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Heat and sweat transport in fibrous media with radiation

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The paper is concerned with heat and sweat transport in porous textile media with a non-local thermal radiation and phase change. The model, based on a combination of these classical heat transfer mechanisms (convection, conduction and radiation), is governed by a nonlinear, degenerate and strongly coupled parabolic system. The thermal radiative flow is described by a radiation transport equation and characterized by the thermal absorptivity and emissivity of fibre. A conservative boundary condition is introduced to describe the radiative heat flux interacting with environment. With the conservative boundary condition, we prove the global existence of positive/non-negative weak solutions of a nonlinear parabolic system. A typical clothing assembly with a polyester batting material sandwiched in two laminated covers is investigated numerically. Numerical results show that the contribution of radiative heat transfer is comparable with that of conduction/convection in the sweating system.

Key words: Heat and moisture transfer; Radiative heat transfer; Porous media; Global weak solution

1 Introduction

The heat and sweat transport in fibrous porous media has attracted much attention in the last decades due to its applications in modern textile industry [4, 8, 11, 13, 15, 17, 18, 23, 26, 27, 33–35]. A typical application is a clothing assembly, consisting of a thick porous fibrous batting sandwiched in two thin cover fabrics. The outer cover of the assembly is exposed to a cold environment with fixed temperature and relative humidity, while the inner cover is exposed to a mixture of air and vapour at higher temperature and relative humidity (e.g. human skin; see Figure 1 for the schematic diagram). Some earlier works can be found in [6, 26] with relative simple models. More precise models have been described recently by many authors as single-component compressible fluid flows [9, 12, 23, 24], multi-component compressible fluid flows [15, 34, 35] or incompressible fluid flows [1, 5, 31] through porous textile media. In these models, the air/vapour gas mixture moves through the porous textile media by convection and diffusion, which are induced by pressure and concentration gradients. Heat is transferred by convection in gas and by conduction in all phases (liquid, fibre and gas), and phase changes occur in the form of evaporation/condensation. Mathematically, the heat and sweat transport in porous textile media is governed by a system of nonlinear, degenerate and strongly

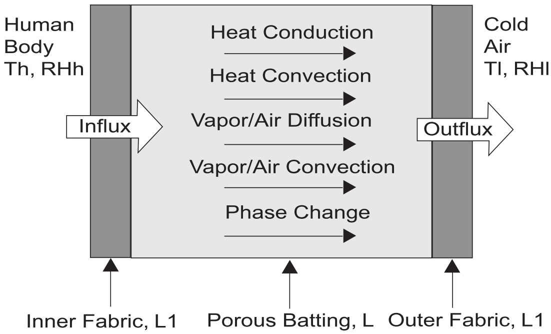

FIGURE 1. Schematic diagram of the porous textile assembly.

coupled parabolic equations. Mathematical analysis for the system of nonlinear parabolic equations can be found in our recent works [19, 20, 22].

The radiative heat transfer in porous fibre materials was studied by several authors. The approaches depend heavily upon applications involved. A simple and classical way is to consider radiative transfer as a 'radiative conductivity', with which the radiative flux was assumed to be proportional to the temperature gradient. This approach is valid in general when the penetration depth for radiation into fibre materials is small compared with the sample thickness. A more precise approximation of the radiative thermal conductivity was presented in [7]. On the other hand, the thermal radiation due to fresh fire (short time) was studied in [28, 30, 34] in terms of the Beer's law to describe an in-depth absorption of radiation in fibrous garment, where the radiative heat flux at the depth x was given by $q_r = q_{rf} e^{-\gamma x}$. The effect of radiative heat transfer through textile batting of large porosity was investigated in [10], where the radiative and conductive heat flows were assumed to be of the same size. In [10], the radiative heat flux was divided into two parts, from all angles in the right-hand and left-hand half spaces. The fluxes were governed by a system of differential equations with certain mixed boundary conditions. Both absorption and emission of the fibre were considered. Recently, this approach was used by several authors [3,9,23] to investigate heat and moisture transport through porous textile media with typical 95–99% porosity. Fan *et al.* [9] applied the radiative thermal transfer approach to study heat and vapour transport through clothing assemblies. In addition to the radiative heat transfer, the vapour diffusion and mixture conduction were also included in their model. An analytic solution of the system of mixed boundary value problems for radiative heat fluxes was obtained, with which the heat and moisture transport was described by a system of nonlinear parabolic equations with a non-local radiative source. Cheng and Wang [3] proposed the Galerkin finite element method for a simplified one-dimensional model with such a radiative source, in which only vapour diffusion and heat conductive processes were included and the vapour bulk motion was neglected. However, in all these previous works [3,9,21,23], the radiative heat transfer was not included in the boundary conditions for energy equation. The non-conserved boundary conditions result in an unnecessary difficulty in the analysis and possible non-existence of solution for a nonlinear parabolic system.

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In this paper, we will study the heat and moisture flow through porous textile materials with the radiative thermal transfer proposed in [10], which is governed mathematically by a system of nonlinear, degenerate and strongly coupled parabolic partial differential equations (PDEs) [9,15], with a non-local radiative source. A conservative boundary condition is proposed for the energy (temperature) equation. With the conservative boundary condition and a closed form of radiative source, we prove the existence of weak solutions for the system of nonlinear parabolic equations.

2 Mathematical equations

Here we consider a single component model of heat and sweat transport system in a one-dimensional setting since the thickness of clothing assemblies is often smaller compared with the sizes of other two dimensions. The model described below is mainly based on the works in [19, 33], which can also be viewed as a generalization of models developed earlier in [4, 17, 23, 26].

2.1 Mathematical equations

Based on the conservation of mass and energy, the model can be described by

$$\frac{\partial}{\partial t}(\epsilon C) + \frac{\partial}{\partial x}(u_g \epsilon C) = -\Gamma_{ce}, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\bar{C}_h T) + \frac{\partial}{\partial x}(\epsilon \bar{C}_g u_g C T) = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) + \lambda M \Gamma_{ce} + \Gamma_r, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho_f (1-\epsilon')W) = M\Gamma_{ce}. \tag{2.3}$$

Here C is the vapour concentration (mol/m^3), T is the temperature (K), W is the mass of liquid water relative to the mass of fibre (%), M is the molecular weight of water and λ is the latent heat of evaporation/condensation in the wet zone, while in the frozen zone, it represents the latent heat of sublimation. In this case, the volume content of liquid water on the fibre surface is given by $\frac{\rho_f}{\rho_w} W(1-\epsilon')$, and therefore the porosities with liquid water content (ϵ) and without liquid water content (ϵ') are related by

$$\epsilon = \epsilon' - \frac{\rho_f}{\rho_w} W(1-\epsilon'),$$

where ρ_f and ρ_w are the densities of fibre and water, respectively.

The effective heat capacity $\bar{C}_h$ is defined as in [35] by the conservative form,

$$\bar{C}_h = \epsilon \bar{C}_g C + (1-\epsilon)\bar{C}_s,$$

where $\bar{C}_g$ and $\bar{C}_s$ are the molar heat capacities of gas and the fibre/liquid water, respectively.

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The effective heat conductivity κ is defined by

$$\kappa = \epsilon\kappa_g + (1-\epsilon)\kappa_s, \tag{2.4}$$

where κ_g and κ_s are the thermal conductivities of gas and fibre/liquid water, respectively.

The vapour velocity (volumetric discharge) is given by the Darcy's law,

$$u_g = -\frac{k_g k_{rg}}{\mu_g C}\frac{\partial P}{\partial x}, \tag{2.5}$$

where k_g and k_{rg} denote the permeability and relative permeability of vapour, respectively, and μ_g is the viscosity of vapour. Usually, the relative permeability is defined by $k_{rg} = (1-s)^3$, where $s = 1-\epsilon/\epsilon'$ denotes the volumetric liquid saturation in the inner fibre void space. More detailed descriptions can be found in [15, 34].

