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Stability and existence of multiple periodic solutions for a quasilinear differential equation with maxima

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We study the stability of periodic solutions of the scalar delay differential equation

$$x' = -\delta x(t) + p \max_{u \in [t-h,t]} x(u) + f(t), \qquad (*)$$

where $f(t)$ is a periodic forcing term and $\delta, p \in \mathbb{R}$. We study stability in the first approximation showing that the non-smooth equation $(*)$ can be linearized along some 'non-singular' periodic solutions. Then the corresponding variational equation together with the Krasnosel'skij index are used to prove the existence of multiple periodic solutions to $(*)$. Finally, we apply a generalization of Halanay's inequality to establish conditions for global attractivity in equations with maxima.

1. Introduction

As a new object of investigation, differential equations with maxima have appeared since the early 1960s, due to various reasons. For example, the system

$$u'(t) = -\delta u(t) + p \max_{t-h \leqslant s \leqslant t} u(s) + f(t), \qquad (1.1)$$

where δ and p are constants, has appeared in the theory of automatic control (see, for example, [20] and references therein). Equation (1.1) plays an important role in the study of stability of differential systems with delay [5, 7, 11]. Note that, by its appearance, equation (1.1) is very close to the celebrated Mackey–Glass type delay differential equations [22]. Moreover, several forms similar to (1.1) were considered earlier as examples of simple systems with rather complicated dynamics. As an illustration of the last remark, we indicate here the Hausrath equation [13]:

$$u'(t) = -\delta u(t) + \delta \max_{t-h \leqslant s \leqslant t} |u(s)|, \quad \delta > 0, \ t \geqslant 0$$

and a model describing the vision process in the compound eye [10]:

$$u'(t) = -\delta u(t) + p \max\{u(\tau(t)), c\}, \quad \delta, p \in \mathbb{R}, \ c < 0.$$

Other examples could be found in [1, 7, 12, 20]. Throughout this paper we will demonstrate the particular simplicity of (1.1), showing that its dynamics is essentially low-dimensional. Actually, the 'effective' dimension in our examples is one.

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This allows us to investigate such equations in depth. On the other hand, it is relevant to mention here the opinion of Myshkis that 'the specific character of these equations is not yet sufficiently clear'. In his survey [23] he also distinguishes the equations with maxima as differential equations with deviating argument of complex structure. We underline also the evident significance of the nonlinear non-smooth 'norm' functional $\max_{-h\leqslant s\leqslant 0} u(s)$ which enters in many important formulations of the theory, including the well-known 3/2 criterion of Myshkis–Yorke [12, 16, 17, 26] and a large variety of integral inequalities with delaying term [6].

Recently, there has been increasing interest towards equations which contain maxima. Undoubtedly, equation (1.1) is the simplest and better known among them due to some properties which make it very close to linear equations with one delaying term. A periodic boundary value problem (BVP) for (1.1) was approached recently in [20, 25] and especially in [2, 3, 24], where in particular the following existence and stability results were established.

Theorem A. *Let $p \neq \delta$. Then there is at least one T-periodic solution to equation (1.1). Moreover, if either $h \geqslant T$, or $|p| < \delta$, or $|p|h < 1/2$, $p < \delta$, then this solution is unique. Furthermore, if $|p| < \delta e^{-\delta h}$ or $e^{(\delta-p)h}|p|h < 1/2$, $p < \delta$, then the T-periodic solution is globally exponentially stable.*

Note that the existence of solutions was proved only for the periodic BVP, while the stability and uniqueness part is valid also for almost periodic (bounded) perturbations in (1.1). It would be interesting to investigate the existence of almost periodic (or even bounded) solutions to (1.1) under the condition $p \neq \delta$, when $f(t) \in C(\mathbb{R}, \mathbb{R})$ is an almost periodic (bounded) function.

In this paper we continue the study of the periodic BVP to (1.1). In particular, we will demonstrate that regardless of the similarity of (1.1) to a linear delay differential equation and the uniqueness of a T-periodic solution to (1.1) for all sufficiently small and large values of hT^{-1}, in the general case this equation can have multiple T-periodic solutions.

Theorem 1.1. *The 2π-periodic system*

$$u'(t) = -\max_{t-3\pi/2\leqslant s\leqslant t} u(s) + f(t), \tag{1.2}$$

where $f(t) = -\sin t + \max_{t-3\pi/2\leqslant \tau\leqslant t} \cos\tau$, has at least two different 8π-periodic solutions.

The 'rotation' number of completely continuous vector fields and some 'variational' equations for (1.1) are the main tools in proving theorem 1.1. In this regard, we continue the study of differentiability of solutions with respect to parameters in state-dependent delay equations approached recently by various authors [4, 14, 15]. In fact, the second term on the right-hand side of (1.1) can be written in the form $u(t - \tau(t, u))$. However, the usual condition of differentiability of $\tau(t, u)$ fails for equations with maxima. Consequently, the 'variational' system for the non-smooth equation (1.1) could be obtained only by assuming some rather strict hypothesis on the 'linearized' periodic solution. In fact, in this paper we consider solutions with only two critical points in the period and with non-degenerate maximum. We believe that it is possible to extend considerably the set of periodic solutions along

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which equation (1.1) could be linearized. Nevertheless, we will consider here only the indicated simplest case of 'non-singular' periodic solutions. The following stability theorem could be considered a by-product of techniques used to prove the non-uniqueness result.

THEOREM 1.2. *Let $T > h$ and suppose that $x^*(t)$ is a T-periodic solution of equation (1.1) with exactly two critical points $t_{\max}, t_{\min}$ in the period and set $x^*(\tau(t)) = \max_{s\in[t-h,t]} x^*(s)$. If $x^*(t)$ has a non-degenerate maximum at the corresponding point $t_{\max}$ (that is, $x^{*\prime\prime}(t_{\max}) \neq 0$), then the Poincaré map for the linear T-periodic variational equation $v'(t) = -\delta v(t) + pv(\tau(t))$ along $x^*(t)$ has a unique non-zero characteristic multiplier χ (which can be calculated explicitly), and $x^*(t)$ is an exponentially stable (unstable) solution if and only if $|\chi| < 1$ ($|\chi| > 1$).*

Apart from proving this local stability result, we use the following theorem to approach global attractivity [8, 9] in (1.1).

THEOREM 1.3. *One of the following three conditions,*

(i) $|p| < \delta$,

(ii) $|p|h < 1/2$, $p < \delta$,

(iii) $0 < -ph < 3/2$ *and* $\delta = 0$,

is sufficient for the global exponential stability and uniqueness of a T-periodic solution to (1.1).

