Proceedings of the Royal Society of Edinburgh, **145A**, 1053–1090, *2015*
DOI:10.1017/S0308210515000268

Cylindrically symmetric travelling fronts in a periodic reaction–diffusion equation with bistable nonlinearity

Zhi-Cheng Wang
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China (wangzhch@lzu.edu.cn)

(MS received 4 September 2013; accepted 12 August 2014)

This paper is concerned with the existence, non-existence and qualitative properties of cylindrically symmetric travelling fronts for time-periodic reaction–diffusion equations with bistable nonlinearity in $\mathbb{R}^m$ with $m \geqslant 2$. It should be mentioned that the existence and stability of two-dimensional time-periodic V-shaped travelling fronts and three-dimensional time-periodic pyramidal travelling fronts have been studied previously. In this paper we consider two cases: the first is that the wave speed of a one-dimensional travelling front is positive and the second is that the one-dimensional wave speed is zero. For both cases we establish the existence, non-existence and qualitative properties of cylindrically symmetric travelling fronts. In particular, for the first case we furthermore show the asymptotic behaviours of level sets of the cylindrically symmetric travelling fronts.

Keywords: reaction–diffusion equation; time-periodic nonlinearity; bistable nonlinearity; cylindrically symmetric travelling front

2010 *Mathematics subject classification:* Primary 35C07
Secondary 35K57; 35B10; 35B08; 92D25

1. Introduction

This paper is concerned with the following time-periodic reaction–diffusion equation:

$$\frac{\partial}{\partial t}u(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + f(u(\boldsymbol{x},t),t), \quad \boldsymbol{x} \in \mathbb{R}^m,\ t > 0, \tag{1.1}$$

where $m \geqslant 2$. Throughout this paper, we assume that $f \in C^{2,1}(\mathbb{R}^2,\mathbb{R})$ satisfies the following hypotheses.

(H1) There exists $T > 0$ such that $f(u,t) = f(u,t+T)$ for all $(u,t) \in \mathbb{R}^2$.

(H2) The period map $P(\alpha) := w(\alpha,T)$ has exactly three fixed points α^-, α^0 and α^+ satisfying $\alpha^- < \alpha^0 < \alpha^+$, where $w(\alpha,t)$ is the solution of

$$w_t = f(w,t), \quad t \in \mathbb{R},\ w(\alpha,0) = \alpha \in \mathbb{R}.$$

Furthermore, they are non-degenerate and the $\alpha^\pm$ are stable, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}P(\alpha^\pm) < 1 < \frac{\mathrm{d}}{\mathrm{d}\alpha}P(\alpha^0).$$

(H3) There exists $\nu_0 > 0$ such that $\nu^+ + \nu^- + f_u(W^{\pm}(t), t) > \nu_0$ for any $t \in [0, T]$, where

$$\nu^{\pm} := -\frac{1}{T}\int_0^T f_u(W^{\pm}(\lambda), \lambda)\mathrm{d}\lambda, \qquad W^{\pm}(t) := w(\alpha^{\pm}, t)$$

and

$$W^0(t) := w(\alpha^0, t).$$

(H4) There exist constants $r_0 > 0$ and $\epsilon \in (0, \min_{t\in[0,T]}(W^0(t) - W^-(t)))$ such that

$$\bar{f}(u, t) \geqslant r_0 u(\epsilon - u) \quad \text{for any } u \in (0, \epsilon) \text{ and } t \in [0, T],$$

where $\bar{f}(u, t) := f(W^0(t), t) - f(W^0(t) - u, t)$.

It is known from [1] (see also [34, 35]) that if $f(u, t) \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ satisfies hypotheses (H1) and (H2), then there exists a unique solution pair (c, U) to (1.1) in one-dimensional space satisfying

$$\left.\begin{aligned} U_t = U_{\eta\eta} - cU_\eta + f(U, t), \quad (\eta, t) \in \mathbb{R}^2, \\ U(\pm\infty, t) = \lim_{\eta\to\pm\infty} U(\eta, t) = W^{\pm}(t), \quad t \in \mathbb{R}, \\ U(\cdot, \cdot + T) = U(\cdot, \cdot), \quad U(0, 0) = \alpha^0, \end{aligned}\right\} \tag{1.2}$$

where the function $U(\cdot, \cdot)\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the wave profile and the constant $c \in \mathbb{R}$ is the wave speed. In addition, (c, U) has the following properties.

(i) $U(\cdot, t)$ is increasing with respect to the moving coordinate for each t. Namely, $U_\eta(\cdot, \cdot) > 0$ in $\mathbb{R} \times \mathbb{R}$.

(ii) There exist positive constants C_1 and β_1 satisfying

$$|U(\pm\eta, t) - W^{\pm}(t)| + |U_\eta(\pm\eta, t)| + |U_{\eta\eta}(\pm\eta, t)| \leqslant C_1 \mathrm{e}^{-\beta_1\eta}, \quad \eta \geqslant 0,\ t \in \mathbb{R}. \tag{1.3}$$

That is, U exponentially approaches its limits as $\eta \to \pm\infty$.

A typical example of f satisfying (H1)–(H3) is the cubic potential $f(u, t) := (1-u^2)(2u-\rho(t))$, where $\rho(t) \in (-2, 2)$ is T-periodic. In this case, $W^{\pm}(t) = \pm 1$ and $-1 < W^0(t) < 1$ for any $t \in [0, T]$. In addition, if we restrict $\max_{t\in[0,T]} |\rho(t)| < \frac{2}{5}\sqrt{5}$, then $f(u, t) := (1 - u^2)(2u - \rho(t))$ satisfies assumption (H4). Here we give a simple proof. For any $t \in \mathbb{R}$, we have

$$\begin{aligned} f(W^0(t), t) &- f(W^0(t) - u, t) \\ &= 2u(1 - u^2) - 6W^0(t)u(W^0(t) - u) + (2W^0(t) - u)\rho(t)u \\ &= u[2(1 - u^2) - 6W^0(t)(W^0(t) - u) + (2W^0(t) - u)\rho(t)]. \end{aligned}$$

Let $g(u, t) := 2(1 - u^2) - 6W^0(t)(W^0(t) - u) + (2W^0(t) - u)\rho(t)$. It follows that

$$g(0, t) = 2[1 - 3(W^0(t))^2 + W^0(t)\rho(t)].$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

Let $t_1, t_2 \in [0,T]$ satisfy

$$W^0(t_1) = \min_{t\in[0,T]} W^0(t) \quad \text{and} \quad W^0(t_2) = \max_{t\in[0,T]} W^0(t).$$

Since $(\mathrm{d}/\mathrm{d}t)W^0(t) = (1-(W^0(t))^2)(2W^0(t)-\rho(t))$, we have that $W^0(t_1) = \frac{1}{2}\rho(t_1)$ and $W^0(t_2) = \frac{1}{2}\rho(t_2)$. Therefore, $\max_{t\in[0,T]}|W^0(t)| \leqslant \frac{1}{2}\max_{t\in[0,T]}|\rho(t)|$. Consequently, we have

$$g(0,t) \geqslant 2\Big(1 - \frac{5}{4}\max_{t\in[0,T]}|\rho(t)|^2\Big) > 0$$

if $\max_{t\in[0,T]}|\rho(t)| < \frac{2}{5}\sqrt{5}$. Thus, we have that (H4) holds.

In fact, $f(u,t) := (1-u^2)(2u-\rho(t))$ is a particular case of the following more general example (see [1])

$$f(u,t) = p(u)(-p'(u) - \rho(t)),$$

where $\rho \in C^1$ and $p \in C^3$ satisfy $\rho(\cdot + T) = \rho(\cdot)$, and $p(\pm 1) = 0$, $p(\cdot) > 0$ in $(-1,1)$. In this case, the wave speed c can be directly calculated from

$$c = \frac{1}{T}\int_0^T \rho(t)\,\mathrm{d}t.$$

Recently, Wang and Wu [43] and Sheng *et al*. [37] studied two-dimensional V-shaped travelling fronts and high-dimensional pyramidal travelling fronts of (1.1) under assumptions (H1)–(H3) and established the existence, uniqueness and stability of the travelling fronts. We note that the results established by [37, 43] for the non-autonomous equation (1.1) can be regarded as an extension of the results established by [22,31,32,39,40] for the autonomous Allen–Cahn equation. Besides V-form travelling fronts and pyramidal travelling fronts, here we would like to mention that there have been many studies concerned with cylindrically symmetric travelling fronts in the autonomous Allen–Cahn equation (see [3,4,12,14,16–18,41]). Moreover, we refer the reader to [7, 11, 13, 15, 19, 20, 26, 30, 38, 42] for more results on multi-dimensional travelling wave solutions. For the non-autonomous reaction–diffusion equation, we would like to mention more results on one-dimensional travelling wavefronts; see [8, 34–36] for bistable nonlinearity and [21, 23, 24, 28, 29] for monostable nonlinearity. However, there is no contribution on cylindrically symmetric travelling fronts of time-heterogeneous equations in multi-dimensional space, even for the time-periodic case. Resolving this issue is the main purpose of our current study.

The study of this paper contains two parts: the first part is concerned with the case in which $c > 0$ and the second part is concerned with the case in which $c = 0$. Assume that $c \geqslant 0$. For any $s > c$, a cylindrically symmetric travelling front of (1.1) means a classical solution $u(\boldsymbol{x},t) = v(\boldsymbol{x}', x_m + st, t)$ such that

$$\left.\begin{aligned}
\frac{\partial}{\partial t}v = \Delta v - s\frac{\partial}{\partial x_m}v + f(v,t), \quad \boldsymbol{x} = (\boldsymbol{x}', x_m) = (x_1,\dots,x_m) \in \mathbb{R}^m,\ t\in\mathbb{R},\\
v(\boldsymbol{x}_1', x_m, t) = v(\boldsymbol{x}_2', x_m, t) \quad \forall \boldsymbol{x}_1', \boldsymbol{x}_2' \in \mathbb{R}^{m-1} \text{ with } |\boldsymbol{x}_1'| = |\boldsymbol{x}_2'|,\ x_m\in\mathbb{R},\ t\in\mathbb{R},\\
v(\boldsymbol{x}, t+T) = v(\boldsymbol{x},t) \quad \forall \boldsymbol{x}\in\mathbb{R}^m,\ t\in\mathbb{R},\\
\lim_{x_m\to\pm\infty} v(\boldsymbol{x}', x_m, t) = W^{\pm}(t) \quad \forall \boldsymbol{x}'\in\mathbb{R}^{m-1},\ t\in\mathbb{R}.
\end{aligned}\right\} \tag{1.4}$$

In the following we give the main results of this paper. The first part is concerned with the case in which $c > 0$.

THEOREM 1.1. *Assume that (H1)–(H4) hold. Suppose that $c > 0$. Then for any $s > c$ there exists a function $W(\boldsymbol{x}, t)$ satisfying (1.4). In addition, one has:*

(i) $W(\boldsymbol{x}', x_m, t) = W(\boldsymbol{x}'', x_m, t)$ *for* $t \in \mathbb{R}$, $x_m \in \mathbb{R}$ *and* $\boldsymbol{x}', \boldsymbol{x}'' \in \mathbb{R}^{m-1}$ *with* $|\boldsymbol{x}'| = |\boldsymbol{x}''|$;

(ii) *for any* $(\boldsymbol{x}_0', x_m') \in \mathbb{R}^m$ *with* $x_m' \geqslant m_* |\boldsymbol{x}_0'|$,

$$W(\boldsymbol{x}' + \boldsymbol{x}_0', x_m, t) \leqslant W(\boldsymbol{x}', x_m + x_m', t) \quad \forall (\boldsymbol{x}', x_m) \in \mathbb{R}^m, \ t \in \mathbb{R},$$

where $m_* = \sqrt{s^2 - c^2}/c$;

(iii) $\dfrac{\partial}{\partial x_m} W(\boldsymbol{x}, t) > 0$ *for any* $\boldsymbol{x} \in \mathbb{R}^m$ *and* $t \in \mathbb{R}$;

(iv) $\dfrac{\partial}{\partial x_i} W(\boldsymbol{x}, t) > 0$ *on* $x_i \in (0, \infty)$, $i = 1, 2, \dots, m-1$;

(v) *we have*

$$\lim_{x_m \to \infty} \|W(\cdot, x_m, t) - W^+(t)\|_{C(\mathbb{R}^{m-1})} = 0$$

and

$$\lim_{x_m \to -\infty} \|W(\cdot, x_m, t) - W^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^{m-1})} = 0$$

uniformly on $t \in \mathbb{R}$;

(vi) $\dfrac{\partial}{\partial \nu} W(\boldsymbol{x}, t) > 0$ *for any* $\boldsymbol{x} \in \mathbb{R}^m$ *and* $t \in \mathbb{R}$, *where*

$$\nu = \frac{1}{\sqrt{1 + \sum_{j=1}^{m-1} \nu_j^2}} (\nu_1, \nu_2, \dots, \nu_{m-1}, 1)$$

satisfies

$$\sqrt{\nu_1^2 + \nu_2^2 + \dots + \nu_{m-1}^2} \leqslant \frac{1}{m_*}.$$

Define

$$\Psi(\rho, z, t) = \Psi(|\boldsymbol{x}'|, x_m, t) := W(\boldsymbol{x}, t) \tag{1.5}$$

for any $(\boldsymbol{x}', x_m) \in \mathbb{R}^m$ and $t \in \mathbb{R}$, where $\rho = |\boldsymbol{x}'|$ and $z = x_m$. Define a function $\phi(\rho)$ by

$$\Psi(\rho, \phi(\rho), 0) = \theta_0,$$

where $\theta_0 \in (\alpha^-, \alpha^0)$ is a given constant. By a shift, let $U(0, 0) = \theta_0$. We then have the following theorem.

THEOREM 1.2. *Assume that (H1)–(H4) hold. Assume that $c > 0$. Let $\Psi(\rho, z, t)$ be defined by (1.5). Then $\Psi(\rho, z, t)$ satisfies*

$$\frac{\partial}{\partial t}\Psi = \frac{\partial^2}{\partial \rho^2}\Psi + \frac{\partial^2}{\partial z^2}\Psi + \frac{m-2}{\rho}\frac{\partial}{\partial \rho}\Psi - s\frac{\partial}{\partial z}\Psi + f(\Psi(\rho, z, t), t) \quad \forall \rho > 0,\ z \in \mathbb{R},\ t \in \mathbb{R}.$$

Moreover, one has

$$\begin{aligned}
\frac{\partial}{\partial \rho}\Psi(\rho, z, t) > 0 \quad &\forall \rho > 0,\ z \in \mathbb{R},\ t \in \mathbb{R},\\
\frac{\partial}{\partial z}\Psi(\rho, z, t) > 0 \quad &\forall \rho \geqslant 0,\ z \in \mathbb{R},\\
\lim_{z \to -\infty} \|\Psi(\cdot, z, t) - W^-(t)\|_{C([0,\omega])} = 0 \quad &\textit{uniformly in } t \in \mathbb{R} \textit{ for any } \omega > 0,\\
\lim_{z \to +\infty} \|\Psi(\cdot, z, t) - W^+(t)\|_{C([0,+\infty))} = 0 \quad &\textit{uniformly in } t \in \mathbb{R},\\
\frac{\partial}{\partial \nu}\Psi(\rho, z, t) > 0 \quad &\forall \rho > 0,\ z > 0,\ t \in \mathbb{R},\\
\lim_{\rho \to \infty} \phi'(\rho) &= -m_*,\\
\lim_{\rho \to \infty} \Psi_\rho(\rho, \phi(\rho), 0) &= \frac{cm_*}{s}U_\eta(0, 0),\\
\lim_{\rho \to \infty} \Psi_z(\rho, \phi(\rho), 0) &= \frac{c}{s}U_\eta(0, 0)
\end{aligned}$$

and

$$\lim_{\rho \to \infty} \left\| \Psi(\rho + x, \phi(\rho) + z, t) - U\left(\frac{s}{c}(z + m_* x), t\right) \right\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R})} = 0,$$

where $\nu = \dfrac{1}{\sqrt{1 + (\nu')^2}}\begin{pmatrix}\nu' \\ 1\end{pmatrix}$ *is a given constant vector with* $\nu' \geqslant -\dfrac{1}{m_*}$.

THEOREM 1.3. *Assume that (H1)–(H4) hold. Assume that $c > 0$. Let $s > c > 0$ and denote $W(\boldsymbol{x}, t)$ defined in theorem 1.1 by $W^s(\boldsymbol{x}, t)$. Let $W^s(\mathbf{0}, 0) = U(0, 0)$. Then one has*

$$\lim_{s \to c} \|W^s(\boldsymbol{x}, t) - U(x_m, t)\|_{C^2_{\mathrm{loc}}(\mathbb{R}^m)} = 0$$

uniformly in $t \in \mathbb{R}$.

THEOREM 1.4. *Assume that (H1)–(H4) hold. Assume that $c > 0$. For $s > c > 0$ there is no function $W(\boldsymbol{x}, t)$ satisfying*

$$\frac{\partial}{\partial t}W = \Delta W - s\frac{\partial}{\partial x_m}W + f(W, t)$$

for any $(\boldsymbol{x}, t) \in \mathbb{R}^m \times \mathbb{R}$, $W(\boldsymbol{x}, t + T) = W(\boldsymbol{x}, t)$ for any $(\boldsymbol{x}, t) \in \mathbb{R}^m \times \mathbb{R}$, $\lim_{x_m \to \pm\infty} W(\mathbf{0}, x_m, t) = W^{\pm}(t)$ uniformly in $t \in \mathbb{R}$, and

$$\frac{\partial}{\partial x_m}W(\boldsymbol{x}, t) \geqslant 0, \qquad \frac{\partial^2}{\partial x_i^2}W(\boldsymbol{x}, t)\bigg|_{\boldsymbol{x}' = \mathbf{0}} \leqslant 0, \quad i = 1, 2, \ldots, m-1.$$

Theorem 1.5. *Assume that (H1)–(H4) hold. Assume that $c > 0$. For $s < c$ there is no function $W(\boldsymbol{x}, t)$ satisfying*

$$\frac{\partial}{\partial t} W = \Delta W - s\frac{\partial}{\partial x_m} W + f(W, t)$$

for any $(\boldsymbol{x}, t) \in \mathbb{R}^m \times \mathbb{R}$, $W(\boldsymbol{x}, t+T) = W(\boldsymbol{x}, t)$ *for any* $(\boldsymbol{x}, t) \in \mathbb{R}^m \times \mathbb{R}$, $\lim_{x_m \to \pm\infty} W(\mathbf{0}, x_m, t) = W^{\pm}(t)$ *uniformly in* $t \in \mathbb{R}$, *and*

$$\frac{\partial}{\partial x_m} W(\boldsymbol{x}, t) \geqslant 0, \qquad \frac{\partial^2}{\partial x_i^2} W(\boldsymbol{x}, t)\bigg|_{\boldsymbol{x}'=\mathbf{0}} \geqslant 0, \quad i = 1, 2, \dots, m-1.$$

In the following we consider the case in which $c = 0$. In this case, we furthermore assume that the following condition holds.