The condensation/evaporation rate Γ_{ce} is often described in terms of the Hertz–Knudsen equation [16],

$$H := \frac{(1-\epsilon)E}{R_f}\sqrt{\frac{1}{2\pi MR}}\left(\frac{P - P_{\text{sat}}(T)}{\sqrt{T}}\right), \tag{2.6}$$

where R is the universal gas constant, R_f is the radius of fibre and E is the non-dimensional phase change coefficient. The vapour pressure is given by $P = RCT$ because of the ideal gas assumption. The saturation pressure P_{sat} is determined from experimental measurements [8]. Physically, the condensation occurs when $H > 0$ (i.e. $P > P_{\text{sat}}$), and the evaporation occurs only when $H < 0$ and simultaneously the amount of liquid water in the core void is positive. For certain clothing assembling cases presented in [9, 23], numerical simulations show that the physical process in the whole batting area is in condensation and no dry zone exists after a short period. In these models, the condensation/evaporation rate was defined directly by $\Gamma_{\text{ce}} = H$. On the other hand, a pure evaporation process was considered in [1, 5] for a motorcycle helmet model and [14] for a bread baking model, in which

$$\Gamma_{\text{ce}} = \text{h}(\text{W})H, \tag{2.7}$$

where $h(\cdot)$ was the Heaviside function. More precise formulation was given in [35] by

$$\Gamma_{\text{ce}} := H^+ - a(W)H^-, \tag{2.8}$$

where $H^\pm = (|H| \pm H)/2$ and $a(W)$ is a monotonically increasing function with $a(0) = 0$ and $0 \leqslant a(W) \leqslant 1$. The above formulation is valid for simultaneous condensation/evaporation cases.

2.2 The radiation

In this paper, we will pay more attention to the radiative heat transfer rate Γ_r. Based on the works in [7, 9], the radiative heat transfer rate can be defined by

$$\Gamma_r = \frac{\partial F_L}{\partial x} - \frac{\partial F_R}{\partial x}, \tag{2.9}$$

where F_L and F_R, the total thermal radiation fluxes travelling to the left and right, respectively, satisfy the differential equations

$$\frac{\partial F_L}{\partial x} = \beta(F_L - \sigma T^4), \tag{2.10}$$

$$\frac{\partial F_R}{\partial x} = -\beta(F_R - \sigma T^4) \tag{2.11}$$

with the coupled boundary conditions

$$(1-\epsilon_1)F_L(0,t) + \epsilon_1 \sigma T^4(0,t) = F_R(0,t), \tag{2.12}$$

$$(1-\epsilon_2)F_R(L,t) + \epsilon_2 \sigma T^4(L,t) = F_L(L,t), \tag{2.13}$$

where $0 < \epsilon_i < 1$ $(i = 1,2)$ denote the emissivities of the covers, β is the absorptivity/emissitivity of the batting [2] and σ is the Stefan–Boltzmann constant.

The solution of differential equations (2.10) and (2.11) was obtained analytically in [9]. A closed form is given by

$$F_L = -\sigma e^{\beta x}\left[g_l(x,t) + f_l\right], \tag{2.14}$$

$$F_R = \sigma e^{-\beta x}\left[g_r(x,t) + f_r\right], \tag{2.15}$$

where

$$g_l(x,t) = \int_0^x \beta e^{-\beta\xi} T^4(\xi,t)d\xi, \quad g_r(x,t) = \int_0^x \beta e^{\beta\xi} T^4(\xi,t)d\xi.$$

With the boundary conditions, we have

$$\begin{aligned} f_r &= \epsilon_1 T^4(0,t) - (1-\epsilon_1) f_l \\ &= -\frac{(1-\epsilon_1)\left[(1-\epsilon_2)e^{-\beta L} g_r(L,t) + e^{\beta L} g_l(L,t) + \epsilon_2 T^4(L,t)\right] + \epsilon_1 e^{\beta L} T^4(0,t)}{(1-\epsilon_2)(1-\epsilon_1)e^{-\beta L} - e^{\beta L}}, \\ f_l &= \frac{(1-\epsilon_2)e^{-\beta L} g_r(L,t) + e^{\beta L} g_l(L,t) + \epsilon_1(1-\epsilon_2)e^{-\beta L} T^4(0,t) + \epsilon_2 T^4(L,t)}{(1-\epsilon_2)(1-\epsilon_1)e^{-\beta L} - e^{\beta L}}. \end{aligned}$$

We note that $f_r \geqslant 0, f_l \leqslant 0$. Then

$$\Gamma_r = \beta(F_L + F_R) - 2\beta\sigma T^4(x,t). \tag{2.16}$$

2.3 Initial/boundary conditions

Since the thickness of cover layers is much smaller than that of the batting layer, the properties of heat and sweat transfer in the covers were often described by simple resistances to heat and vapour transfer. A class of the commonly used Robin-type boundary conditions were used by several authors, in which boundary conditions were defined by a combined simulation of cover layers and ambient environment. However, the effect of the radiative heat transfer was not considered in these boundary conditions, although it was introduced in the equations. Such non-conservative boundary conditions may make the theoretical analysis more difficult or impossible.

Table 1. *Physical parameters for the cover material*

Properties	Laminated	Unit
Thickness	5.15×10^{-4}	m
Density	4.27×10^{3}	kg/m^3
Porosity	0.1	
Thermal resistance	3.16×10^{-2} (R_t)	km^2/W
Resistance to vapour	1.4379×10^{2} (R_g)	s/m

Here we introduce conservative boundary conditions. For mass conservation,

$$u_g \epsilon C|_{x=0} = s_w, \tag{2.17}$$

$$u_g \epsilon C|_{x=L} = \frac{1}{R_g^o + 1/H_g^o}(C - C^o), \tag{2.18}$$

where s_w denotes the local sweating rate, which is a function of the local skin temperature, the average of skin temperature and the body temperature in general [25], and C^o is the vapour concentration of the outer environment. R_g^o is the resistance of the outer cover to gas and H_g^o denotes the mass transfer coefficient in the outer environment for gas. For energy conservation,

$$\left(u_g \epsilon \bar{C}_g C T - \kappa \frac{\partial T}{\partial x} - (F_L - F_R)\right)\bigg|_{x=0} = \bar{C}_g T s_w + \frac{1}{R_t^i + 1/H_t^i}(T^i - T) + \epsilon_1 \sigma(|T^i|^4 - T^4), \tag{2.19}$$

$$\left(u_g \epsilon \bar{C}_g C T - \kappa \frac{\partial T}{\partial x} - (F_L - F_R)\right)\bigg|_{x=L} = \frac{\bar{C}_g}{R_g^o + 1/H_g^o}(C - C^o)T + \frac{1}{R_t^o + 1/H_t^o}(T - T^o) + \epsilon_2 \sigma(T^4 - |T^o|^4), \tag{2.20}$$

where R_t^i and R_t^o are the resistances of the inner and outer covers to heat, H_t^i and H_t^o denote the mass transfer coefficients in the inner and outer environment for heat, and T^i and T^o are the temperature of the inner and outer environment respectively.

The initial conditions are given by

$$T = 25^\circ C, \quad C = 65\% \frac{P_{\text{sat}}(T)}{RT}, \quad W = 0, \quad \text{for } t = 0. \tag{2.21}$$

3 Numerical results and discussion

In this section, computational results are presented for heat/sweat transfer in a clothing assembly with a 10-pile polyester batting sandwiched in two laminated covers. Physical parameter values for both cover and batting materials are presented in Tables 1 and 2, and other parameter values can be found in [9, 15, 34]. Experimental measurements for water accumulated in the cloth assembly were given in [8].

