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HEREDITARILY STRUCTURALLY COMPLETE POSITIVE LOGICS

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Abstract. Positive logics are $\{\wedge, \vee, \rightarrow\}$-fragments of intermediate logics. It is clear that the positive fragment of Int is not structurally complete. We give a description of all hereditarily structurally complete positive logics, while the question whether there is a structurally complete positive logic which is not hereditarily structurally complete, remains open.

§1. Introduction. The notion of an admissible rule evolved from the notion of an auxiliary rule: if a formula B can be derived from a set of formulas $A_1, \dots, A_n$ in a given calculus (deductive system) DS, one can shorten derivations by using a rule $A_1, \dots, A_n/B$. The application of such a rule does not extend the set of theorems, i.e., such a rule is admissible (permissible). In (Lorenzen, 1955, p. 19) P. Lorenzen called the rules not extending the class of the theorems "zulässing," and the latter term was translated as "admissible," the term we are using nowadays.

Independently (see (Novikov, 1977, p. 30)[1]), in the lectures on mathematical logic, for a given calculus DS, P. S. Novikov considered the rules $\mathcal{A}_1, \dots, \mathcal{A}_n/\mathcal{B}$ (where $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ are variable formulas of some type) such that $\vdash_{\mathsf{DS}} \mathcal{B}$ holds every time when $\vdash_{\mathsf{DS}} \mathcal{A}_1, \dots, \vdash_{\mathsf{DS}} \mathcal{A}_n$ hold. He also distinguished between two types of such rules: a rule is strong, if $\vdash_{\mathsf{DS}} \mathcal{A}_1 \rightarrow (\mathcal{A}_2 \rightarrow \cdots (\mathcal{A}_n \rightarrow \mathcal{B}) \cdots)$ holds, otherwise it is weak.

For classical propositional calculus (CPC), the use of admissible rules is merely a matter of convenience, because every admissible in CPC rule $A_1, \dots, A_n/B$ is derivable, that is $A_1, \dots, A_n \vdash_{\mathsf{CPC}} B$ (see, for instance Belnap, Jr., Leblanc, & Thomason (1963)). It was observed by R. Harrop in Harrop (1960) that the rule $\neg p \rightarrow (q \vee r)/(\neg p \rightarrow q) \vee (\neg p \rightarrow r)$ is admissible in intuitionistic propositional logic (Int), but is not derivable: the corresponding formula is not a theorem of Int. Later, in mid 1960s, A. V. Kuznetsov observed that the rule $(\neg\neg p \rightarrow p) \rightarrow (p \vee \neg p)/((\neg\neg p \rightarrow p) \rightarrow \neg p) \vee ((\neg\neg p \rightarrow p) \rightarrow \neg\neg p)$ is also admissible in Int, but not derivable. Another example of an admissible for IPC not derivable rule was found in 1971 by G. Mints (see Mints (1976)): the following rule is admissible but not derivable in Int

$$(p \rightarrow q) \rightarrow (p \vee r)/((p \rightarrow q) \rightarrow p) \vee ((p \rightarrow q) \rightarrow r). \tag{1}$$

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[1] This book was published in 1977, but it is based on the notes of a course that P.S. Novikov taught in 1950th; A.V. Kuznetsov was recalling that P.S. Novikov considered such rules much earlier, in this lectures in 1940th.

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Fragment		Reference
$\{\to\}$	*HSCpl*	Prucnal (1972b)
$\{\to, \bot\}$	Not *HSCpl* (the smallest *HSCpl* fragment exists)	Cintula & Metcalfe (2010)
$\{\to, \wedge\}$	*HSCpl*	Nemitz & Whaley (1973)
$\{\to, \wedge, \bot\}$	*HSCpl*	Wroński (1986)
$\{\to, \wedge, \vee\}$	Not *HSCpl* (the smallest *HSCpl* fragment exists)	this article
$\{\to, \wedge, \vee, \bot\}$	Not *HSCpl* (the smallest *HSCpl* extension exists)	Citkin (1978)

Table 1. *Hereditary structural completeness of Fragments of Intermediate Logics*

Following Pogorzelski (1974), the logics in which every admissible rule is derivable are called structurally complete, and if every extension of a structurally complete logic is structurally complete, such a logic is hereditarily structurally complete (*HSCpl* for short). Thus, CPC is structurally complete, while Int is not.[2]

Very soon (cf. Dzik & Wroński (1973)), it was discovered that the Dummett's Logic LC and all its extensions are structurally complete, and in Prucnal (1972a) T. Prucnal proved that $\to$-fragment of any intermediate logic, is structurally complete, i.e., these fragments do not have admissible not-derivable rules.

Naturally, the questions about admissibility of rules in Int and about structural completeness of intermediate logics (the consistent extensions of Int) and their fragments arose.

In terms of hereditary structural completeness the aforementioned results can be rephrased as follows: LC and the $\to$-fragment of Int are hereditarily structurally complete. Curiously enough, implication-negation (or implication-falsity) fragment of Int is not structurally complete (cf. Wroński (1986)). Cintula and Metcalfe proved that every structurally complete $\to, \neg$-fragment of any intermediate logic is hereditarily structurally complete, and there is the smallest (hereditarily) structurally complete implication-negation fragment of Int (cf. Cintula & Metcalfe (2010)). All hereditarily structurally complete intermediate logics were described by author in Citkin (1978). In (Rybakov, 1995, Theorem 4.5) Rybakov obtained a similar description for the extensions of normal modal logic **K4**, and as a consequence, he gave an alternative proof of the criterion on hereditary structural completeness for intermediate logics (cf. (Rybakov, 1995, Theorem 4.7)).

The situation with hereditary structural completeness of fragments of intermediate logics is summarized in Table 1.

An interesting sufficient condition for positive predicate logic to be hereditarily structurally complete was proved by Dzik (cf. (Dzik, 2004, Theorem 3)).

1.1. Main results. We consider intuitionistic propositional logic Int with connectives $\wedge, \vee, \to, \bot$. The (propositional) formulas which have no occurrences of $\bot$ are called *positive*. Clearly, the set Int^+ of all positive formulas from Int is closed under application of rules modus ponens (denoted by MP) and (simultaneous) substitution (denoted by Sb). By ExtInt^+ we denote the set of all extensions of Int^+, closed under MP and Sb, and we refereed to them as *positive logics*.

To study structural completeness of logics from ExtInt^+, we employ the algebraic methods (and we assume that the reader is familiar with basic notions of universal algebra (cf. Bergman (2012) or Burris & Sankappanavar (1981))).

[2] More information about structural and hereditary structural completeness the reader can find in Rybakov (1995) or Raftery (2016).

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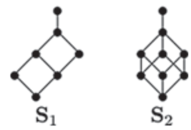

Fig. 1. Non-projective Algebras.

We recall that algebraic semantics for Int and its extensions are Heyting algebras: an algebra $\langle \mathsf{A}; \wedge, \vee, \rightarrow, \mathsf{1}, \mathsf{0}\rangle$, where $\langle \mathsf{A}; \wedge, \vee\rangle$ is a bounded distributive lattice with greatest element $\mathsf{1}$ and smallest element $\mathsf{0}$, and $\rightarrow$ is a relative pseudo-complementation is a *Heyting algebra*. We assume that the reader is familiar with properties of Heyting algebras Rasiowa & Sikorski (1970), where Heyting algebras are called pseudo-Boolean algebras.

Algebraic semantics for positive logics are Brouwerian algebras:[3] an algebra $\langle \mathsf{A}; \wedge, \vee, \rightarrow, \mathsf{1}\rangle$, where $\langle \mathsf{A}; \wedge, \vee, \mathsf{1}\rangle$ is a distributive lattice with a greatest element $\mathsf{1}$, and $\rightarrow$ is a relative pseudo-complementation, is a *Brouwerian algebra*.

In a regular way we define validness of a formula in a given algebra: a formula A is *valid* in a given (Brouwerian) algebra $\mathbf{A}$ (in symbols, $\mathbf{A} \models A$) if $v(A) = \mathsf{1}$ for every valuation v in $\mathbf{A}$. With each logic $\mathsf{L} \in \mathsf{ExtInt}^+$ we associate a class $\mathsf{V}(\mathsf{L})$ of all algebras in which every formula from L is valid:

$$\mathsf{V}(\mathsf{L}) := \{\mathbf{A} \mid \mathbf{A} \models A, \text{ for every } A \in \mathsf{L}\}.$$

The class $\mathsf{V}(\mathsf{L})$ forms a variety (a.k.a. equational class) and elements of $\mathsf{V}(\mathsf{L})$ are being referred to as *models* of L.

On the other hand, every variety $\mathcal{V}$ of Brouwerian algebras defines a positive logic:

$$\mathsf{L}(\mathcal{V}) := \{A \mid \mathbf{A} \models A \text{ for every } \mathbf{A} \in \mathcal{V}\}.$$

THEOREM 1.1 (Main theorem). *A positive logic L is hereditarily structurally complete if and only if Brouwerian algebras $\mathbf{S}_1$ and $\mathbf{S}_2$ depicted in Figure 1 are not models of L.*

It is known (see, e.g., Olson, Raftery, & van Alten (2008)) that the logic L is *HSCpl* if and only if variety $\mathsf{V}(\mathsf{L})$ is *primitive*, that is every subquasivariety of $\mathsf{V}(\mathsf{L})$ is a variety. Thus, the above theorem is equivalent to the following theorem which we prove in §6.

THEOREM 1.2 (Main theorem: Algebraic version). *A variety $\mathcal{V}$ of Brouwerian algebras is primitive if and only if $\mathbf{S}_1, \mathbf{S}_2 \notin \mathcal{V}$ (see Figure 1).*

A proof of the above theorem follows from the following lemmas.

LEMMA 1.3 (A proof can be found in §4). *Any variety that contains algebra $\mathbf{S}_1$ or $\mathbf{S}_2$ is not primitive.*

Recall that a variety $\mathcal{V}$ is said to be *locally finite* if any finitely generated algebra from $\mathcal{V}$ is finite. The below lemma gives an easy sufficient condition for local finiteness.

LEMMA 1.4 (A proof can be found in §4). *Any variety of Brouwerian algebras not containing the algebra $\mathbf{S}_1$ is locally finite.*

[3] We follow Galatos, Jipsen, Kowalski, & Ono (2007) and call these algebras Brouwerian. Some authors are using different names, for instance, implicative lattices Odintsov (2008), lattice with pseudocomplementation Rasiowa & Sikorski (1970).

