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NOTES AND PROBLEMS

A NOTE ON TESTING RESTRICTIONS FOR THE COINTEGRATION PARAMETERS OF A VAR WITH I(2) VARIABLES

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We give a brief introduction to the vector autoregressive model for cointegrated I(2) variables and show how some plausible economic relations can be formulated in the I(2) framework in such a way that likelihood ratio tests for their validity are asymptotically χ^2 distributed.

1. INTRODUCTION AND MOTIVATION

The p-dimensional cointegrated vector autoregressive (VAR) model for I(2) variables, without deterministic terms and just two lags, is given by the error correction model

$$\Delta^2 X_t = \alpha(\rho'\tau' X_{t-1} + \psi'\Delta X_{t-1}) + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau'\Delta X_{t-1} + \varepsilon_t, \quad \textbf{(1)}$$

where the ε_t are independent and identically distributed (i.i.d.) $(0,\Omega)$. The freely varying parameters are

$$\alpha_{p\times r}, \rho_{(r+s)\times r}, \tau_{p\times(r+s)}, \psi_{p\times r}, \kappa_{(r+s)\times(p-r)}, \Omega_{p\times p}.$$

As usual, $\alpha_\perp$ denotes an orthogonal complement of α, and we define the $p \times r$ matrix $\beta = \tau\rho$. Notice that the column dimension r of β is between 0 and p and the same holds for the row dimension $r + s$ of ρ. Hence, $s < p - r$. In

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the analysis of the I(2) model it will be important to specify r and s such that $r = \text{rk}(\alpha\beta')$ and $s = \text{rk}(\tau\rho_\perp)$. Under suitable conditions on the parameters (see Johansen, 1997), the equations in (1) have a solution of the form

$$X_t = C_2 \sum_{i=1}^{t} \sum_{j=1}^{i} \varepsilon_j + C_1 \sum_{i=1}^{t} \varepsilon_i + A_1 + tA_2 + U_t,$$

where U_t is stationary and the coefficient matrices satisfy the relations

$$\tau' C_2 = 0, \qquad \beta' C_1 + \psi' C_2 = 0, \qquad \tau'(A_1, A_2) = 0, \qquad \beta' A_1 + \psi' A_2 = 0,$$

so that the processes $\Delta^2 X_t = C_2 \varepsilon_t + C_1 \Delta\varepsilon_t + \Delta^2 U_t$ and $\tau' \Delta X_t = \tau' C_1 \varepsilon_t + \tau' \Delta U_t$ are stationary. Thus the solution is an I(2) process, and there are $r + s$ cointegrating relations given by the I(1) process $\tau' X_t$. The model also allows for multicointegration (see Engle and Yoo, 1991), that is, cointegration between the levels and the differences because $\beta' X_t + \psi' \Delta X_t = \beta' U_t + \psi' C_1 \varepsilon_t + \psi' \Delta U_t$ is stationary. Equivalently one can show, because $\tau' \Delta X_t$ is stationary, that $\beta' X_t + \delta \tau_\perp' \Delta X_t$ is stationary, where $\delta = \psi' \tau_\perp (\tau_\perp' \tau_\perp)^{-1}$ is the so-called multicointegration parameter.

The theory of the I(2) model is developed by Boswijk (2000), Johansen (1997, 2005), Kongsted (2005), Paruolo (1996), Paruolo and Rahbek (1999), and Rahbek, Kongsted, and Jørgensen (1999).

It has been shown that the likelihood ratio test for the ranks r and s has an asymptotic distribution that can be expressed in terms of Brownian motions and integrated Brownian motions and that has to be tabulated by simulation. Moreover, the asymptotic distribution of the maximum likelihood estimator of the cointegrating parameters τ, ρ, and β is quite involved, as it is not mixed Gaussian. However, many hypotheses on these parameters can be tested using asymptotic χ^2 tests (see Boswijk, 2000; Johansen, 2005). We give subsequently an example of such hypotheses that can be formulated and tested in the I(2) model.