Theorem 1.3 strengthens the stability part of theorem A. While proving it, we use a modification of the Gronwall–Bellman lemma from [2], generalizing Halanay's lemma [11]. Clearly, equation (1.2) can satisfy neither of the assumptions of theorem A or theorem 1.3. We note that condition (i) of theorem 1.3 could also be obtained in the context of Razumikhin's theory [13], so that application of Lyapunov functionals in the study of (1.1) would be a rather effective complement to our methods.

2. Notation and an example

The concept of rotation number of a completely continuous vector field is one of the most important tools to prove theorem 1.1. We will not reproduce preliminaries and basic facts of the corresponding theory, and throughout the paper while using rotation and index numbers, we will always refer to the book by Krasnosel'skij and Zabreiko [19], which is an important and complete source of all necessary facts on the subject. In particular, the notation $\gamma(F, D)$ for the rotation number of a completely continuous field F non-vanishing on the boundary ∂D of an open connected set D (that is, $F(x) \neq 0$ for all $x \in \partial D$) is adopted from [19]. By definition, the index $\operatorname{ind}(z, F)$ of a unique zero $z \in D$ with respect to F is equal to the rotation number $\gamma(F, D)$. Finally, we note that the symbol C^1 is reserved in the paper for the functional space $C^1([-h, 0], \mathbb{R})$ of continuously differentiable initial functions endowed with the norm $|\psi|_{C^1} = \max\{|\psi|_0, |\psi'|_0\}$, where $|\psi|_0 = \sup_{t\in[-h,0]} |\psi(t)|$. We use here C^1 as a phase space for (1.1) instead of the more

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usual space $C = C([-h,0],\mathbb{R})$ of continuous functions. This choice is far from arbitrary. In fact, the topology of C is not sufficiently strong to follow the changes of critical points of smooth functions. We can also reject $C^2([-h,0),\mathbb{R})$ since, in general, solutions to (1.1) are not twice differentiable [14,15]. The identity operator in C^1 will be denoted by E. The initial interval $[-h+a,a]$ we will denote by I_a or simply $I+a$, assigning the letter I for $I = I_0$.

Our study starts with one particular case of (1.1) when $f(t) \equiv 0$. Note that the case $\delta \neq p$, $f(t) \equiv$ const. can also be reduced to this 'homogeneous' form:

$$u'(t) = -\delta u(t) + p \max_{t-h\leqslant s\leqslant t} u(s). \tag{2.1}$$

When $\delta = p$ we have $u'(t) \geqslant 0$ or $u'(t) \leqslant 0$ for all $t \geqslant 0$. For example, if $\delta > 0$ then $u'(t) \geqslant 0$ and (2.1) takes the form $u'(t) = 0$ for all $t \geqslant h$. In another case $\delta < 0$ and (2.1) is equivalent to $u'(t) = \delta(u(t-h) - u(t))$, $t \geqslant h$.

Of course, if $u(t)\colon [\tau - h, +\infty) \to \mathbb{R}$ is a decreasing solution, then it satisfies the delay differential equation

$$u'(t) = -\delta u(t) + pu(t-h), \quad t \geqslant \tau. \tag{2.2}$$

THEOREM 2.1. *Let $p \neq \delta$ and $u(t)$ be a solution of (2.1) such that $u(t) \not\equiv 0$ on every interval of the form $[\rho,+\infty)$. Then there exists $\tau > h$ such that the function $u(t)$ is strictly montone for $t \geqslant \tau - h$. If $u(t)$ on $[\tau - h, +\infty)$ is increasing, then $u(t) = u(\tau)\exp(-(\delta - p)(t-\tau))$ for all $t \geqslant \tau$. If $u(t)\colon [\tau-h,+\infty) \to \mathbb{R}$ is decreasing and $ph\exp(\delta h) \neq -\exp(-1)$, then*

$$\lim_{t\to+\infty} u(t)\exp(-\lambda t) = -\operatorname{sgn}(\lambda)\alpha, \tag{2.3}$$

where α is some positive constant and λ is a real root of the characteristic equation

$$\lambda + \delta = p\mathrm{e}^{-\lambda h}. \tag{2.4}$$

In particular, when (2.4) has no real roots, the space of all solutions to (2.1) is eventually one-dimensional.

Proof. Let us consider the solution $u(s)$ to (2.1) in the interval $s \in J_h(t) = [t, t+h]$, $t \geqslant 0$. The function $M(t) = \max_{J_h(t)} u(s)\colon [0,+\infty) \to \mathbb{R}$ have to meet one of the following options:

(i) $M(t)$ is strictly decreasing. In this case $M(t) = u(t) > u(s)$, $s \in (t, t+h]$, for all $t \geqslant 0$, and therefore $u(t)$ is also strictly decreasing.

(ii) There are points $t_2 > t_1 > 0$ such that $M(t_2) > M(t_1)$. Here $\exists t^{\#}\colon M(t^{\#}) = u(t^{\#} + h)$, $u'(t^{\#} + h) > 0$. Finally, the solution

$$u(t) = M(t^{\#})\exp(-(\delta - p)(t - t^{\#} - h)), \qquad t \geqslant h$$

increases.

(iii) $M(t)$ is decreasing and there are intervals of constancy for $M(t)$.

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We will prove that, in this case (iii), the solution $u(t)$ is also decreasing or eventually trivial. On the contrary, let us suppose that there exists eventually a non-trivial and non-monotone solution $u(t)$ of (2.1). Let d be a point of its local maximum such that $u(t)$ is different from constant in some left neighbourhood of d. We claim that $\max_{s\in[d-h,d]} u(s) = u(d-h) \geqslant u(d)$. Indeed, in the opposite case, there is a point $e \in (d-h, d)$ such that $\max_{s\in[d-h,d]} u(s) = u(e)$, and therefore $u(t)$ satisfies the ordinary differential equation $u'(t) = -\delta u(t) + pu(e)$, $u'(d) = 0$ in some neighbourhood U of d. Thus $u(t)$ is analytic in U where $u(t)$ has all its derivatives equal to 0, a contradiction with the choice of d.

Next, if $\max_{s\in[d-h,d]} u(s) = u(d-h) = u(d)$, then $u(d) = 0$ and, by the uniqueness theorem, $u(s) \equiv 0$ on $[d, \infty)$.