(H5) There exists a sequence of T-periodic functions $\{f_n(u, t)\} \subset C^{2,1}(\mathbb{R}^2, \mathbb{R})$ such that the following hold.

(1) For each $n \in \mathbb{N}$, f_n satisfies (H1)–(H4). In particular, for each $n \in \mathbb{N}$ the periodic map $P_n(\alpha) := w_n(\alpha, T)$ has exactly three fixed points α_n^-, α_n^0, α_n^+ with $\alpha_n^- < \alpha_n^0 < \alpha_n^+$, where $w_n(\alpha, t)$ is the solution of

$$w_t = f_n(w, t), \quad t \in \mathbb{R}, \ w(\alpha, 0) = \alpha \in \mathbb{R}.$$

Furthermore, they are non-degenerate and the $\alpha_n^{\pm}$ are stable, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} P_n(\alpha_n^{\pm}) < 1 < \frac{\mathrm{d}}{\mathrm{d}\alpha} P_n(\alpha_n^0).$$

(2) There hold $\lim_{n\to\infty} \|f_n(\cdot,\cdot) - f(\cdot,\cdot)\|_{C^1([-M,M]\times[0,T])} = 0$ and $\alpha_n^- \to \alpha^-$, $\alpha_n^0 \to \alpha^0$ and $\alpha_n^+ \to \alpha^+$ as $n \to \infty$, where

$$M = \max_{n\in\mathbb{N}} \Big\{ \max_{t\in[0,T]} |w_n(\alpha_n^-, t)|, \max_{t\in[0,T]} |w_n(\alpha_n^+, t)| \Big\}. \tag{1.6}$$

Consequently, $W_n^{\pm}(t) \to W^{\pm}(t)$ and $W_n^0(t) \to W^0(t)$ uniformly in $t \in \mathbb{R}$ as $n \to \infty$, where $W_n^{\pm}(t) = w_n(\alpha_n^{\pm}, t)$ and $W_n^0(t) = w_n(\alpha_n^0, t)$.

(3) Let (c_n, U_n) be the unique travelling wave solution defined by (1.2) with $f_n(u, t)$, which connects two periodic solutions $W_n^{\pm}(t)$. Suppose that $c_n > 0$ and that $c_n \to 0$ as $n \to \infty$.

We note that $f(u, t) = (1-u^2)(u - \rho(t))$ with $\int_0^T \rho(t)\,\mathrm{d}t = 0$ satisfies (H5), where $\rho(t) \in (-2, 2)$ is T-periodic. Let $f_n(u, t) = (1-u^2)(2u - (\varepsilon/n + \rho(t)))$ for some small $\varepsilon > 0$. Let $c_n = (1/T)\int_0^T (\varepsilon/n + \rho(t))\,\mathrm{d}t = \varepsilon/n > 0$. Clearly, $c_n \to 0$ as $n \to \infty$. By the previous argument, we have $|W_n^0(t)| < \varepsilon/n + \max_{t\in[0,T]} |\rho(t)|$ for any $t \in [0, T]$. Up to the extraction of a subsequence, let

$$W_n^0(t) \to W_*^0(t) \quad \text{uniformly in } t \in [0, T] \text{ as } n \to \infty.$$

Then $(\mathrm{d}/\mathrm{d}t)W_*^0(t) = f(W_*^0(t), t)$. It is clear that $-1 < W_*^0(t) < 1$. Therefore, we have $W_*^0(0) = \alpha^0$ and $W_*^0(t) \equiv W^0(t)$.

Theorem 1.6. *Assume that (H1), (H2), (H4) and (H5) hold. Suppose that $c = 0$. Then for any $s > c = 0$ there exists a function $W_0(\boldsymbol{x}, t)$ satisfying (1.4). In addition, one has:*

(i) $W_0(\boldsymbol{x}', x_m, t) = W_0(\boldsymbol{x}'', x_m, t)$ *for* $t \in \mathbb{R}$, $x_m \in \mathbb{R}$ *and* $\boldsymbol{x}', \boldsymbol{x}'' \in \mathbb{R}^{m-1}$ *with* $|\boldsymbol{x}'| = |\boldsymbol{x}''|$;

(ii) $\dfrac{\partial}{\partial x_m} W_0(\boldsymbol{x}, t) > 0$ *for any* $\boldsymbol{x} \in \mathbb{R}^m$ *and* $t \in \mathbb{R}$;

(iii) $\dfrac{\partial}{\partial x_i} W_0(\boldsymbol{x}, t) > 0$ *on* $x_i \in (0, \infty)$, $i = 1, 2, \dots, m-1$;

(iv) *we have*

$$\lim_{x_m \to \infty} \|W_0(\cdot, x_m, t) - W^+(t)\|_{C(\mathbb{R}^{m-1})} = 0$$

and

$$\lim_{x_m \to -\infty} \|W_0(\cdot, x_m, t) - W^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^{m-1})} = 0$$

uniformly on $t \in \mathbb{R}$;

(v) *the conclusions of theorems 1.3–1.5 remain valid for* $W_0(\boldsymbol{x}, t)$.

This paper is organized as follows. In § 2 we list some preliminaries on two-dimensional V-shaped travelling fronts and three-dimensional pyramidal travelling fronts of (1.1), which are needed for the proof of theorem 1.1. In § 3, we prove theorems 1.1–1.5. To get the expected cylindrically symmetric travelling front, we use the results of Sheng *et al*. [37] to construct a sequence of pyramidal travelling fronts of (1.1), and then take a limit for the sequence of pyramidal travelling fronts. Thus, the limit function is just the expected solution. Consequently, we show qualitative properties of the cylindrically symmetric travelling front by a series of arguments with contradictions. In § 4 we prove theorem 1.6. In § 5 we give a discussion to end the paper.

2. Preliminaries

In this section we state the existence results on two-dimensional V-shaped travelling fronts and three-dimensional pyramidal travelling fronts of (1.1) when $c > 0$, which were established by Wang and Wu [43] and Sheng *et al*. [37], respectively. Moreover, we show some properties of the pyramidal travelling fronts that are very important when establishing the cylindrically symmetric travelling fronts in next section. Let (c, U) be defined by (1.2). Assume that $c > 0$.

2.1. Two-dimensional V-shaped travelling fronts

Let $\tilde{v}(\xi, \eta, t; \tilde{v}_0)$ be the solution of the following Cauchy problem:

$$\left.\begin{aligned} \tilde{v}_t = \tilde{v}_{\xi\xi} + \tilde{v}_{\eta\eta} + f(\tilde{v}, t) \quad &\forall (\xi, \eta) \in \mathbb{R}^2,\ t > 0, \\ \tilde{v}(\xi, \eta, 0) = \tilde{v}_0(\xi, \eta) \quad &\forall (\xi, \eta) \in \mathbb{R}^2. \end{aligned}\right\} \tag{2.1}$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

The following theorem and lemma were established by Wang and Wu [43, theorem 1.1, lemma 3.4].

Theorem 2.1. *Assume that (H1)–(H3) holds. For any* $\tilde{s} > c$ *there exists a unique* $\tilde{V}(\xi, \eta, t; \tilde{s})$ *satisfying*

$$\begin{aligned} \tilde{V}_t = \tilde{V}_{\xi\xi} + \tilde{V}_{\eta\eta} - \tilde{s}\tilde{V}_\eta + f(\tilde{V}, t) \quad &\forall (\xi, \eta) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ \tilde{V}(\xi, \eta, t + T; \tilde{s}) = \tilde{V}(\xi, \eta, t; \tilde{s}) \quad &\forall (\xi, \eta) \in \mathbb{R}^2,\ t \in \mathbb{R}, \\ U\left(\frac{c}{\tilde{s}}\left(\eta + \frac{\sqrt{\tilde{s}^2 - c^2}}{c}|\xi|\right), t\right) < \tilde{V}(\xi, \eta, t) \quad &\forall (\xi, \eta) \in \mathbb{R}^2,\ t \in \mathbb{R} \end{aligned}$$

and

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} \left|\tilde{V}(\xi, \eta, t; \tilde{s}) - U\left(\frac{c}{\tilde{s}}\left(\eta + \frac{\sqrt{\tilde{s}^2 - c^2}}{c}|\xi|\right), t\right)\right| = 0 \quad \text{uniformly in } t \in \mathbb{R}.$$

Furthermore, for any initial function $\tilde{v}_0(\xi, \eta) \in C(\mathbb{R}^2)$ *with*

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} |\tilde{v}_0(\xi, \eta) - \tilde{V}(\xi, \eta, 0; \tilde{s})| = 0,$$

we have

$$\lim_{t\to\infty} \|\tilde{v}(\cdot, \cdot, t; \tilde{v}_0) - \tilde{V}(\cdot, \cdot + \tilde{s}t, t; \tilde{s})\|_{C(\mathbb{R}^2)} = 0.$$

Lemma 2.2. *There exists a positive constant* $\delta_* > 0$*, a positive constant* ρ *sufficiently large and a positive constant* β *small enough such that, for any* $\delta \in (0, \delta_*]$*,* w^+ *and* w^- *defined by*

$$\tilde{v}^+(\xi, \eta, t) = \tilde{V}(\xi, \eta + \tilde{s}t \pm \rho\delta(1 - \mathrm{e}^{-\beta t}), t) \pm \delta a(t)$$

are a supersolution and a subsolution of (2.1), respectively, where

$$a(t) = \exp\left\{\left(\nu^+ + \nu^- - \frac{\nu_0}{4}\right)t + \int_0^t f_u(W^+(\tau), \tau)\, \mathrm{d}\tau + \int_0^t f_u(W^-(\tau), \tau)\, \mathrm{d}\tau\right\}$$

and the constants ν_0*,* ν^+ *and* ν^- *are defined in (H3).*

Remark 2.3. Following Wang and Wu [43], we have that the positive constant δ_* depends only on the nonlinearity f. In addition, we have that $\tilde{V}(\xi, \eta, t; \tilde{s})$ furthermore satisfies

$$\begin{aligned} \tilde{V}(\xi, \eta, t) = \tilde{V}(-\xi, \eta, t) \quad &\forall (\xi, \eta, t) \in \mathbb{R}^3, \\ \tilde{V}_\xi(\xi, \eta, t) > 0 \quad &\forall (\xi, \eta, t) \in (0, +\infty) \times \mathbb{R}^2, \\ \tilde{V}_\eta(\xi, \eta, t) > 0 \quad &\forall (\xi, \eta, t) \in \mathbb{R}^3 \end{aligned}$$

and

$$\tilde{V}(\xi + \xi_0, \eta, t) \leqslant \tilde{V}(\xi, \eta + \eta_0, t) \quad \forall (\xi, \eta, t) \in \mathbb{R}^3,\ \xi_0, \eta_0 \in \mathbb{R} \text{ with } \eta_0 \geqslant \frac{\sqrt{\tilde{s}^2 - c^2}}{c}|\xi_0|.$$

2.2. Three-dimensional pyramidal travelling fronts

Fix $s > c > 0$. Assume that the solutions travel towards the $-x_3$ direction without loss of generality. Take

$$u(\boldsymbol{x}, t) = v(\boldsymbol{x}', x_3 + st, t), \qquad \boldsymbol{x}' = (x_1, x_2), \qquad \boldsymbol{x} = (\boldsymbol{x}', x_3) = (x_1, x_2, x_3).$$

We then have the initial-value problem

$$\left.\begin{aligned} \frac{\partial}{\partial t} v(\boldsymbol{x}, t) = \Delta v(\boldsymbol{x}, t) - s \frac{\partial}{\partial x_3} v(\boldsymbol{x}, t) + f(v(\boldsymbol{x}, t), t), \\ v(\boldsymbol{x}, 0) = v^0(\boldsymbol{x}), \end{aligned}\right\} \tag{2.2}$$

where $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$.

Let $n \geqslant 3$ be a given integer and let

$$m_* = \sqrt{s^2 - c^2}/c.$$

Let $\{\boldsymbol{A}_j = (A_j, B_j)\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^2$ such that

$$A_j B_{j+1} - A_{j+1} B_j > 0, \quad j = 1, 2, \dots, n-1, \qquad A_n B_1 - A_1 B_n > 0.$$

It is obvious that $(m_* \boldsymbol{A}_j, 1) \in \mathbb{R}^3$ is the normal vector of $\{\boldsymbol{x} \in \mathbb{R}^3 \mid -x_3 = m_*(\boldsymbol{A}_j, \boldsymbol{x}')\}$. Let

$$h_j(\boldsymbol{x}') = m_*(\boldsymbol{A}_j, \boldsymbol{x}') \quad \text{and} \quad h(\boldsymbol{x}') = \max_{1 \leqslant j \leqslant n} h_j(\boldsymbol{x}') = m_* \max_{1 \leqslant j \leqslant n} (\boldsymbol{A}_j, \boldsymbol{x}')$$

for $\boldsymbol{x}' \in \mathbb{R}^2$. We call $\{\boldsymbol{x} = (\boldsymbol{x}', x_3) \in \mathbb{R}^3 \mid -x_3 = h(\boldsymbol{x}')\}$ a three-dimensional pyramid in $\mathbb{R}^3$. Letting

$$\Omega_j = \{\boldsymbol{x}' \in \mathbb{R}^2 \mid h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\}$$

for $j = 1, \dots, n$, we have $\mathbb{R}^2 = \bigcup_{j=1}^n \Omega_j$. Denote the boundary of Ω_j by $\partial\Omega_j$. Let

$$E = \bigcup_{j=1}^n \partial\Omega_j.$$

Set

$$S_j = \{\boldsymbol{x} \in \mathbb{R}^3 \mid -x_3 = h_j(\boldsymbol{x}') \text{ for } \boldsymbol{x}' \in \Omega_j\}$$

for $j = 1, \dots, n$, and call $\bigcup_{j=1}^n S_j \subset \mathbb{R}^3$ the lateral faces of the pyramid. Let

$$\Gamma_j = S_j \cap S_{j+1}, \quad \Gamma_n = S_n \cap S_1, \quad j = 1, \dots, n-1.$$

Then $\Gamma := \bigcup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. Define

$$v^-(\boldsymbol{x}, t) = U\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}')), t\right) = \max_{1 \leqslant j \leqslant n} U\left(\frac{c}{s}(x_3 + h_j(\boldsymbol{x}')), t\right).$$

Define

$$D(\gamma) = \left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \operatorname{dist}\left(\boldsymbol{x}, \bigcup_{j=1}^n \Gamma_j\right) > \gamma\right\} \quad \text{for } \gamma > 0.$$

Let $v(\boldsymbol{x}, t; v^-)$ be the solution of (2.2) with $v^0 = v^-$. There then exists a function $V(\boldsymbol{x}, t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ such that

$$V(\boldsymbol{x}, t) = \lim_{k\to\infty} v(\boldsymbol{x}, t + kT; v^-).$$

The following theorem comes from Sheng *et al*. [37, theorem 1.1].

THEOREM 2.4. *Assume that $s > c > 0$ holds. Then, under assumptions (H1)–(H3), there exists a function $V(\boldsymbol{x}, t)$ such that $V(\boldsymbol{x}, t + T) = V(\boldsymbol{x}, t)$ for any $(\boldsymbol{x}, t) \in \mathbb{R}^4$, $U((c/s)(x_3 + h(\boldsymbol{x}')), t) < V(\boldsymbol{x}, t) < W^+(t)$ for any $(\boldsymbol{x}, t) \in \mathbb{R}^4$, $(\partial/\partial x_3)V(\boldsymbol{x}, t) > 0$ for all $(\boldsymbol{x}, t) \in \mathbb{R}^4$ and*

$$\lim_{\gamma\to+\infty} \sup_{\boldsymbol{x}\in D(\gamma),\, t\in\mathbb{R}} \left| V(\boldsymbol{x}, t) - U\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}')), t\right)\right| = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}V(\boldsymbol{x}, t) = \Delta V(\boldsymbol{x}, t) - s\frac{\partial}{\partial x_3}V(\boldsymbol{x}, t) + f(V(\boldsymbol{x}, t), t) \quad \forall(\boldsymbol{x}, t) \in \mathbb{R}^4.$$

If we furthermore assume that

$$\lim_{\gamma\to+\infty} \sup_{\boldsymbol{x}\in D(\gamma)} \left| v_0(\boldsymbol{x}) - U\left(\frac{c}{s}(x_3 + h(\boldsymbol{x}')), 0\right)\right| = 0 \tag{2.4}$$

holds, then the solution $v(\boldsymbol{x}, t; v_0)$ to (2.2) satisfies

$$\lim_{t\to+\infty} \|v(\cdot, t) - V(\cdot, t)\|_{C(\mathbb{R}^3)} = 0,$$

or equivalently

$$\lim_{k\to+\infty} \|v(\cdot, \cdot + kT) - V(\cdot, \cdot)\|_{C(\mathbb{R}^3\times\mathbb{R})} = 0.$$

Using (1.2), (2.3) and $U((c/s)(x_3 + h(\boldsymbol{x}')), t) < V(\boldsymbol{x}, t) < W^+(t)$ for $(\boldsymbol{x}, t) \in \mathbb{R}^4$, we furthermore obtain

$$\lim_{x_3\to\infty} \|V(\cdot, x_3, t) - W^+(t)\|_{C(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{x_3\to-\infty} \|V(\cdot, x_3, t) - W^-(t)\|_{C_{\text{loc}}(\mathbb{R}^2)} = 0 \tag{2.5}$$

uniformly on $t \in \mathbb{R}$. The next two lemmas show the monotonicity of the pyramidal travelling front V. The proofs are very similar to those of Taniguchi [41, lemmas 2.5 and 3.4] and we omit them.