2. AN EXAMPLE OF HYPOTHESES ALLOWING FOR ASYMPTOTIC χ^2 TESTS

Denote by m_t the log nominal money stock, by p_t the log price level, by y_t log real income, and by R_t a long-term interest rate and define $X_t = (m_t, p_t, y_t, R_t)'$. Suppose $m_t \sim \text{I}(2)$, $p_t \sim \text{I}(2)$, $y_t \sim \text{I}(1)$, and $R_t \sim \text{I}(1)$. Moreover consider the following cointegration relations:

- $m_t - p_t \sim \text{I}(1)$ (i.e., log real money is I(1))
- $m_t - p_t - y_t + \beta_R R_t \sim \text{I}(0)$ (i.e., there exists a stationary money demand relation with unit income elasticity)
- $R_t - \Delta p_t \sim \text{I}(0)$ (i.e., the "Fisher effect" holds, meaning that the real interest rate is stationary)

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The unit root and cointegration properties of the variables are in line with those found in Lütkepohl and Wolters (2003) for a system of German quarterly data except for some of the values assumed for the cointegration parameters. Therefore one may wish to formulate these hypotheses in the cointegrated model for I(2) variables and develop the likelihood ratio test to check whether the structure is compatible with the data.

We want to show that the hypotheses discussed earlier can be formulated as hypotheses on the parameters ρ, τ, and ψ that have the property that likelihood ratio tests of the restrictions are asymptotically χ^2.

Under the assumption that the process X_t is I(2) it holds that $\rho'\tau' X_{t-1} + \delta\tau_\perp'\Delta X_{t-1}$ and $\tau'\Delta X_t$ are I(0). Therefore we want to express the preceding relations in terms of the parameters τ, $\beta = \tau\rho$, ψ, and $\delta = \psi'\tau_\perp(\tau_\perp'\tau_\perp)^{-1}$. For our system $p = 4$, and from the cointegrating relations

$$\begin{aligned} \Delta m_t - \Delta p_t &\sim \mathrm{I}(0), \\ m_t - p_t - y_t + \beta_R R_t &\sim \mathrm{I}(0), \\ R_t - \Delta p_t &\sim \mathrm{I}(0) \end{aligned}$$

we find that τ is 4×3, so that $r + s = 3$. We see that there are two relations that involve levels, so that $r = 2$, and β has dimension 4×2 and is given by

$$\beta' = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This shows that the hypotheses formulated previously imply that

$$\tau = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau_\perp = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \rho = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the relations fully specify the matrices τ, ρ, and $\beta = \tau\rho$ in this case. We denote the specific τ, ρ, and β matrices by τ_0, ρ_0, and β_0, respectively. With this notation the model reduces to

$$\Delta^2 X_t = \alpha(\beta_0' X_{t-1} + \psi'\Delta X_{t-1}) + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau_0'\Delta X_{t-1} + \varepsilon_t,$$

and we next find the implications of the assumptions for the 4×2 parameter ψ.

Because $\tau_0'\Delta X_{t-1}$ is stationary we decompose $\psi'\Delta X_{t-1}$ as

$$\begin{aligned} \psi'\Delta X_{t-1} &= \delta\tau_{0\perp}'\Delta X_{t-1} + \xi\tau_0'\Delta X_{t-1}, \\ \delta &= \psi'\tau_{0\perp}(\tau_{0\perp}'\tau_{0\perp})^{-1} = (\delta_1, \delta_2)', \\ \xi &= \psi'\tau_0(\tau_0'\tau_0)^{-1}, \end{aligned}$$

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where we used that δ is 2×1 and that $\tau_0(\tau_0'\tau_0)^{-1}\tau_0' + \tau_{0\perp}(\tau_{0\perp}'\tau_{0\perp})^{-1}\tau_{0\perp}'$ is equal to the identity matrix. Hence we can rewrite the equations in (1) as

$$\Delta^2 X_t = \alpha(\beta_0' X_{t-1} + \delta\tau_{0\perp}'\Delta X_{t-1}) + (\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa' + \alpha\xi)\tau_0'\Delta X_{t-1} + \varepsilon_t.$$

Notice that the coefficients δ are identified by the choice of $\beta = \beta_0$ and $\tau_\perp = \tau_{0\perp}$ and that

$$\tau_{0\perp}'\Delta X_{t-1} = \Delta m_t + \Delta p_t.$$

Now consider the stationary (multicointegrating) relation

$$\beta_0' X_t + \delta'\tau_{0\perp}'\Delta X_t = \begin{pmatrix} m_t - p_t - y_t + \delta_1(\Delta m_t + \Delta p_t) \\ R_t + \delta_2(\Delta m_t + \Delta p_t) \end{pmatrix}.$$