Recall also that an algebra $\mathbf{A}$ is *weakly projective* in a given class of algebras $\mathcal{K}$ if for any algebra $\mathbf{B} \in \mathcal{K}$, such that $\mathbf{A} \in \mathbb{H}\mathbf{B}$, we have $\mathbf{A} \in \mathbb{S}\mathbf{B}$.

LEMMA 1.5 (A proof can be found in §5). *In any variety not containing the algebras* $\mathbf{S}_1, \mathbf{S}_2$ *every finite subdirectly irreducible algebra is weakly projective in the class* $\mathcal{V}_{\mathsf{fin}}$ *of all finite algebras from* $\mathcal{V}$.

It is clear that Lemma 1.3 gives just a necessary condition of hereditary structural completeness, while the sufficient condition follows immediately from Lemmas 1.4 and 1.5 and the below Proposition (see (Gorbunov, 1998, Prop. 5.1.24)).

PROPOSITION 1.6. *Let* $\mathcal{V}$ *be a locally finite variety of a finite signature. Then,* $\mathcal{V}$ *is primitive if and only if every finite subdirectly irreducible algebra from* $\mathcal{V}$ *is weakly projective in* $\mathcal{V}_{\mathsf{fin}}$.

We prove also the following corollary (see §7).

COROLLARY 1.7 (Main corollary). *The following holds*:

(a) *There is the smallest HSCpl positive logic and it is finitely axiomatized*;
(b) *The set of all HSCpl positive logics is countable*;
(c) *Every HSCpl positive logic is finitely axiomatizable*;
(d) *There are infinitely many HSCpl intermediate logics whose positive fragment is not HSCpl.*

Note that (c) follows from (a), (b) and the below theorem that holds for any congruence distributive variety of finite signature and it is interesting in its own right (for proof see §6).

THEOREM 1.8. *Let* $\mathcal{V}$ *be a locally finite finitely based congruence distributive variety of finite signature. Then, every subvariety of* $\mathcal{V}$ *is finitely based if and only if* $\mathcal{V}$ *has at most countably many subvarieties.*

In algebraic terms, Corollary 1.7(a) means that there is the largest primitive variety of Brouwerian algebras. This variety can be described in the following way (see §5 for a proof).

THEOREM 1.9. *The largest primitive variety of Brouwerian algebras is generated by Brouwerian reducts of finite projective Heyting algebras.*

As it is known from Balbes & Horn (1970), finite projective Heyting algebras are precisely the subdirectly irreducible coalesced sums of two and four-element Boolean algebras.

§2. Basic information about Brouwerian algebras.

First, let us recall the basic facts about Brouwerian algebras.

2.1. Brouwerian algebras. An algebra $\langle \mathsf{A}; \wedge, \vee, \to, \mathsf{1} \rangle$, where $\langle \mathsf{A}; \wedge, \vee, \mathsf{1} \rangle$ is a distributive lattice with the greatest element and $\to$ is a pseudocomplementation, is called a Brouwerian algebra. If a Brouwerian algebra $\mathbf{A}$ has the smallest element, we say that $\mathbf{A}$ is *bounded*. By $\mathbf{2}$ we denote the two-element Brouwerian algebra, and define $\mathbf{4} := \mathbf{2} \times \mathbf{2}$. We also assume that the reader is familiar with basic properties of Brouwerian algebras

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(e.g., from Rasiowa & Sikorski (1970) where Brouwerian algebras are called lattices with pseudocomplementation).

From now on by "algebra" we mean Brouwerian algebra, unless otherwise indicated.

Let $\mathbf{A}$ be a (Brouwerian) algebra and $a \in \mathbf{A}$ be an element. Then, set $[a) := \{b \in \mathbf{A} \mid a \leq b\}$ forms a filter and a subalgebra of $\mathbf{A}$ denoted by $\mathbf{A}[a)$, that is, $\mathbf{A}[a) = \langle [a); \wedge, \vee, \to, 1\rangle$. A set $(a] := \{b \in \mathbf{A} \mid a \geq b\}$, as a lattice, is isomorphic to $\mathbf{A}/[a)$ and we abbreviate it to $\mathbf{A}(a]$.

It is easy to see that every bounded Brouwerian algebra forms a Heyting algebra. If $\mathbf{A} = (A; \wedge, \vee, \to, 1)$ is a bounded Brouwerian algebra, by $\mathbf{A}^\circ$ we denote Heyting algebra $\mathbf{A} = (A; \wedge, \vee, \to, 1, 0)$. On the other hand, if $\mathbf{B} = (B; \wedge, \vee, \to, 1, 0)$ is a Heyting algebra, by $\mathbf{B}^+$ we denote its *Brouwerian reduct* $(B; \wedge, \vee, \to, 1)$.

PROPOSITION 2.1. *Every finitely generated Brouwerian algebra is bounded.*

Proof. By a simple induction on number of generators one can prove that meet of all generators of algebra $\mathbf{A}$ is indeed the smallest element of $\mathbf{A}$. □

2.2. Nodes. Let $\mathbf{A}$ be an algebra. An element $a \in \mathbf{A}$ is said to be a *node* if a is comparable with every element of $\mathbf{A}$. Clearly, 1 is always a node. If $\mathbf{A}$ is bounded, the smallest element is also a node. The greatest and smallest (if exists) elements are *trivial* nodes. We say that a nontrivial algebra is *nodeless* just in case it does not contain nontrivial nodes.

PROPOSITION 2.2. *Let a_0, a_1 be nodes of algebra $\mathbf{A}$ and $a_0 \leq a_1$. Then, a set of elements $A' := (a_0] \cup [a_1)$ forms a subalgebra.*

Proof. First, it is not hard to see that A' is closed under $\wedge$ and $\vee$ and contains 1. Thus, we need only to verify that A' is closed under $\to$.

Indeed, let $a, b \in A'$. If $b \in [a_1)$, we have

$$a_1 \leq b \leq a \to b.$$

Hence, $a \to b \in [a_1) \subseteq A'$.

Now, assume that $b \in (a_0]$ and $b \notin [a_1)$, that is, $b < a_1$. Consider two remaining possibilities:

(a) $a \in [a_1)$;
(b) $a \in (a_0]$.

Case (a). Suppose that $a \in [a_1)$, that is, $a_1 \leq a$. Recall that a_1 is a node. Hence, $(a \to b) \geq a_1$ or $(a \to b) < a_1$.

Observe that the former case is impossible. Indeed, if

$$a_1 \leq a \text{ and } a_1 \leq (a \to b), \text{ then } a_1 \leq a \wedge (a \to b) \leq b,$$

and this contradicts the assumption that $b < a_1$.

Now, suppose that $a_1 \leq a$ and $a \to b < a_1$. Hence, $a \to b < a_1 \leq a$, and therefore,

$$a \wedge (a \to b) = a \to b. \tag{2}$$

On the other hand, because $b \in (a_0]$, we have

$$a \wedge (a \to b) \leq b \leq a_0. \tag{3}$$

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and from (2) and (3) we get

$$a \to b = a \wedge (a \to b) \leq b \leq a_0, \tag{4}$$

that is, $a \to b \in [a_0) \subseteq A'$.

Case (b). Suppose that $a, b \in (a_0]$. Hence, $a \leq a_0$ and $b < a_0$. Consider element $a \to b$. Because a_0 is a node, $a \to b \leq a_0$ (and we have nothing to prove), or $a \to b > a_0$.

Suppose that $a, b \leq a_0$ and $a_0 < a \to b$. Then, $a \leq a_0 < a \to b$, and hence,

$$a = a \wedge (a \to b) \leq b.$$

Therefore, $a \to b = 1 \in A'$. □

COROLLARY 2.3. *If* **A** *is an algebra with a nontrivial node* a, *then* $(a] \cup \{1\}$ *forms a subalgebra.*

COROLLARY 2.4. *Let* **A** *be an algebra generated by a set of elements* G, *and let* $a < b < c$ *be nodes. Then, there is a generator* $g \in G$ *such that* $a < g < c$.

Proof. Indeed, by Proposition 2.2, $(a] \cup [c)$ forms a subalgebra and $b \notin (a] \cup [c)$ means that $(a] \cup [c)$ is a proper subalgebra. Hence, there is a generator $g \in G$ such that $g \notin (a] \cup [c)$, and because a and c are nodes, we have $a < g < c$. □

COROLLARY 2.5. *Any finitely generated Brouwerian algebra contains finitely many nodes.*

The following corollary gives more precise bound for the number of nodes.

COROLLARY 2.6. *Any n-generated Brouwerian algebra* **A** *has at most* $2n + 2$ *nodes.*

Proof. Let **A** be an algebra generated by elements $g_0, \dots, g_{n-1}$. By Corollary 2.5, algebra **A** has just a finite set of nodes. Assume that for contradiction that **A** contains more then $2n + 2$ nodes. Let

$$a_0 < a_1 < \cdots < a_{2n} < a_{2n+1} < a_{2n+2}$$

be nodes. Then, by Corollary 2.4, for every $j \leq n + 1$, there is a generator g_s such that $a_{2j} < g_s < a_{2j+2}$, which is impossible. □

To simplify notation, we use the following abbreviations: if $a, g_0, \dots, g_{n-1}$ are elements of algebra **A**, we abbreviate $g_0, \dots, n_{n-1}$ by $\overline{g}$, and we abbreviate $g_0 \vee a, \dots, g_{n-1} \vee a$ by $\overline{g} \vee a$.

The following proposition is an extension of Kuznetsov's Theorem (cf. Kuznetsov (1973)) to Brouwerian algebras and nodes. The idea of the proof is borrowed from (Citkin, 1986, Lemma 3) (it was also used in (Bezhanishvili & Grigolia, 2005, Lemma 2.2)).

PROPOSITION 2.7. *Let* **A** *be an n-generated algebra with a nontrivial node* $a \in \mathbf{A}$. *Then, algebra* $\mathbf{A}[a)$ *is also n-generated.*

Proof. Suppose that elements $g_0, \dots, g_{n-1}$ generate algebra **A**. We will prove that elements $g_0 \vee a, \dots, g_{n-1} \vee a$ generate $[a)$.

To prove the proposition we will show that for each term t such that $a \leq t(\overline{g})$, there is a term t' such that $t(\overline{g}) = t'(\overline{g} \vee a)$. We prove this claim by induction on length of term t.

If t is a variable, that is for some $i < n$, $t(\overline{g}) = g_i$, we have $g_i = g_i \vee a$, because by our assumption $t(\overline{g}) \geq a$.

