. The proof of the following remark is the same as [14, Claim 2, p. 1128].

Remark 4.1 If $M \in \mathrm{Tf}_\Lambda$ is R-divisible, then M is injective as a Λ-module.

This allows us to deduce that all $M \in \mathrm{Tf}_\Lambda$ decompose as the direct sum of the divisible part D_M of M and an R-reduced module. Explicitly, let

$$D_M := \{m \in M \mid \pi^n | m \text{ for all } n \in \mathbb{N}\}.$$

It is easy to check that D_M is R-divisible. So, since R-divisible R-torsion-free Λ-modules are injective, D_M is a direct summand of M. Hence, $M \cong D_M \oplus M/D_M$. Now, note that if $m \in M$ and $\pi^n | m + D_M$ for all $n \in \mathbb{N}$, then $\pi^n | m$ for all $n \in \mathbb{N}$. Thus, M/D_M is R-reduced.

Lemma 4.2 *Let S be a ring, $C, M, E \in$ Mod-S, and E injective. Suppose that $C, E \subseteq M$ and $C \cap E = \{0\}$. There exists $N' \subseteq M$ such that $C \subseteq N'$ and $N' \oplus E = M$.*

Proof Using injectivity of E, there is an $f : M \to E$ such that $f|_C = 0$ and $f|_E = \mathrm{Id}_E$. So $C \subseteq \ker f$ and $M = E \oplus \ker f$. ∎

Lemma 4.3 *If $C \in Tf_\Lambda$ is R-reduced, then $H(C)$ is R-reduced.*

Proof Since $Q\Lambda$ is separable, $H(C) = N \oplus D_{H(C)}$. Since C is pure in $H(C)$ and C is reduced, $C \cap D_{H(C)} = \{0\}$. By Lemma 4.2, there exists $N' \subseteq H(C)$ such that $N' \oplus D_{H(C)} = H(C)$ and $C \subseteq N'$. Since N and N' are isomorphic, N' is reduced. Since N' is a direct summand of $H(C)$ and $C \subseteq N' \subseteq H(C)$, $N' = H(C)$. Thus, $H(C)$ is R-reduced. ∎

Definition 4.1 If M is a Λ-module, then let M^* denote the inverse limit along the canonical maps $M/M\pi^{n+1} \to M/M\pi^n$.

Remark 4.4 If $M \in$ Mod-Λ is R-reduced and pure-injective as an R-module, then the canonical map $\nu : M \to M^*$, induced by the quotient maps from M to $M/M\pi^n$, is an isomorphism of Λ-modules.

Proof Since M is R-reduced, ν is an embedding. Since M is pure-injective as an R-module (equivalently, algebraically compact), ν is surjective. ∎

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Theorem 4.5 *Let $M \in Tf_\Lambda$. Then M is pure-injective if and only if*

(1) *$M/M\pi^k$ is pure-injective for all $k \in \mathbb{N}$ and*
(2) *M is pure-injective as an R-module.*

Proof Certainly, if M is pure-injective, then conditions (1) and (2) hold.

So suppose that (1) and (2) hold. We know that M is isomorphic to $D_M \oplus N$ and that D_M is injective. Thus, M is pure-injective if and only if N is pure-injective. Moreover, if conditions (1) and (2) hold for M, then they also hold of N. Let $H(N)$ be the pure-injective hull of N. Since $N/N\pi^k$ is pure-injective, Theorem 3.8 implies that $H(N)/H(N)\pi^k = N/N\pi^k$. By Lemma 4.3, $H(N)$ is reduced, and hence is isomorphic to $H(N)^* \cong N^*$. Since N is reduced and pure-injective as an R-module, $N \cong N^*$. Thus, $N \cong H(N)$ and is hence pure-injective. Thus, $M = D_M \oplus N$ is also pure-injective. ∎

Theorem 4.6 *Let $M \in Tf_\Lambda$ be R-reduced, and suppose that $M/M\pi^n$ is pure-injective for all $n \in \mathbb{N}$. Then the canonical map $v : M \to M^*$ is the pure-injective hull of M.*

Proof Let $u : M \to H(M)$ be a pure-injective hull of M. For each $k \in \mathbb{N}$, let $u_k : M/M\pi^k \to H(M)/H(M)\pi^k$ be the homomorphism induced by u. For each $k \geq k_0 + 1$, $u_k : M/M\pi^k \to H(M)/H(M)\pi^k$ is the pure-injective hull of $M/M\pi^k$. Since $M/M\pi^k$ is pure-injective, u_k is an isomorphism. The maps u_k induce an isomorphism $w : M^* \to H(M)^*$. Since M and hence, by Lemma 4.3, $H(M)$ are reduced, $H(M) \cong H(M)^*$. Viewing $H(M)^*$ as a submodule of $\prod_{i\in\mathbb{N}} H(M)/H(M)\pi^i$, for all $m \in M$, $wv(m) = (u(m) + H(M)\pi^i)_{i\in\mathbb{N}}$. Thus, $v = w^{-1}u$. ∎

The same argument as used in the proof above shows that for any R-reduced $M \in \mathrm{Tf}_\Lambda$, the pure-injective hull of M is $\varprojlim H(M/M\pi^i)$ along some surjective homomorphisms $p_i : H(M/M\pi^{i+1}) \to H(M/M\pi^i)$. Unfortunately, it is not clear how to explicitly describe the homomorphisms p_i beyond saying that $\ker p_i = H(M/M\pi^{i+1})\pi^i$.

For the rest of this section, we focus on an application of Theorem 4.6. We will calculate the pure-injective hull of the direct limit at the "top" of a generalized tube in Latt_Λ. This will allow us to describe certain points of Zg_Λ^{tf} as modules when $\Lambda = \widehat{\mathbb{Z}}_{(2)}C_2 \times C_2$ and answer the questions at the end of [20].

Following Krause in [12], we define a *generalized tube* in mod-S to be a sequence of tuples $\mathcal{T} := (M_i, f_i, g_i)_{i\in\mathbb{N}_0}$ where $M_i \in$ mod-S, $M_0 = 0$, $f_i : M_{i+1} \to M_i$, and $g_i : M_i \to M_{i+1}$ such that for every $i \in \mathbb{N}$,

$$\begin{array}{ccc} M_i & \xrightarrow{f_{i-1}} & M_{i-1} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle g_{i-1}} \\ M_{i+1} & \xrightarrow{f_i} & M_i \end{array}$$

is a pushout and a pullback.

We will show that if $\mathcal{T}$ is a generalized tube in Latt_Λ, then its image, denoted $\mathcal{T}_k$, in mod-Λ_k is a generalized tube.

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Recall that a diagram

$$\begin{array}{ccc} B & \xrightarrow{b} & L \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & P \end{array}$$

is a pushout and a pullback if and only if

$$0 \longrightarrow B \xrightarrow{\binom{a}{b}} M \oplus L \xrightarrow{(f\ -g)} P \longrightarrow 0$$

is an exact sequence.

We say a generalized tube $(M_i, f_i, g_i)_{i\in\mathbb{N}_0}$ is *trivial* if $M_i = 0$ for all $i \in \mathbb{N}_0$.

Remark 4.7 If $(M_i, f_i, g_i)_{i\in\mathbb{N}_0}$ is a nontrivial generalized tube, then there exists $n \in \mathbb{N}$ such that g_i is not an epimorphism for all $i \geq n$.

Proof Let $n \in \mathbb{N}$ be least such that $M_n \neq 0$. Then g_{n-1} is not an epimorphism. Since the pushout of an epimorphism is an epimorphism, g_i is not an epimorphism for all $i \geq n-1$. ■

The following remark seems like it should be false because certainly Maranda's functor does not send monomorphisms between lattices to monomorphisms. Consider the exact sequence below. Since M is projective as an R-module and β is surjective, there exists $\gamma \in \mathrm{Hom}_R(M, N)$ such that $\beta\gamma = \mathrm{Id}_M$. Thus, the exact sequence is split when viewed as an exact sequence of R-modules. Therefore, the second sequence is a split exact sequence of R_k-modules. Hence, it is an exact sequence of Λ_k-modules.

