Probability in the Engineering and Informational Sciences, **17**, 2003, 137–142. Printed in the U.S.A.

ON STOCHASTIC ORDERS FOR THE LIFETIME OF A *k*-OUT-OF-*n* SYSTEM

RAMESH KORWAR
University of Massachusetts
Amherst, MA 01003
E-mail: korwar@math.umass.edu

Let $\tau_{k|n}$ denote the lifetime of a k-out-of-n system, where the n components have independent lifetimes T_i with completely arbitrary distribution $F_i, i = 1,\dots,n$. It is shown that $\tau_{k+1|n} \le_{\rm hr} \tau_{k|n}$, $\tau_{k|n} \le_{\rm hr} \tau_{k-1|n-1}$, and $\tau_{k|n-1} \le_{\rm hr} \tau_{k|n}$ if $T_i \le_{\rm hr} T_n$, $i = 1,\dots,n-1$; $\tau_{k+1|n} \le_{\rm rh} \tau_{k|n}$, $\tau_{k-1|n} \le_{\rm rh} \tau_{k|n}$, and $\tau_{k|n} \le_{\rm rh} \tau_{k-1|n-1}$ if $T_n \le_{\rm rh} T_i$, $i = 1,\dots,n-1$. These results are available in the literature for the special case of F_i's being absolutely continuous. Also, even in this case, the proofs are often tedious and use the concept of "totally positive of order infinity in differences of k." In contrast, the proofs given here are simple and elegant and do not use the above concept.

1. INTRODUCTION

Let $\tau_{k|n}$ denote the lifetime of a k-out-of-n system whose components have independent lifetimes T_i with an arbitrary distribution function (d.f.) F_i (not necessarily identical and not necessarily absolutely continuous), $i = 1,\dots,n$. Recall that a k-out-of-n system functions if and only if at least k $(1 \le k \le n)$ out of the n components function. In the literature, various stochastic orders for $\tau_{k|n}$ are considered. The known results all assume that the F_i's are absolutely continuous. In contrast, the results in Theorem 1.1 make no such assumption. Since $\tau_{k|n} = T_{n-k+1:n}$, where $T_{1:n} \le \cdots \le T_{n:n}$ are the order statistics corresponding to $T_i, i = 1,\dots,n$, the results in Theorem 1.1 have important consequences for order statistics. First, we need the following definitions of stochastic orders.

Let X and Y be two random variables (r.v.'s) with d.f.'s F and G, survival functions $\bar F = 1 - F$ and $\bar G = 1 - G$, and probability functions (p.f.'s) or probability density functions (p.d.f.'s) f and g (whenever they exist), respectively. Then, we have the following definitions.

 0269-9648/03 \$12.50

Definition 1.1: *The r.v. X is said to be stochastically smaller than Y, written* $X \le_{\text{st}} Y$, *if* $\bar{F}(t) \le \bar{G}(t)$ *for all t.*

Definition 1.2: *The r.v. X is said to be smaller than Y in the hazard rate order, written* $X \le_{\text{hr}} Y$, *if* $\bar{G}(t)/\bar{F}(t)$ *is increasing in t.*

Definition 1.3: *The r.v. X is said to be smaller than Y in the reversed hazard rate order, written* $X \le_{\text{rh}} Y$, *if* $G(t)/F(t)$ *is increasing in t.*

If F and G have p.d.f.'s f and g, respectively, then Definition 1.2 is equivalent to $g(t)/\bar{G}(t) \le f(t)/\bar{F}(t)$ for all t. If, however, X and Y are integer-valued, taking values 0,1,..., then Definition 1.2 is equivalent to $P(Y = k)/P(Y \ge k) \le P(X = k)/P(X \ge k)$, for $k = 0,1,\ldots$. Similar remarks apply to Definition 1.3.

Definition 1.4: *Let F and G have p.f.'s or p.d.f.'s f and g, respectively. The r.v. X is said to be smaller than Y in the likelihood ratio order, written* $X \le_{\ell\text{r}} Y$, *if* $g(t)/f(t)$ *is increasing in t or, equivalently, if*

$$g(t_2)f(t_1) \ge g(t_1)f(t_2) \quad \textit{for all } t_1 \le t_2.$$

We are now ready to state the main results.