Suppose that for every term t of length less than m, if $t(\overline{\mathsf{g}}) \geq \mathsf{a}$, there is a term t' such that $t(\overline{\mathsf{g}}) = t'(\overline{\mathsf{g}} \vee \mathsf{a})$.

Let t be a term of length m and $t(\overline{\mathsf{g}}) \geq \mathsf{a}$. First, let us note that, because $\mathsf{b} := \mathsf{g}_0 \wedge \cdots \wedge \mathsf{g}_{n-1}$ is the smallest element of $\mathbf{A}$, and hence, $\mathsf{b} \leq \mathsf{a}$, by distributivity, we have

$$(\mathsf{g}_0 \vee \mathsf{a}) \wedge \cdots \wedge (\mathsf{g}_{n-1} \vee \mathsf{a}) = (\mathsf{g}_0 \wedge \cdots \wedge \mathsf{g}_{n-1}) \vee \mathsf{a} = \mathsf{b} \vee \mathsf{a} = \mathsf{a}. \tag{5}$$

Thus, if $t(\overline{\mathsf{g}}) = \mathsf{a}$, we can take $t' := x_0 \wedge \cdots \wedge x_{n-1}$.

Now, assume that $t(\overline{\mathsf{g}}) > \mathsf{a}$ and we have the following cases:

(a) $t = t_0 \wedge t_1$;
(b) $t = t_0 \vee t_1$;
(c) $t = t_0 \to t_1$,

where t_0, t_1 are terms of length $< m$.

Case (a). Because $\mathsf{a} \leq t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \wedge t_0(\overline{\mathsf{g}})$, for every $i < 2$ we have $\mathsf{a} \leq t_i(\overline{\mathsf{g}})$, and we can simply apply the induction assumption.

Case (b). Suppose that $\mathsf{a} < t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \vee t_1(\overline{\mathsf{g}})$. Let us consider the following possibilities.

(i) $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$;
(ii) $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$ (or $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_1(\overline{\mathsf{g}}) \in [a)$);
(iii) $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \in [a)$.

(i) Suppose $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_0(\overline{\mathsf{g}}) \notin [a)$. Then, because a is a node, $t_0(\overline{\mathsf{g}}) \leq \mathsf{a}$ and $t_1(\overline{\mathsf{g}}) \leq \mathsf{a}$. Hence, $t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \vee t_1(\overline{\mathsf{g}}) \leq \mathsf{a}$, and this contradicts the assumption that $t(\overline{\mathsf{g}}) > \mathsf{a}$. Thus, this case is impossible.

(ii) Suppose that $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$. Then, because a is a node, we have

$$t_1(\overline{\mathsf{g}}) < \mathsf{a} \leq t_0(\overline{\mathsf{g}}), \text{ hence, } t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \vee t_1(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}),$$

and we can apply the induction assumption.

(iii) In this case, $\mathsf{a} \leq t_0(\overline{\mathsf{g}})$ and $\mathsf{a} \leq t_1(\overline{\mathsf{g}})$ and we can apply the induction assumption to t_1 and t_2.

Case (c). Suppose that $\mathsf{a} < t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \to t_1(\overline{\mathsf{g}})$. First, let us observe that $t_1(\overline{\mathsf{a}}) \in [\mathsf{a})$. Let us consider the following possibilities.

(i) $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$;
(ii) $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$;
(iii) $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_1(\overline{\mathsf{g}}) \in [a)$;
(iv) $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \in [a)$.

(i) If $t_0(\overline{\mathsf{g}}) \notin [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$, then by Corollary 2.3,

$$t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \to t_1(\overline{\mathsf{g}}) \in (\mathsf{a}] \text{ or } t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \to t_1(\overline{\mathsf{g}}) = \mathsf{1}.$$

The case where $t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \to t_1(\overline{\mathsf{g}}) \in (\mathsf{a}]$ is impossible, because by the assumption, $t(\overline{\mathsf{g}}) > \mathsf{a}$. Hence, $t(\overline{\mathsf{g}}) = \mathsf{1}$ and we can take $t' := \mathsf{1}$.

(ii) Suppose that $t_0(\overline{\mathsf{g}}) \in [a)$ and $t_1(\overline{\mathsf{g}}) \notin [a)$. Then, because a is a node, we have

$$t_1(\overline{\mathsf{g}}) \leq \mathsf{a} \leq t_0(\overline{\mathsf{g}}), \text{ and hence, } t(\overline{\mathsf{g}}) = t_0(\overline{\mathsf{g}}) \to t_1(\overline{\mathsf{g}}) \leq \mathsf{a} \to t_1(\overline{\mathsf{g}}). \tag{6}$$

By Corollary 2.3, $\mathsf{a} \to t_1(\overline{\mathsf{g}}) \leq \mathsf{a}$ or $\mathsf{a} \to t_1(\overline{\mathsf{g}}) = \mathsf{1}$. The former case is impossible, because we would have $t(\overline{\mathsf{g}}) \leq \mathsf{a} \to t_1(\overline{\mathsf{g}}) \leq \mathsf{a}$, while, by our assumption, $\mathsf{a} < t(\overline{\mathsf{g}})$.

(iii) Suppose that $t_0(\mathsf{g}) \notin [a)$ and $t_1(\mathsf{g}) \in [a)$. Then, because a is a node, we have

$$t_0(\mathsf{g}) \leq \mathsf{a} \leq t_1(\mathsf{g}), \text{ and hence, } t(\mathsf{g}) = t_0(\mathsf{g}) \to t_1(\mathsf{g}) = \mathsf{1}.$$

(iv) If $t_0(\mathsf{g}) \in [a)$ and $t_1(\mathsf{g}) \in [a)$, we can simply apply the induction assumption and complete the proof. □

2.3. Coalesced ordinal sums. A notion of coalesced sum is very useful for the study of finitely generated algebras. The notion of the sum of Heyting algebras was introduced in Troelstra (1965) and it is known under different names: ordinal coalesced sum Galatos *et al.*, (2007), Troelstra sum Skura (1991), sequential sum Kuznetsov & Gerčiu (1970), star sum Balbes & Horn (1970), horizontal sum Day (1973), and vertical sum Bezhanishvili (2006). Coalesced ordinal sums where extensively used in all these and many other articles. In this section we use them for Brouwerian algebras.

Let $\mathbf{A} = \langle \mathsf{A}; \wedge, \vee, \to, \mathsf{1}\rangle$ and $\mathbf{B} = \langle \mathsf{B}; \wedge, \vee, \to, \mathsf{1}\rangle$ be bounded algebras. A *coalesced ordinal sum* of algebras $\mathbf{A}$ and $\mathbf{B}$ (a sum for short) is algebra $\mathbf{A} + \mathbf{B}$ that consists of sublattices $\mathbf{A}'$ and $\mathbf{B}'$ isomorphic respectfully to $\langle \mathsf{A}; \wedge, \vee, \mathsf{1}\rangle$ and $\langle \mathsf{B}; \wedge, \vee, \mathsf{1}\rangle$ and such that $\mathbf{A}' \cap \mathbf{B}'$ contains a single element: the greatest in $\mathbf{A}'$ and the smallest in $\mathbf{B}'$.

Roughly speaking, a Hasse diagram of $\mathbf{A} + \mathbf{B}$ can be obtained by putting the diagram of $\mathbf{B}$ on top of the diagram of $\mathbf{A}$ and by identifying the top element of $\mathbf{A}$ with the bottom element of $\mathbf{B}$.

If $\mathbf{A}$ is a bounded algebra with a nontrivial node a, then $\mathbf{A} = \mathbf{A}(\mathsf{a}] + \mathbf{A}[\mathsf{a})$. Moreover, if $\mathbf{A}$ is a bounded algebra and $\mathsf{a}_0 < \mathsf{a}_1 < \cdots < \mathsf{a}_n < \mathsf{n+1} = \mathsf{1}$ are all nodes of $\mathbf{A}$, then

$$\mathbf{A} = \sum_{i=0}^{n} \mathbf{A}[\mathsf{a}_i, \mathsf{a}_{i+1}],$$

where $\mathbf{A}[\mathsf{a}_i, \mathsf{a}_{i+1}]$ is algebra with universe $\{\mathsf{b} \in \mathbf{A} \mid \mathsf{a}_i \leq \mathsf{b} \leq \mathsf{a}_{i+1}\}$. Thus, from Corollary 2.6 we obtain the following.

PROPOSITION 2.8. *Let* $\mathbf{A}$ *be a finitely generated nontrivial algebra. Then, for some* $n \geq 0$,

$$\mathbf{A} = \sum_{i=0}^{n} \mathbf{A}_i,$$

where $\mathbf{A}_i$ *are nodeless nontrivial algebras.*

Let us also recall that an algebra $\mathbf{A}$ is subdirectly irreducible (s.i. for short) if and only if it contains a nontrivial node a such that $[\mathsf{a}, 1] = \{\mathsf{a}, \mathsf{1}\}$. In other words, algebra $\mathbf{A}$ is s.i. if and only if $\mathbf{A} = \mathbf{B} + \mathbf{2}$ for some (perhaps trivial) algebra $\mathbf{B}$.

PROPOSITION 2.9. *Let* $\mathbf{A}$ *be a finitely generated s.i. algebra. Then,*

$$\mathbf{A} = \mathbf{A}_0 + \cdots + \mathbf{A}_n + \mathbf{2} \tag{7}$$

for some nodeless nontrivial algebras $\mathbf{A}_i$.

Let us observe that if $\mathbf{A}$ is like in (7), then every algebra $\mathbf{A}_i + \mathbf{2}$, $i \leq n$ is a subalgebra of $\mathbf{A}$. For Heyting algebras the situation is different: because every subalgebra contains 0, only $\mathbf{A}_0 + \mathbf{2}$ is a subalgebra of $\mathbf{A}$, while for $i > 0$, $\mathbf{2} + \mathbf{A}_i + \mathbf{2}$, but not $\mathbf{A}_i + \mathbf{2}$, is a subalgebra of $\mathbf{A}$.

The following proposition give us a ground to prove Lemmas 1.4 and 1.5.