Now it remains to assume that $\max_{s\in[d-h,d]} u(s) = u(d-h) > u(d)$. In this case we can indicate $\varepsilon \in (0, h)$ such that $u(d-\varepsilon) = u(d)$, $u'(d-\varepsilon) \leqslant 0$, and $u(d-h) = M(d-h) < M(d-h-\varepsilon)$. Evaluating (2.1) at points $d-\varepsilon$, d, we get immediately $p < 0$.

On the other hand, we can consider local minimum point $a \in (d-\varepsilon, d)$ for $u(t)$. Clearly, such a point exists and we can assume that $u(t)$ is different from constant in some left vicinity of a. Repeated application of the above arguments enables us to write that $\max_{s\in[a-h,a]} u(s) = u(a-h) > u(a)$. Now, since

$$0 = u'(a) = -\delta u(a) + pu(a-h), \qquad 0 = u'(d) = -\delta u(d) + pu(d-h),$$

and $u(a) < u(d)$, $u(d-h) < u(a-h)$, we have $\delta > 0$ and $u(d-h) > 0 > u(d)$. This means that $u(t)$ cannot have non-trivial local maximum for $t > d+h$, a contradiction.

Finally, if $ph\exp(\delta h) \neq -\exp(-1)$, then (2.4) has only simple real roots and (2.3) follows immediately from [4, theorem 4.3] applied to (2.2). The theorem is proved. □

Theorem 2.1 says that the dynamics in (2.1) is at most three-dimensional (there is an attracting subset in a d-dimensional subspace of C, $d = 1, 2, 3$, generated by real exponential solutions of (2.1) and (2.2)) and that the asymptotic behaviours of solutions to (2.1) are completely determined by the sign of $\delta - p$ and by real roots of (2.4). In particular, if (2.4) has no real non-negative solutions, then $\delta - p > 0$ and the zero solution to (2.1) should be exponentially stable.

Corollary 2.2. *Equation (2.1) is uniformly exponentially stable if and only if equation (2.4) has no real non-negative solutions.*

Proof. Necessity is evident. On the other hand, this condition implies the stability of (2.1). Indeed, in the opposite case, there is an unbounded solution of this equation, in contradiction with (2.3). Now, stability and asymptotic attractivity of the trivial solution imply its uniform asymptotic stability which, for the homogeneous equation, is equivalent to uniform exponential stability [13]. □

Remark 2.3. Recently, Ivanov *et al.* [17] have established a criterion for the uniform exponential stability in (1.1) with $\delta \geqslant 0$.

3. Proof of theorem 1.1

Let us consider the Cauchy problem $u_{-\pi/16}(\varphi,\varepsilon)(s)=\varphi(s)\in C^1(I^*)$, where $I^*=[-25\pi/16,-\pi/16]$ is the initial interval for the equation

$$u'(t)=-\max_{t-3\pi/2\leqslant s\leqslant t}u(s)+\varepsilon f(t),\qquad \varepsilon\in[0,1]. \tag{3.1}$$

Let $u_t(\varphi,\varepsilon)\colon[-\pi/16,\omega)\to C^1(I^*)$ be a non-extendable solution to this problem. The map $\psi\to P(\varphi,\varepsilon)=u_{8\pi-\pi/16}(\varphi)$ is the Poincaré map for (3.1) considered as an 8π-periodic equation. Applying standard arguments such as the Gronwall–Bellman inequality for equations with maxima [2], the Ascoli–Arzela compactness criterion, the theorem about extendability of solutions to functional differential equations [13] and the sublinear character of nonlinearity in (1.1), it is easily seen that $\omega=+\infty$ and that $P(\varphi,\varepsilon)$ is a continuous compact map on $C^1(I^*)$.

LEMMA 3.1. *Set* $B_\rho=\{\psi\in C^1(I^*)\colon|\psi|_{C^1}\leqslant\rho\}$. *Then*

$$P(B_\rho,0)\subset\{z\exp(-t)\mid-\rho\exp(-3\pi/2)\leqslant z\leqslant 0\}\subset B_\rho\quad\textit{for all }\rho>0.$$

In particular, $\operatorname{ind}(E-P(\cdot,0),0)=1$.

Proof. We can use here any interval I of length $h=3\pi/2$ instead of I^*. To simplify calculations, we set $I=[-3\pi/2,0]$. We will prove the lemma taking into account all initial data from B_ρ. First consider $\psi\in B_\rho$ such that $\psi(0)\leqslant\max_I\psi\leqslant 0$. Clearly, $x(t,\psi)$ is a non-decreasing function while $x(t,\psi)\leqslant 0$. Let us suppose that $\lambda\geqslant 0$ is the smallest value where $x(\lambda,\psi)=0$. Then $\max_{I+\lambda}x(u,\psi)=0$, $x'(\lambda,\psi)=0$ and therefore $x(t)\equiv 0$ for all $t\geqslant\lambda$. Note that $\lambda\leqslant h$ since otherwise $\max x(u,\psi)=x(h,\psi)<0$ and $x(t)=x(h)\exp(-t+h)$ for all $t\geqslant h$. Finally, $x(t,\psi)=-a\exp(-(t-h))$, where $a\in[0,\rho]$, $t\geqslant h$.

Now we consider the case $\psi(0)\leqslant\max_{u\in I}\psi(u)>0$. If $\psi(0)>0$ then $\max_{I_t}x(u)\geqslant\psi(0)>0$ for all $t\in[0,h]$ and therefore $x(t)-\psi(0)\leqslant-\psi(0)t$. Thus $x(1)\leqslant 0$. Denote by τ, where $1\geqslant\tau>0$, the smallest real number τ such that $x(\tau)=0$. Then $x(t)$ is strictly decreasing on $[0,\tau+h]$, strictly increasing on $[\tau+h,\tau+2h]$, and $x(\tau+2h)\geqslant x(\tau+h)\geqslant-\rho h$. Now, if $x(\tau+2h)\geqslant 0$, then $x(t)\equiv 0$ for all $t\geqslant 2h+\tau$. If $x(\tau+2h)<0$, then $x(\tau+2h)=\max_{[\tau+h,\tau+2h]}x(u)$. Therefore

$$x(t)\equiv x(\tau+2h)\exp(\tau+2h)\exp(-t)\geqslant-\rho h\exp(\tau+2h)\exp(-t),\quad t\geqslant 2h+\tau.$$

Therefore we find that

$$P(\psi,0)(s)=z\exp(2h+\tau)\exp(-8\pi)\exp(-s),$$

where $s\in I$, $z\in[-\rho h,0]$, thus $z\exp(2h+\tau)\exp(-8\pi)\in[-\rho\exp(-3\pi/2),0]$.