LEMMA 2.5. *For any $(x_0, y_0, z_0) \in \mathbb{R}^3$ with $z_0 \geqslant h(x_0, y_0)$, one has*

$$V(x_1 + x_0, x_2 + y_0, x_3, t) \leqslant V(x_1, x_2, x_3 + z_0, t) \quad \text{for any } (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } t \in \mathbb{R}.$$

LEMMA 2.6. *Let*

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2}}\begin{pmatrix}\nu_1\\ \nu_2\\ 1\end{pmatrix}$$

be a given constant vector with $\sqrt{\nu_1^2 + \nu_2^2} \leqslant 1/m_*$. Then one has

$$\frac{\partial}{\partial\nu}V(\boldsymbol{x}, t) > 0 \quad \forall\boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

Let $w(t;M)$ be the solution of the following equation:

$$\begin{aligned} w'(t) &= f(w(t),t), \quad t>0, \\ w(0) &= M \in \mathbb{R}. \end{aligned}$$

Since the Poincaré map $P(\alpha)$ is monotonic and has only three fixed points with $\alpha^{\pm}$ being stable, $P(\alpha) > \alpha$ for all $\alpha < \alpha^-$, and $P(\alpha) < \alpha$ for all $\alpha > \alpha^+$ (see also [1]). Then for any $M^+, M^- \in \mathbb{R}$ with $M^- \leqslant \alpha^- < \alpha^+ \leqslant M^+$, we have

$$\lim_{k\to\infty} w(t+kT; M^{\pm}) = W^{\pm}(t) \quad \text{uniformly for } t \in [0,T].$$

By the comparison principle (see [5, theorem 25.6]), we have

$$w(t;M^-) \leqslant v(\boldsymbol{x},t) \leqslant w(t;M^+),$$

and hence

$$\begin{aligned} W^-(t) &\leqslant \liminf_{k\to\infty} \inf_{\boldsymbol{x}\in\mathbb{R}^3} v(\boldsymbol{x}, t+kT) \\ &\leqslant \limsup_{k\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^3} v(\boldsymbol{x}, t+kT) \\ &\leqslant W^+(t) \end{aligned}$$

provided that $M^- \leqslant v^0(\boldsymbol{x}) \leqslant M^+$ for any $\boldsymbol{x} \in \mathbb{R}^3$.

In the following lemma we show that if the initial value v^0 is even on x_1, then the solution $v(\boldsymbol{x},t;v^0)$ is also even on x_1. Furthermore, if v^0 is non-decreasing in $x_1 \geqslant 0$, then the solution $v(\boldsymbol{x},t;v^0)$ is also non-decreasing in $x_1 \geqslant 0$. The proof is completely similar to that of [45, lemma 2.5].

Lemma 2.7. *Assume that $v^0(\boldsymbol{x}) \in C(\mathbb{R}^3,\mathbb{R})$ is even on x_1, uniformly continuous and bounded in $\boldsymbol{x} \in \mathbb{R}^3$, and non-decreasing in $x_1 \in [0,\infty)$. There then exists a unique solution $v(\boldsymbol{x},t) \in C(\mathbb{R}^3\times[0,\infty),\mathbb{R}) \cap C^{2,1}(\mathbb{R}^3\times(0,\infty),\mathbb{R})$ of (2.2) such that $v(\boldsymbol{x},t)$ is even on x_1 and non-decreasing in $x_1 \in [0,\infty)$.*

Corollary 2.8. *Suppose that $v^-(\boldsymbol{x})$ is even on $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, respectively. Then the pyramidal travelling front $V(\boldsymbol{x},t)$ defined by theorem 2.4 satisfies*

$$\begin{aligned} V(x_1,x_2,x_3,t) &= V(-x_1,x_2,x_3,t) \quad \forall \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ V(x_1,x_2,x_3,t) &= V(x_1,-x_2,x_3,t) \quad \forall \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ \frac{\partial}{\partial x_1} V(\boldsymbol{x},t) &> 0 \quad \forall \boldsymbol{x} \in (0,+\infty)\times\mathbb{R}^2,\ t \in \mathbb{R}, \\ \frac{\partial}{\partial x_2} V(\boldsymbol{x},t) &> 0 \quad \forall \boldsymbol{x} \in \mathbb{R}\times(0,+\infty)\times\mathbb{R},\ t \in \mathbb{R}. \end{aligned}$$

3. Cylindrically symmetric travelling fronts when $c > 0$

In this section we prove theorems 1.1–1.5. We first give the details of the proof in $\mathbb{R}^3$, then generalize the result to $\mathbb{R}^m$ with $m \geqslant 4$ and $m = 2$.

3.1. Proof of theorem 1.1 in $\mathbb{R}^3$

In this section we prove theorem 1.1, namely, we establish the existence of time-periodic cylindrically symmetric travelling fronts of (1.1) in $\mathbb{R}^3$. The method is to take the limit of a sequence of pyramidal travelling fronts. Subsequently, we show some important qualitative properties of the cylindrically symmetric travelling fronts.

Let

$$h^k(x_1, x_2) = m_* \max_{1 \leqslant i \leqslant 2^k} \left\{ x_1 \cos \frac{2(i-1)\pi}{2^k} + x_2 \sin \frac{2(i-1)\pi}{2^k} \right\}, \quad k = 1, 2, \dots.$$

It is not difficult to show that the plane

$$x_3 = m_* \left(x_1 \cos \frac{2(i-1)\pi}{2^k} + x_2 \sin \frac{2(i-1)\pi}{2^k} \right)$$

is tangent to the rotating surface

$$x_3 = m_* \sqrt{x_1^2 + x_2^2}$$

for any $k \in \mathbb{N}$ and $1 \leqslant i \leqslant 2^k$. Replacing $h(\boldsymbol{x}')$ by $h^k(\boldsymbol{x}')$ in theorem 2.4, we obtain a sequence of time-periodic pyramidal travelling fronts of (1.1), namely,

$$V^1, V^2, \dots, V^k, \dots,$$

where

$$V^k(\boldsymbol{x}, t) = \lim_{t \to \infty} v(\boldsymbol{x}, t + kT; v_0^{k,-}), \quad v_0^{k,-}(\boldsymbol{x}) = U\left(\frac{c}{s}(x_3 + h^k(\boldsymbol{x}')), 0 \right).$$

Denote the edge of the pyramid $x_3 = h^k(\boldsymbol{x}')$ by Γ^k and

$$D^k(\gamma) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \operatorname{dist}\left(\boldsymbol{x}, \bigcup_{j=1}^{2^k} \Gamma_j^k \right) > \gamma \right\} \quad \text{for } \gamma > 0.$$

Since $v_0^{-,k}(\boldsymbol{x}, 0)$ is non-decreasing in $x_1 \in (0, \infty)$ and in $x_2 \in (0, \infty)$, and is even on $x_1 \in \mathbb{R}$ and on $x_2 \in \mathbb{R}$, by theorem 2.4, lemma 2.6 and corollary 2.8, we obtain

$$\begin{aligned} V^1 \leqslant V^2 \leqslant \cdots \leqslant V^k \leqslant \cdots \quad & \forall \boldsymbol{x} \in \mathbb{R}^3, \ t \in \mathbb{R}, \\ \frac{\partial}{\partial x_1} V^k(\boldsymbol{x}, t) > 0 \quad & \forall \boldsymbol{x} \in (0, \infty) \times \mathbb{R}^2, \ t \in \mathbb{R}, \\ \frac{\partial}{\partial x_2} V^k(\boldsymbol{x}, t) > 0 \quad & \forall \boldsymbol{x} \in \mathbb{R} \times (0, \infty) \times \mathbb{R}, \ t \in \mathbb{R}, \\ \frac{\partial}{\partial \nu} V^k(\boldsymbol{x}, t) > 0 \quad & \forall \boldsymbol{x} \in \mathbb{R}^3, \ t \in \mathbb{R}, \end{aligned}$$

where

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2}} (\nu_1, \nu_2, 1) \quad \text{satisfies } \sqrt{\nu_1^2 + \nu_2^2} \leqslant \frac{1}{m_*}.$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

Since

$$h^k(x_1, x_2) = h^k\left(x_1 \cos\frac{\pi}{2^{k-1}} + x_2 \sin\frac{\pi}{2^{k-1}}, -x_1 \sin\frac{\pi}{2^{k-1}} + x_2 \cos\frac{\pi}{2^{k-1}}\right),$$

we have

$$V^k(\boldsymbol{x}, t) = V^k(\boldsymbol{x}', x_3, t) = V^k(\boldsymbol{B}_k\boldsymbol{x}', x_3, t) \quad \forall \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R},$$

where

$$\boldsymbol{B}_k = \begin{pmatrix} \cos\dfrac{\pi}{2^{k-1}} & \sin\dfrac{\pi}{2^{k-1}} \\ -\sin\dfrac{\pi}{2^{k-1}} & \cos\dfrac{\pi}{2^{k-1}} \end{pmatrix}.$$

Take $x_3^k \in \mathbb{R}$ such that $x_3^k \geqslant x_3^{k+1}$ and $V^k(0, 0, x_3^k, 0) = \theta_0$, where $\theta_0 \in (\alpha^-, \alpha^0)$ is a given constant. Let

$$\tilde{V}^k(\boldsymbol{x}, t) = V^k(\boldsymbol{x}', x_3 + x_3^k, t) \quad \forall \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

By (2.5), lemmas 2.5 and 2.6, and corollary 2.8 we have that $\tilde{V}^k(\boldsymbol{x}, t)$ satisfies the following.

(a) $\tilde{V}^k(\boldsymbol{0}, 0) = \theta_0$.

(b) $(\partial/\partial\nu)\tilde{V}^k(\boldsymbol{x}, t) > 0$ for any $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$, where $k \in \mathbb{N}$ and

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2}}(\nu_1, \nu_2, 1) \quad \text{satisfies } \sqrt{\nu_1^2 + \nu_2^2} \leqslant \frac{1}{m_*}.$$

(c) For any $(x_0, y_0, z_0) \in \mathbb{R}^3$ with $z_0 \geqslant h^k(x_0, y_0)$, there holds

$$\tilde{V}^k(x_1 + x_0, x_2 + y_0, x_3, t) \leqslant \tilde{V}^k(x_1, x_2, x_3 + z_0, t) \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

(d) $\tilde{V}^k(\boldsymbol{x}', x_3, t) = \tilde{V}^k(\boldsymbol{B}_k\boldsymbol{x}', x_3, t)$ for all $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$.

(e) There hold $(\partial/\partial x_1)\tilde{V}^k(\boldsymbol{x}, t) > 0$ in $x_1 \in (0, \infty)$ and $(\partial/\partial x_2)\tilde{V}^k(\boldsymbol{x}, t) > 0$ in $x_2 \in (0, \infty)$, where $k \in \mathbb{N}$.

(f) $\tilde{V}^k(\boldsymbol{x}, t) = \tilde{V}^k(\boldsymbol{x}, t + T)$ for all $x \in \mathbb{R}^3$, $t \in \mathbb{R}$.

(g) We have

$$\lim_{x_3 \to \infty} \|\tilde{V}^k(\cdot, x_3, t) - W^+(t)\|_{C(\mathbb{R}^2)} = 0$$

and

$$\lim_{x_3 \to -\infty} \|\tilde{V}^k(\cdot, x_3, t) - W^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^2)} = 0$$

uniformly on $t \in \mathbb{R}$.

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

Since $\tilde{V}^k$ satisfies $W^-(t) < \tilde{V}^k(\boldsymbol{x},t) < W^+(t)$ and

$$\frac{\partial}{\partial t}\tilde{V}^k(\boldsymbol{x},t) = \Delta\tilde{V}^k(\boldsymbol{x},t) - s\frac{\partial}{\partial x_3}\tilde{V}^k(\boldsymbol{x},t) + f(\tilde{V}^k(\boldsymbol{x},t),t)$$

for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, using an argument similar to that of Wang *et al*. [44, proposition 4.3] we have that there exists a positive constant K_1 such that

$$\|\tilde{V}^k(\cdot,t)\|_{C^1(\mathbb{R}^3)} \leqslant K_1 \quad \forall k \in \mathbb{N},\ t \in [T, 2T].$$

By the periodicity of $\tilde{V}^k(\boldsymbol{x},t)$ in $t \in \mathbb{R}$, we have

$$\|\tilde{V}^k(\cdot,t)\|_{C^1(\mathbb{R}^3)} \leqslant K_1 \quad \forall k \in \mathbb{N},\ t \in \mathbb{R}.$$

Applying [27, theorems 5.1.3 and 5.1.4] we have that there exists a positive constant K such that

$$\|\tilde{V}^k(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3\times\mathbb{R})} \leqslant K \quad \forall k \in \mathbb{N} \tag{3.1}$$

for some $\alpha \in (0,1)$. There then exists a function $W(\boldsymbol{x},t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ (up to the extraction of some subsequence) satisfying

$$\tilde{V}^k(\boldsymbol{x},t) \to W(\boldsymbol{x},t) \quad \text{in } \|\cdot\|_{C^{2,1}_{\text{loc}}(\mathbb{R}^3\times\mathbb{R})} \text{ as } k \to \infty.$$

Furthermore, we have the following theorem for the function $W(\boldsymbol{x},t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$.

THEOREM 3.1. *Assume that (H1)–(H4) hold. Suppose that $c > 0$. Then for any $s > c > 0$ there exists a function $W(\boldsymbol{x},t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ satisfying $W(\boldsymbol{x},t+T) = W(\boldsymbol{x},t)$ and*

$$\frac{\partial}{\partial t}W(\boldsymbol{x},t) = \Delta W(\boldsymbol{x},t) - s\frac{\partial}{\partial x_3}W(\boldsymbol{x},t) + f(W(\boldsymbol{x},t),t) \tag{3.2}$$

for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. In addition, one has:

(i) $W(\boldsymbol{0},0) = \theta_0$;

(ii) $W(\boldsymbol{x}_1',x_3,t) = W(\boldsymbol{x}_2',x_3,t)$ *for all* $\boldsymbol{x}_1',\boldsymbol{x}_2' \in \mathbb{R}^2$ *with* $|\boldsymbol{x}_1'| = |\boldsymbol{x}_2'|$, $x_3 \in \mathbb{R}$, $t \in \mathbb{R}$;

(iii) *for any* $(x_0,y_0,z_0) \in \mathbb{R}^3$ *with* $z_0 \geqslant m_*\sqrt{x_0^2+y_0^2}$, *there holds*

$$W(x_1+x_0,x_2+y_0,x_3,t) \leqslant W(x_1,x_2,x_3+z_0,t) \quad \forall (x_1,x_2,x_3) \in \mathbb{R}^3;$$

(iv) $\dfrac{\partial}{\partial x_3}W(\boldsymbol{x},t) > 0$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$;

(v) $\dfrac{\partial}{\partial x_i}W(\boldsymbol{x},t) > 0$ *for* $x_i \in (0,\infty)$, $x_j \in \mathbb{R}$, $x_3 \in \mathbb{R}$ *and* $t \in \mathbb{R}$, $i,j = 1,2$, $i \neq j$;

(vi) *we have*

$$\lim_{x_3\to\infty} \|W(\cdot,x_3,t) - W^+(t)\|_{C(\mathbb{R}^2)} = 0$$

and

$$\lim_{x_3 \to -\infty} \|W(\cdot, x_3, t) - W^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^2)} = 0$$

uniformly on $t \in \mathbb{R}$*;*

(vii) $\dfrac{\partial}{\partial \nu} W(\boldsymbol{x}, t) > 0$ *for any* $\boldsymbol{x} \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$*, where*

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2}}(\nu_1, \nu_2, 1) \quad \textit{with} \quad \sqrt{\nu_1^2 + \nu_2^2} \leqslant \frac{1}{m_*}.$$

It is easy to show that $W(\boldsymbol{x}, t)$ satisfies (3.2) and theorem 3.1(i). In view of $h^k(x_1, x_2) \leqslant m_* \sqrt{x_1^2 + x_2^2}$ for any $(x_1, x_2) \in \mathbb{R}^2$ and $h^k(x_1, x_2) \to m_* \sqrt{x_1^2 + x_2^2}$ in $C_{\mathrm{loc}}(\mathbb{R}^2)$ as $k \to +\infty$, we can easily prove theorem 3.1(ii) and (iii). In the following we prove theorem 3.1(iv)–(vii) by a sequence of lemmas. Following from properties (a)–(g) of $\tilde{V}^k(\boldsymbol{x}, t)$, we have:

(I) $\dfrac{\partial}{\partial x_3} W(\boldsymbol{x}, t) \geqslant 0$ for $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$;

(II) we have that

$$\frac{\partial}{\partial x_1} W(0, x_2, x_3, t) = 0 \quad \text{for } (x_2, x_3) \in \mathbb{R}^2 \text{ and } t \in \mathbb{R},$$

and

$$\frac{\partial}{\partial x_2} W(x_1, 0, x_3, t) = 0 \quad \text{for } (x_1, x_3) \in \mathbb{R}^2 \text{ and } t \in \mathbb{R};$$

(III) $(\partial/\partial x_1) W(\boldsymbol{x}, t) \geqslant 0$ for $\boldsymbol{x} \in (0, +\infty) \times \mathbb{R}^2$ and $t \in \mathbb{R}$ and $(\partial/\partial x_2) W(\boldsymbol{x}, t) \geqslant 0$ for $\boldsymbol{x} \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$ and $t \in \mathbb{R}$.

Lemma 3.2. *$W(\cdot, 0) \not\equiv \theta_0$.*

Proof. On the contrary, we assume that $W(\cdot, 0) \equiv \theta_0$. Since $W(\boldsymbol{x}, t)$ satisfies (3.2) and $W(\boldsymbol{x}, t + T) = W(\boldsymbol{x}, t)$, we have that $W(\boldsymbol{x}, t)$ is independent of $\boldsymbol{x} \in \mathbb{R}^3$ and $W(t) \equiv W(\boldsymbol{x}, t)$ is a solution of the following equation:

$$w_t = f(w, t).$$

In particular, we have $W(T) = W(0) = \theta_0$, which implies that the period map $P(\alpha) := w(\alpha, T)$ has a fixed point θ_0 that is different from α^-, α^0 and α^+. This is a contradiction.