Remark 4.8 If

$$0 \longrightarrow L \xrightarrow{\alpha} N \xrightarrow{\beta} M \longrightarrow 0$$

is an exact sequence of Λ-lattices, then

$$0 \longrightarrow L_k \xrightarrow{\overline{\alpha}} N_k \xrightarrow{\overline{\beta}} M_k \longrightarrow 0$$

is an exact sequence of Λ_k-modules.

It follows that if $\mathcal{T}$ is a generalized tube of Λ-lattices, then $\mathcal{T}_k := ((M_i)_k, \overline{f_i}, \overline{g_i})_{i\in\mathbb{N}_0}$ is a generalized tube of finitely presented Λ_k-modules.

Given a generalized tube $\mathcal{T} = (M_i, f_i, g_i)_{i\in\mathbb{N}_0}$, define $\mathcal{T}[\infty]$ to be the direct limit along the embeddings $g_i : M_i \to M_{i+1}$. Note that if $\mathcal{T}$ is trivial, then $\mathcal{T}[\infty] = 0$.

Recall that a module $M \in \text{Mod-}S$ is *Σ-pure-injective* if $M^{(\kappa)}$ is pure-injective for every cardinal κ. Equivalently [17, Theorem 4.4.5], M is Σ-pure-injective if and only if $\mathrm{pp}_S^1 M$ has the descending chain condition.

Proposition 4.9 *Let* $\mathcal{T} = (M_i, f_i, g_i)_{i\in\mathbb{N}_0}$ *be a nontrivial generalized tube in* Latt_Λ. *Then*

(i) $\mathcal{T}[\infty]$ *is R-torsion-free and R-reduced,*
(ii) $\mathcal{T}[\infty]$ *is not pure-injective,*

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(iii) *for all* $k \in \mathbb{N}$, $\mathcal{T}[\infty]/\mathcal{T}[\infty]\pi^k$ *is* Σ*-pure-injective, and*
(iv) $\mathcal{T}[\infty]^*$ *is the pure-injective hull of* $\mathcal{T}[\infty]$.

Proof (i) and (ii): As a direct limit of lattices, $\mathcal{T}[\infty]$ is R-torsion-free. Each g_i is split when viewed as a homomorphism of R-modules. Since $\mathcal{T}$ is nontrivial, there exists an $n \in \mathbb{N}$ such that g_i is not an isomorphism for all $i \geq n$. Therefore, $\mathcal{T}[\infty]$ is isomorphic to $R^{(\aleph_0)}$ as an R-module. So $\mathcal{T}[\infty]$ is reduced. Since R is not Σ-pure-injective as a module over itself [17, Theorem 4.4.8], $R^{(\aleph_0)}$ is not pure-injective as an R-module, and hence $\mathcal{T}[\infty]$ is not pure-injective as a Λ-module.

(iii): Krause shows [12, Proposition 8.3] that if $\mathcal{T}$ is a generalized tube in the category of finitely presented modules over an Artin algebra, then $\mathcal{T}[\infty]$ is Σ-pure-injective. Since Maranda's functor commutes with direct limits and sends generalized tubes to generalized tubes, if $\mathcal{T} = (M_i, f_i, g_i)_{i\in\mathbb{N}_0}$ is a generalized tube in Latt_Λ, then $\mathcal{T}_k[\infty] = \mathcal{T}[\infty]/\mathcal{T}[\infty]\pi^k$. Thus, $\mathcal{T}[\infty]/\mathcal{T}[\infty]\pi^k$ is Σ-pure-injective.

(iv): Follows directly from (i), (iii), and Theorem 4.6. ∎

When R is complete and $Q\Lambda$ is a separable Q-algebra, the category of Λ-lattices has almost split sequences (see [22]). A stable tube is an Auslander–Reiten component of the form $\mathbb{Z}A_\infty/\tau^n$, and we call n the rank of the tube. Explicitly, a stable tube of rank n has points $S_i[j]$ for $1 \leq i \leq n$ and $j \in \mathbb{N}$. We read the index i mod n. For all $i, j \in \mathbb{N}$, a stable tube has a single (trivially valued) arrow $S_i[j] \to S_i[j+1]$ and a single (trivially valued) arrow $S_i[j+1] \to S_{i+1}[j]$. We will identify the points with (the isomorphism type of) the Λ-lattice they represent. As for Artin algebras, generalized tubes can be constructed from stable tubes using the following two facts.

- If $A, B, C \in \mathrm{Latt}_\Lambda$ are indecomposable and pairwise nonisomorphic and $u : A \to B$ and $v : A \to C$ are irreducible morphisms, then there is $w : A \to D$ such that $(u\ v\ w)^T : A \to B \oplus C \oplus D$ is left minimal almost split.
- If $u : S_i[j] \to S_i[j+1]$ is an irreducible map, $w : S_i[j] \to W$, and $W \in \mathrm{Latt}_\Lambda$ is indecomposable and is not isomorphic to any of $S_i[j], S_{i+1}[j-1], \ldots, S_{i+(j-1)}[1]$, then there exists $y : S_i[j+1] \to W$ such that $w = yu$.

Krause [12, Theorem 9.1] showed that if $\mathcal{T}$ is a stable tube (of rank n) in the module category of an Artin algebra, with the labeling of modules as above, then for each $1 \leq i \leq n$, the direct limit $\varinjlim S_i[j]$ is an indecomposable pure-injective. For stable tubes in categories of lattices, we know (Proposition 4.9) that $\oplus_{i=1}^n \varinjlim S_i[j]$ has a pure-injective hull $(\oplus_{i=1}^n \varinjlim S_i[j])^*$. Hence, the pure-injective hull of $\varinjlim S_i[j]$ is $(\varinjlim S_i[j])^*$. This raises the following question.

Question Let R be a complete discrete valuation domain with field of fractions Q, and let Λ be an order R such that $Q\Lambda$ is a separable Q-algebra. If T is a direct limit up a ray of irreducible monomorphisms in a stable tube in Latt_Λ, then is T^* indecomposable? ∎

We are able to answer this question positively for the $\widehat{\mathbb{Z}_2}$-order $\Gamma := \widehat{\mathbb{Z}_2}C_2 \times C_2$. The torsion-free part of the Ziegler spectrum of Γ was described in [20]. However, the points were not described as modules.

We start by explaining the setup. Let e_1, e_2, e_3, e_4 be the primitive orthogonal idempotents as in [20]. Using these idempotents, Butler [2] defined a full functor Δ

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from the category of b-reduced Γ-lattices to the category of finite-dimensional vector spaces over $\mathbb{F}_2$ with four distinguished subspaces. A $\widehat{\mathbb{Z}_2}$-torsion-free Γ-module M is *b-reduced* if $M \cap Me_i = 2Me_i$ for all $1 \le i \le 4$. Note that, since $e_i \notin \widehat{\mathbb{Z}_2}C_2 \times C_2$, Me_i and $M2e_i$ are calculated inside $\widehat{\mathbb{Q}_2}M$. Puninski and Toffalori extended this functor to the category of b-reduced $\widehat{\mathbb{Z}_2}$-torsion-free modules and showed [20, Theorem 5.4] that it is full on $\widehat{\mathbb{Z}_2}$-torsion-free b-reduced pure-injective Γ-modules.

Let M be a b-reduced $\widehat{\mathbb{Z}_2}$-torsion-free $\widehat{\mathbb{Z}_2}C_2 \times C_2$-module. Define $M^\star := Me_1 \oplus \cdots \oplus Me_4$. Then $\Delta(M) := (V; V_1, V_2, V_3, V_4)$, where $V := M^\star/M$ and $V_i := Me_i + M/M \cong Me_i/M \cap Me_i = Me_i/2Me_i$.