Theorem 1.1: *Let* $\tau_{k|n}$ *be the lifetime of a k-out-of-n* $(1 \le k \le n)$ *system whose n components have independent lifetimes* T_i *with arbitrary d.f.'s* F_i, *not necessarily identical and not necessarily absolutely continuous,* $i = 1,\ldots,n$. *Then, we have the following:*

(a) *(i)* $\tau_{k+1|n} \le_{\text{hr}} \tau_{k|n}$, $k = 1,\ldots,n-1$;
(ii) $\tau_{k|n} \le_{\text{hr}} \tau_{k-1|n-1}$, $k = 2,\ldots,n$;
(iii) *if* $T_i \le_{\text{hr}} T_n$, $i = 1,\ldots,n-1$, *then* $\tau_{k|n-1} \le_{\text{hr}} \tau_{k|n}$, $k = 1,\ldots,n-1$.
(b) *(i)* $\tau_{k+1|n} \le_{\text{rh}} \tau_{k|n}$, $k = 1,\ldots,n-1$;
(ii) $\tau_{k|n-1} \le_{\text{rh}} \tau_{k|n}$, $k = 1,\ldots,n-1$;
(iii) *if* $T_n \le_{\text{rh}} T_i$, $i = 1,\ldots,n-1$, *then* $\tau_{k|n} \le_{\text{rh}} \tau_{k-1|n-1}$, $k = 2,\ldots,n$.

Results a(i) and a(iii) were proved by Boland, El-Newihi, and Proschan [3]. They also proved a(ii) under the restriction $T_n \le_{\text{hr}} T_i$, $i = 1,\ldots,n-1$. Results b(i) and b(iii) were derived by Block, Savits, and Singh [2]. Results a(ii) and b(ii) were proved by Hu and He [4]. Bapat and Kochar [1] and Hu, Zhu, and Wei [5] have extended the results in Results a and b to the (stronger) likelihood ratio order when the F_i's are absolutely continuous with a differentiable p.d.f. f_i and $T_1 \le_{\ell\text{r}} T_2 \le_{\ell\text{r}} \cdots \le_{\ell\text{r}} T_n$. Using methods similar to those in this article, the author can prove the same results when $F_i = F$, $i = 1,\ldots,n$, where F is completely arbitrary.

The results in Theorem 1.1 are available only for F_i's absolutely continuous. Also, the proofs in this case are often tedious and use the concept of "totally positive of order infinity in differences of k." The aim of this article is twofold: (1) to remove the absolute continuity requirement of F_i's from these results and (2) to give elementary and elegant proofs of the results without using the concept of total positivity of order infinity in differences of k.

Note that since $\tau_{k|n} = T_{n-k+1:n}$, the results in Theorem 1.1 can be translated to give results for order statistics. For example,

$$T_{i:n} \leq_{\text{hr}} T_{i+1:n}, \qquad i = 1,\ldots,n-1$$

if the T_i's are independent.

2. PROOFS

The proof of Theorem 1.1 rests on the following result. Let T_i, $i = 1,\ldots,n$, be independent r.v.'s, T_i having distribution F_i. Define for two arbitrary numbers $t_1 \leq t_2$,

$$X_i(t_j) = \begin{cases} 1 & \text{if } T_i > t_j \\ 0 & \text{if } T_i \leq t_j, \; j = 1,2, \; i = 1,\ldots,n. \end{cases} \tag{2.1}$$

Then, we have the following.