Thus, we complete the proof. □

Lemma 3.3. *$(\partial/\partial x_1) W(\boldsymbol{x}, t) > 0$ for any $\boldsymbol{x} \in (0, +\infty) \times \mathbb{R}^2$ and $t \in \mathbb{R}$, and $(\partial/\partial x_2) W(\boldsymbol{x}, t) > 0$ for any $\boldsymbol{x} \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$ and $t \in \mathbb{R}$.*

Proof. We note that there hold $(\partial/\partial x_1) W(\boldsymbol{x}, t) \geqslant 0$ for any $\boldsymbol{x} \in (0, +\infty) \times \mathbb{R}^2$ and $t \in \mathbb{R}$, and $(\partial/\partial x_2) W(\boldsymbol{x}, t) \geqslant 0$ for any $\boldsymbol{x} \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$ and $t \in \mathbb{R}$. We first show that $(\partial/\partial x_1) W(\boldsymbol{x}, t) > 0$ for any $x_1 \in (0, +\infty) \times \mathbb{R}^2$ and $t \in \mathbb{R}$. For a

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

contradiction, we assume that there exists some $(\boldsymbol{x}^0, t_0) \in \mathbb{R}^4$ with $x_1^0 > 0$ such that

$$\frac{\partial}{\partial x_1} W(\boldsymbol{x}^0, t_0) = 0.$$

Due to the parabolic strong maximum principle (see [33, ch. 3, theorem 5]) we have

$$\frac{\partial}{\partial x_1} W(\boldsymbol{x}, t_0) \equiv 0 \quad \text{for } t < t_0 \text{ and } \boldsymbol{x} \in \mathbb{R}^3 \text{ with } x_1 > 0.$$

Since $W(x_1, x_2, x_3, t) = W(-x_1, x_2, x_3, t)$ for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

$$\frac{\partial}{\partial x_1} W(\boldsymbol{x}, t) \equiv 0 \quad \text{for } t < t_0 \text{ and } \boldsymbol{x} \in \mathbb{R}^3.$$

By theorem 3.1(ii), we also have

$$\frac{\partial}{\partial x_2} W(\boldsymbol{x}, t) \equiv 0 \quad \text{for } t < t_0 \text{ and } \boldsymbol{x} \in \mathbb{R}^3.$$

It follows from the T-periodicity of $W(\boldsymbol{x}, \cdot)$ that

$$\frac{\partial}{\partial x_i} W(\boldsymbol{x}, t) \equiv 0 \quad \text{for } (\boldsymbol{x}, t) \in \mathbb{R}^4 \text{ and } i = 1, 2,$$

which implies that $W(\boldsymbol{x}, t)$ only depends on $x_3 \in \mathbb{R}$ and $t \in \mathbb{R}$. We rewrite $W(\boldsymbol{x}, t)$ as $W(x_3, t)$ and denote $W(x_3, t)$ by $W(z, t)$ with $z = x_3$. By lemma 3.2, we have $W(z, 0) \not\equiv \theta_0$ and $W(0, 0) = \theta_0$, which implies that $(\partial/\partial z)W(z, t) \gneqq 0$ on $(z, t) \in \mathbb{R}^2$. It follows from the parabolic maximum principle (see [33]) that $(\mathrm{d}/\mathrm{d}z)W(z, t) > 0$ for any $(z, t) \in \mathbb{R}^2$.

Let

$$W(-\infty, t) = \omega^-(t), \qquad W(+\infty, t) = \omega^+(t),$$
$$W^-(t) \leqslant \omega^-(t) < \omega^+(t) \leqslant W^+(t)$$

for all $t \in \mathbb{R}$. In particular, we have $\omega^-(0) < \theta_0$. In this case we rewrite (3.2) as

$$\frac{\partial}{\partial t} W(z, t) = \frac{\partial^2}{\partial z^2} W(z, t) - s\frac{\partial}{\partial z} W(z, t) + f(W(z, t), t).$$

Obviously, $\omega^-(t)$ and $\omega^+(t)$ are two solutions of the following ordinary differential equation:

$$w_t = f(w(t), t).$$

In particular, we have $\alpha^- \leqslant \omega^-(T) = \omega^-(0) < \theta_0 < \omega^+(0) = \omega^+(T) \leqslant \alpha^+$. Therefore, we have either $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) \equiv W^+(t)$, or $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) \equiv W^0(t)$ for any $t \in \mathbb{R}$. We first show that it is impossible that $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) \equiv W^+(t)$. Otherwise, (1.1) admits a one-dimensional periodic travelling front $W(z+st, t)$ with wave speed $s > c$ connecting two stable equilibria $W^-(t)$ and $W^+(t)$, which contradicts the uniqueness of the one-dimensional periodic travelling front (U, c) of (1.1).

It remains to show that the case in which $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) = W^0(t)$ is also impossible. Suppose on the contrary that $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) = W^0(t)$ for all $t \in \mathbb{R}$. Let $\psi(z,t) = W^0(t) - W(-z,t)$. Then we have

$$\frac{\partial}{\partial t}\psi(z,t) = \frac{\partial^2}{\partial z^2}\psi(z,t) + s\frac{\partial}{\partial z}\psi(z,t) - f(W^0(t) - \psi(z,t),t) + f(W^0(t),t) \quad (3.3)$$

and

$$\psi(-\infty,t) = 0, \qquad \psi(+\infty,t) = W^0(t) - W^-(t).$$

Equation (3.3) implies that the reaction–diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - f(W^0(t) - u(x,t),t) + f(W^0(t),t) \quad (3.4)$$

admits a time-periodic travelling wavefront $\psi(x - st, t)$ connecting 0 and $W^0(t) - W^-(t)$. In particular, we have $(\partial/\partial z)\psi(z,t) > 0$ for any $z \in \mathbb{R}$ and $t \in \mathbb{R}$.

Let Q_t denote the solution semi-flow of (3.4) and let Q_T denote the corresponding Poincaré map. Let $\beta = W^0(0) - W^-(0)$ and $\mathcal{C}_\beta = \{u(\cdot) \in C(\mathbb{R})\colon 0 \leqslant u(x) \leqslant \beta \ \forall x \in \mathbb{R}\}$. It follows from assumptions (H1) and (H2) that the Poincaré map $Q_T\colon \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies assumptions (A1)–(A5) of [24]. Thus, it follows from [24, theorem 2.1] that there exists a positive number ν^*, which is called the asymptotic speed of spread of the Poincaré map Q_T, such that for any $\nu \in (0, \nu^*/T)$ and $\sigma \in (0,\beta)$ there is a positive number r_σ such that if $u \in \mathcal{C}_\beta$ and $u(x) > \sigma$ for $x \in [-r_\sigma, r_\sigma]$, then

$$\lim_{t\to\infty} \inf_{|x|\leqslant \nu t}(Q_t[u](x) - (W^0(t) - W^-(t))) = 0.$$

Now we specially take $\sigma_0 = \beta/2$. Let $u^0(\cdot) \in \mathcal{C}_{2\beta/3}$ satisfy $u^0(x) > \sigma_0$ in $x \in [-r_{\sigma_0}, r_{\sigma_0}]$ and $\operatorname{supp} u^0(\cdot)$ is compact. We then have

$$u(0,t;u^0) \to W^0(t) - W^-(t) \quad \text{as } t \to +\infty, \quad (3.5)$$

where $u(x,t;u^0) = Q_t[u^0](x)$. On the other hand, there exists an $x_0 > 0$ such that $\psi(x + x_0, 0) \geqslant u^0(x)$ for any $x \in \mathbb{R}$. By the comparison principle (see [5, theorem 25.6]), we have $u(x,t;u^0) = Q_t[u^0](x) \leqslant \psi(x + x_0 - st, t)$ for any $x \in \mathbb{R}$ and $t > 0$, which implies that

$$u(0,t;u^0) \leqslant \psi(x_0 - st, t) \to 0 \quad \text{as } t \to +\infty.$$

This contradicts (3.5), which implies that the case in which $\omega^-(t) \equiv W^-(t)$ and $\omega^+(t) = W^0(t)$ is impossible.

Finally, we conclude that $(\partial/\partial x_1)W(\boldsymbol{x},t) > 0$ for any $\boldsymbol{x} \in (0,+\infty) \times \mathbb{R}^2$ and $(\partial/\partial x_2)W(\boldsymbol{x},t) > 0$ for any $\boldsymbol{x} \in \mathbb{R} \times (0,+\infty) \times \mathbb{R}$. The proof is complete. □

Following lemma 3.3, we have $W(x_1,x_2,0,t) > W(0,0,0,t)$ for any $x_1 > 0$, $x_2 > 0$ and $t \in \mathbb{R}$. By theorem 3.1(iii), we have that

$$W(0,0,x_3,t) \geqslant W(x_1,x_2,0,t) > W(0,0,0,t) \quad \text{for any } x_3 > m_*\sqrt{x_1^2 + x_2^2} > 0,$$

which implies that $(\partial/\partial x_3)W(\boldsymbol{x},t) \geqslant 0$ and $(\partial/\partial x_3)W(\boldsymbol{x},t) \not\equiv 0$ on $\boldsymbol{x} \in \mathbb{R}^3$. Furthermore, the parabolic maximum principle (see [33]) yields that $(\partial/\partial x_3)W(\boldsymbol{x},t) > 0$ for $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Thus, we have proved theorem 3.1(iv) and (v).

Lemma 3.4. *One has*

$$\lim_{x_3\to-\infty}\|W(\cdot,\cdot,x_3,t)-W^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^2)}=0\quad \textit{uniformly in } t\in\mathbb{R}$$

and

$$\lim_{x_3\to+\infty}\|W(\cdot,\cdot,x_3,t)-W^+(t)\|_{C(\mathbb{R}^2)}=0\quad \textit{uniformly in } t\in\mathbb{R}.$$

Proof. Since

$$W^-(t)\leqslant W(\boldsymbol{x},t)\leqslant W^+(t)\quad\text{and}\quad W(\boldsymbol{x},t+T)=W(\boldsymbol{x},t)\quad\text{for any } (\boldsymbol{x},t)\in\mathbb{R}^4,$$

using an argument similar to that of [44, proposition 4.3], we have that there exists a positive constant K_2 such that

$$\|W(\cdot,t)\|_{C^1(\mathbb{R}^3)}\leqslant K_2\quad \forall t\in\mathbb{R}.$$

Applying [27, theorems 5.1.3 and 5.1.4], we have that there exists a positive constant $K>0$ such that

$$\|W(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3\times\mathbb{R})}\leqslant K \tag{3.6}$$

for some $\alpha\in(0,1)$. By theorem 3.1(iii)–(v), we have that

$$W(0,0,x_3,t)\leqslant W(x_1,x_2,x_3,t)\leqslant W(0,0,x_3+m_*\sqrt{x_1^2+x_2^2},t) \tag{3.7}$$

for any $(x_1,x_2,x_3)\in\mathbb{R}^3$ and $t\in\mathbb{R}$. In view of $(\partial/\partial x_3)W(\boldsymbol{x},t)>0$ for any $(\boldsymbol{x},t)\in\mathbb{R}^4$, we take

$$\tilde{W}^-(t):=\lim_{x_3\to-\infty}W(0,0,x_3,t),\quad \tilde{W}^+(t):=\lim_{x_3\to+\infty}W(0,0,x_3,t)\quad\forall t\in\mathbb{R}.$$

It follows from (3.6), (3.7) and the periodicity of $W(\boldsymbol{x},t)$ in $t\in\mathbb{R}$ that

$$\lim_{x_3\to-\infty}\|W(\cdot,\cdot,x_3,t)-\tilde{W}^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^2)}=0\quad\text{uniformly in } t\in\mathbb{R}$$

and

$$\lim_{x_3\to+\infty}\|W(\cdot,\cdot,x_3,t)-\tilde{W}^+(t)\|_{C(\mathbb{R}^2)}=0\quad\text{uniformly in } t\in\mathbb{R}.$$

It follows that there exist two sequences $\{z_n^+\}$ and $\{z_n^-\}$ with $z_n^+\to+\infty$ and $z_n^-\to-\infty$ as $n\to\infty$ such that

$$\lim_{n\to+\infty}\|W(x_1,x_2,x_3+z_n^-,t)-\tilde{W}^-(t)\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R})}=0$$

and

$$\lim_{n\to+\infty}\|W(x_1,x_2,x_3+z_n^+,t)-\tilde{W}^+(t)\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R})}=0.$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

For any $\phi(x_1, x_2, x_3) \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} [W(x_1, x_2, x_3 + z_n^-, t + \Delta t) - W(x_1, x_2, x_3 + z_n^-, t)]\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&\quad = \int_t^{t+\Delta t}\int_{\mathbb{R}^3} \frac{\partial}{\partial r} W(x_1, x_2, x_3 + z_n^-, r)\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r \\
&\quad = \int_t^{t+\Delta t}\int_{\mathbb{R}^3} \bigg[\bigg(\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} - s\frac{\partial}{\partial x_3}\bigg) W(x_1, x_2, x_3 + z_n^-, r)\bigg] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r \\
&\qquad + \int_t^{t+\Delta t}\int_{\mathbb{R}^3} f(W(x_1, x_2, x_3 + z_n^+, r), r)\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r \\
&\quad = \int_t^{t+\Delta t}\int_{\mathbb{R}^3} \bigg[\bigg(\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + s\frac{\partial}{\partial x_3}\bigg) \phi(x_1, x_2, x_3)\bigg] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times W(x_1, x_2, x_3 + z_n^-, r)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r \\
&\qquad + \int_t^{t+\Delta t}\int_{\mathbb{R}^3} f(W(x_1, x_2, x_3 + z_n^+, r), r)\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} [\tilde{W}^-(t + \Delta t) - \tilde{W}^-(t)]\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&\quad = \int_t^{t+\Delta t}\int_{\mathbb{R}^3} \bigg[\bigg(\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + s\frac{\partial}{\partial x_3}\bigg) \phi(x_1, x_2, x_3)\bigg] \tilde{W}^-(r)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r \\
&\qquad + \int_t^{t+\Delta t}\int_{\mathbb{R}^3} f(\tilde{W}^-(r), r)\phi(x_1, x_2, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}r,
\end{aligned}$$

which implies that the function $\tilde{W}^-(t)$ satisfies

$$\tilde{W}^-(t + \Delta t) - \tilde{W}^-(t) = \int_t^{t+\Delta t} f(\tilde{W}^-(r), r)\,\mathrm{d}r \quad \forall t \in \mathbb{R},\ \Delta t \in \mathbb{R}.$$

Then $\tilde{W}^-(t)$ satisfies $\tilde{W}^-(t) = \tilde{W}^-(t + T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{W}^-(t) = f(\tilde{W}^-(t), t) \quad \forall t \in \mathbb{R}.$$

Similarly, $\tilde{W}^+(t)$ satisfies $\tilde{W}^+(t) = \tilde{W}^+(t + T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{W}^+(t) = f(\tilde{W}^+(t), t) \quad \forall t \in \mathbb{R}.$$

Since $\alpha^- < W(\mathbf{0}, 0) = \theta_0 < \alpha^0$, we have $\alpha^- \leqslant \tilde{W}^-(0) < \theta_0 < \alpha^0$ and $\theta_0 < \tilde{W}^+(0) \leqslant \alpha^+$. It follows from (H2) that either $\tilde{W}^-(0) = \alpha^-$ and $\tilde{W}^+(0) = \alpha^+$, or $\tilde{W}^-(0) = \alpha^-$ and $\tilde{W}^+(0) = \alpha^0$. If the former holds, then the lemma has been proved. In the following we show that the latter is impossible.

Assume to the contrary that $\tilde{W}^-(0) = \alpha^-$ and $\tilde{W}^+(0) = \alpha^0$. We then have $\tilde{W}^-(t) = W^-(t)$ and $\tilde{W}^+(t) = W^0(t)$. Therefore,

$$W^-(t) \leqslant W(\boldsymbol{x}, t) \leqslant W^0(t) \quad \forall (\boldsymbol{x}, t) \in \mathbb{R}^4.$$

Let

$$\bar{W}(x_1, x_2, x_3, t) := W^0(t) - W(x_1, x_2, -x_3, t) \quad \forall (x_1, x_2, x_3, t) \in \mathbb{R}^4.$$

Then $\bar{W}(x_1, x_2, x_3, t)$ satisfies

$$\frac{\partial}{\partial t}\bar{W}(\boldsymbol{x}, t) = \Delta \bar{W}(\boldsymbol{x}, t) + s\frac{\partial}{\partial x_3}\bar{W}(\boldsymbol{x}, t) + \bar{f}(\bar{W}(\boldsymbol{x}, t), t),$$

where $\bar{f}(u, t) = f(W^0(t), t) - f(W^0(t) - u, t)$, which implies that $\bar{W}(x_1, x_2, x_3 - st, t)$ is a time-periodic travelling wavefront of the following equation:

$$\frac{\partial}{\partial t}\bar{u}(\boldsymbol{x}, t) = \Delta \bar{u}(\boldsymbol{x}, t) + \bar{f}(\bar{u}(\boldsymbol{x}, t), t). \tag{3.8}$$

It is obvious that

$$\begin{aligned}
\frac{\partial}{\partial x_1}\bar{W}(x_1, x_2, x_3, t) < 0 \quad &\text{for } x_1 > 0 \text{ and } (x_2, x_3, t) \in \mathbb{R}^3,\\
\frac{\partial}{\partial x_2}\bar{W}(x_1, x_2, x_3, t) < 0 \quad &\text{for } x_2 > 0 \text{ and } (x_1, x_3, t) \in \mathbb{R}^3,\\
\frac{\partial}{\partial x_3}\bar{W}(x_1, x_2, x_3, t) > 0 \quad &\text{for } (x_1, x_2, x_3, t) \in \mathbb{R}^4,\\
\bar{W}(-x_1, x_2, x_3, t) = \bar{W}(x_1, x_2, x_3, t) \quad &\text{for } (x_1, x_2, x_3, t) \in \mathbb{R}^4,\\
\bar{W}(x_1, -x_2, x_3, t) = \bar{W}(x_1, x_2, x_3, t) \quad &\text{for } (x_1, x_2, x_3, t) \in \mathbb{R}^4,\\
\bar{W}(0, 0, 0, 0) = \alpha^0 - \theta_0.&
\end{aligned}$$

Let

$$\begin{aligned}
\hat{W}(x_1, x_2, x_3, t) &= \min\{\bar{W}(x_1, x_2, x_3 - st, t), \bar{W}(x_1, x_2, -x_3 - st, t)\}\\
&= \bar{W}(x_1, x_2, -|x_3| - st, t)
\end{aligned}$$

for any $(x_1, x_2, x_3, t) \in \mathbb{R}^4$. Then $\hat{W}(x_1, x_2, x_3, t)$ is a supersolution of (3.8). In particular, we have

$$\lim_{t \to +\infty} \hat{W}(x_1, x_2, x_3, t) = 0 \quad \text{uniformly in } (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{3.9}$$