The category of finite-dimensional vector spaces over $\mathbb{F}_2$ with four distinguished subspaces may be identified with a full subcategory of modules over the path algebra $\mathbb{F}_2\widetilde{D}_4$. The only indecomposable representations which are not in this full subcategory are the simple injective $\mathbb{F}_2\widetilde{D}_4$-modules. We will make this identification and consider Δ as a functor to Mod-$\mathbb{F}_2\widetilde{D}_4$.

As observed by Puninski and Toffalori, just from the construction, one can see that Δ is an interpretation functor. Note that if M is b-reduced and $\widehat{\mathbb{Z}_2}$-torsion-free, then $\Delta(M) = 0$ if and only if M is $\widehat{\mathbb{Z}_2}$-divisible.

Dieterich, in [6], showed that Δ induced an isomorphism from the Auslander–Reiten quiver of $\mathbb{F}_2\widetilde{D}_4$ with all projective points removed and all simple injective modules removed and the Auslander–Reiten quiver of Latt_Γ restricted to the b-reduced lattices. Using this, he was able [6, Proposition 3.4] to compute the full Auslander–Reiten quiver of $\text{Latt}_{\widehat{\mathbb{Z}_2}C_2\times C_2}$. Moreover, see the proof of [6, Lemma 2.2] and [6, Proposition 3.4], Δ induces a bimodule isomorphism between $\text{Irr}_{\text{Latt}_\Gamma}(M, L)$ and $\text{Irr}_{\mathbb{F}_2\widetilde{D}_4}(\Delta(M), \Delta(L))$ for all L, M indecomposable b-reduced Γ-lattices. In particular, Δ sends irreducible morphisms between indecomposable b-reduced Γ-lattices to irreducible morphisms in mod-$\mathbb{F}_2\widetilde{D}_4$. This implies that the Auslander–Reiten quiver of $\text{Latt}_{\widehat{\mathbb{Z}_2}C_2\times C_2}$ has infinitely many stable tubes of rank 1 and rank 3 stable tubes of rank 2 and Δ sends each stable tube in $\text{Latt}_{\widehat{\mathbb{Z}_2}C_2\times C_2}$ to a stable tube in mod-$\mathbb{F}_2\widetilde{D}_4$.

Keeping our notation as above, let $S_i[j]$ be the lattices in a stable tube of rank $n = 1$ or $n = 2$ in $\text{Latt}_{\widehat{\mathbb{Z}_2}C_2\times C_2}$. Fix $1 \le i \le n$ and for each $j \in \mathbb{N}$ let $w_j : S_i[j] \to S_i[j+1]$ be an irreducible map. Let $S_i[\infty] := \varinjlim S_i[j]$ be the direct limit along the maps w_j. Then $\Delta S_i[\infty] = \varinjlim \Delta S_i[j]$ is pure-injective and indecomposable by [12, Theorem 9.1] since Δ sends stable tubes to stable tubes. Since Δ is full on pure-injective modules, by [16, Lemmas 3.15 and 3.16],[2] it preserves pure-injective hulls. Thus, $\Delta(S_i[\infty]) \cong \Delta(S_i[\infty]^*)$. Since $S_i[\infty]^*$ is reduced and $\Delta(S_i[\infty]^*)$ is indecomposable, $S_i[\infty]^*$ is indecomposable.

So, finally, for each quasi-simple S at the base of a tube (i.e., $S_i[1]$ for some stable tube), the S-prüfer point in [20, Theorem 6.1] is $S[\infty]^*$, where $S[\infty]$ is the direct limit up a ray of irreducible monomorphisms starting at S.

The module T in Question 6.2 of [20] is indecomposable but not pure-injective; however, its pure-injective hull is indecomposable (and $\widehat{\mathbb{Z}_2}$-reduced).

[2] The proof of the required part of [16, Lemma 3.16] is a little unclear. Lemma 3.7 clears this up.

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5 Duality

Throughout this section, let R be a Dedekind domain which is not a field, Q its field of fractions, Λ an R-order, and $Q\Lambda$ a separable Q-algebra. The main aim of this section is to show that the lattice of open sets of Zg_Λ^{tf} is isomorphic to the lattice of open sets of ${}_\Lambda\mathrm{Zg}^{tf}$. We will also show, by other methods, that the m-dimension of $\mathrm{pp}^1_\Lambda(\mathrm{Tf}_\Lambda)$ is equal to the m-dimension of ${}_\Lambda\mathrm{pp}^1({}_\Lambda\mathrm{Tf})$ and that the Krull–Gabriel dimension of $(\mathrm{Latt}_\Lambda, \mathrm{Ab})^{fp}$ is equal to the Krull–Gabriel dimension of $({}_\Lambda\mathrm{Latt}, \mathrm{Ab})^{fp}$.

5.1 Duality for the *R*-reduced part of Zg_Λ^{tf} when *R* is a discrete valuation domain

Throughout this subsection, R will be a discrete valuation domain, k will be a natural number strictly greater than Maranda's constant for Λ as an R-order, and $I : \mathrm{Tf}_\Lambda \to \text{Mod-}\Lambda_k$ (respectively, $I : {}_\Lambda\mathrm{Tf} \to \Lambda_k\text{-Mod}$) will be Maranda's functor.

Maranda's functor $I : \mathrm{Tf}_\Lambda \to \text{Mod-}\Lambda_k$ is an interpretation functor. The kernel of I is the definable subcategory of R-divisible modules. Since $Q\Lambda$ is separable, by Remark 4.1 and the discussion just below it, all indecomposable pure-injective modules in Tf_Λ are either R-reduced or R-divisible modules. When Λ is an order over a discrete valuation domain R, we will write $\mathrm{Zg}_\Lambda^{rtf}$ for the subset of R-reduced modules in Zg_Λ^{tf}. We have shown in Section 3 that if $N, M \in \mathrm{Tf}_\Lambda$ are R-reduced and pure-injective, then $IN \cong IM$ implies $N \cong M$ and that if N is also indecomposable, then so is IN. Thus, Proposition 2.7 gives us the following theorem.

Theorem 5.1 *The map which sends $N \in \mathrm{Zg}_\Lambda^{rtf}$ to $N/N\pi^k \in \mathrm{Zg}_{\Lambda_k}$ induces a homeomorphism onto its image which is closed.*

In theory, the above theorem could be used to give a description of $\mathrm{Zg}_\Lambda^{rtf}$ and hence Zg_Λ^{tf} based on a description of Zg_{Λ_k}. However, as explained in Section 1, Zg_{Λ_k} is generally much more complicated than Zg_Λ^{tf}.

Based on Prest's duality for pp formulas, Herzog defined a lattice isomorphism between the lattice of open subsets of Zg_S and the lattice of open subsets of ${}_S\mathrm{Zg}$.

Theorem 5.2 [9] *There is a lattice isomorphism D between that lattice of open subsets of Zg_S (respectively, ${}_S\mathrm{Zg}$) and the lattice of open subsets of ${}_S\mathrm{Zg}$ (respectively, Zg_S), which is given on basic open sets by*

$$(\phi/\psi) \mapsto (D\psi/D\phi)$$

for ϕ, ψ pp-1-formulas. Moreover, D^2 is the identity map.

It is unknown if this lattice isomorphism is always induced by a homeomorphism.

If X is a closed subset of Zg_S, then we will write DX for ${}_S\mathrm{Zg}\backslash D(\mathrm{Zg}_S\backslash X)$. Since closed subsets of Zg_S are in correspondence with the definable subcategories of Mod-S, this isomorphism also defines an inclusion preserving bijection between the definable subcategories of Mod-S and S-Mod. If $\mathcal{X} \subseteq \text{Mod-}S$ is a definable subcategory, then we will write $D\mathcal{X}$ for the corresponding definable subcategory of S-Mod.

Herzog's duality can be applied to closed subspaces of Zg_S as follows. Let X be a closed subset of Zg_S. Open subsets of Zg_S containing $\mathrm{Zg}_S\backslash X$ are in bijective

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correspondence with open subsets of X equipped with the subspace topology via the map $U \mapsto U \cap X$. If U is an open subset of Zg_S containing $\mathrm{Zg}_S \backslash X$, then DU is an open subset of ${}_S\mathrm{Zg}$ containing ${}_S\mathrm{Zg} \backslash DX$. Thus, D induces a lattice isomorphism between the lattice of open sets of X and the lattice of open sets of DX both equipped with the appropriate subspace topology.