Table 2. *Physical parameters for the batting material*

Properties	Polyester	Unit
Thickness (L)	$4.92 \times 10^{-3} \times 10$	m
Fibre density (ρ_f)	1.39×10^3	kg/m^3
Porosity (ϵ')	0.993	
Phase change coefficient (E)	2.4×10^{-6}	
Radius of fibre (R_f)	1.03×10^{-5}	m

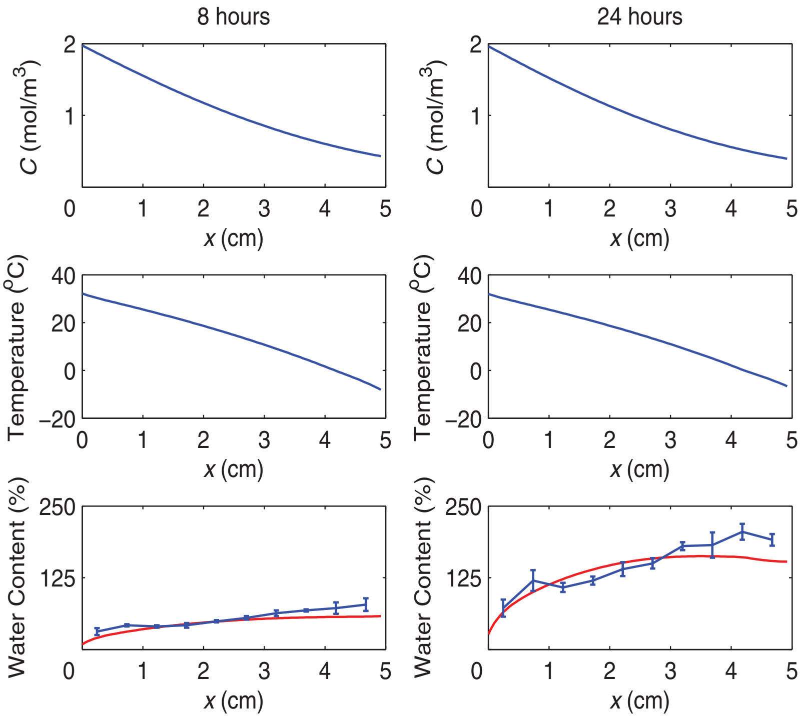

FIGURE 2. (Colour online) Numerical results with the Robin boundary conditions ($\beta = 400$).

To compare with the experimental measurements done in [8], we solve the system (2.1)–(2.3) with the Robin boundary conditions and the initial conditions given in (2.21). Here all numerical results are obtained by using the finite volume method presented in [29, 34] with $\Delta t = 10$ s, $\Delta x = L/100$ (1% of the batting length). Numerical simulations with smaller time and spatial steps are also made to ensure the convergence of numerical results.

We present in Figure 2 numerical results at 8 hours and 24 hours, respectively, in which C, T and W are the vapour concentration, temperature and water content respectively. The comparisons with experimental measurements of water content done in [8] are given in last two figures. In Figures 3 and 4, we present numerical results with a normal human sweating rate ($s_w = 30\,\text{mL}\,\text{m}^{-2}\,\text{h}^{-1}$) and an extreme sweating rate ($s_w = 300\,\text{mL}\,\text{m}^{-2}\,\text{h}^{-1}$), respectively. In the normal case, since the sweating rate is relatively low and the vapour

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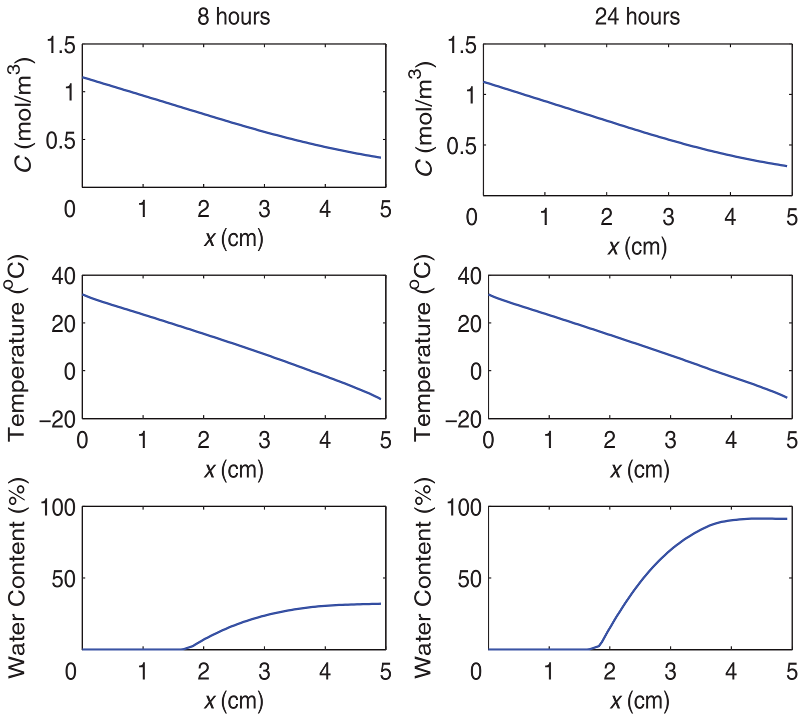

FIGURE 3. (Colour online) Numerical results of normal human sweating system ($\beta = 400$).

density is much small, the condensation ($\Gamma_{ce} \geqslant 0$) mainly occurs near to the outer cover. In the extremal case, due to the increase in sweating rate, the water content is much larger than that in the normal case and a stronger condensation near the inner cover can be observed.

To see more clearly the effect of radiative heat transfer, in Figure 5, we present the heat fluxes of convection/conduction and radiation, and the source functions from condensation/evaporation and radiation, for the normal human sweating system at 8 hours with $\beta = 400$ and 1400, respectively. In the case of $\beta = 400$, one can see that the convection/conduction flux is dominant near the two covers, mainly due to large vapour velocity near the inner cover induced by the coming sweat and the relatively large conductivity of ice near the outer cover. Apart from this, the radiation flux is comparable with the convection/conduction flux in general. Since β defines the absorptivity/emissivity of radiation, the radiation flux decreases as β increases. Also, one can see in the last two figures that there are large variations in the radiative heat transfer rate (source) near the inner and outer covers since it is a function of T^4 and the temperature arrives its maximum and minimum there, respectively. Moreover, in the case of $\beta = 1400$, the convection/conduction flux is absolutely dominant everywhere.

In Figure 6, we present numerical results of an extremal human sweating system at 1 hour with $\beta = 400$ and 1400. Due to the large amount of sweat coming from the inner cover, a stronger condensation occurs near the inner cover, which results in lower convection/conduction flux near the cover.

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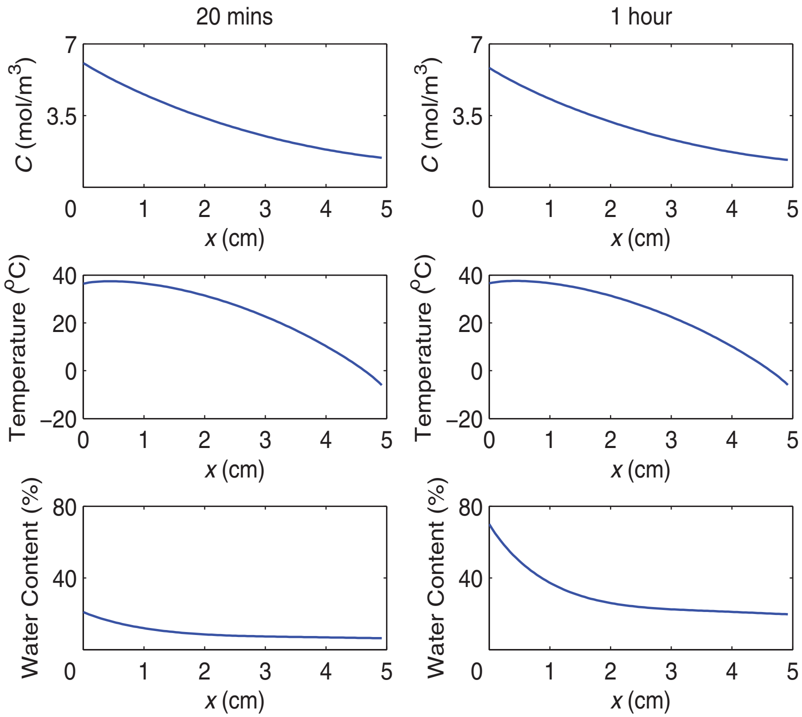

FIGURE 4. (Colour online) Numerical results of an extremal human sweating system ($\beta = 400$).