Let now $\psi(t)\leqslant 0$ for all $t\in[-r,0]$ and let positive $r\leqslant h$ be the largest number with such property. Then $x(t)$ is strictly decreasing on $[0,h-r]$, increasing on $[h-r,2h-r]$ and therefore

$$x(2h-r)\geqslant x(h-r)\geqslant-\rho(h+1-r)\geqslant-\rho(h+1).$$

Analogously, we obtain that

$$P(\psi,0)(s)=z\exp(-(t-2h-r))=z\exp(2h+r)\exp(-8\pi)\exp(-s),$$

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where $s \in I$, $z \in [-\rho(h+1), 0]$; it follows that

$$z \exp(2h + r) \exp(-8\pi) \in [-\rho \exp(-3\pi/2), 0].$$

Lemma 3.1 is proved. □

LEMMA 3.2. *There exists a ball* $B_k = \{\varphi \in C^1(I^*) \colon |\varphi|_{C^1} \leqslant k\}$ *containing all fixed points of the operator* $P_\varepsilon = P(\cdot, \varepsilon)$, $\varepsilon \in [0, 1]$ *and such that the rotation number* $\gamma(E - P_\varepsilon, B_{k+1}) = 1$.

Proof. Let $p(t, \varepsilon)$ be an 8π-periodic solution to (3.1) and $\tau(\varepsilon)$ be a point where $p(t, \varepsilon)$ attains its absolute maximum. Then $p'(\tau(\varepsilon), \varepsilon) = 0$, and

$$M(\varepsilon) = \max_{u \in \mathbb{R}} p(u, \varepsilon) = p(\tau(\varepsilon), \varepsilon) = \varepsilon f(\tau(\varepsilon)).$$

Since $p(t, \varepsilon)$, $t \geqslant \tau(\varepsilon)$ is also a solution of the initial value problem $p(s, \varepsilon) = M(\varepsilon)$, where $s \in [\tau(\varepsilon) - 3\pi/2, \tau(\varepsilon)]$ for equation (3.1), it satisfies the integral equation

$$p(t, \varepsilon) = M(\varepsilon) - \int_{\tau(\varepsilon)}^{t} \max_{u \in [s-h, s]} p(u, \varepsilon) \, \mathrm{d}s + \varepsilon \int_{\tau(\varepsilon)}^{t} f(s) \, \mathrm{d}s, \quad t \geqslant \tau(\varepsilon).$$

Hence $|p(t, \varepsilon)| \leqslant |f|_0 = \max_{t \in \mathbb{R}} |f(t)|$ on $[\tau(\varepsilon) - 3\pi/2, \tau(\varepsilon)]$ and

$$p(t, \varepsilon) \leqslant |f|_0 + \int_{\tau(\varepsilon)}^{t} \max_{u \in [s-h, s]} p(u, \varepsilon) \, \mathrm{d}s + 8\pi |f|_0, \quad t \in [\tau(\varepsilon), \tau(\varepsilon) + 8\pi].$$

Now a slight modification of the Gronwall–Bellman lemma implies that, for all $\varepsilon \in [0, 1]$,

$$|p(t, \varepsilon)| \leqslant |f|_0 (1 + 8\pi) \exp(8\pi), \quad \forall t \in [\tau(\varepsilon), \tau(\varepsilon) + 8\pi],$$

and consequently

$$|p(t, \varepsilon)|_0 \leqslant |f|_0 (1 + 8\pi) \exp(8\pi), \quad |p'(t, \varepsilon)|_0 \leqslant k,$$

where

$$k = |f|_0 [1 + (1 + 8\pi) \exp(8\pi)].$$

Since the compact continuous maps $P(\varphi, \varepsilon) \colon C^1(I^*) \times [0, 1] \to C^1(I^*)$ have no fixed points on the boundary of B_{k+1} for all $\varepsilon \in [0, 1]$ and since the set $P(B_{k+1}, [0, 1])$ is compact, the fields $E - P(\varphi, \varepsilon)$ must have the same rotation number on the boundary of B_{k+1} for all $\varepsilon \in [0, 1]$ [19, theorem 20.1]. Now an application of lemma 3.1 ends the proof of lemma 3.2. □

It is obvious that the right-hand side of (1.2) was chosen in such a way as to have $\theta(t) = \cos t$ as the periodic solution of (1.2) or $\theta \in C^1$ as fixed point to $P(\cdot, 1)$. Below we will prove that θ is an isolated fixed point for $P(\cdot, 1)$ and consequently the index $\operatorname{ind}(E - P(\cdot, 1), \theta)$ is well defined. We will need the exact value of it; the corresponding work will be done in the same assertion. Let us first outline its proof.

We begin with the observation that, for some neighbourhood B_δ of $\theta \in C^1$, the image $P(B_\delta(\theta), 1)$ is a curve in C^1. Therefore, we can reduce the index problem from the infinite dimensional to the scalar level. Some technical work is needed to find the value the derivative of the scalar return map associated with $P(\cdot, 1)$, in the fixed point corresponding to θ. Computing this derivative, we get the above variational equation along $\theta(t) = \cos t$. The detailed proof follows.

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Lemma 3.3. *$\theta(t) = \cos t \in C^1(I^*)$ is an isolated fixed point of $P_1 = P(\cdot,1)$ and $\operatorname{ind}(E - P(\cdot,1),\theta) = -1$.*

Proof. Notice that by the theorem about continuous dependence of solutions of delay differential equations on initial conditions, we can choose $\delta > 0$ so small that the inequality $|\gamma - \theta|_{C^1} \leqslant \delta$, $\gamma \in C^1(I^*)$ implies the existence of $\tau_0 < 8\pi - \pi/16$ such that

$$\max_{s\in I^*} \gamma(s) = \gamma(-\pi/16) \quad \text{and} \quad \max_{s\in I^*} u_\sigma(\gamma,1)(s) = u_\sigma(\gamma,1)(-\pi/16)$$

for all $\sigma \in \Sigma = [\tau_0, 8\pi - \pi/16]$. These relations imply that, in fact, the values of $P(\gamma,1)$ depend only on values of $\gamma(s)$ evaluated at the point $s = -\pi/16$ and that the function $P(\gamma,1)(t - 8\pi)$ coincides with a solution of $x(t,\gamma)$ of the initial value problem

$$u'(t) = -u(t) + f(t), \quad u(8\pi - \pi/16) = P(\gamma,1)(-\pi/16), \tag{3.2}$$

on the interval $t \in \Sigma$ for all $\gamma \in B_\delta(\theta) = \{\gamma \in C^1(I^*)\colon |\gamma - \theta|_{C^1} \leqslant \delta\}$. In fact, $P(B_\delta(\theta),1)$ is a curve in the space $C^1(I^*)$ and $P(\gamma,1)(-\pi/16)$ is determined completely by the value of $z = \gamma(-\pi/16)$. This suggests considering the scalar map $z \to \Lambda(z) = u(8\pi - \pi/16, z)$ defined in some neighbourhood of the point $\mu = \cos(\pi/16)$, where $u(t,z)\colon [-\pi/16, +\infty) \to \mathbb{R}$ is an ordinary solution for the initial value problem $u(s,z) = z$, $s \in I^*$ to (1.2).