Herzog's isomorphism D sends the definable subcategory Tf_Λ to the definable subcategory of R-divisible Λ-modules. Thus, directly applying Herzog's duality does not give an isomorphism between the open subsets of Zg_Λ^{tf} and ${}_\Lambda\mathrm{Zg}^{tf}$. With this in mind, we instead use the right module version of Maranda's functor I to move to $\mathrm{Mod}\text{-}\Lambda_k$, then we apply D there, and then we use the left module version of Maranda's functor to move back to ${}_\Lambda\mathrm{Tf}$. This will give us an isomorphism between the lattice of open subsets of $\mathrm{Zg}_\Lambda^{rtf}$ and ${}_\Lambda\mathrm{Zg}^{rtf}$.

Our first step is to show that $\langle I\mathrm{Tf}_\Lambda \rangle = D\langle I_\Lambda \mathrm{Tf}\rangle$.

The contravariant functor

$$\mathrm{Hom}_R(-, R) : \mathrm{Mod}\text{-}\Lambda \to \Lambda\text{-}\mathrm{Mod}$$

induces an equivalence between the category of right Λ-lattices and the opposite of the category of left Λ-lattices (see [21, Section IX 2.2]). If M is right Λ-lattice, denote the left Λ-lattice $\mathrm{Hom}_R(M, R)$ by $M^\dagger$.

The ring $\Lambda/\pi^n\Lambda$ is an $R/\pi^n R$-Artin algebra. For all S-Artin algebras $\mathcal{A}$, there is a duality between mod-$\mathcal{A}$ and $\mathcal{A}$-mod given by $\mathrm{Hom}_S(-, E)$ where E is the injective hull of $S/\mathrm{rad}(S)$. We will write M^* for $\mathrm{Hom}(M, E)$. If $S = R/\pi^n R$, then $S/\mathrm{rad}(S) = R/\pi R$. One can check, using Baer's criterion, that $R/\pi^n R$ is injective as an S-module. The map which sends $a + \pi R \in R/\pi R$ to $a\pi^{n-1} + \pi^n R \in R/\pi^n R$ embeds $R/\pi R$ into the socle of $R/\pi^n R$ which is simple. Thus, $E = R/\pi^n R$ is the injective hull of $S/\mathrm{rad}(S) = R/\pi R$.

We will now show that if L is a right Λ-lattice, then $(IL)^* = IL^\dagger$.

Lemma 5.3 *If M is a right Λ-lattice and $n \in \mathbb{N}$, then*

$$Hom_R(M, R)/\pi^n Hom_R(M, R) \cong Hom_{R/\pi^n}(M/M\pi^n, R/\pi^n R).$$

Proof For $f \in \mathrm{Hom}_R(M, R)$, let $\overline{f} : M/M\pi^n \to R/\pi^n R \in \mathrm{Hom}_{R/\pi^n R}(M/M\pi^n, R/\pi^n R)$ be the homomorphism which sends $m + M\pi^n$ to $f(m) + \pi^n R$.

Let $\Phi : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_{R/\pi^n R}(M/M\pi^n, R/\pi^n R)$ be defined by $\Phi(f) = \overline{f}$. It is clear that Φ is a homomorphism of left Λ-modules. Since M is projective as an R-module, Φ is surjective.

If $\Phi(f) = 0$, then for all $m \in M$, $\pi^n | f(m)$. For all $m \in M$, let $g(m) \in M$ be such that $g(m)\pi^n = f(m)$. Since M is R-torsion-free, the choice of $g(m)$ is unique. From this, it follows easily that g is a homomorphism of R-modules. Thus, if $\Phi(f) = 0$, then $f \in \pi^n \mathrm{Hom}_R(M, R)$. ∎

The next remark follows from the fact (see [17, Corollary 1.3.13] for instance) that if $\mathcal{A}$ is an Artin algebra, ϕ/ψ is a pp-pair, and M is a finite length $\mathcal{A}$-module, then $\phi(M) = \psi(M)$ if and only if $D\phi(M^*) = D\psi(M^*)$.

Remark 5.4 Suppose that $\mathcal{A}$ is an Artin algebra and $\{M_i \mid i \in I\}$ is a set of finite length right $\mathcal{A}$-modules. Then

$$D\langle M_i \mid i \in I\rangle = \langle M_i^* \mid i \in I\rangle.$$

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Lemma 5.5 *The following equalities hold.*

(1) $\langle I Tf_\Lambda \rangle = \langle IL \mid L \text{ is an indecomposable right } \Lambda\text{-lattice}\rangle$

(2) $= \langle IM^\dagger \mid M \text{ is an indecomposable left } \Lambda\text{-lattice}\rangle$

(3) $= \langle (IM)^* \mid M \text{ is an indecomposable left } \Lambda\text{-lattice}\rangle$

(4) $= D\langle IM \mid M \text{ is an indecomposable left } \Lambda\text{-lattice}\rangle$

(5) $= D\langle I_\Lambda Tf\rangle.$

Proof (1) and (5). These hold because all $N \in \mathrm{Tf}_\Lambda$ are direct limits of Λ-lattices, all Λ-lattices are direct sums of indecomposable Λ-lattices, and I commutes with direct limits.

(2) For all (right) Λ-lattices $L^{\dagger\dagger} \cong L$ and $L^\dagger$ is a (left) Λ-lattice. (3) holds by Lemma 5.3 and (4) holds by Remark 5.4. ∎

Herzog's duality D gives an isomorphism from the lattice of open sets of $\mathrm{Zg}(\langle I\mathrm{Tf}_\Lambda\rangle)$ to the lattice of open sets of $\mathrm{Zg}(D\langle I\mathrm{Tf}_\Lambda\rangle)$. By Lemma 5.5, $D\langle I\mathrm{Tf}_\Lambda\rangle = \langle I_\Lambda \mathrm{Tf}\rangle$.

If U is an open subset of $\mathrm{Zg}_\Lambda^{rtf}$ (respectively, ${}_\Lambda\mathrm{Zg}^{rtf}$), then write IU for the set of all IN where $N \in U$.

Definition 5.1 Let U be an open subset of $\mathrm{Zg}_\Lambda^{rtf}$. Define

$$dU := \{N \in {}_\Lambda\mathrm{Zg}^{rtf} \mid IN \in DIU\}.$$

By Theorem 5.1, IU is an open subset of $\mathrm{Zg}(\langle I\mathrm{Tf}_\Lambda\rangle)$. So DIU is an open subset of $\mathrm{Zg}(\langle I_\Lambda \mathrm{Tf}\rangle)$. Again by Theorem 5.1, the set of $N \in {}_\Lambda\mathrm{Zg}^{rtf}$ such that $IN \in DIU$ is an open subset of ${}_\Lambda\mathrm{Zg}^{rtf}$.