By a linear transformation of the rows, which can be absorbed in α, we can eliminate $\delta_1(\Delta m_t + \Delta p_t)$ from the first equation and find that the model implies stationarity of the linear combinations

$$\begin{pmatrix} m_t - p_t - y_t - (\delta_1/\delta_2)R_t \\ R_t + \delta_2(\Delta m_t + \Delta p_t) \end{pmatrix}.$$

Because $\Delta m_t + \Delta p_t = 2\Delta p_t + \Delta m_t - \Delta p_t$ and $\Delta m_t - \Delta p_t$ is stationary, we have that the model, with τ_0 and β_0 as given before, allows the stationary relations

$$\begin{pmatrix} m_t - p_t - y_t - (\delta_1/\delta_2)R_t \\ R_t + 2\delta_2\Delta p_t \end{pmatrix}.$$

Hence we see that we can define $\beta_R = -\delta_1/\delta_2$, and the only extra restriction we need to test is the hypothesis $\delta_2 = -0.5$. Thus the restrictions formulated previously can be tested successively as the hypotheses

$$\mathcal{H}_0 : r = 2, s = 1,$$

$$\mathcal{H}_1 : \beta = \beta_0,$$

$$\mathcal{H}_2 : \tau = \tau_0,$$

$$\mathcal{H}_3 : \delta_2 = -0.5.$$

The first hypothesis is a test on cointegrating ranks, and the asymptotic distribution is nonstandard and tabulated by simulation (see Johansen, 1997). It follows from the results in the same paper (see also Boswijk, 2000; Johansen, 2005) that $-2\log LR(\mathcal{H}_1|\mathcal{H}_0)$ and $-2\log LR(\mathcal{H}_2|\mathcal{H}_1)$ are asymptotically distributed as $\chi^2(4)$ and $\chi^2(1)$, respectively. In general one cannot expect hypotheses on the coefficient δ to give asymptotic χ^2 tests (see Paruolo, 2000), but $\mathcal{H}_2$ specifies $\tau_\perp$ completely, and when that is the case, one can in fact show that a test on δ, and hence $-2\log LR(\mathcal{H}_3|\mathcal{H}_2)$, is asymptotically distributed as $\chi^2(1)$.

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This can be seen by the "nominal-to-real" transformation (see Kongsted, 2005),

$$Z_t = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \tau_0' X_t \\ \tau_{0\perp}' \Delta X_t \end{pmatrix},$$

which satisfies a model of the form

$$\begin{aligned}(\tau_0(\tau_0'\tau_0)^{-1}, \tau_{0\perp}(\tau_{0\perp}'\tau_{0\perp})^{-1})\Delta Z_t \\ = \alpha \begin{pmatrix} \rho_0 \\ \delta' \end{pmatrix}' Z_{t-1} + (\tau_0(\tau_0'\tau_0)^{-1} + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa' + \alpha\xi) \\ \times (I_{r+s}, 0)\Delta Z_{t-1} + \varepsilon_t.\end{aligned}$$

Premultiplying with $(\tau_0, \tau_{0\perp})'$ gives the I(1) cointegration model

$$\Delta Z_t = \tilde{\alpha}\tilde{\beta}' Z_{t-1} + \tilde{\Gamma}_1 \Delta Z_{t-1} + \tilde{\varepsilon}_t$$

with parameters

$$\tilde{\alpha} = \begin{pmatrix} \tau_0'\alpha \\ \tau_{0\perp}'\alpha \end{pmatrix}, \qquad \tilde{\beta} = \begin{pmatrix} \rho_0 \\ \delta' \end{pmatrix},$$

$$\tilde{\Gamma}_1 = \begin{pmatrix} I_{r+s} + \tau_0'(\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa' + \alpha\xi) & 0 \\ \tau_{0\perp}'(\Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa' + \alpha\xi) & 0 \end{pmatrix},$$

and $\tilde{\varepsilon}_t = (\varepsilon_t'\tau_0, \varepsilon_t'\tau_{0\perp})'$. Thus the transformed model is an I(1) model with linear restrictions on $\tilde{\Gamma}_1$ and $\tilde{\beta}$ partly known. A hypothesis on δ is therefore a hypothesis on $\tilde{\beta}$ in an I(1) model, which is known to give asymptotic χ^2 tests (see Johansen, 1991).

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