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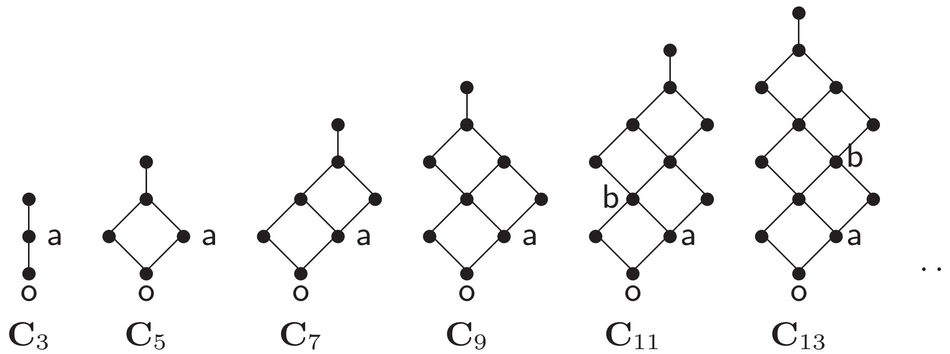

Fig. 2. Two-generated Brouwerian algebras.

PROPOSITION 2.10. *Let $\mathcal{V}$ be a variety that does not contain algebra $\mathbf{S}_1$. Then, every s.i. finitely generated algebra $\mathbf{A} \in \mathcal{V}$ is of form*

$$\mathbf{A} = \mathbf{A}_1 + \cdots + \mathbf{A}_n + \mathbf{2},$$

where for every $1 \leq i \leq n$, $\mathbf{A}_i^\circ$ (that is, $\mathbf{A}_i$ regarded as a Heyting algebra) is a Boolean algebra.

Proof. By virtue of Proposition 2.9, algebra $\mathbf{A}$ is of form (7). Hence, it suffices to prove that all algebras $\mathbf{A}_i^\circ$ are Boolean. For this, we consider subalgebras $\mathbf{A}_i + \mathbf{2}$ and we prove that for every nontrivial nodeless algebra $\mathbf{B}$, such that $\mathbf{B} + \mathbf{2} \in \mathcal{V}$, Heyting algebra $\mathbf{B}^\circ$ is Boolean.

Indeed, let o be a smallest element of $\mathbf{B}$. Suppose that $\mathsf{a} \in \mathbf{B}$ is such that $\mathsf{o} < \mathsf{a} \leq 1_{\mathbf{B}}$. Then, elements o and a generate a subalgebra of $\mathbf{B}+\mathbf{2}$ which is one of the algebras depicted in Figure 2. Let us observe that because $\mathsf{o} < \mathsf{a}$, these two elements generate an algebra $\mathbf{B}$ such that $\mathbf{B}^\circ$ is a homomorphic image of the Rieger-Nishimura lattice.

Let us observe that for every $k > 2$ algebra $\mathbf{C}_{2k+1}$ cannot be in $\mathcal{V}$ for the following reason: $\mathbf{C}_7 \cong \mathbf{S}_1$, $\mathbf{C}_9[\mathsf{a}) \cong \mathbf{S}_1$, and for all $k > 2$, subalgebra $\mathbf{C}_{2k+1}[\mathsf{b})$ is isomorphic to $\mathbf{S}_1$, while $\mathbf{S}_1 \notin \mathcal{V}$. Thus, the subalgebra of $\mathbf{B} + \mathbf{2}$ generated by a is either $\mathbf{C_3} = \mathbf{2} + \mathbf{2}$ or $\mathbf{C}_5 = \mathbf{4} + \mathbf{2}$. In any case, $\mathsf{a} \vee (\mathsf{a} \rightarrow \mathsf{o}) = 1_{\mathbf{B}}$. Hence, in $\mathbf{B}^\circ$ we have $\mathsf{a} \vee \neg\mathsf{a} = 1_{\mathbf{B}}$, that is, $\mathbf{B}^\circ$ is a Boolean algebra. □

§3. Proof of Lemma 1.3. We say that an algebra $\mathbf{A}$ is *totally non-projective* if $\mathbf{A}$ is not weakly projective in the variety $\mathsf{V}(\mathbf{A})$ generated by algebra $\mathbf{A}$.

Proof. Our goal is to prove that algebras $\mathbf{S}_1$ and $\mathbf{S}_2$ (see Figure 3) are totally non-projective. To this end, it is enough to prove that for both $i = 1, 2$

$$\mathbf{S}'_i \in \mathsf{V}(\mathbf{S}_i) \text{ and } \mathbf{S}'_i \in \mathbb{HS}'_i, \text{ while } \mathbf{S}_i \notin \mathbb{SS}'_i.$$

First, let us observe that $\mathbf{C} \in \mathsf{V}(\mathbf{S}_i)$, $i = 1, 2$, because $\mathbf{C} \cong \mathbf{S}_i[\mathsf{a}) \in \mathbb{SS}_i \subseteq \mathsf{V}(\mathbf{S}_i)$.

Second, let us observe that $\mathbf{S}'_i$ is a subdirect product of $\mathbf{S}_i$ and $\mathbf{C}$. Indeed, in $\mathbf{S}'_1$ we have $\mathsf{a} \vee \mathsf{b} = 1$ and $\mathbf{S}_1(\mathsf{a}] \cong \mathbf{S}_1$ and $\mathbf{S}_1(\mathsf{c}] \cong \mathbf{C}$. Similarly, in $\mathbf{S}'_2$ we have $\mathsf{b} \vee \mathsf{c} = 1$ and $\mathbf{S}_2(\mathsf{b}] \cong \mathbf{B}$ and $\mathbf{S}_2(\mathsf{c}] \cong \mathbf{C}$. Thus, we established that $\mathbf{S}'_i \in \mathsf{V}(\mathbf{S}_i)$ and $\mathbf{S}_i \in \mathbb{HS}'_i$.

Now, let us demonstrate that $\mathbf{S}_1 \notin \mathbb{SS}'_1$. The proof follows from the observation that

$$((\mathsf{a} \rightarrow \mathsf{b}) \rightarrow \mathsf{b}) \rightarrow \mathsf{a} = \mathsf{a} \vee (\mathsf{a} \rightarrow \mathsf{b}) \text{ and } ((\mathsf{a} \rightarrow \mathsf{b}) \rightarrow \mathsf{b}) \vee (\mathsf{a} \rightarrow \mathsf{b}) \neq 1,$$

while algebra $\mathbf{S}'_1$ does not contain such a pair of elements.

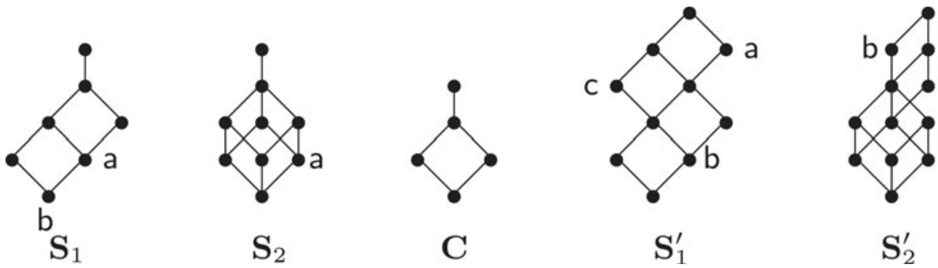

Fig. 3. Examples of Brouwerian algebras.

To prove that $\mathbf{S}_2$ is not embedded in $\mathbf{S}_2'$, note that any embedding sends any three mutually incomparable elements of $\mathbf{S}_2$ in three mutually incomparable elements of $\mathbf{S}_2'$, but whole algebra $\mathbf{S}_2'$ is generated by any its three mutually incomparable elements (see Figure 3). □

§4. Proof of Lemma 1.4. Let $\mathcal{V}$ be a variety excluding algebra $\mathbf{S}_1$ (see Figure 1). We need to prove that $\mathcal{V}$ is locally finite.

First, let us recall that a set of algebras $\mathcal{K}$ is said to be *uniformly locally finite* if for each finite m there exists a finite $n = f(m)$ such that any m elements of any algebra from $\mathcal{K}$ generate a subalgebra of cardinality that does not exceed n. To prove the claim we will demonstrate that the set $\mathcal{V}_{\mathsf{fgsi}}$ of all finitely generated s.i. algebras from $\mathcal{V}$ is uniformly locally finite, and then we can use the following proposition and complete the proof.

PROPOSITION 4.1 ((Mal'cev, 1973, sec. 14 Theorem 3)). *A variety of finite signature is locally finite if and only if it is generated by a uniformly locally finite family of algebras.*

Indeed, let $\mathbf{A} \in \mathcal{V}_{\mathsf{fgsi}}$ be an m-generated algebra s.i. algebra from $\mathcal{V}$. Then, by Proposition 2.9,

$$\mathbf{A} = \sum_{i=0}^{k} \mathbf{A}_i + \mathbf{2}, \text{ where } k \leq 2m,$$

and, by Proposition 2.10, every $\mathbf{A}_i^\circ$ is Boolean.

From Proposition 2.7 it follows immediately that for each $i \leq k$ algebra $\mathbf{A}_i$ is m-generated: the algebra $\mathbf{A}_i$ is a homomorphic image of subalgebra $\mathbf{A}[\mathsf{a}_i)$, where a_i is the bottom element of $\mathbf{A}_i$.

Now, let us recall that m-generated Boolean algebra contains at most 2^{2^m} elements. Hence, $|\mathbf{A}_i| \leq 2^{2^m}$ for each i, and therefore,

$$|\mathbf{A}| \leq (2m)2^{2^m} + 1$$

and consequently, the set $\mathcal{V}_{\mathsf{fgsi}}$ is uniformly locally finite. Thus, the conditions of Proposition 4.1 are satisfied, and we can complete the proof.

§5. Proof of Lemma 1.5. Let $\mathcal{V}$ be a variety not containing algebras $\mathbf{S}_1, \mathbf{S}_2$ (see Figure 1). We need to prove that every algebra $\mathbf{A} \in \mathcal{V}_{\mathsf{fnsi}}$, where $\mathcal{V}_{\mathsf{fnsi}}$ is a set of all finite s.i. algebras from $\mathcal{V}$, is weakly projective in $\mathcal{V}_{\mathsf{fin}}$.

Indeed, suppose that $\mathbf{A} \in \mathcal{V}_{\mathsf{fnsi}}$. Because $\mathbf{S}_1 \notin \mathcal{V}$, we can apply Proposition 2.10 and conclude that

$$\mathbf{A} = \sum_{i=1}^{n} \mathbf{A}_i + \mathbf{2}, \tag{8}$$

where $\mathbf{A}_i^\circ$ are Boolean algebras. Let us prove that $|\mathbf{A}_i| \leq 4$ for all $i = 1, \ldots, n$.