Let $u_0(\boldsymbol{x}) \in C_0(\mathbb{R}^3)$ satisfy $0 \leqslant u_0(\boldsymbol{x}) \leqslant \hat{W}(\boldsymbol{x}, 0)$ for $\boldsymbol{x} \in \mathbb{R}^3$, where $C_0(\mathbb{R}^3)$ denotes the set of all continuous functions in $\mathbb{R}^3$ with compact supports. By the comparison principle (see [5, theorem 25.6]), we have that

$$0 \leqslant \bar{u}(\boldsymbol{x}, t; u_0(\cdot)) \leqslant \hat{W}(\boldsymbol{x}, t) \quad \text{for any } \boldsymbol{x} \in \mathbb{R}^3,\ t > 0.$$

Due to (3.9), we have

$$\lim_{t \to +\infty} \bar{u}(\boldsymbol{x}, t; u_0(\cdot)) = 0 \quad \text{uniformly in } \boldsymbol{x} \in \mathbb{R}^3. \tag{3.10}$$

Therefore, there exists a $k_0 \in \mathbb{N}$ large enough such that

$$0 \leqslant \bar{u}(\boldsymbol{x}, t + k_0 T; u_0(\cdot)) \leqslant \epsilon \quad \forall \boldsymbol{x} \in \mathbb{R}^3,\ t > 0,$$

where $\epsilon > 0$ is defined in assumption (H4). In addition, it follows from (H4) that

$$\begin{aligned}\frac{\partial}{\partial t}\bar{u}(\boldsymbol{x}, t + k_0 T) &= \Delta \bar{u} + \bar{f}(\bar{u}(\boldsymbol{x}, t + k_0 T), t + k_0 T)\\ &= \Delta \bar{u} + \bar{f}(\bar{u}(\boldsymbol{x}, t + k_0 T), t)\\ &\geqslant \Delta \bar{u} + r_0 \bar{u}(\boldsymbol{x}, t + k_0 T)(\epsilon - \bar{u}(\boldsymbol{x}, t + k_0 T)),\end{aligned}$$

which implies that $\hat{u}(\boldsymbol{x}, t) = \bar{u}(\boldsymbol{x}, t + k_0 T; u_0(\cdot))$ is a supersolution of the following equation:

$$\frac{\partial}{\partial t} v(\boldsymbol{x}, t) = \Delta v + r_0 v(\boldsymbol{x}, t)(\epsilon - v(\boldsymbol{x}, t)). \tag{3.11}$$

It is due to (3.10) that

$$\lim_{t \to +\infty} \hat{u}(\boldsymbol{x}, t) = 0 \quad \text{uniformly in } \boldsymbol{x} \in \mathbb{R}^3. \tag{3.12}$$

Let $v_0(\boldsymbol{x}) \in C_0(\mathbb{R}^3)$ satisfy $0 \leqslant v_0(\boldsymbol{x}) \leqslant \hat{u}(\boldsymbol{x}, 0) \leqslant \epsilon$. The comparison principle (see [5, theorem 25.6]) yields that

$$0 \leqslant v(\boldsymbol{x}, t; v_0(\cdot)) \leqslant \hat{u}(\boldsymbol{x}, t) \quad \text{for any } \boldsymbol{x} \in \mathbb{R}^3 \text{ and } t > 0,$$

where $v(\boldsymbol{x}, t; v_0(\cdot))$ denotes the solution of (3.11) with initial value $v_0(\boldsymbol{x}) \in C_0(\mathbb{R}^3)$. Using (3.12) yields

$$\lim_{t \to +\infty} v(\boldsymbol{x}, t; v_0(\cdot)) = 0 \quad \text{uniformly in } \boldsymbol{x} \in \mathbb{R}^3. \tag{3.13}$$

However, using the result of Aronson and Weinberger [2, corollary 1], for any $0 < \nu < \nu^* := 2\sqrt{r_0 \epsilon}$, we have

$$\lim_{t \to +\infty} \inf_{|x| \leqslant \nu t} v(\boldsymbol{x}, t; v_0(\cdot)) = \epsilon,$$

which implies that

$$\lim_{t \to +\infty} v(\boldsymbol{x}, t; v_0(\cdot)) = \epsilon \quad \text{locally uniformly in } \boldsymbol{x} \in \mathbb{R}^3.$$

This contradicts (3.13). This contradiction implies that

$$\tilde{W}^+(t) \equiv W^+(t) \quad \text{for } t \in \mathbb{R}.$$

The proof is complete. □

The proof of theorem 3.1(vii) directly follows from property (b) of V^k, the fact that $(\partial/\partial\nu) W(\boldsymbol{x}, t) \not\equiv 0$ and the parabolic strong maximum principle, where

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2}}(\nu_1, \nu_2, 1) \quad \text{with } \sqrt{\nu_1^2 + \nu_2^2} \leqslant \frac{1}{m_*}.$$

Here we omit the details of the proof. The following theorem gives a proof of theorem 1.3 in $\mathbb{R}^3$.

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

THEOREM 3.5. *Let $s > c > 0$ and denote the cylindrically symmetric travelling front $W(\boldsymbol{x},t)$ defined in theorem 3.1 by $W^s(\boldsymbol{x},t)$. Let $U(0,0) = W^s(\mathbf{0},0) = \theta_0 \in (\alpha^-, \alpha^0)$. Then one has*

$$\lim_{s\to c+0} \|W^s(\boldsymbol{x},t) - U(x_3,t)\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^4)} = 0.$$

Proof. Observing estimate (3.6) for $W(x,t)$, we note that there exists $K > 0$ such that

$$\|W^s(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3\times\mathbb{R})} < K$$

for any $s \in (c, c+1)$, where $\alpha \in (0,1]$ is a constant. Let $\{s_n\}$ satisfy $s_n < s_{n+1} < c+1$ and let $s_n \to c$ as $n \to \infty$. There then exists a function $\hat{U}(\cdot,\cdot) \in C^{2,1}(\mathbb{R}^2)$ such that

$$W^{s_n}(0,0,\cdot,\cdot) \to \hat{U}(\cdot,\cdot) \quad \text{under the norm } \|\cdot\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^2)} \text{ as } n\to\infty.$$

By theorem 3.1(iii), we have

$$W^{s_n}(0,0,x_3,t) \leqslant W^{s_n}(x_1,x_2,x_3,t) \leqslant W^{s_n}(0,0,x_3 + m_*^n\sqrt{x_1^2+x_2^2},t)$$

for all $(x_1,x_2,x_3) \in \mathbb{R}^3$ and $t \in \mathbb{R}$, where $m_*^n = \sqrt{(s_n^2 - c^2)/c}$. Since $m_*^n \to 0$ as $n \to \infty$, we have that $W^{s_n}(x_1,x_2,x_3,t)$ converges to $\hat{U}(x_3,t)$ uniformly in any compact subset of $\mathbb{R}^4$ as $n \to \infty$. Consequently, we have that $W^{s_n}(x_1,x_2,x_3,t)$ converges to $\hat{U}(x_3,t)$ in the sense of $\|\cdot\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R})}$ as $n \to \infty$. Thus, we have that $\hat{U}(\cdot,\cdot) \in C^{2,1}(\mathbb{R}^2)$ satisfies

$$\frac{\partial}{\partial t}\hat{U}(x,t) = \frac{\partial^2}{\partial x^2}\hat{U}(x,t) - c\frac{\partial}{\partial x}\hat{U}(x,t) + f(t,\hat{U}(x,t)) \quad \forall x \in \mathbb{R},\ t \in \mathbb{R}.$$

In view of $\hat{U}(0,0) = \theta_0$ and $(\partial/\partial x)\hat{U}(x,t) \geqslant 0$ for any $(x,t) \in \mathbb{R}^2$, similar to the proof of theorem 3.1, we can show that $\hat{U}(+\infty,t) = W^+(t)$, $\hat{U}(-\infty,t) = W^-(t)$ and $(\partial/\partial x)\hat{U}(x,t) > 0$ for any $(x,t) \in \mathbb{R}^2$. It then follows from the uniqueness of one-dimensional time-periodic travelling fronts of (1.1) connecting two T-periodic solutions $W^-(t)$ and $W^+(t)$ that $\hat{U}(x,t) \equiv U(x,t)$ in $(x,t) \in \mathbb{R}^2$. This completes the proof. □

3.2. Proof of theorem 1.2 in $\mathbb{R}^3$

By theorem 3.1(ii), we define

$$\Psi(\rho,z,t) = \Psi(|\boldsymbol{x}'|,x_3,t) := W(\boldsymbol{x},t) \tag{3.14}$$

for any $(\boldsymbol{x}',x_3) \in \mathbb{R}^3$ and $t \in \mathbb{R}$, where $\rho = |\boldsymbol{x}'|$ and $z = x_3$.

THEOREM 3.6. *Let $\Psi(\rho,z,t)$ be defined by (3.14). Then $\Psi(\rho,z,t)$ satisfies*

$$\frac{\partial}{\partial t}\Psi = \frac{\partial^2}{\partial \rho^2}\Psi + \frac{\partial^2}{\partial z^2}\Psi + \frac{1}{\rho}\frac{\partial}{\partial \rho}\Psi - s\frac{\partial}{\partial z}\Psi + f(\Psi(\rho,z,t),t) \quad \forall \rho > 0,\ z \in \mathbb{R},\ t \in \mathbb{R}. \tag{3.15}$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

Moreover, one has

$$\frac{\partial}{\partial\rho}\Psi(\rho,z,t)>0 \quad \forall\rho>0,\ z\in\mathbb{R},\ t\in\mathbb{R},$$

$$\frac{\partial}{\partial z}\Psi(\rho,z,t)>0 \quad \forall\rho\geqslant 0,\ z\in\mathbb{R},$$

$$\lim_{z\to-\infty}\|\Psi(\cdot,z,t)-W^-(t)\|_{C([0,\omega])}=0 \quad \textit{uniformly in } t\in\mathbb{R} \textit{ for any } \omega>0,$$

$$\lim_{z\to+\infty}\|\Psi(\cdot,z,t)-W^+(t)\|_{C([0,+\infty))}=0 \quad \textit{uniformly in } t\in\mathbb{R}$$

and

$$\frac{\partial}{\partial\nu}\Psi(\rho,z,t)>0 \quad \forall\rho>0,\ z>0,\ t\in\mathbb{R},$$

where $\nu=\dfrac{1}{\sqrt{1+(\nu')^2}}\begin{pmatrix}\nu'\\1\end{pmatrix}$ *is a given constant vector with* $\nu'\geqslant-\dfrac{1}{m_*}$.

Theorem 3.6 directly follows from theorem 3.1, we omit the details of the proof. Now we give an estimate for $\Psi(\rho,z,t)$. Applying an argument similar to that of [44, proposition 4.3], we have that there exists a positive constant K_1' such that $\|W(\cdot,t)\|_{C(\mathbb{R}^3)}\leqslant K_1'$ for all $t\in\mathbb{R}$. Since $(\partial/\partial z)\Psi(\rho,z,t)=(\partial/\partial x_3)W(\boldsymbol{x},t)$ and $|(\partial/\partial\rho)\Psi(\rho,z,t)|\leqslant|(\partial/\partial x_1)W(\boldsymbol{x},t)|+|(\partial/\partial x_2)W(\boldsymbol{x},t)|$, applying the interior L^p estimate of parabolic differential equations (see [25, proposition 7.18]) we have that there exists a positive constant K_2' such that

$$\|\Psi(\cdot,\cdot)\|_{W_p^{2,1}(Q((\rho_0,z_0,4(1+T)),2\sqrt{1+T}))}\leqslant K_2'$$

for any $(\rho_0,z_0)\in(1+2\sqrt{1+T},+\infty)\times\mathbb{R}$, where $p>1$ is a constant, and

$$Q((\rho_0,z_0,t_0),R) \\ :=\{(\rho,z,t)\in\mathbb{R}^3\colon \sqrt{(\rho-\rho_0)^2+(z-z_0)^2}<R,\ |t-t_0|^{1/2}<R,\ t<t_0\}.$$

Differentiating (3.15) twice, once with respect to ρ and once with respect to z, and applying the above argument to the equations for $(\partial/\partial\rho)\Psi$ and $(\partial/\partial z)\Psi$, respectively, we have that there exists a positive constant K_3' such that

$$\|\,\mathrm{D}_i\,\Psi(\cdot,\cdot)\|_{W_p^{2,1}(Q_{(\rho_0,z_0)})}\leqslant K_3'$$

for any $(\rho_0,z_0)\in(1+2\sqrt{1+T},+\infty)\times\mathbb{R}$, $i=1,2$, where $p>1$ is a constant, $\mathrm{D}_1:=\partial/\partial\rho$, $\mathrm{D}_2:=\partial/\partial z$, and

$$Q_{(\rho_0,z_0)}:=Q((\rho_0,z_0,4(1+T)),\sqrt{1+T}).$$

Fix $p>4$. Using the embedding theorem (see [46, theorem 1.4.1]), we have that there exists a constant $K_4'>0$ such that

$$\|\Psi(\cdot,\cdot)\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q_{(\rho_0,z_0)}})}\leqslant K_4' \quad \text{and} \quad \|\,\mathrm{D}_i\,\Psi(\cdot,\cdot)\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q_{(\rho_0,z_0)}})}\leqslant K_4'$$

for any $(\rho_0,z_0)\in(1+2\sqrt{1+T},+\infty)\times\mathbb{R}$ and $i=1,2$, where $1+\alpha=2-4/p$ is a constant. In view of the arbitrariness of $(\rho_0,z_0)\in(1+2\sqrt{1+T},+\infty)\times\mathbb{R}$ and the periodicity of $\Psi(\rho, z, t)$ in $t \in \mathbb{R}$, applying [27, theorem 5.1.20] to (3.15) we have that there exists a positive constant K' such that

$$\|\Psi(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}([1+2\sqrt{1+T},\infty)\times\mathbb{R}\times\mathbb{R})} \leqslant K'. \tag{3.16}$$

We now list a very useful lemma on the Harnack inequalities of cooperative parabolic systems, which was given in [9] and is needed in what follows.

LEMMA 3.7 (Földes and Poláčik [9] and Zhao and Ruan [47]). *Let the differential operators*

$$\mathrm{L}_k := \sum_{i,j=1}^{n} a_{i,j}^k(t,\boldsymbol{x})\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} + \sum_{i=1}^{n} b_i^k \frac{\partial}{\partial_{x_i}} - \frac{\partial}{\partial t}, \quad k = 1,2,\dots,l,$$

be uniformly parabolic in an open domain $(\tau, M) \times \Omega$ of $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n$, that is, there is $\alpha_0 > 0$ such that $a_{i,j}^k(t,\boldsymbol{x})\xi_i\xi_j \geqslant \alpha_0 \sum_{i=1}^n \xi_i^2$ for any n-tuples of real numbers $(\xi_1, \xi_2, \dots, \xi_n)$, where $-\infty < \tau < M \leqslant +\infty$ and Ω is open and bounded. Suppose that $a_{i,j}^k, b_i^k \in C((\tau, M) \times \Omega, \mathbb{R})$ and

$$\max_{(t,\boldsymbol{x})\in(\tau,M)\times\Omega} |b_i^k(t,\boldsymbol{x})| + |a_{i,j}^k(t,\boldsymbol{x})| \leqslant \beta_0$$

for some $\beta_0 > 0$. Assume that

$$\boldsymbol{w} = (w_1, w_2, \dots, w_l) \in C((\tau, M) \times \bar{\Omega}, \mathbb{R}^l) \cap C^{1,2}((\tau, M) \times \Omega, \mathbb{R}^l)$$

satisfies

$$\sum_{s=1}^{l} c^{k,s}(t,\boldsymbol{x})w_s + \mathrm{L}_k\, w_k \leqslant 0, \quad (t,\boldsymbol{x}) \in (\tau, M) \times \Omega,\ k = 1,2,\dots,l, \tag{3.17}$$

where $c^{k,s} \in C((\tau, M) \times \Omega, \mathbb{R})$ and $c^{k,s} \geqslant 0$ if $k \neq s$, and

$$\max_{(t,\boldsymbol{x})\in(\tau,M)\times\Omega} |c^{k,s}(t,\boldsymbol{x})| \leqslant \gamma_0$$

$(k, s = 1, 2, \dots, l)$ for some $\gamma_0 > 0$. Let D and U be domains in Ω such that $D \subset\subset U$, $\operatorname{dist}(\bar{D}, \partial U) > \varrho$, and $|D| > \varepsilon$ for certain positive constants ϱ and ε. Let θ be a positive constant with $\tau + 4\theta < M$. There then exist positive constants p, ω_1 and ω_2 determined only by α_0, β_0, γ_0, ϱ, ε, n, $\operatorname{diam}\Omega$ and θ, such that

$$\inf_{(\tau+3\theta,\tau+4\theta)\times D} w_k \geqslant \omega_1\|(w_k)^+\|_{L^p((\tau+\theta,\tau+2\theta)\times D)} - \omega_2 \max_{j=1,\dots,k} \sup_{\partial_P((\tau,\tau+4\theta)\times U)} (w_j)^-.$$

Here $(w_k)^+ = \max\{w_k, 0\}$, $(w_k)^- = \max\{-w_k, 0\}$ and $\partial_P((\tau, \tau + 4\theta) \times U) = \tau \times U \cup [\tau, \tau + 4\theta] \times \partial U$. Moreover, if all inequalities in (3.17) are replaced by equalities, then the conclusion holds with $p = \infty$ and with ω_1, ω_2 independent of ε.

We now prove the remainder of theorem 1.2. Define a function $\phi \in C^2([0, \infty), \mathbb{R})$ by $\Psi(\rho, \phi(\rho), 0) = \theta_0$ for any $\rho \in [0, \infty)$, where $\theta_0 \in (\alpha^-, \alpha^0)$ is a constant and satisfies $\theta_0 < \alpha^- + \delta_*$, where $\delta_* > 0$ is defined in lemma 2.2. We then have

$$-m^* \leqslant \phi'(\rho) < 0 \quad \forall \rho \in (0, \infty),$$
$$\phi(0) = \theta_0, \quad \phi'(0) = 0,$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

and

$$\phi'(\rho) = -\frac{(\partial/\partial\rho)\Psi(\rho,\phi(\rho),0)}{(\partial/\partial z)\Psi(\rho,\phi(\rho),0)} \quad \forall \rho \in [0,\infty).$$

In the following we show the asymptotic behaviour of the level set $\phi(\rho)$ as $\rho \to \infty$. Here we use a strategy similar to that in [41].