We claim that Λ is differentiable in μ and that $D = \Lambda'(\mu) > 1$. This immediately implies that $u = \cos t$ is an unstable isolated solution for (1.2) and that $\operatorname{ind}(E - P_1, \theta)$ is well defined. Moreover, we will show later that

$$\operatorname{ind}(I - P_1, \theta) = \operatorname{ind}(1 - \Lambda, \mu) = -1. \tag{3.3}$$

Indeed, notice that $\sigma^{-1}(u(-\pi/16, \mu + \sigma) - u(-\pi/16, \mu)) = 1$, and therefore

$$\frac{u(t,\mu+\sigma) - \cos t}{\sigma} = 1 - \int_{-\pi/16}^{t} \frac{\max_{x\in I_s} u(x,\mu+\sigma) - \max_{x\in I_s} \cos x}{\sigma}\,\mathrm{d}s, \tag{3.4}$$

where $\sigma \neq 0$ is sufficiently small.

If $t \in [-\pi/16, 0]$, then for every $s \in [-\pi/16, t)$, we have that

$$\max_{x\in I_s} u(x,\mu+\sigma) = u(s,\mu+\sigma)$$

for sufficiently small σ. Therefore, equation (3.4) implies the existence of $w(t) = u'_z(t,\mu)$ satisfying the initial value problem

$$w'(t) = -w(t), \quad w(-\pi/16) = 1, \quad t \in [-\pi/16, 0].$$

In particular, $u'_z(0,\mu) = w(0) = \exp(-\pi/16)$.

Now, $u(t,\mu+\sigma)$ is a convex function on $U = [-\pi/16, +\pi/16]$ since $f(t)$ decreases strictly on U, $\max_{s\in I_t} u(s,\sigma+\mu)$ is non-decreasing in $t \in U$, and

$$u'(t_2,\sigma+\mu) - u'(t_1,\sigma+\mu) = f(t_2) - f(t_1) + \max_{s\in I_{t_1}} u(s,\sigma+\mu) - \max_{s\in I_{t_2}} u(s,\sigma+\mu) < 0$$

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for all $t_2 > t_1$. The convexity of $u(t, \sigma + \mu)$ on the interval U implies, for small σ, the existence of exactly one maximum of this function attained at the point $M(\sigma) \in U$, with $M(0) = 0$.

Moreover, $u(M(\sigma), \mu + \sigma) = f(M(\sigma))$ and since for all $t \in [-\pi/16, M(\sigma)]$

$$u(t, \sigma + \mu) = \exp[-(t + \pi/16)](\sigma + \mu) + \int_{-\pi/16}^{t} \exp[-(t - s)] f(s) \, ds,$$

we get the following equation to determine $M = M(\sigma)$:

$$f(M) \exp(M) - \int_{-\pi/16}^{M} \exp(s) f(s) \, ds = \exp(-\pi/16)(\sigma + \mu).$$

Denoting the left-hand side of the last equation by $R(M)$, we obtain $R'(0) = f'(0) = \cos''(0) = -1 \neq 0$ and therefore $M(\sigma)$ is correctly defined in some neighbourhood of 0, $M(\sigma) \to 0$ as $\sigma \to 0$, and $M'(0) = -\exp(-\pi/16)$.

Consider $t \in [0, 3\pi/2]$. For every $s \in (0, t)$ and sufficiently small σ we have $\max_{x \in I_s} u(x, \mu + \sigma) = u(M(\sigma), \mu + \sigma)$ and therefore $\sigma \to 0$,

$$\frac{\max_{x \in I_s} u(x, \mu + \sigma) - \max_{x \in I_s} \cos x}{\sigma} = \frac{f(M(\sigma)) - 1}{\sigma} \to f'(0) M'(0) = w(0).$$

Hence for $t \in [0, 3\pi/2]$ there exists a derivative $w(t) = u'_z(t, \mu)$, which is a solution of the initial value problem

$$w'(t) = -w(0), \quad w(0) = u'_z(0, \mu), \quad t \in [0, 3\pi/2].$$

Now, the integral equation (3.4) for $u(t, \mu + \sigma)$ on the interval $[3\pi/2, 7\pi/4]$ has the form

$$\frac{u(t, \mu + \sigma) - \cos t}{\sigma} = \frac{u(3\pi/2, \mu + \sigma) - \cos(3\pi/2)}{\sigma} + \int_{3\pi/2}^{t} \frac{\max_{x \in I_s} u(x, \mu + \sigma) - \max_{x \in I_s} \cos x}{\sigma} \, ds.$$

Since

$$\lim_{\sigma \to 0} \frac{u(3\pi/2, \mu + \sigma) - \cos(3\pi/2)}{\sigma} = u'_z(3\pi/2, \mu)$$

and

$$\max_{x \in I_s} u(x, \mu + \sigma) - \max_{x \in I_s} \cos x = u(s - 3\pi/2, \mu + \sigma) - \cos(s - 3\pi/2)$$

for sufficiently small values of σ and $s \in (3\pi/2, 7\pi/4]$, we deduce the existence of the derivative $w(t) = u'_z(t, \mu)$ on the interval $[3\pi/2, 7\pi/4]$ satisfying the initial value problem

$$w'(t) = -w(t - 3\pi/2), \quad w(3\pi/2) = u'_z(3\pi/2, \mu), \quad t \in [3\pi/2, 7\pi/4].$$

Finally, proceeding analogously, we establish the existence of $w(t) = u'_z(t, \mu)$ satisfying the initial value problem

$$w'(t) = -w(t), \quad w(7\pi/2) = u'_z(7\pi/4), \quad t \in [7\pi/4, 31\pi/16].$$

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Calculating $w(t)$, we find that $u'_z(31\pi/16, \mu) = (1 - 7\pi/4 + \pi^2/32)\exp(-\pi/4) = -1.91\ldots$. By the periodicity of (1.2), $\Lambda'(\mu) = (u'_z(31\pi/16, \mu))^4 > 1$ and therefore $\operatorname{ind}(1 - \Lambda, \mu) = -1$.