LEMMA 2.1: *For $t_1 \leq t_2$, ($X_i(t_1)$ and $X_i(t_2)$), $i = 1,\ldots,n$, are independent pairs of increasing failure rate (IFR) r.v.'s, and for $m = 1,\ldots,n$,*

$$\sum_{i=1}^{m} X_i(t_2) \leq_{\text{hr}} \sum_{i=1}^{m} X_i(t_1). \tag{2.2}$$

PROOF: For $t_1 \leq t_2$, we have $\bar{F}_i(t_2) \leq \bar{F}_i(t_1)$ and it follows from the definition of $X_i(t_1)$ and $X_i(t_2)$ that $X_i(t_2) \leq_{\text{st}} X_i(t_1)$. However, for Bernoulli r.v.'s, stochastically smaller implies hazard rate smaller. Thus, $X_i(t_2) \leq_{\text{hr}} X_i(t_1)$, $i = 1,\ldots,n$. Now, the result follows from Theorem 1.B.6 of Shaked and Shanthikumar [7]. ■

PROOF OF THEOREM 1.1: Part a(i). Since

$$\frac{P(\tau_{k|n} > t)}{P(\tau_{k+1|n} > t)} = \frac{P\left(\sum_{i=1}^{n} X_i(t) \geq k\right)}{P\left(\sum_{i=1}^{n} X_i(t) \geq k+1\right)},$$

we must show that for $t_1 \leq t_2$,

$$\frac{P\left(\sum_{i=1}^{n} X_i(t_1) \geq k\right)}{P\left(\sum_{i=1}^{n} X_i(t_1) \geq k+1\right)} \leq \frac{P\left(\sum_{i=1}^{n} X_i(t_2) \geq k\right)}{P\left(\sum_{i=1}^{n} X_i(t_2) \geq k+1\right)},$$

https://doi.org/10.1017/S0269964803171082 Published online by Cambridge University Press

or, equivalently,

$$\frac{P\left(\sum_{i=1}^{n} X_i(t_1) \ge k\right)}{P\left(\sum_{i=1}^{n} X_i(t_2) \ge k\right)} \le \frac{P\left(\sum_{i=1}^{n} X_i(t_1) \ge k+1\right)}{P\left(\sum_{i=1}^{n} X_i(t_2) \ge k+1\right)}. \tag{2.3}$$

However, (2.3) is true, since, from (2.2),

$$\frac{P\left(\sum_{i=1}^{n} X_i(t_1) \ge k\right)}{P\left(\sum_{i=1}^{n} X_i(t_2) \ge k\right)} \uparrow k.$$

Part a(ii). We must prove that

$$\begin{aligned}\frac{P(\tau_{k-1|n-1} > t)}{P(\tau_{k|n} > t)} &= \frac{P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right)}{P\left(\sum_{i=1}^{n} X_i(t) \ge k\right)} \\ &= \frac{P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right)}{\bar{F}_n(t) P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right) + F_n(t) P\left(\sum_{i=1}^{n-1} X_i(t) \ge k\right)} \\ &= \frac{1}{1 - F_n(t) P\left(\sum_{i=1}^{n-1} X_i(t) = k-1\right) \Big/ P\left(\sum_{i=1}^{n-1} X_i(t) \ge k-1\right)}\end{aligned}$$

is increasing in t. This is true since (1) $F_n(t)$ is increasing in t and (2) from (2.2) $P(\sum_{i=1}^{n-1} X_i(t) = k-1)/P(\sum_{i=1}^{n-1} X_i(t) \ge k-1)$ is increasing in t.

Part a(iii). As in part a(ii), we can show that

$$\frac{P(\tau_{k|n} > t)}{P(\tau_{k|n-1} > t)} = 1 + \frac{\bar{F}_n(t) P\left(\sum_{i=1}^{n-1} X_i(t) = k-1\right)}{P\left(\sum_{i=1}^{n-1} X_i(t) \ge k\right)}. \tag{2.4}$$

It will now be shown, by mathematical induction, that the right-hand side of (2.4) is increasing in t. For this purpose, let us recast the problem.

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Let $T_i \leq_{\text{hr}} T^*$, $i = 1,\ldots,m$, and let $\tau^*_{k|m}$ denote the lifetime of the k-out-of-$m+1$ system made up of the m components and the component with lifetime $T^* \sim F^*$.