Proposition 5.6 *The map d between the lattice of open sets of Zg_Λ^{rtf} and ${}_\Lambda Zg^{rtf}$ is a lattice isomorphism.*

Proof The homeomorphism from Theorem 5.1 sends an open subset U of $\mathrm{Zg}_\Lambda^{rtf}$ to $IU \subseteq \mathrm{Zg}(\langle I\mathrm{Tf}_\Lambda\rangle)$. So the map sending U to IU is a lattice isomorphism. By Lemma 5.5, Herzog's duality gives a lattice isomorphism between the open subsets of $\mathrm{Zg}(\langle I\mathrm{Tf}_\Lambda\rangle)$ and the lattice of open subset of $\mathrm{Zg}(\langle I_\Lambda \mathrm{Tf}\rangle)$. Thus, the map which sends an open subset U of $\mathrm{Zg}_\Lambda^{rtf}$ to $DIU \subseteq \mathrm{Zg}(\langle I_\Lambda \mathrm{Tf}\rangle)$ is a lattice isomorphism. Finally, the inverse of the homeomorphism from Theorem 5.1 sends an open subset of $W \subseteq \mathrm{Zg}(\langle I_\Lambda \mathrm{Tf}\rangle)$ to the set of all $N \in {}_\Lambda\mathrm{Zg}^{rtf}$ such that $IN \in W$. So this map is also a lattice isomorphism. Since d is the composition of these three lattice isomorphisms, d is also a lattice isomorphism. ∎

If Λ is an order over a complete discrete valuation domain, then the Λ-lattices are pure-injective (see [8, Proposition 2.2] for instance). When R is not complete, we can instead consider the lattices over the $\widehat{R}$-order $\widehat{\Lambda}$. Then the $\widehat{\Lambda}$-lattices are pure-injective as $\widehat{\Lambda}$-modules and hence also as Λ-modules. Moreover, if L is an indecomposable $\widehat{\Lambda}$-lattice, then, since L is R-reduced, L is also indecomposable as a Λ-module (see [14, Remark 1] for a proof over group rings that also works in our context).

Proposition 5.7 *Let R be a discrete valuation domain and Λ an R-order. If L is an indecomposable right $\widehat{\Lambda}$-lattice, then for all open sets $U \subseteq Zg^{rtf}_{\Lambda}$, $L \in U$ if and only if $L^{\dagger} \in dU$ where $L^{\dagger} := Hom_{\widehat{R}}(L, \widehat{R})$.*

Proof First note that IL is finite-length as a Λ_k-module. Since Λ_k is an Artin algebra, if $M \in \mathrm{Zg}(\langle I\mathrm{Tf}_{\Lambda}\rangle)$ is finite-length, then for all open subsets U of $\mathrm{Zg}(\langle I\mathrm{Tf}_{\Lambda}\rangle)$, $M \in U$ if and only if $M^* \in DU$ (see [17, Corollary 1.3.13]). So, if L is an indecomposable right $\widehat{\Lambda}$-lattice, then $L \in U$ if and only if $IL \in IU$ and $IL \in IU$ if and only if $(IL)^* \in DIU$. By Lemma 5.3, $(IL)^* = IL^{\dagger}$, so $(IL)^* \in DIU$ if and only if $L^{\dagger} \in dU$. So $L \in U$ if and only if $L^{\dagger} \in dU$. ■

5.2 Duality for $\mathrm{Zg}^{tf}_{\Lambda}$

We now work to extend Proposition 5.6 in two ways concurrently. We extend the isomorphism to an isomorphism between the lattices of open subsets of $\mathrm{Zg}^{tf}_{\Lambda}$ and ${}_{\Lambda}\mathrm{Zg}^{tf}$ and we extend the statement to the case where R is a Dedekind domain (which is not a field).

In order to do this, we need to recall some key features of $\mathrm{Zg}^{tf}_{\Lambda}$ from [8]. As explained in [8, Section 3], for each $P \in \mathrm{Max}R$, the canonical homomorphism $\Lambda \to \Lambda_P$ induces, via restriction of scalars, an embedding of $\mathrm{Zg}^{tf}_{\Lambda_P}$ into $\mathrm{Zg}^{tf}_{\Lambda}$ and the image of this embedding is closed. We identify $\mathrm{Zg}^{tf}_{\Lambda_P}$ with its image. Moreover, for all $N \in \mathrm{Zg}^{tf}_{\Lambda}$, there exists a $P \in \mathrm{Max}R$ such that $N \in \mathrm{Zg}^{tf}_{\Lambda_P}$. So

$$\mathrm{Zg}^{tf}_{\Lambda} = \bigcup_{P \in \mathrm{Max}R} \mathrm{Zg}^{tf}_{\Lambda_P}.$$

Finally, if $N \in \mathrm{Zg}_{\Lambda_P}$ for all $P \in \mathrm{Max}R$, then N is R-divisible and hence may be viewed as a module over $Q\Lambda$. Since $Q\Lambda$ is separable, hence semisimple, all indecomposable R-divisible modules, when viewed as $Q\Lambda$-modules, are simple.

For each $P \in \mathrm{Max}R$, let $P|x$ denote the pp formula $\exists y_1, \ldots, y_n\ x = \sum_{i=1}^{n} y_i r_i$, where $r_1, \ldots r_n$ generate P. In all Λ-modules M, $P|x$ defines the subset MP. If $P, P' \in \mathrm{Max}R$ are not equal, then $(x = x/P|x) \cap (x = x/P'|x)$ is empty. For all $N \in \mathrm{Zg}^{tf}_{\Lambda}$, either N is R-divisible or $N \in (x = x/P|x)$ for some $P \in \mathrm{Max}R$. So

$$\mathrm{Zg}^{tf}_{\Lambda} = \mathrm{Zg}_{Q\Lambda} \cup \bigcup_{P \in \mathrm{Max}R} (x = x/P|x).$$

Note that $(x = x/P|x) = \mathrm{Zg}^{tf}_{\Lambda_P} \setminus \mathrm{Zg}_{Q\Lambda}$. Under the assumption that $Q\Lambda$ is a semisimple Q-algebra, this means that $(x = x/P|x)$ is the set of R_P-reduced indecomposable pure-injective Λ_P-modules. For this reason, we will write $\mathrm{Zg}^{rtf}_{\Lambda_P}$ for this set. Note that this notation matches that of the previous section when Λ is an order over a discrete valuation domain.

Theorem 5.8 [8, Theorem 3.1] *Let R be a Dedekind domain with field of fractions Q, and Λ an R-order such that $Q\Lambda$ is semisimple. If $N \in Zg^{tf}_{\Lambda}$, then either*

- *N is a simple $Q\Lambda$-module or*
- *there is some maximal ideal P of R such that $N \in Zg^{tf}_{\widehat{\Lambda_P}}$ and N is R_P-reduced.*

Moreover, if $N \in Zg^{tf}_{\widehat{\Lambda_P}}$ is R_P-reduced, then $N \in Zg^{tf}_{\Lambda}$.

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This theorem means that if $Q\Lambda$ is separable, then the R_P-reduced points of Zg_Λ^{tf} can be identified with the $\widehat{R_P}$-reduced (equivalently, R_P-reduced) points of $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$. Following [14], it is shown in [8, Theorem 3.3] that the topology on the set of R_P-reduced points of Zg_Λ^{tf} is the same whether it is viewed as a subspace of $\mathrm{Zg}_{\Lambda_P}^{tf}$ or $\mathrm{Zg}_{\widehat{\Lambda_P}}^{tf}$. Thus, we may identify $\mathrm{Zg}_{\Lambda_P}^{rtf}$ and $\mathrm{Zg}_{\widehat{\Lambda_P}}^{rtf}$.

We have already mentioned in Section 5.1 that a $\widehat{\Lambda_P}$-lattice is pure-injective. Therefore, the restrictions of indecomposable $\widehat{\Lambda_P}$-lattices to Λ are points in Zg_Λ^{tf}.

From now on, if $P \in \mathrm{Max}R$, then let d_P denote the isomorphism between the lattice of open subsets of $\mathrm{Zg}_{\Lambda_P}^{rtf}$ and of ${}_{\Lambda_P}\mathrm{Zg}^{rtf}$ induced by d for Λ_P. Patching the d_P together as $P \in \mathrm{Max}R$ varies will give us an isomorphism between the open subset of $\bigcup_{P\in\mathrm{Max}R} \mathrm{Zg}_{\Lambda_P}^{rtf} \subseteq \mathrm{Zg}_\Lambda^{tf}$ and the open subsets of $\bigcup_{P\in\mathrm{Max}R}{}_{\Lambda_P}\mathrm{Zg}^{rtf} \subseteq {}_\Lambda\mathrm{Zg}^{tf}$. Thus, we just need to know what to do with open subsets which contain R-divisible points.