4 Analysis

Theoretical analysis for the above system without radiative heat transfer was given in [22]. In this section, we present *a priori* estimates for the solution of the system, from which, together with the standard construction of approximate solution and fixed-point theory, the existence of weak solution follows immediately. In the proof, the conservative form of boundary conditions plays an important role.

For a textile model, the water content in the batting area usually is relatively small and one often assumes that all these physical parameters involved in the system (2.1)–(2.2) are positive constants. For simplicity, we also neglect evaporation/condensation in phase change. With these techniques used in [19, 22], it is not difficult to extend the approach to more general cases. Under the above assumptions, with non-dimensionalization, the system (2.1)–(2.2) reduces to

$$\rho_t - (\zeta(\rho\theta)_x)_x = 0, \tag{4.1}$$

$$(\rho\theta)_t + \eta\theta_t - (\zeta(\rho\theta)_x\theta)_x - (\kappa(\rho)\theta_x)_x = \Gamma_r(\theta), \tag{4.2}$$

for $x \in \Omega := (0, 1)$, $t > 0$, where $(\cdot)_\mu = \frac{\partial}{\partial\mu}$ for $\mu = x, t$, $\rho = \rho(x, t)$ and $\theta = \theta(x, t)$ represent the vapour density and the temperature, respectively, $\Gamma_r = \beta(F_L(\theta) + F_R(\theta)) - 2\beta\sigma\theta^4$ as defined in (2.10)–(2.13) with $L = 1$, η and ζ are positive constants, and $\kappa(\rho) = \kappa_1 + \kappa_2\rho^2$ is the heat conductivity coefficient with κ_i $(i = 1, 2)$ being positive constants. A more general

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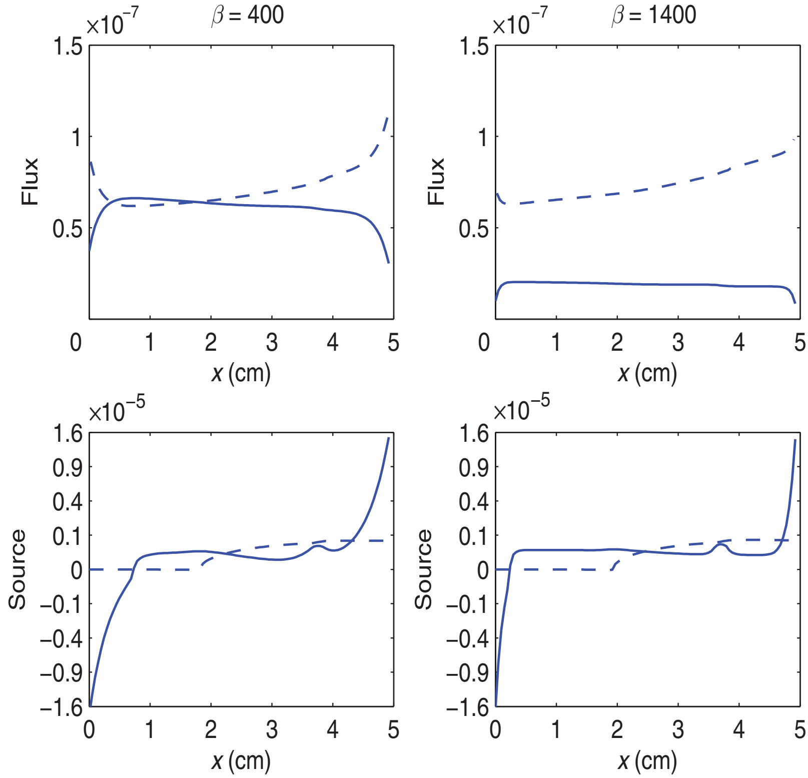

FIGURE 5. (Colour online) Fluxes and sources for a normal human sweating system at 8 hours. In the top subfigures, - - - represents convection/conduction heat fluxes, and — represents radiative heat flux. In the bottom subfigures, - - - represents condensation/evaporation source (Γ_{ce}), and — represents radiative heat transfer rate (Γ_r).

form of $\kappa(\rho)$ can be found in [32]. The corresponding boundary conditions are given by

$$-\zeta(\rho\theta)_x|_{x=0} = s_0, \tag{4.3}$$

$$-\zeta(\rho\theta)_x|_{x=1} = \alpha^1(\rho(1,t) - \bar{\rho}_1), \tag{4.4}$$

and

$$-\kappa(\rho)\theta_x|_{x=0} - (F_L - F_R)|_{x=0} = \beta^0(\bar{\theta}_0 - \theta(0,t)) + \epsilon_1\sigma(\bar{\theta}_0^4 - \theta^4(0,t)), \tag{4.5}$$

$$-\kappa(\rho)\theta_x|_{x=1} - (F_L - F_R)|_{x=1} = \beta^1(\theta(1,t) - \bar{\theta}_1) + \epsilon_2\sigma(\theta^4(1,t) - \bar{\theta}_1^4), \tag{4.6}$$

and the initial conditions are

$$\rho(x,0) = \rho_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in (0,1). \tag{4.7}$$

We assume that all the above parameters are positive constants, ρ_0 is non-negative and θ_0 is strictly positive. To simplify the notations, we denote by C a generic positive constant which solely depends upon the physical parameters involved in the equations as well as in the initial and boundary conditions, and δ is a small generic positive constant. In [19,22],

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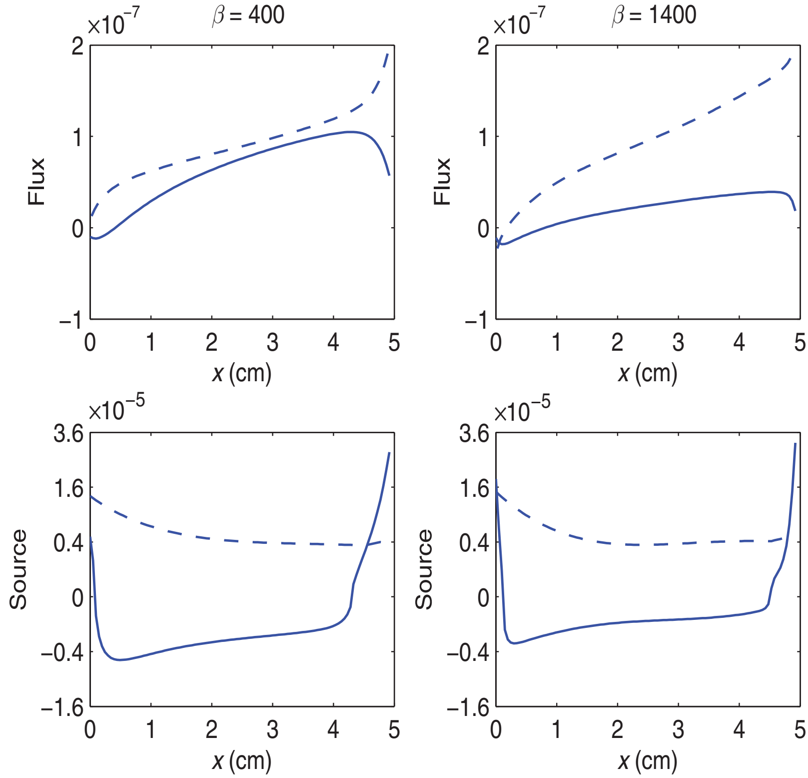

FIGURE 6. (Colour online) Fluxes and sources for an extremal human sweating system at 1 hour. In the top subfigures, - - - represents convection/conduction heat fluxes, and — represents radiative heat flux. In the bottom subfigures, - - - represents condensation/evaporation source (Γ_{ce}), and — represents radiative heat transfer rate (Γ_r).

the existence of a weak solution for the above system without radiative heat transfer was proved by constructing a sequence of approximate solution $(\rho,\ \theta) \in W_2^{2,1}(Q_T)$, where $Q_T := \Omega \times [0, T]$ for some positive constant T. We present *a priori* estimates for the solution of the system (4.1)–(4.2), where we assume that $(\rho,\ \theta) \in W_2^{2,1}(Q_T)$.