LEMMA 3.8. *One has*

$$\limsup_{\rho\to\infty} \phi'(\rho) < 0.$$

Proof. We prove the lemma by way of a contradiction argument. We assume that there exists a sequence $\rho_i \in (0,\infty)$ such that $\lim_{i\to\infty} \rho_i = +\infty$ and $\lim_{i\to\infty} \phi'(\rho_i) = 0$. Consequently, we have $\lim_{i\to\infty} \Psi_\rho(\rho_i,\phi(\rho_i),0) = 0$. For any given $r > 1$ we take $N \in \mathbb{N}$ so that $\rho_i > r+1$ when $i > N$. Direct calculation yields

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial\rho^2} - \frac{\partial^2}{\partial z^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + s\frac{\partial}{\partial z}\right)\Psi_\rho = \left(f_u(\Psi,t) - \frac{1}{\rho^2}\right)\Psi_\rho$$

for any $(\rho,z,t) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}$. Note that $\Psi_\rho(\rho,z,t) > 0$ for any $(\rho,z,t) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}$. Applying lemma 3.7 to the last equation with $\theta = T$ and $\tau = -T$, we obtain

$$\begin{aligned} 0 < \sup_{B((\rho_i,\phi(\rho_i));r)\times(0,T)} \Psi_\rho(\rho,z,t) &\leqslant C_0 \inf_{B((\rho_i,\phi(\rho_i));r)\times(2T,3T)} \Psi_\rho(\rho,z,t) \\ &= C_0 \min_{\overline{B((\rho_i,\phi(\rho_i));r)}\times[2T,3T]} \Psi_\rho(\rho,z,t) \\ &\leqslant C_0\Psi_\rho(\rho_i,\phi(\rho_i),0), \end{aligned}$$

where $C_0 > 0$ is a constant independent of i and

$$B((x_0,z_0);r) = \{(x,z)\in\mathbb{R}^2 \mid \sqrt{(x-x_0)^2+(z-z_0)^2} < r\}.$$

Note that $\Psi(\rho,z,t+T) = \Psi(\rho,z,t)$ and $\Psi_\rho(\rho,z,t+T) = \Psi_\rho(\rho,z,t)$ for any $(\rho,z,t) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}$. Letting $i\to\infty$ (up to extraction of a subsequence if necessary), we have that there exists a function $\Phi(\rho,z,t)$ that is C^2 both in $\rho > 0$ and $z \in \mathbb{R}$, is C^1 in $t\in\mathbb{R}$ and satisfies

$$\Psi(\rho+\rho_i, z+\phi(\rho_i), t) \to \Phi(\rho,z,t) \quad \text{locally uniform in } C^{2,1}((0,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$$

as $i\to\infty$. Since

$$0 < \sup_{B((\rho_i,\phi(\rho_i));r)\times(0,T)} \Psi_\rho(\rho,z,t) \leqslant C_0\Psi_\rho(\rho_i,\phi(\rho_i),0) \to 0$$

as $i\to\infty$, we have that the limit function Φ is independent of $\rho \geqslant 0$. Therefore, we denote $\Phi(\rho,z,t)$ by $\Phi(z,t)$. In addition, we have

$$\Phi(0,0) = \theta_0, \quad \Phi_z(z,t) \geqslant 0, \quad \Phi(z,t+T) = \Phi(z,t) \quad \forall (z,t)\in\mathbb{R}^2.$$

For any $\varphi(\rho, z) \in C_0^\infty((0,\infty)\times\mathbb{R},\mathbb{R})$, we have

$$\begin{aligned}
&\int_{(0,\infty)\times\mathbb{R}} \frac{\partial}{\partial t}\Psi(\rho+\rho_i, z+\phi(\rho_i), t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z \\
&\quad= \int_{(0,\infty)\times\mathbb{R}} \left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{\rho+\rho_i}\frac{\partial}{\partial\rho}-s\frac{\partial}{\partial z}\right)\Psi(\rho+\rho_i, z+\phi(\rho_i), t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z \\
&\qquad+\int_{(0,\infty)\times\mathbb{R}} f(\Psi(\rho+\rho_i, z+\phi(\rho_i), t), t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z \\
&\quad= \int_{(0,\infty)\times\mathbb{R}} \left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho+\rho_i}\frac{\partial}{\partial\rho}+s\frac{\partial}{\partial z}+\frac{1}{(\rho+\rho_i)^2}\right)\varphi(\rho,z) \\
&\qquad\qquad\times\Psi(\rho+\rho_i, z+\phi(\rho_i), t)\,\mathrm{d}\rho\,\mathrm{d}z \\
&\qquad+\int_{(0,\infty)\times\mathbb{R}} f(\Psi(\rho+\rho_i, z+\phi(\rho_i), t), t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z.
\end{aligned}$$

Letting $i\to\infty$, we obtain

$$\begin{aligned}
\int_{(0,\infty)\times\mathbb{R}} \frac{\partial}{\partial t}\Phi(z,t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z &= \int_{(0,\infty)\times\mathbb{R}} \left(\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}+s\frac{\partial}{\partial z}\right)\varphi(\rho,z)\Phi(z,t)\,\mathrm{d}\rho\,\mathrm{d}z \\
&\quad+\int_{(0,\infty)\times\mathbb{R}} f(\Phi(z,t),t)\varphi(\rho,z)\,\mathrm{d}\rho\,\mathrm{d}z,
\end{aligned}$$

which implies that the function $\Phi(z,t)$ satisfies

$$\frac{\partial}{\partial t}\Phi(z,t) = \frac{\partial^2}{\partial z^2}\Phi(z,t) - s\frac{\partial}{\partial z}\Phi(z,t) + f(\Phi(z,t),t) \quad \forall (z,t)\in\mathbb{R}^2.$$

Due to $\Phi(0,0)=\theta_0$ and $\Phi_z(z,t)\geqslant 0$ for any $z\in\mathbb{R}$ and $t\in\mathbb{R}$, we have that $\Phi(z,t)$ is a travelling wave solution of (1.1) either connecting two periodic solutions $W^-(t)$ and $W^0(t)$ with wave speed $s>0$ or connecting two periodic solutions $W^-(t)$ and $W^+(t)$ with wave speed $s>0$. Similar to the proof of lemma 3.3, we conclude that both cases are impossible. Hence, we have

$$\limsup_{\rho\to\infty}\phi'(\rho)<0.$$

This completes the proof. □

Lemma 3.9. *One has* $\liminf_{\rho\to\infty}\dfrac{\phi(\rho)}{\rho} = -m^*$.

Proof. Note that

$$0 \geqslant \frac{\phi(\rho)}{\rho} = \frac{1}{\rho}\int_0^\rho \phi'(\rho)\,\mathrm{d}\rho \geqslant -m_*.$$

We prove the lemma by way of a contradiction argument. Assume on the contrary that

$$-m_* < -\tau := \liminf_{\rho\to\infty}\frac{\phi(\rho)}{\rho} < 0.$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

There then exists a positive and increasing sequence $\{\rho_i\}$ satisfying $\phi(\rho_i)/\rho_i \to -\tau$ and $\rho_i \to +\infty$ as $i \to +\infty$. Set $s_* \in (c, s)$ such that

$$\sqrt{s_*^2 - c^2}/c = \tau.$$

Define

$$\tilde{\Psi}(x, z, t; \rho_i) := \Psi(\rho_i + x, \phi(\rho_i) + z, t) \quad \forall (x, z, t) \in (-\rho_i, +\infty) \times \mathbb{R} \times \mathbb{R}.$$

It is obvious that there holds

$$\frac{\partial}{\partial t}\tilde{\Psi} = \frac{\partial^2}{\partial x^2}\tilde{\Psi} + \frac{\partial^2}{\partial z^2}\tilde{\Psi} + \frac{1}{\rho_i + x}\frac{\partial}{\partial x}\tilde{\Psi} - s\frac{\partial}{\partial z}\tilde{\Psi} + f(\tilde{\Psi}(x, z, t; \rho_i), t),$$
$$\tilde{\Psi}(0, 0, 0) = \theta_0,$$

for any $(x, z, t) \in (-\rho_i, +\infty) \times \mathbb{R} \times \mathbb{R}$. Then there exists a function $\tilde{\Psi}_0 \in C^{2,1}(\mathbb{R}^2 \times \mathbb{R})$ such that

$$\tilde{\Psi}(x, z, t; \rho_i) \to \tilde{\Psi}_0(x, z, t) \quad \text{in } C^{2,1}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$$

as $i \to \infty$ (up to extraction of a subsequence if necessary). In particular, $\tilde{\Psi}_0(x, z, t)$ satisfies

$$\frac{\partial}{\partial t}\tilde{\Psi}_0 = \frac{\partial^2}{\partial x^2}\tilde{\Psi}_0 + \frac{\partial^2}{\partial z^2}\tilde{\Psi}_0 - s\frac{\partial}{\partial z}\tilde{\Psi}_0 + f(\tilde{\Psi}_0(x, z, t), t),$$
$$\tilde{\Psi}_0(0, 0, 0) = \theta_0,$$

for any $(x, z, t) \in \mathbb{R}^3$. Since $(\partial/\partial x)\tilde{\Psi}(x, z, t; \rho_i) > 0$ and $(\partial/\partial z)\tilde{\Psi}(x, z, t; \rho_i) > 0$ for any $(x, z, t) \in (-\rho_i, +\infty) \times \mathbb{R}^2$, we have

$$\frac{\partial}{\partial x}\tilde{\Psi}_0(x, z, t) \geqslant 0 \quad \text{and} \quad \frac{\partial}{\partial z}\tilde{\Psi}_0(x, z, t) \geqslant 0 \quad \forall (x, z, t) \in \mathbb{R}^3.$$

In the following we show that

$$\frac{\partial}{\partial x}\tilde{\Psi}_0(x, z, t) > 0 \quad \text{and} \quad \frac{\partial}{\partial z}\tilde{\Psi}_0(x, z, t) > 0 \quad \forall (x, z, t) \in \mathbb{R}^3.$$

Firstly, due to $\tilde{\Psi}_0(0, 0, 0) = \theta_0$ and $\tilde{\Psi}_0(x, z, t) = \tilde{\Psi}_0(x, z, t+T)$ for all $(x, z, t) \in \mathbb{R}^3$, we have that it is impossible that both $(\partial/\partial x)\tilde{\Psi}_0(x, z, t) \equiv 0$ and $(\partial/\partial z)\tilde{\Psi}_0(x, z, t) \equiv 0$ on $(x, z, t) \in \mathbb{R}^3$. We consider the other two cases:

(1) $\dfrac{\partial}{\partial x}\tilde{\Psi}_0(x, z, t) \not\equiv 0$ and $\dfrac{\partial}{\partial z}\tilde{\Psi}_0(x, z, t) \equiv 0$ on $(x, z, t) \in \mathbb{R}^3$,

(2) $\dfrac{\partial}{\partial x}\tilde{\Psi}_0(x, z, t) \equiv 0$ and $\dfrac{\partial}{\partial z}\tilde{\Psi}_0(x, z, t) \not\equiv 0$ on $(x, z, t) \in \mathbb{R}^3$.

Assume that $(\partial/\partial x)\tilde{\Psi}_0(x, z, t) \not\equiv 0$ and $(\partial/\partial z)\tilde{\Psi}_0(x, z, t) \equiv 0$ on $(x, z, t) \in \mathbb{R}^3$. Then we have $(\partial/\partial x)\tilde{\Psi}_0(x, z, t) > 0$ on $(x, z, t) \in \mathbb{R}^3$ due to the parabolic maximum principle (see [33]). Denote $\tilde{\Psi}_0(x, z, t)$ by $\tilde{\Psi}_0(x, t)$. Then we have

$$\frac{\partial}{\partial t}\tilde{\Psi}_0(x, t) = \frac{\partial^2}{\partial x^2}\tilde{\Psi}_0(x, t) + f(\tilde{\Psi}_0(x, t), t),$$
$$\tilde{\Psi}_0(0, 0) = \theta_0,$$

for any $(x,t) \in \mathbb{R}^2$. Since $(\partial/\partial x)\tilde{\Psi}_0(x,t) > 0$ on $(x,t) \in \mathbb{R}^2$, there exist two T-periodic functions $\tilde{W}^{\pm}(t) \in C^1(\mathbb{R},\mathbb{R})$ such that

$$\tilde{W}^{\pm}(t) = \lim_{x\to\pm\infty} \tilde{\Psi}_0(x,t), \quad t \in \mathbb{R},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{W}^{\pm}(t) = f(\tilde{W}^{\pm}(t),t), \qquad t \in \mathbb{R},$$

and

$$\tilde{W}^-(0) < \theta_0 < \tilde{W}^+(0).$$

Following assumption (H1), we have either $\tilde{W}^-(t) = W^-(t)$ and $\tilde{W}^+(t) = W^+(t)$, or $\tilde{W}^-(t) = W^-(t)$ and $\tilde{W}^+(t) = W^0(t)$, which implies that $\tilde{\Psi}_0(x,t)$ is a time-periodic travelling wave solution of (1.1) with wave speed $c = 0$ either connecting two periodic solutions $W^-(t)$ and $W^+(t)$ or connecting two periodic solutions $W^-(t)$ and $W^0(t)$. As discussed in lemma 3.3, they are all impossible. Therefore, we can rule out the case that $(\partial/\partial x)\tilde{\Psi}_0(x,z,t) \not\equiv 0$ and $(\partial/\partial z)\tilde{\Psi}_0(x,z,t) \equiv 0$ on $(x,z,t) \in \mathbb{R}^3$.

Assume that $(\partial/\partial x)\tilde{\Psi}_0(x,z,t) \equiv 0$ and $(\partial/\partial z)\tilde{\Psi}_0(x,z,t) \not\equiv 0$ on $(x,z,t) \in \mathbb{R}^3$. We have $(\partial/\partial z)\tilde{\Psi}_0(x,z,t) > 0$ on $(x,z,t) \in \mathbb{R}^3$ due to the parabolic maximum principle (see [33]). Denote $\tilde{\Psi}_0(x,z,t)$ by $\tilde{\Psi}_0(z,t)$. We then have

$$\frac{\partial}{\partial t}\tilde{\Psi}_0(z,t) = \frac{\partial^2}{\partial z^2}\tilde{\Psi}_0(z,t) - s\frac{\partial}{\partial z}\tilde{\Psi}_0(z,t) + f(\tilde{\Psi}_0(z,t),t),$$
$$\tilde{\Psi}_0(0,0) = \theta_0,$$

for any $(z,t) \in \mathbb{R}^2$. Since $(\partial/\partial z)\tilde{\Psi}_0(z,t) > 0$ on $(z,t) \in \mathbb{R}^2$, there exist two T-periodic functions $\hat{W}^{\pm}(t) \in C^1(\mathbb{R},\mathbb{R})$ such that

$$\hat{W}^{\pm}(t) = \lim_{z\to\pm\infty} \hat{\Psi}_0(z,t), \quad t \in \mathbb{R},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{W}^{\pm}(t) = f(\tilde{W}^{\pm}(t),t), \qquad t \in \mathbb{R},$$

and

$$\tilde{W}^-(0) < \theta_0 < \tilde{W}^+(0).$$

Following assumption (H1), we have either $\tilde{W}^-(t) = W^-(t)$ and $\tilde{W}^+(t) = W^+(t)$ or $\tilde{W}^-(t) = W^-(t)$ and $\tilde{W}^+(t) = W^0(t)$, which implies that $\tilde{\Psi}_0(z,t)$ is a periodic travelling wave solution of (1.1) with wave speed $s > c > 0$ either connecting two periodic solutions $W^-(t)$ and $W^+(t)$ or connecting two periodic solutions $W^-(t)$ and $W^0(t)$. As discussed in lemma 3.3, they are impossible. Therefore, we can rule out the case that $(\partial/\partial x)\tilde{\Psi}_0(x,z,t) \equiv 0$ and $(\partial/\partial z)\tilde{\Psi}_0(x,z,t) \not\equiv 0$ on $(x,z,t) \in \mathbb{R}^3$.