Let $e_1, \dots, e_n$ be a complete set of centrally primitive orthogonal idempotents for $Q\Lambda$. For each $1 \le i \le n$, $e_iQ\Lambda$ is isomorphic as a right $Q\Lambda$-module to $S_i^{(\alpha_i)}$ for some simple right $Q\Lambda$-module S_i and if $S_i \cong S_j$ then $i = j$.

Lemma 5.9 [8, Lemma 2.7] *Let $N \in Zg_\Lambda^{tf}$ and $S \in Zg_{Q\Lambda}$. If S is a direct summand of QN, then S is in the closure of N. In particular, if N is a closed point in Zg_Λ^{tf}, then $N \in Zg_{Q\Lambda}$.*

Lemma 5.10 *Let D be a Dedekind domain with field of fractions Q, and let Λ be an order over D such that $Q\Lambda$ is semisimple. Let $e \in Q\Lambda$ be a centrally primitive idempotent, let S be the simple right $Q\Lambda$-module corresponding to e, and suppose that $d \in D$ is such that $ed \in \Lambda$. The following are equivalent for all $N \in Zg_\Lambda^{tf}$.*

(1) *$N \in (xd(1-e) = 0/x = 0)$.*
(2) *S is a direct summand of QN.*
(3) *S is in the closure of N.*

Proof (1) ⇒ (2) Suppose $md(1-e) = 0$ and $m \neq 0$. Then, as an element of QN viewed as a $Q\Lambda$-module, $m(1-e) = 0$. Thus, $m = me$. The kernel of the homomorphism from $Q\Lambda$ to QN sending 1 to m contains $(1-e)Q\Lambda$ and thus induces a nonzero homomorphism from $eQ\Lambda$ to QN. Thus, S is a submodule and hence a direct summand of QN.

(2) ⇒ (3) This is Lemma 5.9.

(3) ⇒ (1) Suppose S is in the closure of N. Since $eQ\Lambda(1-e)d = 0$, $S \in (xd(1-e) = 0/x = 0)$. Thus, $N \in (xd(1-e) = 0/x = 0)$. ∎

Note that the above shows that the set of points specializing to a closed point in Zg_Λ^{tf} is an open set. For $S \in \mathrm{Zg}_{Q\Lambda}$, we will write $\mathcal{V}(S)$ for the open set of points whose closure contains S.

Corollary 5.11 *Let U be an open subset of Zg_Λ^{tf}. Then*

$$U = \bigcup_P (U \cap Zg_{\Lambda_P}^{rtf}) \cup \bigcup_{S\in\lambda(U)} \mathcal{V}(S),$$

where $\lambda(U) := U \cap Zg_{Q\Lambda}$.

Proof If $N \in \mathrm{Zg}_\Lambda^{tf}$, then either $N \in \mathrm{Zg}_{\Lambda_P}^{rtf}$ for some $P \in \mathrm{Max}R$ or $N \in \mathrm{Zg}_{Q\Lambda}$. So, since for all $S \in \mathrm{Zg}_{Q\Lambda}$, $S \in \mathcal{V}(S)$, $U \subseteq \bigcup_P (U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_{S \in \lambda(U)} \mathcal{V}(S)$.

Suppose $S \in \lambda(U)$ and $N \in \mathcal{V}(S)$. Then S is in the closure of N. Hence, $N \in U$. Thus, $\mathcal{V}(S) \subseteq U$. So $U \supseteq \bigcup_P (U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_{S \in \lambda(U)} \mathcal{V}(S)$. ■

For each simple $Q\Lambda$-module S, we now consider where to send the open set $\mathcal{V}(S)$. In particular, we need to calculate the image of $\mathcal{V}(S) \cap \mathrm{Zg}_{\Lambda_P}^{rtf}$ under d_P for each $P \in \mathrm{Max}R$.

Lemma 5.12 *Let R be a discrete valuation domain and Λ an R-order. For all $M \in \mathrm{Latt}_\Lambda$, $Q\mathrm{Hom}_R(M, R)$ and $\mathrm{Hom}_Q(MQ, Q)$ are isomorphic as $Q\Lambda$-modules.*

Proof Let $\Delta : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_Q(MQ, Q)$ be defined by setting $\Delta(f)(m \cdot q) = f(m) \cdot q$ for all $m \in M$ and $q \in Q$. A quick computation shows that for all $f \in \mathrm{Hom}_R(M, R)$, $\Delta(f)$ is a well-defined element of $\mathrm{Hom}_Q(MQ, Q)$ and Δ is an injective homomorphism of left Λ-modules. Since $\mathrm{Hom}_Q(MQ, Q)$ is Q-divisible, Δ extends to an injective homomorphism Δ' from $Q\mathrm{Hom}_R(M, R)$ to $\mathrm{Hom}_Q(MQ, Q)$.

Suppose that M is rank n. Then $\dim_Q MQ = \dim_Q \mathrm{Hom}_Q(MQ, Q) = \dim_Q Q\mathrm{Hom}_R(M, R) = n$. Thus, Δ' is an injective homomorphism between two n-dimensional Q-vector spaces and hence is surjective. ■

Lemma 5.13 *Let R be a discrete valuation domain. Let $L \in \mathrm{Latt}_\Lambda$, e a central idempotent of $Q\Lambda$, and $d \in R$ be such that $ed \in \Lambda$. Then $L \in (x(e-1)d = 0/x = 0)$ if and only if $L^\dagger \in ((e-1)dx = 0/x = 0)$.*

Proof Suppose $L \in (x(e-1)d = 0/x = 0)$. Then there exists $a \in QL \backslash \{0\}$ such that $a(e-1) = 0$. By Lemma 5.12, $Q\mathrm{Hom}_R(L, R) \cong \mathrm{Hom}_Q(QL, Q)$. Thus, we need to show that there exists $0 \neq f \in \mathrm{Hom}_Q(QL, Q)$ such that $(e-1) \cdot f = 0$. Since e is central, $QL = QLe \oplus QL(e-1)$ and $QLe \neq 0$. Take $f \in \mathrm{Hom}_Q(QL, Q)$ such that f is zero on $QL(e-1)$ and nonzero on QLe. Then for all $m \in QL$, $(e-1) \cdot f(m) = f(m(e-1)) = 0$, but $f \neq 0$. Thus, there exists $b \in QL^\dagger \backslash \{0\}$ such that $(e-1) \cdot b = 0$. There exists $r \in R \backslash \{0\}$ such that $rb \in L^\dagger$ and $(e-1)d \cdot rb = 0$. So $L^\dagger \in ((e-1)dx = 0/x = 0)$. ■

Lemma 5.14 *Let $a \in \Lambda$. The set of indecomposable $\widehat{\Lambda_P}$-lattices, as $P \in \mathrm{Max}R$ varies, is dense in $\mathrm{Zg}_\Lambda^{tf} \backslash (xa = 0/x = 0)$.*

Proof Suppose that $(\phi/\psi) \cap (\mathrm{Zg}_\Lambda^{tf} \backslash (xa = 0/x = 0)) \neq \varnothing$. Pick $N \in (\phi/\psi) \cap (\mathrm{Zg}_\Lambda^{tf} \backslash (xa = 0/x = 0))$. Since N is a direct union of its finitely generated submodules, there exists a finitely generated submodule L of N such that $\phi(L) \supsetneq \psi(L)$. Since L is a submodule of N, L is R-torsion-free and $\mathrm{ann}_L a = 0$. Thus, $\phi(H(L)) \supsetneq \psi(H(L))$ and $\mathrm{ann}_{H(L)} a = 0$. Since $H(L)$ is isomorphic to $\prod_{P \in \mathrm{Max}R} \widehat{L_P}$ by Lemma 2.1, for all $P \in \mathrm{Max}R$, $\mathrm{ann}_{\widehat{L_P}} a = 0$, and there exists $P \in \mathrm{Max}R$ such that $\phi(\widehat{L_P}) \supsetneq \psi(\widehat{L_P})$. Thus, there exist a $P \in \mathrm{Max}R$ and a $\widehat{\Lambda_P}$-lattice M such that $\phi(M) \supsetneq \psi(M)$ and $\mathrm{ann}_M a = 0$. Since the category of $\widehat{\Lambda_P}$-lattices is Krull–Schmidt, it follows that there exists an indecomposable $\widehat{\Lambda_P}$-lattice with the required properties. ■

The following is proved in the case that R is a discrete valuation domain in [14].