4.1 Positivity/non-negativity

First, we rewrite the system (4.1)–(4.2) by

$$\rho_t - (\zeta(\rho\theta)_x)_x = 0, \tag{4.8}$$

$$(\rho + \eta)\theta_t - \zeta(\rho\theta)_x\theta_x - (\kappa(\rho)\theta_x)_x = \Gamma_r(\theta), \tag{4.9}$$

where the second equation is obtained by adding (4.1) times $-\theta$ to (4.2).

In this section, we will prove that there exists a constant $\theta_{\min} > 0$ such that

$$\theta \geqslant \theta_{\min}, \quad \text{and} \quad \rho \geqslant 0, \quad \text{for } (x, t) \in Q_T. \tag{4.10}$$

The following lemma is useful for our proof.

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Lemma 4.1 *Both thermal radiation fluxes travelling to the left and right are non-negative, i.e.*

$$F_L \geqslant 0, \quad F_R \geqslant 0.$$

Proof The second inequality above can be observed directly from (2.15). By (2.14), we can express F_L by

$$\begin{aligned} F_L &= \sigma e^{\beta x}\left[\int_0^1 \beta e^{-\beta\xi}\theta^4 d\xi - \int_0^x \beta e^{-\beta\xi}\theta^4 d\xi\right] + \sigma e^{\beta x}\left[-\int_0^1 \beta e^{-\beta\xi}\theta^4 d\xi - f_l\right] \\ &= \sigma e^{\beta x}\int_x^1 \beta e^{-\beta\xi}\theta^4 d\xi + \sigma e^{\beta x} f, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} f = \frac{1}{e^{\beta} - (1-\epsilon_2)(1-\epsilon_1)e^{-\beta}} &\big[(1-\epsilon_2)e^{-\beta} g_r(1,t) \\ &+ (1-\epsilon_2)(1-\epsilon_1)e^{-\beta} g_l(1,t) + \epsilon_1(1-\epsilon_2)e^{-\beta}\theta^4(0,t) + \epsilon_2\theta^4(1,t)\big]. \end{aligned}$$

Since $f \geqslant 0$, we can see $F_L \geqslant 0$ immediately. □

Due to the uniform continuity of ρ in Q_T and η being positive, there exists t^* (only depends on η) such that when $|t_1 - t_2| \leqslant t^*$, $|\rho(x,t_1) - \rho(x,t_2)| \leqslant \frac{\eta}{2}$. Since $(\rho,\theta) \in C(\bar{Q}_T)$ and $\rho_0(x)$ is non-negative, we have $\rho(x,t) \geqslant -\frac{\eta}{2}$ in $[0,t^*]$. Now we prove the uniform positivity of θ in Q_{t^*}.

Let $\theta^+ = \max\{\theta, 0\}$ and $\theta^- = \max\{-\theta, 0\}$. Then $\theta = \theta^+ - \theta^-$. Subtracting (4.8) times $|\theta^-|^2$ from (4.9) times $2\theta^-$, and integrating the resulting equation, we obtain

$$\begin{aligned} &\int_0^{t^*}\int_\Omega ((\rho+\eta)|\theta^-|^2)_\tau dx d\tau - \int_0^{t^*} |\theta^-|^2 \cdot \zeta(\rho\theta)_x\Big|_0^1 d\tau + 2\int_0^{t^*} (\kappa\theta_x)\cdot\theta^-\Big|_0^1 d\tau \\ &\quad + 2\int_0^{t^*}\int_\Omega \kappa|\theta_x^-|^2 dx d\tau = -2\int_0^{t^*}\int_\Omega \Gamma_r \cdot \theta^- dx d\tau\,. \end{aligned} \tag{4.12}$$

By the boundary conditions (4.3)–(4.6), we have

$$\begin{aligned} &\int_\Omega (\rho+\eta)|\theta^-|^2 dx + 2\int_0^{t^*}\int_\Omega \kappa|\theta_x^-|^2 dx d\tau + 2\int_0^{t^*}\int_\Omega \beta(F_L+F_R)\theta^- dx d\tau \\ &\quad + 2\int_0^{t^*}\Big\{\theta^-(0,\tau)\big[\beta^0(\theta^-(0,\tau)+\bar\theta_0) + \epsilon_1\sigma\bar\theta_0^4 + \epsilon_1 F_L(0,\tau)\big] \\ &\quad + \theta^-(1,\tau)\big[\beta^1(\theta^-(1,\tau)+\bar\theta_1) + \epsilon_2\sigma\bar\theta_1^4 + \epsilon_2 F_R(1,\tau)\big]\Big\} d\tau \\ &\leqslant 4\|\theta^-\|_{L^\infty(Q_T)}^3 \int_0^{t^*}\int_\Omega \beta\sigma|\theta^-|^2 dx d\tau + \int_0^{t^*}\left[s_0 + \alpha^1\left(\bar\rho_1 + \frac{\eta}{2}\right)\right]\|\theta^-\|_{L^\infty(\Omega)}^2 d\tau \\ &\quad + 4\int_0^{t^*}\sigma(\epsilon_1+\epsilon_2)\|\theta^-\|_{L^\infty(\Omega)}^5 d\tau. \end{aligned} \tag{4.13}$$

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Since $\theta \in W_2^{2,1}(Q_T)$, $\theta \in C(\bar{Q}_T)$ and

$$\|\theta\|_{C(\bar{Q}_T)} \leqslant \|\theta\|_{W_2^{2,1}(Q_T)}.$$

By noting the inequalities

$$\|\theta^-\|^2_{L^\infty(\Omega)} \leqslant \delta\|\theta^-_x\|^2_{L^2(\Omega)} + C(\delta)\|\theta^-\|^2_{L^2(\Omega)},$$
$$\|\theta^-\|^5_{L^\infty(\Omega)} \leqslant C\|\theta^-_x\|^{5/2}_{L^2(\Omega)}\|\theta^-\|^{5/2}_{L^2(\Omega)} + C\|\theta^-\|^5_{L^2(\Omega)},$$

we can arrive at

$$\int_\Omega |\theta^-|^2 dx \leqslant \int_0^{t^*} C^*(\tau)\int_\Omega |\theta^-|^2 dx d\tau,$$

where $C^*(\tau) = C\|\theta^-\|^3_{L^\infty(Q_T)} + C\|\theta^-_x\|^{5/2}_{L^2(\Omega)} + C(\delta)$ is integrable in $(0, T)$. Using the Gronwall's inequality, we derive $\theta^- \equiv 0$, which implies $\theta = \theta^+ \geqslant 0$.

Moreover, let $\tilde{\theta} = \theta e^t - \delta_1$. Subtracting (4.8) times $|\tilde{\theta}^-|^2$ from (4.9) times $2\tilde{\theta}^-$ and integrating the resulting equation again, we obtain

$$\begin{aligned}
&\int_0^{t^*}\int_\Omega ((\rho+\eta)|\tilde{\theta}^-|^2)_\tau dx d\tau - \int_0^{t^*} |\tilde{\theta}^-|^2 \cdot \zeta(\rho\theta)_x\Big|_0^1 d\tau + 2\int_0^{t^*}(\kappa\tilde{\theta}_x)\cdot\tilde{\theta}^-\Big|_0^1 d\tau\\
&\quad + 2\int_0^{t^*}\int_\Omega \kappa|\tilde{\theta}^-_x|^2 dx d\tau + 2\int_0^{t^*}\int_\Omega (\rho+\eta)\delta_1\tilde{\theta}^- dx d\tau\\
&= 2\int_0^{t^*}\int_\Omega (\rho+\eta)|\tilde{\theta}^-|^2 dx d\tau - 2\int_0^{t^*}\int_\Omega e^\tau \Gamma_r \cdot \tilde{\theta}^- dx d\tau, \qquad (4.14)
\end{aligned}$$

with the boundary conditions

$$-\kappa(\rho)\tilde{\theta}_x|_{x=0} - e^t(F_L - F_R)|_{x=0} = \beta^0\left[(e^t\bar{\theta}_0 - \delta_1) - \tilde{\theta}(0,t)\right] + \epsilon_1\sigma e^t\left[\bar{\theta}_0^4 - (e^{-t}(\tilde{\theta}(0,t)+\delta_1))^4\right],$$
$$-\kappa(\rho)\tilde{\theta}_x|_{x=1} - e^t(F_L - F_R)|_{x=1} = \beta^1\left[\tilde{\theta}(1,t) - (e^t\bar{\theta}_1 - \delta_1)\right] + \epsilon_2\sigma e^t\left[(e^{-t}(\tilde{\theta}(1,t)+\delta_1))^4 - \bar{\theta}_1^4\right],$$

and the initial condition $\tilde{\theta}(x,0) = e^t\theta_0(x) - \delta_1$.