To conclude the proof of (3.3), we define a map $Q\colon C^1(I^*) \to C^1(I^*)$ as $Q(\gamma)(s) = x(s + 8\pi, \gamma)$, $s \in I^*$, where $x(t, \gamma)$ is a solution of Cauchy problem (3.2). Q is a well-defined affine map on $C^1(I^*)$ and $(Q\gamma)(s) = P(\gamma, 1)(s)$ if $s \in \Sigma - 8\pi$. This implies that the homotopy $H(s) = sP_1 + (1 - s)Q$, $s \in [0, 1]$ carries Q into P in such a way that the map $H(s)$ has only one fixed point θ in $B_\delta(\theta)$. Thus $\operatorname{ind}(E - P_1, \theta) = \operatorname{ind}(E - Q, \theta)$; see, for example, [19, theorem 20.1]. By definition of rotation number for a one-dimensional map Q defined on a subset B_δ of the infinite-dimensional space $C^1(I^*)$, this number coincides with the index for the restriction $Q\colon Q(B_\delta) \to Q(B_\delta)$ at the point θ [19, § 20]. Identifying $Q(B_\delta)$ with $Q(B_\delta)(8\pi - \pi/16)$, we reduce Q to the map $\gamma(-\pi/16) \to Q(\gamma)(-\pi/16)$, or, in other words, to the map

$$\gamma(-\pi/16) \to P(\gamma, 1)(-\pi/16) = \Lambda(\gamma(-\pi/16)).$$

Equation (3.3) and consequently lemma 3.3 are proved. □

Proof of theorem 1.1. Let us suppose that $u = \cos t$ is a unique 8π-periodic solution to equation (1.2). Then $\theta \in C^1(I^*)$ is a unique fixed point of the Poincaré map $P_1\colon C^1(I^*) \to C^1(I^*)$ and therefore, taking into account relations between the rotation number $\gamma(E - P_1, B_{k+1})$ and indices of fixed points for P_1, we get that

$$\gamma(E - P(\cdot, 1), B_{k+1}) = \operatorname{ind}(E - P(\cdot, 1), \theta).$$

However, lemmas 3.2 and 3.3 make this equality impossible. This contradiction proves the existence of another fixed point for P_1. □

4. Proof of theorem 1.2

Without loss of generality we can assume that $t_{\max} = 0$, $T > t_{\mathrm{mim}} = \kappa > 0$. Thus necessarily there exists a point $\nu \in (\kappa, T)$ such that $x^*(\nu) = x^*(\nu - h)$. Now the situation considered in theorem 1.2 is completely analogous to the situation in lemma 3.3 and it is sufficient to repeat all stages of the second part of the proof of lemma 3.3 to get a proof of theorem 1.2. In particular, the variational equation can be deduced by considering an integral equation for the solution $u(t, \sigma + \mu)$ to the initial value problem

$$u(s, \sigma + \mu) = \sigma + \mu, \qquad s \in [a - h, a],$$

for equation (1.1), where $a \in (0, T - \nu)$ is fixed, $\sigma \in \mathbb{R}$ and $\mu = x^*(a)$. For example, let us calculate $u'(t, \sigma + \mu)$ for t close to 0. Using continuous dependence of $u(t, \sigma + \mu)$ on σ we can find intervals $\Sigma_* = [-\sigma_*, \sigma_*]$, $T_1 = [-t_*, t_*]$ and $T_2 = [-t_* + \kappa, \kappa + t_*]$ such that $u(t, \sigma + \mu)$ is strictly monotone on $[-a, T - a] \backslash (T_1 \cup T_2)$ for all $\sigma \in \Sigma_*$ and furthermore

$$(|p| + |\delta|)|u'(t, \sigma + \mu)| < \min_{s \in T_1} |f'(s)|, \quad \text{for all } \sigma \in \Sigma_*,\ t \in T_1.$$

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This is possible since $f'(0) = x^{*\prime\prime}(0) < 0$ and $\max_{s\in T_1} |u'(s, \sigma + \mu)| \to 0$, as $t_* \to 0$, $\sigma \to 0$. Now, for $t_2 > t_1$, $t_1, t_2 \in T_1$ we obtain

$$\begin{aligned} u'(t_2, \sigma + \mu) - u'(t_1, \sigma + \mu) \\ &= f(t_2) - f(t_1) + p[\max_{s\in I_{t_2}} u(s, \sigma + \mu) - \max_{s\in I_{t_1}} u(s, \sigma + \mu)] \\ &\quad + \delta(u(t_1, \sigma + \mu) - u(t_2, \sigma + \mu)) \\ &\leqslant (f'(\theta_1) - \delta u'(\theta_1, \sigma + \mu))(t_2 - t_1) + |p| u'(\theta_2, \sigma + \mu)(t_2 - t_1) < 0, \end{aligned}$$

where $\theta_1, \theta_2 \in T_1$. This implies the convexity of $u(t, \sigma + \mu)$ on T_1 with similar estimates for $M'(0)$ and $M(\sigma)$.

Finally, the variational equation along $x^*(t)$ takes the form

$$v'(t) = \begin{cases} -\delta v(t) + pv(0), & 0 \leqslant t < h; \\ -\delta v(t) + pv(t - h), & h \leqslant t < \nu; \\ -(\delta - p)v(t), & \nu \leqslant t < T. \end{cases}$$

The explicit form of this equation allows us to calculate the characteristic multiplier $\Lambda'(\mu)$; if $\delta = 0, \nu \leqslant 2h$ we find that $\chi = (1 + p\nu + p^2(\nu - h)^2/2) \exp(p(T - \nu))$.

5. Halanay's inequalities

Theorem 1.1 answers negatively the uniqueness conjecture from [2] and justifies study of the conditions which are sufficient for the existence of a unique periodic solution to (1.1). The aim of the last two sections of this paper is to investigate mutual attractivity of solutions in (1.1), which of course implies the uniqueness of periodic solution.