For contradiction: assume that for some $1 \leq i \leq n$, $|\mathbf{A}_i| > 4$ and consider subalgebra $\mathbf{A}_i + \mathbf{2}$. Let $\mathbf{B}$ be 8-element Boolean algebra. Then, $\mathbf{B}$ embeds in $\mathbf{A}_i$. Thus, $\mathbf{B} + \mathbf{2}$ embeds in $\mathbf{A}_i + \mathbf{2}$: one can easily extend any embedding of $\mathbf{B}$ in $\mathbf{A}_i$ to the embedding of $\mathbf{B} + \mathbf{2}$ in $\mathbf{A}_i + \mathbf{2}$. Clearly, $\mathbf{A}_i + \mathbf{2} \leq \mathbf{A} \in \mathcal{V}$. Hence, $\mathbf{B} + \mathbf{2} \in \mathcal{V}$. Note that $\mathbf{B} + \mathbf{2}$ is isomorphic to $\mathbf{S}_2$, and we arrived to contradiction with the assumption $\mathbf{S}_2 \notin \mathcal{V}$.

Thus, we established that in (8), for all $i = 1, \ldots, n$, $\mathbf{A}_i \in \{\mathbf{2}, \mathbf{4}\}$, and we need to prove that $\mathbf{A}$ is weakly projective in $\mathcal{V}_{\mathsf{fin}}$ - the set of all finite algebras from $\mathcal{V}$.

Suppose that algebra $\mathbf{A}$ of form (8) is a homomorphic image of some algebra $\mathbf{C} \in \mathcal{V}_{\mathsf{fin}}$. Because $\mathbf{A}$ and $\mathbf{C}$ are finite, they can be viewed as Heyting algebras. Moreover, if $\mathbf{A}$ is a Brouwerian homomorphic image of $\mathbf{C}$, $\mathbf{A}^\circ$ is a Heyting homomorphic image of $\mathbf{C}^\circ$. Recall Balbes & Horn (1970) that finite Heyting algebras of form (8) are projective. Thus, taking into account that each Heyting embedding is at the same time a Brouwerian embedding, we conclude that $\mathbf{A}$ is embedded in $\mathbf{C}$, and this proves that $\mathbf{A}$ is weakly projective in $\mathcal{V}_{\mathsf{fin}}$.

A proof of Theorem 1.9 immediately follows from the above proof.

§6. Proof of Theorem 1.8.

In the proof of Corollary 1.7 we use Theorem 1.8. Let us prove it first.

In this section $\mathcal{V}$ is assumed to be a locally finite congruence distributive variety of algebras. By $L_{\mathsf{V}}\mathcal{V}$ we denote the lattice of all subvarieties of $\mathcal{V}$, and by $\mathcal{V}_{\mathsf{fnsi}}$ we denote the set of all finite subdirectly irreducible algebras from $\mathcal{V}$.

We recall from (Day, 1973, Corollary 3.8) that every algebra $\mathbf{A} \in \mathcal{V}_{\mathsf{fnsi}}$ is a splitting algebra, which means that there is a variety $\overline{\mathsf{V}}(\mathbf{A}) \in L_{\mathsf{V}}\mathcal{V}$ such that for every $\mathcal{W} \in L_{\mathsf{V}}\mathcal{V}$,

$$\text{either } \mathsf{V}(\mathbf{A}) \subseteq \mathcal{W} \text{ or } \mathcal{W} \subseteq \overline{\mathsf{V}}(\mathbf{A}). \tag{9}$$

In other words, $\overline{\mathsf{V}}(\mathbf{A})$ is the greatest variety from $L_{\mathsf{V}}\mathcal{V}$ that does not contain $\mathbf{A}$. The variety $\overline{\mathsf{V}}(\mathbf{A})$ can be defined relative to $\mathcal{V}$ by a single identity which we denote by $\mathsf{i}_{\mathbf{A}}$. In case of Heyting or Brouwerian algebras, $\mathsf{i}_{\mathbf{A}}$ can be taken as $X(\mathbf{A}) \approx \mathsf{1}$, where $X(\mathbf{A})$ is a characteristic (Yankov) formula of $\mathbf{A}$ (cf. Jankov (1969)).

On the set $\mathcal{V}_{\mathsf{fnsi}}$ we define a quasi-order: if $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{\mathsf{fnsi}}$, then

$$\mathbf{A} \preceq \mathbf{B} \Longleftrightarrow \mathbf{A} \in \mathsf{V}(\mathbf{B}). \tag{10}$$

Let us note that, because $\mathbf{A}$ and $\mathbf{B}$ are finite s.i. algebras, by Jónsson's Lemma $\mathbf{A} \preceq \mathbf{B}$ if and only if $\mathbf{A} \in \mathbb{HSP}_u\mathbf{B}$. Because $\mathbf{B}$ is finite, $\mathbb{HSP}_u\mathbf{B} = \mathbb{HS}\mathbf{B}$. Hence,

$$\mathbf{A} \preceq \mathbf{B} \text{ yields } |\mathbf{A}| \leq |\mathbf{B}|. \tag{11}$$

Therefore, if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{B} \preceq \mathbf{A}$, we have $|\mathbf{A}| = |\mathbf{B}|$, and because $\mathbf{A}$ and $\mathbf{B}$ are finite, we have $\mathbf{A} \cong \mathbf{B}$. Thus, $\preceq$ is a partial order.

Let us observe that (11) entails that every descending w.r.t. $\preceq$ chain of algebras from $\mathcal{V}_{\mathsf{fnsi}}$ contains only a finite number of non-isomorphic members. That is, $\preceq$ enjoys the descending chain condition. Hence, every set $\{\mathbf{A}_i, i \in I\} \subseteq \mathcal{V}_{\mathsf{fnsi}}$ contains minimal elements $\mathbf{A}_j, j \in J \subseteq I$ and for every $i \in I$ there is an algebra $\mathbf{A}_j$ minimal w.r.t. $\preceq$ and such that $\mathbf{A}_j \preceq \mathbf{A}_i$.

If $\{\mathbf{A}_i, i \in I\}$ are algebras from $\mathcal{V}_{\mathsf{fnsi}}$, we say that $\{\mathbf{A}_i, i \in I\}$ is an *antichain* if for all distinct $i, j \in I$ algebras $\mathbf{A}_i$ and $\mathbf{A}_j$ are incomparable, that is $\mathbf{A}_i \not\preceq \mathbf{A}_j$ and $\mathbf{A}_j \not\preceq \mathbf{A}_i$.

A variety $\mathcal{V}$ is *hereditarily finitely based* or it is a *Specht variety*[4], if $\mathcal{V}$ and all its subvarieties are finitely based. Theorem 1.8 is a trivial consequence of the following theorem which gives a criterion for a locally finite finitely based congruence distributive variety to be a Specht variety.

THEOREM 6.1. *Let $\mathcal{V}$ be a locally finite finitely based congruence distributive variety. Then, the following is equivalent*:

(a) *$\mathcal{V}$ is a Specht variety*;
(b) *$L_{\mathsf{V}}\mathcal{V}$ is at most countable*;
(c) *$\mathcal{V}_{\mathsf{fnsi}}$ has no infinite antichains*;
(d) *$L_{\mathsf{V}}\mathcal{V}$ enjoys the descending chain condition.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c) (comp. (Grätzer & Quackenbush, 2010, Theorem 5.1)) To prove the claim we will show that if $\mathcal{V}_{\mathsf{fnsi}}$ contains an infinite antichain, then $L_{\mathsf{V}}\mathcal{V}$ is not countable.

Indeed, suppose that $\mathsf{C} := \{\mathbf{A}_i, i \in I\} \subseteq L_{\mathsf{V}}\mathcal{V}$ is an infinite antichain. Let us demonstrate that for every distinct subsets $I_0, I_1 \subseteq I$ the varieties $\mathcal{V}_0 := \mathsf{V}(\{\mathbf{A}_i, i \in I_0\})$ and $\mathcal{V}_1 := \mathsf{V}(\{\mathbf{A}_i, i \in I_1\})$ are distinct.

Let I_0 and I_1 be distinct subsets if I. Because $I_0 \neq I_1$, without loosing generality, we can assume that there is $i \in I_0$ such that $i \notin I_1$. Now, let us consider algebra $\mathbf{A}_i$. Because C is an antichain, $\mathbf{A}_i \not\preceq \mathbf{A}_j$ for all $j \neq i$ and we have $\mathbf{A}_i \notin \mathsf{V}(\mathbf{A}_j)$, that is, $\mathsf{V}(\mathbf{A}_i) \nsubseteq \mathsf{V}(\mathbf{A}_j)$. Thus, by (9), we have that for all $j \in I_1$

$$\mathsf{V}(\mathbf{A}_j) \subseteq \overline{\mathsf{V}}(\mathbf{A}_i).$$

Hence $\mathcal{V}_1 \subseteq \overline{\mathsf{V}}(\mathbf{A})$. An observation that $\mathbf{A}_i \notin \overline{\mathsf{V}}(\mathbf{A}_i)$, and therefore, $\mathbf{A}_i \notin \mathcal{V}_1$ completes the proof that $\mathcal{V}_0 \neq \mathcal{V}_1$. As we saw, any $I_0 \subset I$ uniquely defines a subvariety of $\mathcal{V}$, and because I is infinite, there is a continuum distinct subquasivarieties of $\mathcal{V}$.

(c) $\Rightarrow$ (d) For contradiction: assume that $\mathcal{V}_{\mathsf{fnsi}}$ has no infinite antichains and $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots$ is a strongly descending chain of subvarieties of $\mathcal{V}$. Then, for each $i \geq 0$, there is a finitely generated s.i. algebra $\mathbf{A}_i \in \mathcal{V}_i \setminus \mathcal{V}_{i+1}$. Recall that $\mathcal{V}$, and therefore each $\mathcal{V}_i$, $i \geq 0$, is locally finite. Thus, all algebras $\mathbf{A}_i$ are finite and $\mathbf{A}_i \in \mathcal{V}_{\mathsf{fnsi}}$. As we pointed out earlier, the set $\mathbf{A}_i$, $i \geq 0$ contains a subset of minimal relative to $\preceq$ members. Let $\{\mathbf{A}_i, i \in I\}$ be a set of all minimal elements. Then,

$$\text{for any } n \geq 0 \text{ there is } j \in I \text{ such that } \mathbf{A}_j \preceq \mathbf{A}_n,$$

that is,

$$\text{for any } n \geq 0 \text{ there is } j \in I \text{ such that } \mathbf{A}_j \in \mathsf{V}(\mathbf{A}_n),$$

or, equivalently,

$$\text{for any } n \geq 0 \text{ there is } j \in I \text{ such that } \mathsf{V}(\mathbf{A}_j) \subseteq \mathsf{V}(\mathbf{A}_n). \tag{12}$$

Next, we observe that, because of the minimality of its members, the set $\{\mathbf{A}_i, i \in I\}$ forms an antichain, and hence, by our assumption, it is finite. Suppose that I does not

[4] In 1950, W. Specht was investigating whether or not every variety of associative algebras is finitely based. Nowadays, the hereditarily finitely based varieties are often referred to as Specht varieties (cf., e.g., Bahturin & Ol'shanskij. (1991)).