If we take

$$\delta_1 = \min_{x\in[0,1]}\{\bar{\theta}_0, \bar{\theta}_1, \theta_0(x), 1\},$$

the above system has the same structure as the system (4.5)–(4.6). Taking the same approach, we can see that $\tilde{\theta} \geqslant 0$, thus $\theta \geqslant e^{-T}\delta_1$.

Next, we prove the non-negativity of ρ in Q_{t^*}. Multiplying (4.1) by ρ^- and integrating the resulting equation give

$$\begin{aligned}
&\int_\Omega \frac{|\rho^-|^2}{2}dx + \int_0^{t^*}\int_\Omega \zeta\theta|\rho_x^-|^2 dxd\tau \\
&\quad + \int_0^{t^*}\left[\alpha^1\rho^-(1,\tau)(\rho^-(1,\tau)+\bar\rho_1)+\rho^-(0,\tau)s_0\right]d\tau \\
&= -\int_0^{t^*}\int_\Omega \zeta\theta_x\rho^-\rho_x^- dxd\tau \\
&\leqslant \int_0^{t^*}\int_\Omega \left(\frac{\zeta}{2\theta}\theta_x^2|\rho^-|^2+\frac{\zeta}{2}\theta|\rho_x^-|^2\right)dxd\tau \qquad (4.15)\\
&\leqslant \int_0^{t^*}\frac{\zeta e^T}{2\delta_1}\|\theta_x\|^2_{L^\infty(\Omega)}\int_\Omega|\rho^-|^2dxd\tau + \int_0^{t^*}\int_\Omega\frac{\zeta}{2}\theta|\rho_x^-|^2dxd\tau,
\end{aligned}$$

which implies

$$\int_\Omega |\rho^-|^2dx \leqslant \int_0^{t^*}\frac{\zeta e^T}{\delta_1}\|\theta_x\|^2_{L^\infty(\Omega)}\int_\Omega|\rho^-|^2dxd\tau.$$

Since $\theta\in W_2^{2,1}(Q_T)$ and $\|\theta_x\|^2_{L^\infty(\Omega)}$ is integrable, using the Gronwall's inequality again, we derive that $\rho^-\equiv 0$. Thus, $\rho=\rho^+\geqslant 0$ in Q_{t^*}.

Using the same approach, we can prove $\rho\geqslant 0,\ \theta\geqslant e^{-T}\delta_1$ in $[t^*,2t^*]$. Repeat the above process for any given $T>0$, we can see that $\rho\geqslant 0,\ \theta\geqslant e^{-T}\delta_1$ in Q_T.

4.2 Uniform estimates

We start from the estimates for θ. Integrating equation (4.2) over Q_t leads to

$$\int_\Omega(\rho+\eta)\theta\Big|_0^t dx - \int_0^t\zeta(\rho\theta)_x\theta\Big|_0^1 d\tau - \int_0^t(\kappa\theta_x)\Big|_0^1 d\tau = \int_0^t(F_L-F_R)\Big|_0^1 d\tau, \qquad (4.16)$$

which with boundary conditions gives

$$\begin{aligned}
&\int_\Omega(\rho+\eta)\theta dx + \int_0^t a^1\rho(1,\tau)\theta(1,\tau)d\tau \\
&\quad + \int_0^t\left[\beta^0\theta(0,\tau)+\beta^1\theta(1,\tau)+\epsilon_1\sigma\theta^4(0,\tau)+\epsilon_2\sigma\theta^4(1,\tau)\right]d\tau \\
&= \int_\Omega(\rho_0+\eta)\theta_0dx + \int_0^t\left[s_0\theta(0,\tau)+\alpha^1\bar\rho_1\theta(1,\tau)\right]d\tau \\
&\quad + \int_0^t\left[\beta^0\bar\theta_0+\beta^1\bar\theta_1+\epsilon_1\sigma\bar\theta_0^4+\epsilon_2\sigma\bar\theta_1^4\right]d\tau.
\end{aligned}$$

By noting the fact

$$a_1\theta^a \leqslant a_2\theta^b + C^a, \qquad (4.17)$$

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for any $\theta \geqslant 0$, and a, b, a_1, b_1 being positive constants with $a < b$, we have

$$\max_{0\leqslant\tau\leqslant t} \|\theta(\cdot,\tau)\|_{L^1(\Omega)} \leqslant C_T, \quad \int_0^t \theta^4(0,\tau)d\tau \leqslant C_T, \quad \int_0^t \theta^4(1,\tau)d\tau \leqslant C_T, \tag{4.18}$$

where C_T is a constant depending on T.

Lemma 4.2 *Let F_L and F_R be the radiation fluxes defined in (2.14)–(2.15). Then*

$$\begin{aligned}&\epsilon_2\theta^l(1,\tau)F_R(1,\tau) + \epsilon_1\theta^l(0,\tau)F_L(0,\tau)\\ &\quad \leqslant (1+\delta)\epsilon_2\sigma\theta^{l+4}(1,\tau) + (1+\delta)\epsilon_1\sigma\theta^{l+4}(0,\tau) + \delta\|\theta\|^{l+4}_{L^{l+4}(\Omega)} + C(\delta)\end{aligned} \tag{4.19}$$

and

$$\int_0^t\int_\Omega \beta(F_L+F_R)\theta^l dxd\tau \leqslant \delta\int_\Omega \theta^{l+1}dx + \int_0^t \delta\|\theta\|^{l+4}_{L^{l+4}(\Omega)}d\tau + \left(\frac{C_T}{\delta}\right)^{l+1} \tag{4.20}$$

for any $l > 0$.

Proof By (2.14)–(2.15) with $L = 1$, we can see that

$$\begin{aligned}f_l \geqslant -\frac{1}{e^\beta - (1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\Big[&(1-\epsilon_2)\beta\|\theta\|^4_{L^4(\Omega)} + \beta e^\beta\|\theta\|^4_{L^4(\Omega)}\\ &+(1-\epsilon_2)\epsilon_1 e^{-\beta}\theta^4(0,\tau) + \epsilon_2\theta^4(1,\tau)\Big],\end{aligned}$$