Estimating the distance between two solutions of (1.1), we get inequalities with maxima like the following:

$$\left.\begin{aligned} v(t) &\leqslant \alpha \exp(-\nu t) + \beta \int_0^t \exp(-\nu(t - \sigma)) \max_{u\in[\sigma-h,\sigma]} w(u)\, \mathrm{d}\sigma, \\ w(t) &\leqslant \mu v(t) + \kappa \max_{u\in[t-h,t]} w(u), \end{aligned}\right\} \tag{5.1}$$

where α, β, κ are non-negative constants and $\nu, \mu > 0$. While investigating conditions of the mutual exponential convergence of solutions, it is natural to ask about the existence of solutions to (5.1) in the form of decreasing exponential functions. This leads immediately to the characteristic equation

$$\exp(\gamma h) = \frac{\nu - \gamma}{\kappa(\nu - \gamma) + \mu\beta} \tag{5.2}$$

for the exponent γ. Note that if

$$\frac{\beta\mu}{\nu} + \kappa < 1, \tag{5.3}$$

then (5.2) has a unique positive root $\gamma \in (0, \nu)$.

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Lemma 5.1. *Suppose that m is a positive integer and that the piecewise continuous functions $v, w\colon [-h, mh] \to \mathbb{R}_+$ have only discontinuities of the first kind in the set $D_h = \{jh\colon 0 \leqslant j \leqslant m\}$. Let these functions satisfy, for all $t \in [0, mh]$, the system (5.1) and, in addition, $v(t) \leqslant \alpha$, $w(t) \leqslant \rho$ for all $t \in [-h, 0]$. If (5.3) holds, then for every*

$$\alpha' > \alpha,\ \rho' > \rho \quad \textit{such that} \quad \frac{\alpha'}{\rho'} = \beta e^{\gamma h} \tag{5.4}$$

we have that $v(t) \leqslant \alpha' \exp(-\gamma t)$, $w(t) \leqslant \rho' \exp(-\gamma t)$ for all $t \in [0, mh]$.

Proof. Without loss of generality we can assume that relations (5.4) are met. Let X be the Banach space of all piecewise continuous functions $f(t)\colon [-h, mh] \to \mathbb{R}$ with discontinuities of the first kind in D_h endowed with sup norm. Notice that $z(t) = \max_{u\in[t-h,t]} f(u)\colon [0, mh] \to \mathbb{R}$ is also a piecewise continuous function with discontinuities in D_h. Clearly, $(v(t), w(t)) \in X^2$, which is a Banach space with the norm $\|(f, g)\| = \sqrt{|f|_0^2 + |g|_0^2}$, where $|f|_0 = \sup_{t\in[-h,mh]} |f(t)|$. Define the operator $A\colon X^2 \to X^2$ as

$$A(f,g)(t) = \begin{cases} (\alpha \exp(-\gamma t),\ \rho \exp(-\gamma t)), & \text{if } t \in [-h, 0]; \\ \Big(\alpha \exp(-\nu t) + \beta \displaystyle\int_0^t \exp(-\nu(t-\sigma)) \max_{u\in[\sigma-h,\sigma]} g(u)\, d\sigma, & \\ \qquad\qquad \mu f(t) + \kappa \displaystyle\max_{u\in[t-h,t]} g(u)\Big) & \text{if } t \in [0, mh]. \end{cases} \tag{5.5}$$

Then the inequality (5.1) is equivalent to $\omega \leqslant A\omega$, where $\omega = (v, w)$. It is clear that the operator A is monotone with respect to the cone K consisting of all non-negative functions from X^2 and $A\colon K \to K$. Moreover, if inequality (5.3) is satisfied, then A is a contraction after an appropriate renormalization of X^2. Indeed, let $|z|$, $z \in \mathbb{R}^2$, be the Euclidean norm in $\mathbb{R}^2$. This norm is monotone one; that is, for $0 \preceq x \preceq y$ we get $|x| \leqslant |y|$, if the sign $\preceq$ is the comparison relative to coordinates.

Since (5.3) implies that the spectral radius $r(L) = (\kappa + (\kappa^2 + 4\beta\mu/\nu)^{1/2})/2$ of the matrix

$$L = \begin{pmatrix} 0 & \beta/\nu \\ \mu & \kappa \end{pmatrix}$$

is less than 1: $r(L) < 1$, we can prescribe an $\varepsilon > 0$ and a norm $|\cdot|_*$ in $\mathbb{R}^2$, such that $|L|_* \leqslant r(L) + \varepsilon < 1$ [18, lemma 2.2]. Moreover, using the fact that the cone $\mathcal{K} = \{(x, y)\colon x \geqslant 0, y \geqslant 0\}$ generated by the order relation $\preceq$ is invariant under the action of the operator L, we can assume that the norm $|x|_*$ is also monotone one [18, lemma 2.3].

Assume $\zeta = (f, g)$, $\bar{\zeta} = (|f|_0, |g|_0) \in \mathcal{K}$. Then $\|\zeta\|_* = \|\bar{\zeta}\|_*$ is an equivalent norm in the space X^2 because of the equivalence of the norms $|\cdot|$ and $|\cdot|_*$. Now, from (5.5) we deduce that $|(A\zeta_1)(t) - (A\zeta_2)(t)| = 0$ if $t \in [-h, 0]$ and that for $\zeta_i = (f_i, g_i) \in X^2$,

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$i = 1, 2$; $t \geqslant 0$ we have

$$|(A\zeta_1)(t) - (A\zeta_2)(t)| \leqslant \begin{pmatrix} \beta \displaystyle\int_0^t \mathrm{e}^{-\nu(t-s)} \max_{\nu \in [s-h,s]} |g_1(u) - g_2(u)| \, \mathrm{d}s \\ \mu |f_1(t) - f_2(t)| + \kappa \displaystyle\max_{u \in [t-h,t]} |g_1(u) - g_2(u)| \, \mathrm{d}s \end{pmatrix} \leqslant L \begin{pmatrix} |f_1(t) - f_2(t)|_0 \\ |g_1(t) - g_2(t)|_0 \end{pmatrix},$$

or briefly $a \leqslant Lb$. By the monotonicity of the norm $|\cdot|_*$,

$$\|A\zeta_1 - A\zeta_2\|_* = |a|_* \leqslant |L|_* |b|_* \leqslant (r + \varepsilon) \|\zeta_1 - \zeta_2\|_*,$$

so that the monotone operator A is contractive in X^2. Therefore

$$\omega \leqslant A\omega \leqslant A^2\omega \leqslant \cdots \leqslant A^n\omega \leqslant \cdots \leqslant A^{n+k}\omega, \quad k \geqslant 0,$$
$$\lim_{k \to +\infty} A^{n+k}\omega = \omega^* \quad \text{and} \quad A\omega^* = \omega^*.$$

This implies that $\omega(t) \leqslant \omega^*(t)$, where $\omega^*(t) = (v^*, w^*)$ is a unique solution to the system $A\omega^* = \omega^*$:

$$\left.\begin{aligned} v^*(t) &= \alpha \exp(-\nu t) + \beta \int_0^t \exp(-\nu(t - \sigma)) \max_{u \in [\sigma-h,\sigma]} w^*(u) \, \mathrm{d}\sigma, \\ w^*(t) &= \mu v^*(t) + \kappa \max_{u \in [t-h,t]} w^*(u), \quad \text{if } t \in [0, mh], \end{aligned}\right\} \tag{5.6}$$

where $v^*(t) = \alpha \exp(-\gamma t), w^*(t) = \rho \exp(-\gamma t)$, if $t \in [-h, 0]$.