Consequently, we have

$$\frac{\partial}{\partial x}\tilde{\Psi}_0(x,z,t) \not\equiv 0 \quad \text{and} \quad \frac{\partial}{\partial z}\tilde{\Psi}_0(x,z,t) \not\equiv 0$$

on $(x,z,t) \in \mathbb{R}^3$. Using the parabolic maximum principle (see [33]) yields

$$\frac{\partial}{\partial x}\tilde{\Psi}_0(x,z,t) > 0 \quad \text{and} \quad \frac{\partial}{\partial z}\tilde{\Psi}_0(x,z,t) > 0 \quad \forall (x,z,t) \in \mathbb{R}^3.$$

Let

$$\lim_{z\to-\infty}\tilde{\Psi}_0(x,z,t)=\tilde{\Psi}_0^-(x,t)\quad\text{for any fixed }(x,t)\in\mathbb{R}^2.$$

We then have

$$\frac{\partial}{\partial t}\tilde{\Psi}_0^-(x,t)=\frac{\partial^2}{\partial x^2}\tilde{\Psi}_0^-(x,t)+f(\tilde{\Psi}_0^-(x,t),t)$$

and

$$\tilde{\Psi}_0^-(0,0)<\theta_0,\quad \frac{\partial}{\partial x}\tilde{\Psi}_0^-(x,t)\geqslant 0.$$

Similar to the argument above, we have that

$$\tilde{\Psi}_0^-(x,t)\equiv W^-(t)\quad\text{for any fixed }(x,t)\in\mathbb{R}^2.$$

Define a function $\varrho(x)$ on $x\in\mathbb{R}$ by $\tilde{\Psi}_0(x,\varrho(x),0)=\theta_0$. Due to the strict monotonicity of $\tilde{\Psi}_0(x,z,0)$ on x and z, the function $\varrho(x)$ is strictly decreasing. In particular, $\varrho(0)=0$. Since

$$\liminf_{\rho\to\infty}\frac{\phi(\rho)}{\rho}=-\tau\quad\text{and}\quad\lim_{i\to\infty}\frac{\phi(\rho_i)}{\rho_i}=-\tau,$$

it follows from the definition of the function $\tilde{\Psi}_0(x,z,t)$ that $\lim_{i\to\infty}(\phi(x+\rho_i)-\phi(\rho_i))=\varrho(x)$ for any $x\in\mathbb{R}$ and $\varrho(x)\geqslant-\tau x$ for any $x\geqslant 0$ (see [41, p. 1035]). Take $z^*>0$ large enough so that $U((c/s)z^*,0)+\delta_*\geqslant\alpha^+$. Consequently, we obtain

$$\tilde{\Psi}_0(x,z,0)\leqslant U\left(\frac{c}{s_*}(z+z^*+\tau|x|),0\right)+\delta_*\leqslant\tilde{V}(x,z+z^*,0;s_*)+\delta_*\quad\forall(x,z)\in\mathbb{R}^2,$$

where $\tilde{V}(x,z,t;s_*)$ is the two-dimensional V-shaped travelling front defined by theorem 2.1 with $\tilde{s}=s_*$. Using lemma 2.2, we have

$$\tilde{\Psi}_0(x,z+st,t)\leqslant\tilde{V}(x,z+z^*+s_*t+\sigma\delta_*(1-\mathrm{e}^{-\beta t}),t;s_*)+\delta_*a(t)\quad\forall(x,z)\in\mathbb{R}^2,\ t>0,$$

where the positive constants ρ and β, and the function $a(t)$ are defined in lemma 2.2. Keeping $z+skT=0$ and $x=0$ and letting $t=kT\to+\infty$, we have $\tilde{\Psi}_0(0,0,0)=W^-(0)=\alpha^-$, which contradicts the fact $\tilde{\Psi}_0(0,0,0)=\theta_0>\alpha^-$. This completes the proof. □

Following the above discussion, we have $0\geqslant\phi'(\rho)\geqslant-m_*$ for any $\rho>0$ and

$$\liminf_{\rho\to\infty}\int_0^\rho\phi'(r)\,\mathrm{d}r=-m_*.$$

Following from [41, p. 1036, (32)], we have that there exists an increasing sequence $\{\rho_i\}\subset(0,+\infty)$ such that

$$\lim_{i\to\infty}\rho_i=\infty,\qquad\lim_{i\to\infty}\phi'(\rho_i)=-m_*,\qquad\sup_{i\in\mathbb{N}}|\rho_{i+1}-\rho_i|<\infty.$$

Let

$$\nu_0:=\frac{1}{\sqrt{1+m_*^2}}\begin{pmatrix}-1\\ m_*\end{pmatrix}.$$

We then have

$$\lim_{i\to\infty} \frac{\partial}{\partial \nu_0}\Psi(\rho,z,0)\bigg|_{(\rho,z)=(\rho_i,\phi(\rho_i))} = 0.$$

Following from theorem 3.6, we have

$$\frac{\partial}{\partial \nu_0}\Psi(\rho,z,0) > 0 \quad \text{for all } \rho \geqslant 0 \text{ and } z \in \mathbb{R}.$$

Direct calculations yield

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \rho^2} - \frac{\partial^2}{\partial z^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + s\frac{\partial}{\partial z} - f_u(\Psi,t)\right)\frac{\partial}{\partial \nu_0}\Psi = \frac{1}{\rho^2}\Psi_\rho > 0 \tag{3.18}$$

for any $\rho > 0$, $z \in \mathbb{R}$ and $t \in \mathbb{R}$. Applying lemma 3.7 to (3.18), we have that for any given $r > 1$ there exist positive constants $p > 0$ and $C' > 0$ such that

$$\begin{aligned}\left\|\frac{\partial}{\partial \nu_0}\Psi\right\|_{L^p(B((\rho_i,\phi(\rho_i));r)\times(0,T))} &\leqslant C' \inf_{B((\rho_i,\phi(\rho_i));r)\times(2T,3T)} \frac{\partial}{\partial \nu_0}\Psi(\rho,z,t)\\ &\leqslant C'\frac{\partial}{\partial \nu_0}\Psi(\rho_i,\phi(\rho_i),0)\\ &\to 0 \quad \text{as } i\to\infty.\end{aligned}$$

Due to the fact that $\sup_{i\in\mathbb{N}}|\rho_{i+1}-\rho_i| < \infty$, taking $r > 1$ large enough we obtain

$$\lim_{\rho\to\infty}\left\|\frac{\partial}{\partial \nu_0}\Psi\right\|_{L^p(B((\rho,\phi(\rho));r)\times(0,T))} = 0 \tag{3.19}$$

for any $r > 1$ with some $p > 0$.

LEMMA 3.10. *One has*

$$\begin{aligned}\lim_{\rho\to\infty}\phi'(\rho) &= -m_*,\\ \lim_{\rho\to\infty}\Psi_\rho(\rho,\phi(\rho),0) &= \frac{cm_*}{s}U_\eta(0,0),\\ \lim_{\rho\to\infty}\Psi_z(\rho,\phi(\rho),0) &= \frac{c}{s}U_\eta(0,0)\end{aligned}$$

and

$$\lim_{\rho\to\infty}\Psi(\rho+x,\phi(\rho)+z,t) = U\left(\frac{s}{c}(z+m_*x),t\right) \quad \text{in } C^{2,1}_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}).$$

Proof. For $\rho > 0$ we define $(\xi,\eta)\in\mathbb{R}^2$ by

$$\begin{pmatrix} x-\rho \\ z-\phi(\rho)\end{pmatrix} = \xi\begin{pmatrix}\dfrac{-1}{\sqrt{1+m_*^2}}\\ \dfrac{m_*}{\sqrt{1+m_*^2}}\end{pmatrix} + \eta\begin{pmatrix}\dfrac{m_*}{\sqrt{1+m_*^2}}\\ \dfrac{1}{\sqrt{1+m_*^2}}\end{pmatrix}.$$

In view of (3.16), for any increasing positive sequence $\{\sigma_i\}$ with $\sigma_i\to\infty$ as $i\to\infty$ there exists a subsequence (still denoted by $\{\sigma_i\}$, without loss of generality) such

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

that there exists a function $\tilde{U}(\xi,\eta,t) \in C^{2,1}(\mathbb{R}^2 \times \mathbb{R})$ satisfying

$$\lim_{i\to\infty} \left\| \tilde{U}(\xi,\eta,t) - \Psi\left(\sigma_i + \frac{-\xi + m_*\eta}{\sqrt{1+m_*^2}}, \phi(\sigma_i) + \frac{m_*\xi + \eta}{\sqrt{1+m_*^2}}, t\right) \right\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{R})} = 0.$$

We then have

$$\frac{\partial}{\partial t}\tilde{U} = \frac{\partial^2}{\partial \xi^2}\tilde{U} + \frac{\partial^2}{\partial \eta^2}\tilde{U} - c\frac{\partial}{\partial \eta}\tilde{U} + f(\tilde{U}(\xi,\eta,t),t) \quad \forall (\xi,\eta,t) \in \mathbb{R}^3.$$

Here we claim that $(\partial/\partial\xi)\tilde{U}(\xi,\eta,t) \equiv 0$ in $\mathbb{R}^3$. Otherwise, suppose that

$$\frac{\partial}{\partial \xi}\tilde{U}(\xi,\eta,t) \geqslant 0 \quad \text{and} \quad \frac{\partial}{\partial \xi}\tilde{U}(\xi,\eta,t) \not\equiv 0 \quad \text{in } \mathbb{R}^3.$$

Using the parabolic maximum principle (see [33]) yields that $(\partial/\partial\xi)\tilde{U}(\xi,\eta,t) > 0$ for all $(\xi,\eta,t) \in \mathbb{R}^3$. We then have

$$\left\| \frac{\partial}{\partial \xi}\tilde{U} \right\|_{L^p(B((0,0);1)\times(0,T))} > \varsigma,$$

where $\varsigma > 0$ is a constant and p is given in (3.19). Since

$$\lim_{i\to\infty} \left\| \frac{\partial}{\partial \xi}\tilde{U}(\xi,\eta,t) - \frac{\partial}{\partial \nu_0}\Psi\left(\sigma_i + \frac{-\xi + m_*\eta}{\sqrt{1+m_*^2}}, \phi(\sigma_i) + \frac{m_*\xi + \eta}{\sqrt{1+m_*^2}}, t\right) \right\|_{C_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{R})} = 0,$$

there exists $I \in \mathbb{N}$ such that

$$\left\| \frac{\partial}{\partial \nu_0}\Psi\left(\sigma_i + \frac{-\xi + m_*\eta}{\sqrt{1+m_*^2}}, \phi(\sigma_i) + \frac{m_*\xi + \eta}{\sqrt{1+m_*^2}}, t\right) \right\|_{L^p(B((0,0);1)\times(0,T))} > \tfrac{1}{2}\varsigma > 0$$

for any $i > I$. However, this contradicts (3.19). This implies that $\tilde{U}$ is independent of ξ and that $(\partial/\partial\xi)\tilde{U}(\xi,\eta,t) \equiv 0$ in $\mathbb{R}^3$.

Still denote $\tilde{U}(\xi,\eta,t)$ by $\tilde{U}(\eta,t)$. Consequently, we have

$$\frac{\partial}{\partial t}\tilde{U} = \frac{\partial^2}{\partial \eta^2}\tilde{U} - c\frac{\partial}{\partial \eta}\tilde{U} + f(\tilde{U}(\eta,t),t) \quad \forall (\eta,t) \in \mathbb{R}^2$$

and

$$\tilde{U}(0,0) = \theta_0 \in (\alpha^-, \alpha^0), \quad \frac{\partial}{\partial \eta}\tilde{U}(\eta,t) \geqslant 0 \quad \text{for any } (\eta,t) \in \mathbb{R}^2.$$

Similar to the arguments in § 3.1, we have that $(\partial/\partial\eta)\tilde{U}(\eta,t) > 0$ for all $(\eta,t) \in \mathbb{R}^2$ and $\lim_{\eta\to\pm}\tilde{U}(\eta,t) = W^{\pm}(t)$. In particular, we have $\tilde{U}(\cdot,\cdot+T) = \tilde{U}(\cdot,\cdot)$. Then the uniqueness of the one-dimensional travelling front gives $\tilde{U}(\cdot,\cdot) \equiv U(\cdot,\cdot)$. Following the arbitrariness of the sequence of σ_i, we conclude that

$$\lim_{\rho\to\infty} \left\| U(\eta,t) - \Psi\left(\rho + \frac{-\xi + m_*\eta}{\sqrt{1+m_*^2}}, \phi(\rho) + \frac{m_*\xi + \eta}{\sqrt{1+m_*^2}}, t\right) \right\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{R})} = 0. \quad (3.20)$$

This furthermore implies that

$$\lim_{\rho\to\infty} \Psi(\rho + x, \phi(\rho) + z, t) = U\left(\frac{s}{c}(z + m_* x), t\right) \quad \text{in } C^{2,1}_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}).$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

Finally, we show that $\lim_{\rho\to\infty}\phi'(\rho) = -m_*$. Following from (3.20), we have that

$$\lim_{\rho\to\infty}(-\Psi_\rho(\rho,\phi(\rho),0) + m_*\Psi_z(\rho,\phi(\rho),0)) = 0$$

and

$$\lim_{\rho\to\infty}(m_*\Psi_\rho(\rho,\phi(\rho),0) + \Psi_z(\rho,\phi(\rho),0)) = \frac{s}{c}U_\eta(0,0) > 0.$$

Hence, we obtain that

$$\lim_{\rho\to\infty}\Psi_\rho(\rho,\phi(\rho),0) = \frac{cm_*}{s}U_\eta(0,0),$$
$$\lim_{\rho\to\infty}\Psi_z(\rho,\phi(\rho),0) = \frac{c}{s}U_\eta(0,0)$$

and

$$\lim_{\rho\to\infty}\phi'(\rho) = -\lim_{\rho\to\infty}\frac{\Psi_\rho(\rho,\phi(\rho),0)}{\Psi_z(\rho,\phi(\rho),0)} = -m_*.$$

This completes the proof. □

3.3. Non-existence of cylindrically symmetric travelling fronts

In this section we prove theorems 1.4 and 1.5, which imply the non-existence of cylindrically symmetric travelling fronts. Here we give only the proof of theorem 1.4. Theorem 1.5 can be proved similarly. In addition, we only consider the case in which $m = 3$.

Proof of theorem 1.4. . We prove it by way of a contradiction argument. Contrary to the statement of theorem 1.4, we assume that for $s > c > 0$ there exists a cylindrically symmetric travelling front $W(\boldsymbol{x},t)$ satisfying

$$\frac{\partial}{\partial t}W = \Delta W - s\frac{\partial}{\partial x_3}W + f(W,t) \quad \forall(\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R},$$
$$W(\boldsymbol{x},t+T) = W(\boldsymbol{x},t) \quad \forall(\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R},$$
$$\lim_{x_3\to\pm\infty}W(\mathbf{0},x_3,t) = W^{\pm}(t) \quad \text{uniformly in } t\in\mathbb{R}$$

and

$$\frac{\partial}{\partial x_3}W(\boldsymbol{x},t) \geqslant 0, \qquad \frac{\partial^2}{\partial x_i^2}W(\boldsymbol{x},t)\bigg|_{\boldsymbol{x}'=\mathbf{0}} \leqslant 0, \quad i=1,2.$$

Let $\tilde{U}(x_3,t) = W(0,0,x_3,t)$ for any $(x_3,t)\in\mathbb{R}^2$. Then we have

$$\frac{\partial}{\partial t}\tilde{U}(x_3,t) - \frac{\partial^2}{\partial x_3^2}\tilde{U}(x_3,t) + s\frac{\partial}{\partial x_3}\tilde{U}(x_3,t) - f(\tilde{U}(x_3,t),t)$$
$$= \frac{\partial^2}{\partial x_1^2}W(\boldsymbol{x},t)\bigg|_{x_1=x_2=0} + \frac{\partial^2}{\partial x_2^2}W(\boldsymbol{x},t)\bigg|_{x_1=x_2=0} \leqslant 0,$$

which implies that $u^-(x,t) = \tilde{U}(x+st,t)$ is a subsolution of the following equation:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + f(u(x,t),t), \quad x\in\mathbb{R},\ t>0. \tag{3.21}$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

On the other hand, following [43, lemma 4.2] (see also [34, 35]), we obtain that the function

$$u^+(x,t) = U(x + ct + \xi^+ + \sigma\delta(1 - \mathrm{e}^{-\beta t}), t) + \delta a(t)$$

is a supersolution of (3.21), where σ, δ, β are appropriate positive constants, $a(t)$ is defined in lemma 2.2 and $\xi^+ \in \mathbb{R}$ is an arbitrary number. In view of

$$u^+(x,0) = U(x+\xi^+,0) + \delta, \quad u^-(-\infty,0) = \alpha^- \quad \text{and} \quad u^-(+\infty,0) = \alpha^+,$$

there exists a sufficiently large $\xi^+ > 0$ such that

$$u^-(x,0) \leqslant u^+(x,0) \quad \forall x \in \mathbb{R}.$$

Applying the comparison principle (see [5, theorem 25.6]), we have

$$\tilde{U}(x+st,t) = u^-(x,t) \leqslant u^+(x,t) \quad \forall x \in \mathbb{R},\ t > 0.$$

Note that $s > c$. It follows that

$$\begin{aligned} \alpha^- < \tilde{U}(0,0) &= u^-(-skT, kT) \\ &\leqslant u^+(-skT, kT) \\ &= U((c-s)kT + \xi^+ + \rho\delta(1 - \mathrm{e}^{-\beta kT}), kT) + \delta a(kT) \\ &\to \alpha^- \quad \text{as } k \to +\infty, \end{aligned}$$

which is a contradiction. This completes the proof of theorem 1.4. □

REMARK 3.11. Due to theorems 3.1, 3.5 and 3.6, lemma 3.10 and the proof of theorem 1.4, we have proved theorems 1.1–1.5 for the case in which $m = 3$. In this remark we consider the case in which $\boldsymbol{x} \in \mathbb{R}^m$ $(m \geqslant 4)$. Let

$$\begin{aligned} &h^k_{i_1,i_2,\dots,i_{m-1}}(x_1,x_2,\dots,x_{m-1}) \\ &\quad = m_*\biggl(x_1 \cos\frac{(i_1-1)\pi}{2^{k-1}} + x_2 \sin\frac{(i_1-1)\pi}{2^{k-1}} \cos\frac{(i_2-1)\pi}{2^{k-1}} \\ &\qquad\quad + x_3 \sin\frac{(i_1-1)\pi}{2^{k-1}} \sin\frac{(i_2-1)\pi}{2^{k-1}} \cos\frac{(i_3-1)\pi}{2^{k-1}} + \cdots \\ &\qquad\quad + x_{m-2} \sin\frac{(i_1-1)\pi}{2^{k-1}} \sin\frac{(i_2-1)\pi}{2^{k-1}} \cdots \sin\frac{(i_{m-3}-1)\pi}{2^{k-1}} \cos\frac{2(i_{m-2}-1)\pi}{2^{k}} \\ &\qquad\quad + x_{m-1} \sin\frac{(i_1-1)\pi}{2^{k-1}} \sin\frac{(i_2-1)\pi}{2^{k-1}} \cdots \sin\frac{(i_{m-3}-1)\pi}{2^{k-1}} \sin\frac{2(i_{m-2}-1)\pi}{2^{k}}\biggr) \end{aligned}$$

and let

$$h^k(x_1,x_2,\dots,x_{m-1}) = \max_{\substack{1 \leqslant i_1,\dots,i_{m-3} \leqslant 2^{k-1}, \\ 1 \leqslant i_{m-2} \leqslant 2^k}} h^k_{i_1,i_2,\dots,i_{m-1}}(x_1,x_2,\dots,x_{m-1}),$$

where $k \in \mathbb{N}$. The hyperplane

$$x_m = h^k_{i_1,i_2,\dots,i_{m-1}}(x_1,x_2,\dots,x_{m-1})$$

is tangent to the rotating surface

$$x_m = m_*\sqrt{x_1^2 + x_2^2 + \cdots + x_{m-1}^2}$$

https://doi.org/10.1017/S0308210515000268 Published online by Cambridge University Press

for any $k \in \mathbb{N}$, $1 \leqslant i_1, \ldots, i_{m-3} \leqslant 2^{k-1}$ and $1 \leqslant i_{m-2} \leqslant 2^k$. Following from [37, theorem 5.2], we know that there exists a sequence of periodic pyramidal travelling fronts of (1.1), namely,

$$V^1, V^2, \ldots, V^k, \ldots,$$

where

$$V^k(\boldsymbol{x}, t) = \lim_{k\to\infty} v(\boldsymbol{x}, t + kT; v^{k,-}), \quad v^{k,-}(\boldsymbol{x}, t) = U\left(\frac{c}{s}(x_m + h^k(\boldsymbol{x}')), t\right).$$

In particular, $V^k(\boldsymbol{x}, t)$ is even on x_i ($1 \leqslant i \leqslant m-1$), increasing in $x_i \in (0, +\infty)$ ($1 \leqslant i \leqslant m-1$) and increasing in $x_m \in \mathbb{R}$. In particular, $(\partial/\partial\nu)V^k(\boldsymbol{x}, t) > 0$ for $\boldsymbol{x} \in \mathbb{R}^m$, where

$$\nu = \frac{1}{\sqrt{1 + \nu_1^2 + \nu_2^2 + \cdots + \nu_{m-1}^2}}(\nu_1, \nu_2, \ldots, \nu_{m-1}, 1)$$

satisfies $\sqrt{\nu_1^2 + \nu_2^2 + \cdots + \nu_{m-1}^2} \leqslant 1/m_*$. Using arguments similar to those in the $\mathbb{R}^3$ case, we can easily show that theorems 1.1–1.5 hold in $\boldsymbol{x} \in \mathbb{R}^m$ ($m \geqslant 4$).