and moreover

$$\begin{aligned}&\epsilon_2\theta^l(1,\tau)F_R(1,\tau) + \epsilon_1\theta^l(0,\tau)F_L(0,\tau)\\ &= \epsilon_2\beta\sigma e^{-\beta}\theta^l(1,\tau)\int_0^1 e^{\beta\xi}\theta^4(\xi,\tau)d\xi + \epsilon_1\epsilon_2\theta^l(1,\tau)\sigma e^{-\beta}\theta^4(0,\tau)\\ &\quad - \left[\epsilon_2(1-\epsilon_1)\sigma e^{-\beta}\theta^l(1,\tau) + \epsilon_1\sigma\theta^l(0,\tau)\right] f_l\\ &\leqslant \left[\epsilon_2\beta\sigma + \epsilon_2(1-\epsilon_1)\beta\sigma e^{-\beta}\frac{(1-\epsilon_2)+e^\beta}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\right]\theta^l(1,\tau)\|\theta\|^4_{L^4(\Omega)}\\ &\quad + \epsilon_1\beta\sigma\frac{(1-\epsilon_2)+e^\beta}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\theta^l(0,\tau)\|\theta\|^4_{L^4(\Omega)}\\ &\quad + \frac{\epsilon_2(1-\epsilon_1)e^{-\beta}+\epsilon_1}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\epsilon_2\sigma\theta^{l+4}(1,\tau) + \frac{\epsilon_1(1-\epsilon_2)e^{-\beta}+\epsilon_2}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\epsilon_1\sigma\theta^{l+4}(0,\tau)\\ &\leqslant C\theta^l(1,\tau)\|\theta\|^4_{L^4(\Omega)} + C\theta^l(0,\tau)\|\theta\|^4_{L^4(\Omega)} + \epsilon_2\sigma\theta^{l+4}(1,\tau) + \epsilon_1\sigma\theta^{l+4}(0,\tau),\end{aligned} \tag{4.21}$$

where we have noted that

$$\frac{\epsilon_2(1-\epsilon_1)e^{-\beta}+\epsilon_1}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}} \leqslant 1, \quad \frac{\epsilon_1(1-\epsilon_2)e^{-\beta}+\epsilon_2}{e^\beta-(1-\epsilon_1)(1-\epsilon_2)e^{-\beta}} \leqslant 1.$$

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By (4.18), we further have

$$\theta^l(\xi,\tau)\|\theta\|^4_{L^4(\Omega)} \leqslant \theta^l(\xi,\tau)\big(\delta\|\theta\|^4_{L^{l+4}(\Omega)} + C(\delta)\|\theta\|^4_{L^1(\Omega)}\big)$$
$$\leqslant \delta\theta^{l+4}(\xi,\tau) + \delta\|\theta\|^{l+4}_{L^{l+4}(\Omega)} + C(\delta),$$

for $\xi = 0, 1$. (4.19) can be obtained by substituting these inequalities into (4.21).

To prove (4.20), we rewrite the sum of the radiation fluxes as

$$\beta(F_L + F_R) = -\beta^2\sigma e^{\beta x}\int_0^x e^{-\beta\xi}\theta^4(\xi,\tau)d\xi + \beta^2\sigma e^{-\beta x}\int_0^x e^{\beta\xi}\theta^4(\xi,\tau)d\xi$$
$$+\epsilon_1\beta\sigma e^{-\beta x}\theta^4(0,\tau) - \Big(\beta\sigma e^{\beta x} + \beta\sigma e^{-\beta x}(1-\epsilon_1)\Big)f_l.$$

Since

$$-e^{\beta x}\int_0^x e^{-\beta\xi}\theta^4(\xi,\tau)d\xi \leqslant -e^{\beta x}\int_0^x e^{-\beta x}\theta^4(\xi,\tau)d\xi = -\int_0^x \theta^4(\xi,\tau)d\tau$$
$$e^{-\beta x}\int_0^x e^{\beta\xi}\theta^4(\xi,\tau)d\xi \leqslant e^{-\beta x}\int_0^x e^{\beta x}\theta^4(\xi,\tau)d\xi = \int_0^x \theta^4(\xi,\tau)d\tau,$$

we have

$$\int_\Omega \beta(F_L+F_R)\theta^l dx \leqslant \int_\Omega \left[\epsilon_1\beta\sigma e^{-\beta x}\theta^4(0,\tau) - \left(\beta\sigma e^{\beta x} + \beta\sigma e^{-\beta x}(1-\epsilon_1)\right)f_l\right]\theta^l dx$$
$$\leqslant \epsilon_1\beta\sigma\theta^4(0,\tau)\|\theta\|^l_{L^l(\Omega)}$$
$$+\frac{e^\beta + (1-\epsilon_1)}{e^\beta - (1-\epsilon_1)(1-\epsilon_2)e^{-\beta}}\beta\sigma\Big[\big((1-\epsilon_2)\beta + \beta e^\beta\big)\|\theta\|^4_{L^4(\Omega)}$$
$$+\epsilon_1(1-\epsilon_2)e^{-\beta}\theta^4(0,\tau) + \epsilon_2\theta^4(1,\tau)\Big]\|\theta\|^l_{L^l(\Omega)}$$
$$\leqslant C\left[\theta^4(0,\tau)\|\theta\|^l_{L^l(\Omega)} + \theta^4(1,\tau)\|\theta\|^l_{L^l(\Omega)} + \|\theta\|^4_{L^4(\Omega)}\|\theta\|^l_{L^l(\Omega)}\right].$$

By (4.17)–(4.18) we further have

$$\int_0^t \theta^4(\xi,\tau)\|\theta\|^l_{L^l(\Omega)}d\tau \leqslant \left(\int_0^t \theta^4(\xi,\tau)d\tau\right)\|\theta\|^l_{L^l(\Omega)}$$
$$\leqslant \delta\int_\Omega \theta^{l+1}dx + \left(\frac{C_T}{\delta}\right)^{l+1}$$

for $\xi = 0, 1$, and by noting the fact that

$$\|\theta\|_{L^4(\Omega)} \leqslant \|\theta\|^{\delta_0}_{L^1(\Omega)}\|\theta\|^{1-\delta_0}_{L^{l+4}(\Omega)},$$

where $\delta_0 \in (\frac{1}{16}, \frac{1}{4})$,

$$\|\theta\|^4_{L^4(\Omega)}\|\theta\|^l_{L^l(\Omega)} \leqslant (C_T)^{4\delta_0}\|\theta\|^{l+4-4\delta_0}_{L^{l+4}(\Omega)}$$
$$\leqslant \delta\|\theta\|^{l+4}_{L^{l+4}(\Omega)} + \left(\frac{C_T}{\delta}\right)^{l+1}.$$

If follows that

$$\int_0^t \int_\Omega \beta(F_L+F_R)\theta^l dxd\tau \leqslant \delta \int_\Omega \theta^{l+1}dx + \int_0^t \delta \|\theta\|_{L^{l+4}(\Omega)}^{l+4} d\tau + \left(\frac{C_T}{\delta}\right)^{l+1}.$$

The proof is complete. □

By subtracting (4.1) times $l \times \theta^{l+1}$ from (4.2) times $(l+1)\times\theta^l$ and integrating the resulting equation over Q_t, we get

$$\begin{aligned}\int_\Omega (\rho+\eta)\theta^{l+1}dx + \int_0^t I(\tau)d\tau + \int_0^t\int_\Omega \kappa(\rho)l(l+1)\theta^{l-1}|\theta_x|^2dxd\tau \\ = \int_\Omega(\rho_0+\eta)\theta_0^{l+1}dx + \int_0^t\int_\Omega (l+1)\Gamma_r\theta^l dxd\tau,\end{aligned} \tag{4.22}$$

where

$$\begin{aligned} I(\tau) &= -\zeta(\rho\theta)_x\theta^{l+1}\Big|_{x=0}^{x=1} - (l+1)(\kappa(\rho)\theta_x)\theta^l\Big|_{x=0}^{x=1} \\ &= \left[\alpha^1(\rho(1,\tau)-\bar\rho_1)\theta^{l+1}(1,\tau) - s_0\theta^{l+1}(0,\tau)\right] \\ &\quad + (l+1)\left[\beta^1\theta^l(1,\tau)(\theta(1,\tau)-\bar\theta_1) + \beta^0\theta^l(0,\tau)(\theta(0,\tau)-\bar\theta_0)\right] \\ &\quad + (l+1)\left[\epsilon_2\sigma\theta^l(1,\tau)(\theta^4(1,\tau)-\bar\theta_1^4) + \epsilon_1\sigma\theta^l(0,\tau)(\theta(0,\tau)^4-\bar\theta_0^4)\right] \\ &\quad + (l+1)\left[\theta^l(1,\tau)(F_L-F_R)|_{x=1} - \theta^l(0,\tau)|(F_L-F_R)_{x=0}\right].\end{aligned}$$