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contain numbers exceeding k. Then, by our selection of algebras $\mathbf{A}_i$, we have

$$\mathbf{A}_j \notin \mathcal{V}_{k+1} \text{ for all } j \leq k. \tag{13}$$

On the other hand, consider $\mathbf{A}_{k+1}$. By selection, $\mathbf{A}_{k+1} \in \mathcal{V}_{k+1} \setminus \mathcal{V}_{k+2}$, hence, $\mathbf{A}_{k+1} \in \mathcal{V}_{k+1}$ and

$$\mathsf{V}(\mathbf{A}_{k+1}) \subseteq \mathcal{V}_{k+1}. \tag{14}$$

By (12), for some $j \leq k$ there is a minimal algebra $\mathbf{A}_j$ such that

$$\mathsf{V}(\mathbf{A}_j) \subseteq \mathsf{V}(\mathbf{A}_{k+1}). \tag{15}$$

Thus, from (15) and (14), we have

$$\mathbf{A}_j \in \mathsf{V}(\mathbf{A}_j) \subseteq \mathsf{V}(\mathbf{A}_{k+1}) \subseteq \mathcal{V}_{k+1}$$

and this contradicts (13).

(d) $\Rightarrow$ (a) Suppose that $L_{\mathsf{V}}\mathcal{V}$ enjoys the descending chain condition. By the theorem's assumption, $\mathcal{V}$ is finitely based. Thus we need to demonstrate that every proper subvariety $\mathcal{V}' \subset \mathcal{V}$ is finitely based relative to $\mathcal{V}$.

Indeed, let $\{\mathrm{i}_\mu, \mu < \sigma\}$ be a set of all identities such that $\mathcal{V}' \models \mathrm{i}_\mu$ and $\mathcal{V} \not\models \mathrm{i}_\mu$. For each $\kappa < \sigma$, consider variety $\mathcal{V}_\kappa$ defined relative to $\mathcal{V}$ by identities $\mathrm{i}_\mu, \mu \leq \kappa$. It is clear that $\kappa \leq \kappa'$ yields $\mathcal{V}_\kappa \supseteq \mathcal{V}_{\kappa'}$, hence,

$$\{\mathcal{V}_\kappa \mid \kappa < \sigma\} \text{ is a descending chain such that } \bigcap_{m \geq 0} \mathcal{V}_m = \mathcal{V}'. \tag{16}$$

Because $L_{\mathsf{V}}\mathcal{V}$ enjoys the descending chain condition, $\{\mathcal{V}_\kappa \mid \kappa < \sigma\}$ is finite. Hence, by (16), for some $n < \omega$, $\mathcal{V}'$ coincides with $\mathcal{V}_n$, and this means that $\mathcal{V}'$ is defined relative to $\mathcal{V}$ by a finite set of identities, namely by $\mathrm{i}_k, k < n$. □

§7. Proof of main corollary. *Proof of* (a). We need to prove that there is the smallest *HSCpl* positive logic. To prove this, we prove that there is the greatest primitive variety of Brouwerian algebras and this variety is finitely based.

Indeed, the variety of Brouwerian algebras is congruence distributive. By Theorem 1.2, a variety $\mathcal{V}$ is primitive if and only if $\mathbf{S}_1, \mathbf{S}_2 \notin \mathcal{V}$. Moreover, by Lemma 1.4, it is locally finite. Algebras $\mathbf{S}_1$ and $\mathbf{S}_2$ are subdirectly irreducible. Hence, by (9), there is the greatest variety $\overline{\mathsf{V}(\mathbf{A})}$ that does not contain $\mathbf{A}$, and $\overline{\mathsf{V}(\mathbf{A})}$ can be defined by a single identity. Thus, variety $\mathcal{P} = \overline{\mathsf{V}(\mathbf{S}_1)} \cap \overline{\mathsf{V}(\mathbf{S}_1)}$ is the greatest primitive variety of Brouwerian algebras and $\mathcal{P}$ can be defined by two identities, which means that it is finitely based.

Proof of (b). We need to prove that a set of *HSCpl* positive logics is countable. First, we note that the set of all *HSCpl* positive logics is infinite: a locally finite primitive variety $\mathcal{V}$ has a finite set of subvarieties if and only if $\mathcal{V}$ is generated by a finite algebra (see (Gorbunov, 1998, Prop. 5.1.25)), and clearly for primitive varieties of Brouwerian algebras the latter is not the case.

Now, we prove that the set of all primitive varieties of Brouwerian algebras is countable.

Indeed, let $\mathcal{P}$ be the greatest primitive variety of Brouwerian algebras. From Theorem 1.2 we know that $\mathcal{P}$ is locally finite. From Corollary 1.7(a) we know that $\mathcal{P}$ is finitely based. Hence, if we prove that $L_{\mathsf{V}}\mathcal{P}$ does not contain infinite antichains, we will be able to apply Theorem 6.1 and complete the proof.

Recall that a quasi-ordered set $(\mathsf{A}, \preceq)$ is said to be *well quasi-ordered* if it enjoys the descending chain condition and does not contain infinite antichains.

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Recall also from (Huczynska & Ruškuc, 2015, Corollary 1.6) that if $(\mathsf{A}, \preceq)$ is a well quasi-ordered set (an alphabet), then the set A^* of all finite words (strings) over A is well quasi-ordered by the domination ordering:

$$\mathsf{a}_1, \ldots, \mathsf{a}_m \preceq \mathsf{b}_1, \ldots, \mathsf{b}_n \text{ if and only if}$$
$$\exists (1 \leq j_1 < \cdots < j_m \leq n) \forall (i = 1, \ldots, m)(\mathsf{a}_i \preceq \mathsf{b}_{j_i}).$$

For instance, consider a set $\mathbb{N}^*$ of all finite sequences of natural numbers with quasi-order

$$\mathsf{s}_1, \ldots, \mathsf{s}_m \preceq \mathsf{r}_1, \ldots, \mathsf{r}_n \text{ if and only if}$$
$$\exists (1 \leq j_1 < \cdots < j_m \leq n) \forall (i = 1, \ldots, m)(\mathsf{s}_i \leq \mathsf{r}_{j_i}).$$

Then, $(\mathbb{N}^*, \preceq)$ is well quasi-ordered.

It is also known that a class of well quasi-ordered sets is closed under: (i) taking of subsets; (ii) homomorphic images – images of the surjective order-preserving maps (see, e.g., (Huczynska & Ruškuc, 2015, Theorem 1.2)).

In particular, subset $(\widehat{\mathbb{N}}^*, \preceq)$ of all finite sequences that have no subsequent positive components is well quasi-ordered. For instance, $1, 2, 3 \notin \widehat{\mathbb{N}}^*$, while $1, 0, 2, 0, 3 \in \widehat{\mathbb{N}}^*$.

Observe that all algebras in $\mathcal{P}_{\mathsf{fnsi}}$ are weakly projective. Hence, for all $\mathbf{A}, \mathbf{B} \in \mathcal{P}$,

$$\mathbf{A} \preceq \mathbf{B} \text{ if and only if } \mathbf{A} \in \mathbb{HS}\mathbf{B} \text{ if and only if } \mathbf{A} \in \mathbb{S}\mathbf{B} \text{ if and only if } \mathbf{A} \leq \mathbf{B}.$$

We also know that every algebra from $\mathcal{P}_{\mathsf{fnsi}}$ is of form $\mathbf{A}_0 + \cdots + \mathbf{A}_n + \mathbf{2}$, where $\mathbf{A}_i \in \{\mathbf{2}, \mathbf{4}\}$ for all $i \leq n$.

Let us consider a quasi-ordered set $(\mathsf{P}, \leq)$ of all finite subdirectly irreducible Brouwerian algebras of type $\mathbf{A}_0 + \mathbf{A}_1 \cdots + \mathbf{A}_n + \mathbf{2}$, where $\mathbf{A}_i \in \{\mathbf{2}, \mathbf{4}\}$ and $\mathbf{A} \leq \mathbf{B} \leftrightharpoons \mathbf{A} \in \mathbb{S}\mathbf{B}$. For each $m > 0$ by $m\mathbf{4}$ we denote algebra $\underbrace{\mathbf{4} + \cdots + \mathbf{4}}_{m \text{ times}}$, and $0\mathbf{4}$ denotes $\mathbf{2}$.

Let us note that

$$n \leq m, \text{ entails } n\mathbf{4} \leq m\mathbf{4}. \tag{17}$$

We will need the following simple properties of bounded Brouwerian algebras, the proof of which is left to the reader.

PROPOSITION 7.1. *Let* $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2$ *be bounded algebras such that* $\mathbf{B}_1 \leq \mathbf{C}_1$, $\mathbf{B}_2 \leq \mathbf{C}_2$ *and* $\mathbf{A}$ *is nontrivial. Then, the following holds*:

(1) $\mathbf{B}_1 + \mathbf{2} + \mathbf{B}_2 \leq \mathbf{C}_1 + \mathbf{A} + \mathbf{C}_2$;
(2) $\mathbf{B}_1 + \mathbf{A} \leq \mathbf{C}_1 + \mathbf{A}$, *in particular,* $\mathbf{B}_1 + \mathbf{2} \leq \mathbf{C}_1 + \mathbf{2}$;
(3) $\mathbf{B}_1 + \mathbf{2} \leq \mathbf{C}_1 + \mathbf{A}$;
(4) $\mathbf{B}_1 \leq \mathbf{A} + \mathbf{C}_1$;
(5) $\mathbf{B}_1 + \mathbf{2} + \mathbf{B}_2 \leq \mathbf{B}_1 + \mathbf{A} + \mathbf{B}_2$.