Finally, it is easily seen that the pair $v(t) = \alpha \exp(-\gamma t)$, $w(t) = \rho \exp(-\gamma t)$ is a solution to (5.6) if $\alpha = \rho\beta \exp(\gamma h)$ and $\gamma > 0$ satisfies (5.2). Lemma 5.1 is proved. □

Remark 5.2. Note that the proof of lemma 5.1 develops some ideas and constructions from [2]. In the particular case when $\mu = 1$, $\kappa = 0$ and $w(t) \equiv v(t)$, lemma 5.1 coincides with the following lemma.

Lemma 5.3 (Halanay's lemma). *Assume that the continuous function*

$$y(t) \colon [-h, b] \to \mathbb{R}_+$$

satisfies the inequalities

$$y(t) \leqslant \alpha \exp(-\delta t) + \beta \int_0^t \exp(-\delta(t - s)) \max_{u \in [s-h;s]} y(u) \, \mathrm{d}s, \qquad t \in [0, b],$$
$$y(t) \leqslant \alpha, \qquad t \in [-h, 0],$$

where $\alpha, \beta, \delta > 0$ and $\beta < \delta$. Then $y(t) \leqslant \alpha \exp(-\gamma t)$ for all $t \in [-h, b]$, where γ is the unique positive root of the equation $\delta = \gamma + \beta \exp(\gamma h)$.

We have called lemma 5.3 Halanay's lemma since it is in fact equivalent to a result of Halanay [11] concerning the differential inequality

$$u'(t) \leqslant -\delta u(t) + p \max_{t-h \leqslant s \leqslant t} u(s). \tag{5.7}$$

The inequality (5.7) is used frequently in the stability theory for delay differential equations [5,7,11,21] (references [7,21] also consider the multidimensional analogue of (5.7)).

6. Proof of theorem 1.3

(i) Theorem A guarantees the existence of a unique periodic solution $y(t)$ to (1.1). If $z(t)$, $t \geqslant -h$ is another solution to this equation, then the difference $w(t) = y(t) - z(t)$ satisfies the equation

$$w'(t) = -\delta w(t) + p(\max_{u\in[s-h;s]} y(u) - \max_{u\in[t-h;t]} z(u)), \quad t \geqslant 0.$$

Integrating it, we obtain that

$$w(t) = \mathrm{e}^{-\delta t} w(0) + p\int_0^t \mathrm{e}^{-\delta(t-s)}(\max_{u\in[s-h;s]} y(u) - \max_{u\in[s-h;s]} z(u))\,\mathrm{d}s.$$

Furthermore,

$$|w(t)| \leqslant \max_{s\in[-h,0]} |w(s)|\mathrm{e}^{-\delta t} + |p|\int_0^t \mathrm{e}^{-\delta(t-s)} \max_{u\in[s-h;s]} |w(u)|\,\mathrm{d}s.$$

Applying lemma 5.3, we find that for $t \geqslant 0$

$$|w(t)| \leqslant \exp(-\gamma t) \max_{u\in[-h,0]} |w(u)|, \tag{6.1}$$

where γ is a positive root of the characteristic equation given in lemma 5.3. Part (i) of theorem 1.3 is now completely proved.

Remark 6.1. Part (i) of theorem 1.3, without the estimation (6.1), can also be proved by Razumikhin's methods [13, pp. 152–154], [12, theorem 3.1].

(ii) Let $x^*(t)$ be the unique periodic solution to (1.1) and let $y(t)$ be another solution defined for $t \geqslant -2h$. For all $t \geqslant 0$, we have that

$$\begin{aligned} x^*(t) - y(t) &= \mathrm{e}^{-(\delta-p)t}(x^*(0) - y(0)) \\ &\quad + \int_0^t p\,\mathrm{e}^{-(\delta-p)(t-\sigma)}[\max_{u\in[\sigma-h,\sigma]}(x^*(u) - x^*(\sigma)) - \max_{u\in[\sigma-h,\sigma]}(y(u) - y(\sigma))]\,\mathrm{d}\sigma. \end{aligned}$$

Thus

$$\begin{aligned} |x^*(t) - y(t)| &\leqslant \mathrm{e}^{-(\delta-p)t}|x^*(0) - y(0)| + \int_0^t |p|\,\mathrm{e}^{-(\delta-p)(t-\sigma)}[h \max_{u\in[\sigma-h,\sigma]} |x^{*\prime}(u) - y'(u)|]\,\mathrm{d}\sigma. \end{aligned}$$

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Let $\delta - p = \nu$, $|x^{*\prime}(t) - y'(t)| = w(t)$, $|x^*(t) - y'(t)| = v(t)$, $\alpha = \max_{u\in[-h,0]} v(u)$, and $\rho = \max_{u\in[-h,0]} w(u)$. Then, for all $t \in [0, +\infty)$, we get the inequalities

$$\begin{aligned} v(t) &\leqslant v(0)\exp(-\nu t) + \int_0^t \exp(-\nu(t-s))|p|h \max_{u\in[\sigma-h,\sigma]} w(u)\,\mathrm{d}\sigma, \\ w(t) &\leqslant \nu v(t) + |p|h \max_{u\in[t-h,t]} w(u). \end{aligned}$$

Since $v(t) \leqslant \alpha$, $w(t) \leqslant \rho$ for $t \in [-h, 0]$, setting $\kappa = \beta = |p|h$, $\mu = \nu$ and applying lemma 5.1, we obtain that

$$v(t) \leqslant c_1(\alpha,\rho,\mu)\exp(-\gamma t), \quad |x^*(t) - y(t)| \leqslant c_1(\alpha,\rho,\mu)\exp(-\gamma t), \quad t \geqslant 0,$$

where γ is a positive root of (5.2). This proves part (ii) of theorem 1.3

(iii) Finally, we notice that part (iii) of theorem 1.3 is a rather simple consequence of the Myshkis–Yorke 3/2 criterion [26], so the details are left to the reader.

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