For $\frac{l}{2} \geqslant \frac{\alpha^1\bar\rho_1}{\beta^1} + \frac{s_0}{\beta^0}$, by (2.12)–(2.13) and (4.17), we note that

$$\begin{aligned} I(\tau) \geqslant &-(l+1)\left[\left(\frac{2}{\beta^1}\right)^l(\beta^1\bar\theta_1+\epsilon_2\sigma\bar\theta_1^4)^{l+1} + \left(\frac{2}{\beta^0}\right)^l(\beta^0\bar\theta_0+\epsilon_1\sigma\bar\theta_0^4)^{l+1}\right] \\ &+(l+1)\left[\epsilon_2\sigma\theta^{l+4}(1,\tau)+\epsilon_1\sigma\theta^{l+4}(0,\tau)\right] \\ &+(l+1)\theta^l(1,\tau)\left[\epsilon_2\sigma\theta^4(1,\tau)-\epsilon_2F_R(1,\tau)\right] \\ &+(l+1)\theta^l(0,\tau)\left[\epsilon_1\sigma\theta^4(0,\tau)-\epsilon_1F_L(0,\tau)\right].\end{aligned}$$

Then (4.22) reduces to

$$\begin{aligned}&\int_\Omega(\rho+\eta)\theta^{l+1}dx + \int_0^t\int_\Omega\kappa(\rho)l(l+1)\theta^{l-1}|\theta_x|^2dxd\tau \\ &\quad+\int_0^t 2\sigma(l+1)\left[\epsilon_2\theta^{l+4}(1,\tau)+\epsilon_1\theta^{l+4}(0,\tau)\right]d\tau + \int_0^t\int_\Omega 2\beta\sigma(l+1)\theta^{l+4}(x,\tau)dxd\tau \\ &\leqslant C_0^{l+1} + \int_0^t(l+1)\left[\epsilon_2\theta^l(1,\tau)F_R(1,\tau)+\epsilon_1\theta^l(0,\tau)F_L(0,\tau)\right]d\tau \\ &\quad+\int_0^t\int_\Omega(l+1)\beta(F_L+F_R)\theta^l dxd\tau,\end{aligned} \tag{4.23}$$

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where

$$C_0^{l+1} \geqslant \int_\Omega (\rho_0+\eta)\theta_0^{l+1}dx$$
$$+T(l+1)\left[\left(\frac{2}{\beta^1}\right)^l(\beta^1\bar\theta_1+\epsilon_2\sigma\bar\theta_1^4)^{l+1}+\left(\frac{2}{\beta^0}\right)^l(\beta^0\bar\theta_0+\epsilon_1\sigma\bar\theta_0^4)^{l+1}\right].$$

From Lemma 4.2, we see that

$$\int_\Omega\left(\rho+\frac{\eta}{2}\right)\theta^{l+1}dx+\int_0^t\int_\Omega \kappa(\rho)l(l+1)\theta^{l-1}|\theta_x|^2dxd\tau \leqslant C_1^{l+1}, \tag{4.24}$$

where $C_1^{l+1}\geqslant C_0^{l+1}+C(\delta)+(\frac{C_T}{\delta})^{l+1}$. Thus, we have

$$||\theta||_{L^{l+1}(Q_T)}\leqslant C_1.$$

By taking $l\to\infty$,

$$||\theta||_{L^\infty(Q_T)}\leqslant C_1. \tag{4.25}$$

On the other hand, with (4.25), taking $l=1$ in (4.22) gives

$$\int_0^t\int_\Omega \kappa(\rho)|\theta_x|^2dxd\tau\leqslant C_1. \tag{4.26}$$

To estimate ρ, we integrate (4.1) times ρ over Q_t to get

$$\begin{aligned}\int_\Omega\frac{\rho^2}{2}dx&+\int_0^t\int_\Omega\zeta\rho_x^2\theta dxd\tau+\int_0^t\alpha^1\rho^2(1,\tau)d\tau\\
&=\int_\Omega\frac{\rho_0^2}{2}dx+\int_0^t\left[\alpha^1\rho(1,\tau)\bar\rho_1+\rho(0,\tau)s_0\right]d\tau-\int_0^t\int_\Omega\zeta\theta_x\rho\rho_xdxd\tau\\
&\leqslant\int_\Omega\frac{\rho_0^2}{2}dx+\int_0^t\left[\alpha^1\rho(1,\tau)\bar\rho_1+\rho(0,\tau)s_0\right]d\tau\\
&\quad+\int_0^t\int_\Omega\frac{\zeta}{2}\rho_x^2\theta dxd\tau+\int_0^t\int_\Omega\frac{\zeta}{2\theta}\rho^2\theta_x^2dxd\tau.\end{aligned} \tag{4.27}$$

By noting that

$$\|\rho\|^2_{L^\infty(\Omega)}\leqslant\delta\|\rho_x\|^2_{L^2(\Omega)}+C(\delta)\|\rho\|^2_{L^2(\Omega)}+C,$$

from which, with (4.10) and (4.26), we obtain

$$\int_\Omega\rho^2dx\leqslant\int_0^t\int_\Omega C(\delta)\rho^2dx+C_1.$$

Using the Gronwall's inequality, we have

$$\sup_{0\leqslant t\leqslant T}\int_\Omega\rho^2dx\leqslant C_1. \tag{4.28}$$

With the above inequality, (4.27) implies

$$\int_0^T \int_\Omega \rho_x^2 dx d\tau \leqslant C_1. \tag{4.29}$$

With the above *a priori* estimates, the existence of a weak solution can be obtained by a routine argument with the Leray–Schauder fixed-point theorem [19,22]. We summarize the main result of this section into the following theorem.

Theorem 4.1 *If the initial data* (ρ_0, θ_0) *satisfies* $\rho_0 \in L^2(\Omega)$, $\theta_0 \in L^\infty(\Omega)$ *and* $\rho_0 > 0$, $\theta_0 \geqslant \theta_{min}$ *for some positive constant* θ_{min}*, then there exists a global weak solution* (ρ, θ) *to the initial-boundary value problem (4.1)–(4.7) such that*

$$\begin{aligned} &\rho \in L^\infty(0,T; L^2(\Omega)), \quad \rho \in L^4(Q_T), \quad \rho_x \in L^2(Q_T); \\ &\theta, \theta^{-1} \in L^\infty(Q_T), \quad (1+\rho)\theta_x \in L^2(Q_T) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega (-\rho\phi_t + \zeta(\rho\theta)_x \phi_x) dx dt - \int_0^T s_0 \phi(0,t) dt \\ &\quad + \int_0^T \alpha^1(\rho(1,t) - \bar{\rho}_1)\phi(1,t) dt = \int_\Omega \rho_0 \phi(x,0) dx \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \left[-(\rho+\eta)\theta\psi_t + \zeta(\rho\theta)_x \theta\psi_x + \kappa\theta_x\psi_x + (F_L - F_R)\psi_x\right] dx dt \\ &\quad + \int_0^T \left[-s_0\theta(0,t)\psi(0,t) + a^1(\rho(1,t) - \bar{\rho}_1)\theta(1,t)\psi(1,t)\right] dt \\ &\quad + \int_0^T \left[\beta^0(\theta(0,t) - \bar{\theta}_0) + \epsilon_1\sigma\left(\theta^4(0,t) - \bar{\theta}_0^4\right)\right] \psi(0,t) dt \\ &\quad + \int_0^T \left[\beta^1(\theta(1,t) - \bar{\theta}_1) + \epsilon_2\sigma\left(\theta^4(1,t) - \bar{\theta}_1^4\right)\right] \psi(1,t) dt \\ &= \int_\Omega (\rho_0\theta_0 + \eta\theta_0)\psi(x,0) dx \end{aligned} \tag{4.32}$$

for any test functions $\phi, \psi \in C^\infty(\overline{Q}_T)$ *which vanish at* $t = T$.

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