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Corollary 5.15 *The set of indecomposable $\widehat{\Lambda_P}$-lattices, as $P \in MaxR$ varies, is a dense subset of Zg^{tf}_{Λ}, and each $\widehat{\Lambda_P}$-lattice is isolated in Zg^{tf}_{Λ}. Therefore, all isolated points in Zg^{tf}_{Λ} are $\widehat{\Lambda_P}$-lattices for some $P \in MaxR$.*

Proof Density is a special case of Lemma 5.14. It is shown in [8, Lemma 2.4] that the indecomposable $\widehat{\Lambda_P}$-lattices are isolated in $\mathrm{Zg}^{tf}_{\widehat{\Lambda_P}}$. As explained just after Theorem 5.8, we may identify $\mathrm{Zg}^{rtf}_{\widehat{\Lambda_P}}$ with $\mathrm{Zg}^{rtf}_{\Lambda_P}$. Thus, the $\widehat{\Lambda_P}$-lattices are isolated in $\mathrm{Zg}^{rtf}_{\Lambda_P}$. Finally, viewed as a subspace of $\mathrm{Zg}^{tf}_{\Lambda}$, $\mathrm{Zg}^{rtf}_{\Lambda_P}$ is equal to the open set $(x = x/P|x)$. Thus, the indecomposable $\widehat{\Lambda_P}$-lattices are isolated in $\mathrm{Zg}^{tf}_{\Lambda}$. The final statement follows from the first two statements. ∎

Recall that for each $P \in \mathrm{Max}R$, d_P is the isomorphism between the lattice of open subsets of $\mathrm{Zg}^{rtf}_{\Lambda_P}$ and of ${}_{\Lambda_P}\mathrm{Zg}^{rtf}$ defined in Section 5.1.

Lemma 5.16 *For all simple $Q\Lambda$-modules S and all $P \in MaxR$,*

$$d_P(\mathcal{V}(S) \cap Zg^{rtf}_{\Lambda_P}) = \mathcal{V}(S^*) \cap {}_{\Lambda_P}Zg^{rtf}.$$

Proof We first show that if L is an indecomposable right $\widehat{\Lambda_P}$-lattice and S is a simple right $Q\Lambda$-module, then $L \in \mathcal{V}(S)$ if and only if $L^\dagger \in \mathcal{V}(S^*)$. Let e be a centrally primitive idempotent of $Q\Lambda$ corresponding to S. Note that e is central and idempotent as an element of $\widehat{Q_P}\widehat{\Lambda}$. We have shown in Lemma 5.13 that $L \in (x(e-1)d = 0/x = 0)$ if and only if $L^\dagger \in ((e-1)dx = 0/x = 0)$. So it is enough to show that $((e-1)dx = 0/x = 0) = \mathcal{V}(S^*)$. However, this is clear because certainly $(e-1)S^* = 0$, and thus e is a centrally primitive idempotent corresponding to S^*.

Since, by Lemma 5.14, the indecomposable right $\widehat{\Lambda_P}$-lattices are dense in the closed subset $\mathrm{Zg}^{rtf}_{\Lambda_P} \setminus (x(e-1)d = 0/x = 0)$ of $\mathrm{Zg}^{rtf}_{\Lambda_P}$,

$$\mathrm{Zg}^{rtf}_{\Lambda_P} \setminus (x(e-1)d = 0/x = 0) \subseteq \mathrm{Zg}^{rtf}_{\Lambda_P} \setminus d_P(((e-1)dx = 0/x = 0) \cap {}_{\Lambda_P}\mathrm{Zg}^{rtf}).$$

So $d_P(((e-1)dx = 0/x = 0) \cap {}_{\Lambda_P}\mathrm{Zg}^{rtf}) \subseteq (x(e-1)d = 0/x = 0)$. The same argument using left $\widehat{\Lambda_P}$-lattices shows that

$$d_P((x(e-1)d = 0/x = 0) \cap \mathrm{Zg}^{rtf}_{\Lambda_P}) \subseteq ((e-1)dx = 0/x = 0).$$

So, since d_P^2 is the identity,

$$d_P((x(e-1)d = 0/x = 0) \cap \mathrm{Zg}^{rtf}_{\Lambda_P}) = ((e-1)dx = 0/x = 0) \cap {}_{\Lambda_P}\mathrm{Zg}^{rtf}. \qquad ∎$$

Definition 5.2 Let U be an open subset of $\mathrm{Zg}^{tf}_{\Lambda}$. Define

$$dU := \bigcup_{P \in \mathrm{Max}R} d_P(U \cap \mathrm{Zg}^{rtf}_{\Lambda_P}) \cup \bigcup_{S \in \lambda(U)} \mathcal{V}(S^*),$$

where $\lambda(U) := U \cap \mathrm{Zg}_{Q\Lambda}$.

We will also use d to denote the analogous map for open subsets of ${}_{\Lambda}\mathrm{Zg}^{tf}$.

Theorem 5.17 *Let R be a Dedekind domain, Q its field of fractions, and Λ an R-order with $Q\Lambda$ a separable Q-algebra.*

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The mapping d is an isomorphism between the lattice of open sets of Zg_Λ^{tf} and ${}_\Lambda Zg^{tf}$ such that

(1) *if L is an indecomposable right $\widehat{\Lambda_P}$-lattice, then for all open sets $U \subseteq Zg_\Lambda^{tf}$, $L \in U$ if and only if $L^\dagger \in dU$, and*
(2) *for all open sets $U \subseteq Zg_\Lambda^{tf}$, if S is a simple $Q\Lambda$-module, then $S \in U$ if and only if $S^* \in dU$.*

Proof Let U be an open subset of Zg_Λ^{tf}. We start by showing that for all open subsets $U \subseteq \mathrm{Zg}_\Lambda^{tf}$, $d^2U = U$. So

$$\begin{aligned}
d^2U &= d[\bigcup_P d_P(U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_{S\in\lambda(U)} \mathcal{V}(S^*)] \\
&= \bigcup_P d_P[d_P(U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_{S\in U} \mathcal{V}(S^*) \cap {}_{\Lambda_P}\mathrm{Zg}^{rtf}] \cup \bigcup_{S\in\lambda(U)} \mathcal{V}(S) \\
&= \bigcup_P d_P^2(U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_P \bigcup_{S\in\lambda(U)} d_P[\mathcal{V}(S^*) \cap {}_{\Lambda_P}\mathrm{Zg}^{rtf}] \cup \bigcup_{S\in\lambda(U)} \mathcal{V}(S) \\
&= \bigcup_P (U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \cup \bigcup_{S\in\lambda(U)} \mathcal{V}(S) \\
&= U.
\end{aligned}$$

The first two equalities follow from the definition of d. The third is true because each d_P is a lattice homomorphism. The fourth follows from Lemma 5.16 and the fifth follows from Corollary 5.11.

Thus, d gives a bijection between the lattice of open subsets of Zg_Λ^{tf} and ${}_\Lambda\mathrm{Zg}^{tf}$. We now just need to show that d preserves inclusion.

Suppose $U \subseteq W$ are open subsets of Zg_Λ^{tf}. Then $\lambda(U) \subseteq \lambda(W)$ and $U \cap \mathrm{Zg}_{\Lambda_P}^{rtf} \subseteq W \cap \mathrm{Zg}_{\Lambda_P}^{rtf}$ for all $P \in \mathrm{Max}(R)$. So $d_P(U \cap \mathrm{Zg}_{\Lambda_P}^{rtf}) \subseteq d_P(W \cap \mathrm{Zg}_{\Lambda_P}^{rtf})$ for all $P \in \mathrm{Max}R$. For all open sets U, $S \in \lambda(U)$ if and only if $S^* \in \lambda(dU)$. So $\lambda(U) \subseteq \lambda(W)$ implies $\lambda(dU) \subseteq \lambda(dW)$. Therefore, $dU \subseteq dW$.