Then, we want to show that

$$\frac{P(\tau^*_{k|m} > t)}{P(\tau_{k|m} > t)} = 1 + \bar{F}^*(t) r^{(m)}(k-1) \uparrow t \quad \text{for all } k = 1,\ldots,m \text{ and } m = 1,2,\ldots, \tag{2.5}$$

where

$$r^{(m)}(k-1) = \frac{P\left(\sum_{i=1}^{m} X_i(t) = k-1\right)}{P\left(\sum_{i=1}^{m} X_i(t) \geq k\right)}. \tag{2.6}$$

For $m = 1$ and $k = 1$,

$$\bar{F}^*(t) r^{(1)}(0) = \frac{\bar{F}^*(t) P(X_1(t) = 0)}{P(X_1(t) \geq 1)} = \left\{\frac{\bar{F}^*(t)}{\bar{F}_1(t)}\right\} F_1(t),$$

which is certainly increasing in t, since, by hypothesis, $T_1 \leq_{\text{hr}} T^*$ implies $\bar{F}^*(t)/\bar{F}_1(t) \uparrow t$ and $F_1(t) \uparrow t$. Thus, (2.6) is increasing in t and, hence, (2.5) holds for $m = 1$. Next, assume that (2.5) holds for $m = \ell$. Now, we use the identity

$$(\ell - k + 2) P\left(\sum_{i=1}^{\ell+1} X_i(t) = k-1\right) = \sum_{i=1}^{\ell+1} P\left(X_i(t) = 0, \sum_{j=1, j\neq i}^{\ell+1} X_j(t) = k-1\right), \tag{2.7}$$

which can be easily proved by mathematical deduction (also see [3, p. 190]). We have, for each $i = 1,\ldots,\ell+1$,

$$\begin{aligned}
&\frac{\bar{F}^*(t) P\left(X_i(t) = 0, \sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) = k-1\right)}{P\left(\sum_{j=1}^{\ell+1} X_j(t) \geq k\right)} \\
&\quad = \left\{\frac{F_i(t)\bar{F}^*(t) P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) = k-1\right)}{P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) \geq k\right)}\right\}\left\{\frac{1 + \bar{F}_i(t) P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) = k-1\right)}{P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) \geq k\right)}\right\}^{-1} \\
&\quad = \frac{F_i(t)}{1/r_i(k-1)\bar{F}^*(t) + \bar{F}_i(t)/\bar{F}^*(t)},
\end{aligned} \tag{2.8}$$

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where

$$r_i(k-1) = \frac{P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) = k-1\right)}{P\left(\sum_{\substack{j=1\\ j\neq i}}^{\ell+1} X_j(t) \geq k\right)}.$$

Now, the last equality in (2.8) is increasing in t because $\bar{F}_i(t)/\bar{F}^*(t)$ and $1/r_i(k-1) \times \bar{F}^*(t)$ are both decreasing in t by the hypothesis $T_i \leq_{\mathrm{hr}} T^*$ and the induction hypothesis, respectively. Statement (2.5) follows from (2.7) and from mathematical induction. ■

No proof of part b is required since Nanda and Shaked [6] have shown that the results in part b hold whenever those in part a hold.

Remark 2.1: Lemma 2.1 and the identity $\{T_{n-k+1;n} > t\} = \{\sum_{i=1}^{n} X_i(t) \geq k\}$ hold for all t and all T_i's. The stated results for the order statistics $T_{i;n}$ can be derived directly from the above results instead of deriving first the corresponding results for $\tau_{k|n}$ and then using $\tau_{k|n} = T_{n-k+1;n}$. Hence, the results for the order statistics hold for arbitrary independent T_i's and not just for lifetimes T_i's, which are nonnegative.

Acknowledgments

The author wishes to sincerely thank the Editor for pointing out Refs. 1, 5, and 6, and improving the readability of Lemma 2.1. This work was done while the author was visiting the Indian Statistical Institute, Bangalore Center, Bangalore, India. The author wishes to acknowledge the hospitality extended to him by the ISI during the visit.

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