Let us point out that, contrary to the situation with Heyting algebras where $\mathbf{B}_1 \leq \mathbf{C}_1$ and $\mathbf{B}_2 \leq \mathbf{C}_2$ yields $\mathbf{B}_1 + \mathbf{B}_2 \leq \mathbf{C}_1 + \mathbf{C}_2$, for Brouwerian algebras this needs not be true. Nevertheless, for bounded Brouwerian algebras the following holds:

$$\mathbf{B}_1 + \mathbf{2} \leq \mathbf{C}_1 \text{ and } \mathbf{B}_2 \leq \mathbf{C}_2, \text{ then } \mathbf{B}_1 + \mathbf{2} + \mathbf{B}_2 \leq \mathbf{C}_1 + \mathbf{C}_2. \tag{18}$$

PROPOSITION 7.2. *The quasi-ordered set* $(\mathsf{P}, \leq)$ *is well quasi-ordered.*

Proof. Because a homomorphic image of any well quasi-ordered set is well quasi-ordered, to prove that P is well quasi-ordered it is sufficient to demonstrate that the map

$\varphi : (\widehat{\mathbb{N}}^*, \preceq) \longrightarrow (\mathsf{P}, \leq)$ defined as follows:

$$\varphi : k_1, \ldots, k_m \mapsto k_1\mathbf{4} + \cdots + k_m\mathbf{4} + \mathbf{2}, \tag{19}$$

is a homomorphism of $(\widehat{\mathbb{N}}^*, \preceq)$ onto $(\mathsf{P}, \leq)$.

It is not hard to see that φ maps $\widehat{\mathbb{N}}^*$ onto P.

To prove that φ is a homomorphism, we need to demonstrate that

$$k_1, \ldots, k_m \preceq l_1, \ldots, l_n \text{ yields } k_1\mathbf{4} + \cdots + k_m\mathbf{4} + \mathbf{2} \leq l_1\mathbf{4} + \cdots + l_n\mathbf{4} + \mathbf{2}.$$

We prove the claim by induction on m.

Basis. If $m = 1$, that is, $k_1 \preceq l_1, \ldots, l_n$, there is $1 \leq s \leq n$ such that $k_1 \leq l_s$. Thus, we have

$$\sum_{j=1}^{n} l_j\mathbf{4} + \mathbf{2} = \mathbf{A} + l_s\mathbf{4} + \mathbf{C} + \mathbf{2}, \tag{20}$$

where

$$\mathbf{A} := \sum_{j=1}^{i_s-1} l_j\mathbf{4} \text{ and } \mathbf{C} = \sum_{j=i_{s+1}}^{n} l_j\mathbf{4} + \mathbf{2}.$$

Thus, we have

$k_1\mathbf{4} \leq l_s\mathbf{4}$	by (17)
$k_1\mathbf{4} \leq \mathbf{A} + l_s\mathbf{4}$	by Proposition 7.1.4
$k_1\mathbf{4} + \mathbf{2} \leq \mathbf{A} + l_s\mathbf{4} + \mathbf{C} + 2$	by Proposition 7.1.3
$k_1\mathbf{4} + \mathbf{2} \leq l_1\mathbf{4} + \cdots + l_n\mathbf{4} + \mathbf{2}$	by (20).

Assumption. Suppose that for all $s < m$,

$$\text{if } k_1, \ldots, k_s \leq l_1, \ldots, l_n, \text{ then } \sum_{i=1}^{s} k_i\mathbf{4} + \mathbf{2} \leq \sum_{j=1}^{n} l_j\mathbf{4} + \mathbf{2}.$$

Step. Let $k_1, \ldots, k_m \preceq l_1, \ldots, l_n$. Then, by the definition of $\preceq$, there are $1 \leq j_1 < \cdots < j_m \leq n$ such that $k_i \leq l_{j_i}$ for all $i = 1, \ldots, m$. Let us consider two cases: (a) $k_m = 0$ and (b) $k_m > 0$.

Case (a). We have already considered subcase $m = 1$. Let $m > 1$. Therefore, taking into account that $k_m = 0$, that is, $k_m\mathbf{4} = \mathbf{2}$, we have,

$$\sum_{i=1}^{m} k_i\mathbf{4} = \mathbf{B}_1 + \mathbf{2} \text{ and } \sum_{j=1}^{n} l_j\mathbf{4} = \mathbf{C}_1 + l_{j_m}\mathbf{4} + \mathbf{C}_2, \tag{21}$$

where

$$\mathbf{B}_1 := \sum_{i=1}^{m-1} k_i\mathbf{4}, \quad \mathbf{C}_1 := \sum_{j=1}^{j_m-1} l_j\mathbf{4}, \quad \mathbf{C}_2 := \sum_{j=j_m+1}^{n} l_j\mathbf{4}$$

Then,

(a)	$\mathbf{B}_1 + \mathbf{2} \leq \mathbf{C}_1 + \mathbf{2}$	induction assumption
(b)	$(\mathbf{B}_1 + \mathbf{2}) + \mathbf{2} \leq (\mathbf{C}_1 + \mathbf{2}) + \mathbf{2}$	by Proposition 7.1.2
(c)	$(\mathbf{C}_1 + \mathbf{2}) + \mathbf{2} \leq (\mathbf{C}_1 + \mathbf{2}) + \mathbf{C}_2 + \mathbf{2}$	by Proposition 7.1.3
(d)	$\mathbf{C}_1 + \mathbf{2} + (\mathbf{C}_2 + \mathbf{2}) \leq \mathbf{C}_1 + l_{j_m}\mathbf{4} + (\mathbf{C}_2 + \mathbf{2})$	by Proposition 7.1.1
(e)	$\mathbf{B}_1 + \mathbf{2} + \mathbf{2} \leq \mathbf{C}_1 + l_{j_m}\mathbf{4} + \mathbf{C}_2 + \mathbf{2}$	by (b), (c), (d)

And (e) means (see (21)) that $\sum_{i=1}^{m} k_i\mathbf{4} + \mathbf{2} \leq \sum_{j=1}^{m} l_j\mathbf{4} + \mathbf{2}$.

Case (b). Suppose that $k_m > 0$. Recall that word $k_1, \ldots, k_m$ is from $\widehat{\mathbb{N}}^*$. Therefore, $k_{m-1} = 0$. That is,

$$\sum_{i=1}^{m} k_i\mathbf{4} = \sum_{i=1}^{m-2} k_i\mathbf{4} + \mathbf{2} + k_m\mathbf{4} = \mathbf{B}_1 + \mathbf{2} + k_m\mathbf{4}, \tag{22}$$

and

$$\sum_{j=1}^{n} l_j\mathbf{4} = \mathbf{C}_1 + j_{m-1}\mathbf{4} + \mathbf{C}_2' + l_{j_m}\mathbf{4} + \mathbf{C}_2'', \tag{23}$$

where

$$\mathbf{B}_1 := \sum_{i=1}^{m-2} k_i\mathbf{4}, \quad \mathbf{C}_1 := \sum_{j=1}^{j_{m-1}-1} l_j\mathbf{4}, \quad \mathbf{C}_2' := \sum_{j=j_{m-1}+1}^{j_m-1} l_j\mathbf{4}, \quad \mathbf{C}_2' := \sum_{j=j_m+1}^{n} l_j\mathbf{4}.$$

Then,

(a)	$k_m \leq l_{j_m}$	
(b)	$k_m\mathbf{4} \leq l_{j_m}\mathbf{4}$	from (a)
(c)	$k_m\mathbf{4} \leq \mathbf{C}_2' + l_{j_m}\mathbf{4}$	by Proposition 7.1.4
(d)	$k_m\mathbf{4} + \mathbf{2} \leq \mathbf{C}_2' + l_{j_m}\mathbf{4} + \mathbf{C}_2'' + \mathbf{2}$	by Proposition 7.1.3
(e)	$\mathbf{B}_1 + \mathbf{2} \leq \mathbf{C}_1 + \mathbf{2}$	by induction assumption
(f)	$\mathbf{B}_1 + \mathbf{2} + k_m\mathbf{4} + \mathbf{2} \leq \mathbf{C}_1 + \mathbf{2} + \mathbf{C}_2' + l_{j_m}\mathbf{4} + \mathbf{C}_2'' + \mathbf{2}$	from (e), (d)
(g)	$\mathbf{B}_1 + \mathbf{2} + k_m\mathbf{4} + \mathbf{2} \leq \mathbf{C}_1 + l_{j_{m-1}}\mathbf{4} + \mathbf{C}_2' + l_{j_m}\mathbf{4} + \mathbf{C}_2'' + \mathbf{2}$	by Proposition 7.1.5

Thus, by (22) and (23), (g) means that

$$\sum_{i=1}^{m} k_i\mathbf{4} + \mathbf{2} \leq \sum_{j=1}^{n} l_j\mathbf{4} + \mathbf{2}, \tag{24}$$

and therefore, we completed the proof of (b). □

Proof of (c). As we pointed out earlier, (c) follows immediately from (a), (b) and Theorem 1.8.

Proof of (d). We need to prove that there are infinitely many *HSCpl* intermediate logics whose positive fragment is not *HSCpl*.

Indeed, let us consider Heyting algebras $\mathbf{C}_9^\circ$, $(\mathbf{C}_9 + \mathbf{2})^\circ$, $(\mathbf{C}_9 + \mathbf{2} + \mathbf{2})^\circ, \ldots,$ where diagram of $\mathbf{C}_9$ is depicted in Figure 2. These algebras are finite non-isomorphic subdirectly irreducible members of a congruence distributive variety. Hence, by (Jónsson, 1967, Corollary 3.5), the varieties generated by these algebras are distinct. Consequently, the intermediate logics $\mathsf{L}_0, \mathsf{L}_1, \mathsf{L}_2, \ldots$ defined by these algebras are distinct. Also, by virtue of Theorem 5.4.8 from Rybakov (1997) (or from Lemma 15 from Citkin (1987)), all logics L_i, $i \geq 0$ are *HSCpl*. On the other hand, algebra $\mathbf{S}_1$ (see Figure 1) is a subalgebra of each of algebras $\mathbf{C}_9, \mathbf{C}_9 + \mathbf{2}, \mathbf{C}_9 + \mathbf{2} + \mathbf{2}, \ldots,$ and hence, $\mathbf{S}_1$ is a model of positive fragments of each logic L_i, $i \geq 0$. Thus, by Theorem 1.1, and these positive fragments are not *HSCpl*.

§8. Final Remarks. To underscore a difference between structural completeness in the intermediate and positive logics, let us note that for any intermediate logic L and any positive formulas $A_1, \ldots, A_n, B$, admissibility in L of rule $r := A_1, \ldots, A_n/B$ entails that r

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is admissible in L^+. But with regard to structural completeness, the relation between L and L^+ is more complex: from Corollary 1.7(d) it follows that there are structurally complete intermediate logics, positive fragment of which is not structurally complete. Below we give a particularly important example of a structurally complete intermediate logic that has a positive fragment which is not structurally complete.