For the case in which $m = 2$, combining the proof of [43] and the previous arguments for the case in which $m = 3$, we have that theorems 1.1–1.5 remain valid for two-dimensional V-shaped travelling fronts $\tilde{V}$ defined in theorem 2.1 in $\mathbb{R}^2$.

4. Cylindrically symmetric travelling fronts when $c = 0$

In this section we show the existence of time-periodic cylindrically symmetric travelling fronts of (1.1) in $\mathbb{R}^m$ ($m \geqslant 2$) when the planar wave speed $c = 0$, namely, we prove theorem 1.6. In the following we only consider the case in which $m = 3$, the proofs for $m = 2$ and $m \geqslant 4$ are similar and we omit them. The main method is to take the limit of a sequence of cylindrically symmetric travelling fronts with planar wave speeds $c_n > 0$.

Assume that assumption (H5) holds. Fix $s > 0$. Consequently, for any $\delta > 0$, due to $c_n \to 0$, we have that there exists $N \in \mathbb{N}$ such that $0 < c_n < s$ for any $n > N$. Without loss of generality, we assume that $0 < c_n < s$ for $n \in \mathbb{N}$. By theorem 3.1, we have that there exists $W_n(\boldsymbol{x}, t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R})$ satisfying $W_n(\boldsymbol{x}, t + T) = W_n(\boldsymbol{x}, t)$ and

$$\frac{\partial}{\partial t}W_n(\boldsymbol{x}, t) = \Delta W_n(\boldsymbol{x}, t) - s\frac{\partial}{\partial x_3}W_n(\boldsymbol{x}, t) + f_n(W_n(\boldsymbol{x}, t), t)$$

for any $(\boldsymbol{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$. In addition, one has:

(i) $W_n(\mathbf{0}, 0) = \theta_n := \dfrac{\alpha_n^- + \alpha_n^0}{2}$;

(ii) $W_n(\boldsymbol{x}_1', x_3, t) = W_n(\boldsymbol{x}_2', x_3, t)$ for all $\boldsymbol{x}_1', \boldsymbol{x}_2' \in \mathbb{R}^2$ with $|\boldsymbol{x}_1'| = |\boldsymbol{x}_2'|$, $x_3 \in \mathbb{R}$, $t \in \mathbb{R}$;

(iii) $\dfrac{\partial}{\partial x_3}W_n(\boldsymbol{x}, t) > 0$ for any $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$;

(iv) $\dfrac{\partial}{\partial x_i}W_n(\boldsymbol{x}, t) > 0$ for $x_i \in (0, \infty)$, $x_j \in \mathbb{R}$, $x_3 \in \mathbb{R}$ and $t \in \mathbb{R}$, $i, j = 1, 2$, $i \neq j$;

(v) we have that

$$\lim_{x_3\to\infty} \|W_n(\cdot,x_3,t)-W_n^+(t)\|_{C(\mathbb{R}^2)}=0$$

and

$$\lim_{x_3\to-\infty} \|W_n(\cdot,x_3,t)-W_n^-(t)\|_{C_{\mathrm{loc}}(\mathbb{R}^2)}=0$$

uniformly on $t\in\mathbb{R}$.

In view of assumption (H5), especially,

$$\lim_{n\to\infty} \|f_n(\cdot,\cdot)-f(\cdot,\cdot)\|_{C^1([-M,M]\times[0,T])}=0,$$

and the fact that $-M\leqslant W_n^-(t)\leqslant W_n(x,t)\leqslant W_n^+(t)\leqslant M$ for any $(\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R}$ (see (1.6) for the definition of the constant M), by an argument similar to that of Wang *et al*. [44, proposition 4.3], we have that there exists a positive constant K_1'' such that

$$\|W_n(\cdot,t)\|_{C^1(\mathbb{R}^3)}\leqslant K_1'' \quad \forall n\in\mathbb{N},\ t\in[T,2T].$$

Since we have that $W_n(\boldsymbol{x},t+T)=W_n(\boldsymbol{x},t)$ for any $(\boldsymbol{x},t)\in\mathbb{R}^3\times\mathbb{R}$, we have $\|W_n(\cdot,t)\|_{C^1(\mathbb{R}^3)}\leqslant K_1''$ for any $n\in\mathbb{N}$ and $t\in\mathbb{R}$. Consequently, applying [27, theorems 5.1.3 and 5.1.4] and using the assumption

$$\lim_{n\to\infty} \|f_n(\cdot,\cdot)-f(\cdot,\cdot)\|_{C^1([-M,M]\times[0,T])}=0,$$

we have that there exists a positive constant $K''>0$ such that

$$\|W_n(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3\times\mathbb{R})}<K'' \quad \forall n\in\mathbb{N},$$

where $\alpha\in(0,1)$ is a constant. It follows that there exists a function $W_0(\boldsymbol{x},t)\in C^{2,1}(\mathbb{R}^3\times\mathbb{R})$ (up to extraction of a subsequence if necessary) satisfying

$$\|W_0(\boldsymbol{x},t)-W_n(\boldsymbol{x},t)\|_{C^{2,1}_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R})}=0 \quad \text{as } n\to\infty.$$

Following the properties satisfied by $W_n(\boldsymbol{x},t)$, we conclude that $W_0(\boldsymbol{x},t)$ satisfies the following.

(I) For any $\boldsymbol{x}\in\mathbb{R}^3$ and $t\in\mathbb{R}$ there holds $W_0(\boldsymbol{x},t)=W_0(\boldsymbol{x},t+T)$ and

$$\frac{\partial}{\partial t}W_0(\boldsymbol{x},t)=\Delta W_0(\boldsymbol{x},t)-s\frac{\partial}{\partial x_3}W_0(\boldsymbol{x},t)+f(W_0(\boldsymbol{x},t),t).$$

(II) $W_0(\mathbf{0},0)=\theta_0:=\dfrac{\alpha^-+\alpha^0}{2}$.

(III) $W_0(\boldsymbol{x}_1',x_3,t)=W_0(\boldsymbol{x}_2',x_3,t)$ for all $\boldsymbol{x}_1',\boldsymbol{x}_2'\in\mathbb{R}^2$ with $|\boldsymbol{x}_1'|=|\boldsymbol{x}_2'|$, $x_3\in\mathbb{R}$, $t\in\mathbb{R}$.

(IV) $\dfrac{\partial}{\partial x_3}W_0(\boldsymbol{x},t)\geqslant 0$ for any $\boldsymbol{x}\in\mathbb{R}^3$ and $t\in\mathbb{R}$.

(V) $\dfrac{\partial}{\partial x_i}W_0(\boldsymbol{x},t)\geqslant 0$ for $x_i\in(0,\infty)$, $x_j\in\mathbb{R}$, $x_3\in\mathbb{R}$ and $t\in\mathbb{R}$, $i,j=1,2$, $i\neq j$.

Now using arguments similar to those in § 3, we can prove that the function $W_0(\boldsymbol{x}, t)$ satisfies the conclusions in theorem 1.6 in $\boldsymbol{x} \in \mathbb{R}^3$. This completes the proof of theorem 1.6.

5. Discussion

In this paper we have established the existence and non-existence of time-periodic cylindrically symmetric travelling fronts of (1.1) with time-periodic nonlinearity in $\boldsymbol{x} \in \mathbb{R}^m$ $(m \geqslant 2)$ for both of the cases $c > 0$ and $c = 0$; see theorems 1.1 and 1.3–1.6. For the case in which $c > 0$, we have also established the asymptotic behaviour of the level set of cylindrically symmetric travelling fronts; see theorem 1.2. Furthermore, it is worth mentioning that when $c \leqslant 0$, the existence of time-periodic cylindrically symmetric travelling fronts of (1.1) with wave speed $s < c$ in $\boldsymbol{x} \in \mathbb{R}^m$ $(m \geqslant 2)$ can also be obtained by an argument similar to that in [43, theorem 1.2].

However, the property and shape of level sets of cylindrically symmetric travelling fronts of (1.1) with time-periodic nonlinearity when $c = 0$ remain open. As pointed out in the introduction, the level set and uniqueness of cylindrically symmetric travelling fronts of autonomous scalar equations with balanced nonlinearity (namely, $c = 0$ and $f(u, t) = f(u)$ in (1.1)) have been studied by Chen *et al*. [4] and Gui [10], respectively, where the variational method was used. However, due to the influence of time heterogeneity, it seems a difficult problem to study the level set and uniqueness of time-periodic cylindrically symmetric travelling fronts of (1.1) when $c = 0$. In addition, Del Pino *et al*. [6] have found some new types of multi-dimensional travelling wave solutions for the Allen–Cahn equation with balanced nonlinearity (namely, $f(u) = u(1 - u^2)$ in (1.1)). Naturally, it is very interesting to find similar travelling wave solutions for a time heterogeneous (1.1) when $c = 0$. A simple and interesting problem is to consider the nonlinearity $f(u, t) = (1 - u^2)(2u - \rho(t))$, where $\rho(t) \in (-2, 2)$ is T-periodic and satisfies $\int_0^T \rho(t)\,\mathrm{d}t = 0$.

Besides the problems mentioned above, the stability of cylindrically symmetric travelling fronts of (1.1) is also a very important topic and remains open.

Acknowledgements

This work was partly supported by NNSF of China (grant nos 11071105 and 11371179), the Program for New Century Excellent Talents in University (grant no. NCET-10-0470) and the Scientific Research Foundation for Returned Overseas Chinese Scholars.

References

1 N. Alikakos, P. W. Bates and X. Chen. Traveling waves and locating oscillating patterns in multidimensional domains. *Trans. Am. Math. Soc.* **351** (1999), 2777–2805.

2 D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusions arising in population genetics. *Adv. Math.* **30** (1978), 33–76.

3 A. Bonnet and F. Hamel. Existence of non-planar solutions of a simple model of premixed Bunsen flames. *SIAM J. Math. Analysis* **31** (1999), 80–118.

4 X. Chen, J. S. Guo, F. Hamel, H. Ninomiya and J. M. Roquejoffre. Traveling waves with paraboloid like interfaces for balanced bistable dynamics. *Annales Inst. H. Poincaré Analyse Non Linéaire* **24** (2007), 369–393.

5 D. Daners and P. K. Medina. *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematics Series, vol. 279 (New York: Longman, 1992).

6 M. Del Pino, M. Kowalczyk and J. Wei. Traveling waves with multiple and non-convex fronts for a bistable semilinear parabolic equation. *Commun. Pure Appl. Math.* **66** (2013), 481–547.

7 M. El Smaily, F. Hamel and R. Huang. Two-dimensional curved fronts in a periodic shear flow. *Nonlin. Analysis TMA* **74** (2011), 6469–6486.

8 J. Fang and X.-Q. Zhao. Bistable traveling waves for monotone semiflows with applications. *J. Eur. Math. Soc.* (In the press.)

9 J. Földes and P. Poláčik. On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry. *Discrete Contin. Dynam. Syst.* **25** (2009), 133–157.

10 C. Gui. Symmetry of traveling wave solutions to the Allen–Cahn equation in $\mathbb{R}^2$. *Arch. Ration. Mech. Analysis* **203** (2012), 1037–1065.

11 F. Hamel. Bistable transition fronts in $\mathbb{R}^N$. Preprint, 2013. (Avalable at http://arxiv.org/abs/1302.4817.)

12 F. Hamel and R. Monneau. Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets. *Commun. PDEs* **25** (2000), 769–819.

13 F. Hamel and N. Nadirashvili. Travelling fronts and entire solutions of the Fisher–KPP equation in $\mathbb{R}^N$. *Arch. Ration. Mech. Analysis* **157** (2001), 91–163.

14 F. Hamel and N. Nadirashvili. Traveling fronts and entire solutions of the Fisher–KPP equation in $\mathbb{R}^N$. *Arch. Ration. Mech. Analysis* **157** (2001), 91–163.

15 F. Hamel and J. M. Roquejoffre. Heteroclinic connections for multidimensional bistable reaction–diffusion equations. *Discrete Contin. Dynam. Syst.* S **4** (2011), 101–123.

16 F. Hamel, R. Monneau and J. M. Roquejoffre. Stability of travelling waves in a model for conical flames in two space dimensions. *Annales Scient. Éc. Norm. Sup.* **37** (2004), 469–506.

17 F. Hamel, R. Monneau and J. M. Roquejoffre. Existence and qualitative properties of multidimensional conical bistable fronts. *Discrete Contin. Dynam. Syst.* **13** (2005), 1069–1096.

18 F. Hamel, R. Monneau and J. M. Roquejoffre. Asymptotic properties and classification of bistable fronts with Lipschitz level sets. *Discrete Contin. Dynam. Syst.* **14** (2006), 75–92.

19 M. Haragus and A. Scheel. Almost planar waves in anisotropic media. *Commun. PDEs* **31** (2006), 791–815.

20 M. Haragus and A. Scheel. Corner defects in almost planar interface propagation. *Annales Inst. H. Poincaré Analyse Non Linéaire* **23** (2006), 283–329.

21 J. Huang and W. Shen. Speeds of spread and propagation for KPP models in time almost and space periodic media. *SIAM J. Appl. Dyn. Syst.* **8** (2009), 790–821.

22 Y. Kurokawa and M. Taniguchi. Multi-dimensional pyramidal traveling fronts in the Allen–Cahn equations. *Proc. R. Soc. Edinb.* A **141** (2011), 1–24.

23 X. Liang and X.-Q. Zhao. Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. *Commun. Pure Appl. Math.* **60** (2007), 1–40.

24 X. Liang, Y. Yi and X.-Q. Zhao. Spreading speeds and traveling waves for periodic evolution systems. *J. Diff. Eqns* **231** (2006), 57–77.

25 G. M. Lieberman. *Second order parabolic differential equations* (World Scientific, 1998).

26 B. Lou and X. Chen. Traveling waves of a curvature flow in almost periodic media. *J. Diff. Eqns* **247** (2009), 2189–2208.

27 A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems* (Birkhäuser, 1995).

28 G. Nadin. Traveling fronts in space-time periodic media. *J. Math. Pures Appl.* **92** (2009), 232–262.

29 G. Nadin. Existence and uniqueness of the solution of a space-time periodic reaction–diffusion equation. *J. Diff. Eqns* **249** (2010), 1288–1304.

30 W.-M. Ni and M. Taniguchi. Traveling fronts of pyramidal shapes in competition–diffusion equations. *Networks Heterogen. Media* **8** (2013), 379–395.

31 H. Ninomiya and M. Taniguchi. Existence and global stability of traveling curved fronts in the Allen–Cahn equations. *J. Diff. Eqns* **213** (2005), 204–233.

32 H. Ninomiya and M. Taniguchi. Global stability of traveling curved fronts in the Allen–Cahn equations. *Discrete Contin. Dynam. Syst.* **15** (2006), 819–832.

33 M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations* (Englewood Cliffs, NJ: Prentice Hall, 1967).

34 W. Shen. Traveling waves in time almost periodic structures governed by bistable nonlinearities. I. Stability and uniqueness. *J. Diff. Eqns* **159** (1999), 1–54.

35 W. Shen. Traveling waves in time almost periodic structures governed by bistable nonlinearities. II. Existence. *J. Diff. Eqns* **159** (1999), 55–101.

36 W. Shen. Traveling waves in time dependent bistable media. *Diff. Integ. Eqns* **19** (2006), 241–278.

37 W.-J. Sheng, W.-T. Li and Z.-C. Wang. Periodic pyramidal traveling fronts of bistable reaction–diffusion equations with time-periodic nonlinearity. *J. Diff. Eqns* **252** (2012), 2388–2424.

38 W.-J. Sheng, W.-T. Li and Z.-C. Wang. Multidimensional stability of V-shaped traveling fronts in the Allen–Cahn equation. *Sci. China Math.* **56** (2013), 1969–1982.

39 M. Taniguchi. Traveling fronts of pyramidal shapes in the Allen–Cahn equations. *SIAM J. Math. Analysis* **39** (2007), 319–344.

40 M. Taniguchi. The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations. *J. Diff. Eqns* **246** (2009), 2103–2130.

41 M. Taniguchi. Multi-dimensional traveling fronts in bistable reaction–diffusion equations. *Discrete Contin. Dynam. Syst.* **32** (2012), 1011–1046.

42 Z.-C. Wang. Traveling curved fronts in monotone bistable systems. *Discrete Contin. Dynam. Syst.* **32** (2012), 2339–2374.

43 Z.-C. Wang and J. Wu. Periodic traveling curved fronts in reaction–diffusion equation with bistable time-periodic nonlinearity. *J. Diff. Eqns* **250** (2011), 3196–3229.

44 Z.-C. Wang, W.-T. Li and S. Ruan. Entire solutions in bistable reaction–diffusion equations with nonlocal delayed nonlinearity. *Trans. Am. Math. Soc.* **361** (2009), 2047–2084.

45 Z.-C. Wang, H.-L. Niu and S. Ruan. Axisymmetric traveling fronts in the Lotka–Volterra competition–diffusion system. *Discrete Contin. Dynam. Syst.* (In the press.)

46 Z. Wu, J. Yin and C. Wang. *Elliptic and parabolic equations* (World Scientific, 2006).

47 G. Zhao and S. Ruan. Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka–Volterra competition system with diffusion. *J. Math. Pures Appl.* **95** (2011), 627–671.