Finally, (1) holds for d by Proposition 5.6 and (2) holds by definition of d. ■

We finish this section with a different aspect of duality.

Corollary 5.18 *Let R be a discrete valuation domain with maximal ideal generated by π. The lattices $[\pi|x, x = x]_{Tf_\Lambda}$ and $[\pi|x, x = x]_{{}_\Lambda Tf}$ are anti-isomorphic.*

Proof Let $k > k_0$, and let $p = \pi + \pi^k\Lambda$. By Proposition 3.6, $[\pi|x, x = x]_{\mathrm{Tf}_\Lambda}$ is isomorphic to $[p|x, x = x]_{\langle I\mathrm{Tf}_\Lambda\rangle}$ and $[\pi|x, x = x]_{{}_\Lambda\mathrm{Tf}}$ is isomorphic to $[p|x, x = x]_{\langle I_\Lambda\mathrm{Tf}\rangle}$. So, it is enough to show that $[p|x, x = x]_{\langle I\mathrm{Tf}_\Lambda\rangle}$ is anti-isomorphic to $[p|x, x = x]_{\langle I_\Lambda\mathrm{Tf}\rangle}$.

We have seen in Lemma 5.5 that $D\langle I\mathrm{Tf}_\Lambda\rangle = \langle I_\Lambda\mathrm{Tf}\rangle$. Thus, Prest's duality for pp formulas gives an anti-isomorphism between $\mathrm{pp}^1_{\Lambda_k}(\langle I\mathrm{Tf}_\Lambda\rangle)$ and ${}_{\Lambda_k}\mathrm{pp}^1(\langle I_\Lambda\mathrm{Tf}\rangle)$. Thus, $[p|x, x = x]_{\langle I\mathrm{Tf}_\Lambda\rangle}$ is anti-isomorphic to $[x = 0, px = 0]_{\langle I_\Lambda\mathrm{Tf}\rangle}$.

The formula $y = xp^{k-1}$ induces a lattice isomorphism between the intervals $[xp^{k-1} = 0, x = x]$ and $[y = 0, p^{k-1}|y]$ of $\mathrm{pp}^1_{\Lambda_k}$ defined by

$$\phi(x) \mapsto \exists x(y = xp^{k-1} \wedge \phi(x))$$

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(see the proof of Goursat's lemma [24, Lemma 8.9]). On $\langle I_\Lambda \mathrm{Tf}\rangle$, $p^{k-1}x = 0$ is equivalent to $p|x$ and $p^{k-1}|y$ is equivalent to $py = 0$. Thus, $[x = 0, px = 0]_{\langle I_\Lambda \mathrm{Tf}\rangle}$ is isomorphic to $[p|x, x = x]_{\langle I_\Lambda \mathrm{Tf}\rangle}$. ■

For the definition of the m-dimension of a modular lattice, see [17, Section 7.2].

Corollary 5.19 *Suppose R is a Dedekind domain with field of fractions Q, Λ is an R-order, and $Q\Lambda$ is separable. The m-dimensions of $pp^1_\Lambda(Tf_\Lambda)$ and ${}_\Lambda pp^1({}_\Lambda Tf)$ are equal.*

Proof For each $P \in \mathrm{Max}R$, by [8, Corollary 3.8], the m-dimension of $\mathrm{pp}^1_{\Lambda_P}(\mathrm{Tf}_{\Lambda_P})$ is equal to the m-dimension of $[P|x, x = x]_{\mathrm{Tf}_{\Lambda_P}}$ plus 1. Since R_P is discrete valuation domain, by Corollary 5.18, the m-dimension of $[P|x, x = x]_{\mathrm{Tf}_{\Lambda_P}}$ is equal to the m-dimension of $[P|x, x = x]_{{}_{\Lambda_P}\mathrm{Tf}}$. Thus, by [8, Corollary 3.8], ${}_{\Lambda_P}\mathrm{pp}^1({}_{\Lambda_P}\mathrm{Tf})$ has m-dimension equal to the m-dimension of $[P|x, x = x]_{\mathrm{Tf}_{\Lambda_P}}$ plus 1, i.e., equal to the m-dimension of $\mathrm{pp}^1_{\Lambda_P}(\mathrm{Tf}_{\Lambda_P})$.

By [8, Remark 3.9], the m-dimension of $\mathrm{pp}^1_\Lambda(\mathrm{Tf}_\Lambda)$ (respectively, ${}_\Lambda\mathrm{pp}^1({}_\Lambda\mathrm{Tf})$) is equal to the supremum of the m-dimensions of $\mathrm{pp}^1_{\Lambda_P}(\mathrm{Tf}_{\Lambda_P})$ (respectively, ${}_{\Lambda_P}\mathrm{pp}^1({}_{\Lambda_P}\mathrm{Tf})$) where $P \in \mathrm{Max}R$. ■

We now translate the above corollary into a result about the Krull–Gabriel dimensions of $(\mathrm{Latt}_\Lambda, \mathrm{Ab})^{fp}$ and $({}_\Lambda\mathrm{Latt}, \mathrm{Ab})^{fp}$. See [7, Definition 2.1] for a definition of the Krull–Gabriel dimension of a (skeletally) small abelian category.

Recall that a full subcategory $\mathcal{C} \subseteq$ mod-S which is closed under isomorphism, finite direct sums, and direct summands is *covariantly finite* in mod-S if for each $M \in$ mod-S there exists a homomorphism $f_M : M \to M_\mathcal{C}$ with $M_\mathcal{C} \in \mathcal{C}$ such that all homomorphisms $g : M \to L$ with $L \in \mathcal{C}$, factor through f_M. For $M \in$ mod-Λ, let $\mathrm{tor}M$ denote the submodule $\{m \in M \mid$ there exists $r \in R\backslash\{0\}$ with $mr = 0\}$ consisting of R-torsion elements of M. Then $M/\mathrm{tor}M \in \mathrm{Latt}_\Lambda$ and for any $L \in \mathrm{Latt}_\Lambda$ and $g : M \to L$, $\mathrm{tor}M \subseteq \ker g$. Hence, g factors through the canonical surjection $f_M : M \to M/\mathrm{tor}M$. Therefore, Latt_Λ is covariantly finite in mod-Λ.

If $\mathcal{C} \subseteq$ mod-S is a covariantly finite subcategory, then $(\mathcal{C}, \mathrm{Ab})^{fp}$ is equivalent to $(\text{mod-}S, \mathrm{Ab})^{fp}/\mathcal{S}(\mathcal{C})$, the Serre localization of $(\text{mod-}S, \mathrm{Ab})^{fp}$ at the Serre subcategory

$$\mathcal{S}(\mathcal{C}) := \{F \in (\text{mod-}S, \mathrm{Ab})^{fp} \mid FC = 0 \text{ for all } C \in \mathcal{C}\}.$$

See [10] for details.

By [17, Corollary 13.2.2], the Krull–Gabriel dimension of $(\mathcal{C}, \mathrm{Ab})^{fp}/\mathcal{S}(\mathcal{C})$ is equal to the m-dimension of $\mathrm{pp}^1_S(\langle\mathcal{C}\rangle)$.

Applying this to Latt_Λ as a covariantly finite subcategory of mod-Λ, we get that the Krull–Gabriel dimension of $(\mathrm{Latt}_\Lambda, \mathrm{Ab})^{fp}$ is equal to the m-dimension of $\mathrm{pp}^1_\Lambda \mathrm{Tf}_\Lambda$. Thus, we get the following corollary to Corollary 5.19.

Corollary 5.20 *Suppose R is a Dedekind domain with field of fractions Q, Λ is an R-order, and $Q\Lambda$ is separable. The Krull–Gabriel dimension of $(Latt_\Lambda, Ab)^{fp}$ is equal to the Krull–Gabriel dimension of $({}_\Lambda Latt, Ab)^{fp}$.*

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