It is known from Prucnal (1976) that Medvedev's logic ML is structurally complete. Recall from Jankov (1968) that for any intermediate logic L between Int and KC – the logic of week law of the excluded middle – we have $\mathsf{L}^+ = \mathsf{Int}^+$. Also, it was observed that $\mathsf{Int} \subseteq \mathsf{ML} \subseteq \mathsf{KC}$ (see Maksimova, Skvorcov, & Šehtman (1979)). Hence, $\mathsf{ML}^+ = \mathsf{Int}^+$. The latter entails that any positive rule admissible in Int is admissible in ML^+. For instance, Mints' rule (1), which is admissible in Int^+, is admissible in ML^+. On the other hand, Mints' rule is not derivable in Int and therefore, it is not derivable in ML^+. Thus, ML^+ is not structurally complete.

REMARK 8.1. *Mints' rule is admissible in* ML^+ (*because it is admissible in* Int *and* $\mathsf{ML}^+ = \mathsf{Int}^+$) *but it is not admissible in* ML. *Indeed, in* (1) *substitute*

$$p \mapsto \neg\neg p \to p$$
$$q \mapsto \neg p \vee \neg\neg p$$
$$r \mapsto \neg\neg p$$

and we get

$$\frac{((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to ((\neg\neg p \to p) \vee \neg\neg p)}{(((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg\neg p \to p)) \vee (((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to \neg\neg p)} \tag{25}$$

Because $((\neg\neg p \to p) \to (p \vee \neg p)) \to (\neg p \vee \neg\neg p) \in \mathsf{ML}$, *we have*

$$((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to ((\neg\neg p \to p) \vee \neg\neg p) \in \mathsf{ML},$$

On the other hand, note that

$$(((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg\neg p \to p)) \leftrightarrow (\neg\neg p \to p) \in \mathsf{Int},$$

and by the Glivenko Theorem,

$$(((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to \neg\neg p) \leftrightarrow \neg\neg p \in \mathsf{Int}.$$

Hence,

$$(((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg\neg p \to p)) \notin \mathsf{ML} \textit{ and}$$
$$(((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to \neg\neg p) \notin \mathsf{ML}.$$

Thus, because ML *enjoys the disjunction property, Mints' rule is not admissible in* ML.

Note, that ML is the only known structurally complete intermediate logic that is not *HSCpl*. Thus, it is natural to ask the following.

Problem 1. *Is there a structurally complete positive logic that is not hereditarily structurally complete*?

And we would like to remind the following long standing open problem.

Problem 2. *Besides* ML, *is there a structurally complete intermediate logic that is not hereditarily structurally complete*?

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BIBLIOGRAPHY

Bahturin, Y. & Ol'shanskij, A. (1991). Identities. In Kostrikin, A. I. and Shafarevich, I., editors. *Algebra*, Vol. 18. Encyclopedia of Mathematical Sciences. Berlin: Springer, pp. 107–221.

Balbes, R. & Horn, A. (1970). Injective and projective Heyting algebras. *Transactions of the American Mathematical Society,* **148**, 549–559.

Belnap, Jr., N. D., Leblanc, H., & Thomason, R. H. (1963). On not strengthening intuitionistic logic. *Notre Dame Journal of Formal Logic,* **4**, 313–320.

Bergman, C. (2012). *Universal Algebra. Fundamentals and Selected Topics.* Pure and Applied Mathematics (Boca Raton), Vol. 301. Boca Raton, FL: CRC Press.

Bezhanishvili, G. & Grigolia, R. (2005). Locally finite varieties of Heyting algebras. *Algebra Universalis,* **54**(4), 465–473.

Bezhanishvili, N. (2006). Lattices of Intermediate and Cylindric Modal Logics. Ph.D. Thesis, Institute for Logic, Language and Computation University of Amsterdam.

Burris, S. & Sankappanavar, H. P. (1981). *A Course in Universal Algebra*. Graduate Texts in Mathematics, Vol. 78. New York: Springer-Verlag.

Cintula, P. & Metcalfe, G. (2010). Admissible rules in the implication-negation fragment of intuitionistic logic. *Annals of Pure and Applied Logic.* **162**(2), 162–171.

Citkin, A. (1987). Structurally complete superintuitionistic logics and primitive varieties of pseudo-Boolean algebras. *Matematicheskie Issledovaniya,* **98**, 134–151. (A. Tsitkin).

Citkin, A. I. (1978). Structurally complete superintuitionistic logics. *Doklady Akademii Nauk SSSR,* **241**(1), 40–43.

Citkin, A. I. (1986). Finite axiomatizability of locally tabular superintuitionistic logics. *Matematicheskie Zametki,* **40**(3), 407–413, 430.

Day, A. (1973). Splitting algebras and a weak notion of projectivity. In Fajtlowicz, S., and Kaiser, K., editors. *Proceedings of the University of Houston Lattice Theory Conference* (*Houston, Tex., 1973*). University of Houston, Houston, Texas: Department of Mathematics, pp. 466–485.

Dzik, W. (2004). Chains of structurally complete predicate logics with the application of Prucnal's substitution. *Reports on Mathematical Logic,* **38**, 37–48.

Dzik, W. & Wroński, A. (1973). Structural completeness of Gödel's and Dummett's propositional calculi. *Studia Logica,* **32**, 69–75.

Galatos, N., Jipsen, P., Kowalski, T., & Ono, H. (2007). *Residuated Lattices*: *An Algebraic Glimpse at Substructural Logics.* Studies in Logic and the Foundations of Mathematics, Vol. 151. Amsterdam: Elsevier B. V.

Gorbunov, V. A. (1998). *Algebraic Theory of Quasivarieties*. Siberian School of Algebra and Logic. New York: Consultants Bureau. Translated from the Russian.

Grätzer, G. & Quackenbush, R. W. (2010). Positive universal classes in locally finite varieties. *Algebra Universalis,* **64**(1–2), 1–13.

Harrop, R. (1960). Concerning formulas of the types $A \rightarrow B \bigvee C$, $A \rightarrow (Ex)B(x)$ in intuitionistic formal systems. *Journal of Symbolic Logic,* **25**, 27–32.

Huczynska, S. & Ruškuc, N. (2015). Well quasi-order in combinatorics: Embeddings and homomorphisms. In Czumaj, A., editor. *Surveys in Combinatorics 2015*. London Mathematical Society Lecture Note Series, Vol. 424. Cambridge: Cambridge Universtiy Press, pp. 261–293.

Jankov, V. A. (1968). Calculus of the weak law of the excluded middle. *Izvestiya Akademii Nauk SSSR Seriya Matematicheskaya,* **32**, 1044–1051. English translation in *Mathematics of the USSR-Izvestiya*, **2**(5), 997–1004 (1968).

Jankov, V. A. (1969). Conjunctively irresolvable formulae in propositional calculi. *Izvestiya Akademii Nauk SSSR Seriya Matematicheskaya,* **33**, 18–38. English translation in *Mathematics of the USSR-Izvestiya*, **3**(1), 17–35 (1969).

Jónsson, B. (1967). Algebras whose congruence lattices are distributive. *Mathematica Scandinavica,* **21**, 110–121 (1968).

Kuznetsov, A. V. (1973). On finitely generated pseudo-Boolean algebras and finitely approximable varieties. In Starostin, A. I., editor. *Proceedings of the 12th USSR Algebraic Colloquium*. Sverdlovsk, p. 281 (in Russian).

Kuznetsov, A. V. & Gerčiu, V. J. (1970). The superintuitionistic logics and finite approximability. *Doklady Akademii Nauk SSSR,* **195**, 1029–1032. (in Russian).

Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXVIII. Springer-Verlag, Berlin-Göttingen-Heidtem

Maksimova, L. L., Skvorcov, D. P., & Šehtman, V. B. (1979). Impossibility of finite axiomatization of Medvedev's logic of finite problems. *Doklady Akademii Nauk SSSR,* **245**(5), 1051–1054.

Mal'cev, A. (1973). *Algebraic systems*. Die Grundlehren der mathematischen Wissenschaften. Band 192. Berlin-Heidelberg-New York: Springer-Verlag; Berlin: Akademie-Verlag. p. XII, 317.

Mints, G. (1976). Derivability of admissible rules. *Journal of Soviet Mathematics,* **6**, 417–421. Translated from Mints, G. E. Derivability of admissible rules. (Russian) Investigations in constructive mathematics and mathematical logic, V. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 32 (1972), pp. 85–89.

Nemitz, W. & Whaley, T. (1973). Varieties of implicative semilattices II. *Pacific Journal of Mathematics,* **45**, 303–311.

Novikov, P. S. (1977). *Konstruktivnaya matematicheskaya logika s tochki zreniya klassicheskoi* [*Constructive mathematical logic from the point of view of classical logic*]. Izdat. "Nauka", Moscow. With a preface by S. I. Adjan, Matematicheskaya Logika i Osnovaniya Matematiki. [Monographs in Mathematical Logic and Foundations of Mathematics] (in Russian).

Odintsov, S. P. (2008). *Constructive Negations and Paraconsistency*. Trends in Logic—Studia Logica Library, Vol. 26. New York: Springer.

Olson, J. S., Raftery, J. G., & van Alten, C. J. (2008). Structural completeness in substructural logics. *Logic Journal of the IGPL,* **16**(5), 455–495.

Pogorzelski, W. A. (1974). Concerning the notion of completeness of invariant propositional calculi. *Studia Logica,* **33**, 69–72. (errata insert).

Prucnal, T. (1972a). On the structural completeness of some pure implicational propositional calculi. *Studia Logica,* **30**, 45–52.

Prucnal, T. (1972b). Structural completeness of Lewis's system S5. *Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences, Mathematiques, Astronomiques et Physiques,* **20**, 101–103.

Prucnal, T. (1976). Structural completeness of Medvedev's propositional calculus. *Reports on Mathematical Logic,* **6**, 103–105.

Raftery, J. G. (2016). Admissible rules and the Leibniz hierarchy. *Notre Dame Journal of Formal Logic,* **57**(4), 569–606.

Rasiowa, H. & Sikorski, R. (1970). *The Mathematics of Metamathematics* (third edition). Warsaw: PWN—Polish Scientific Publishers. Monografie Matematyczne, Tom 41.

Rybakov, V. V. (1995). Hereditarily structurally complete modal logics. *Journal of Symbolic Logic,* **60**(1), 266–288.

Rybakov, V. V. (1997). *Admissibility of Logical Inference Rules*. Studies in Logic and the Foundations of Mathematics, Vol. 136. Amsterdam: North-Holland.

Skura, T. (1991). On decision procedures for sentential logics. *Studia Logica,* **50**(2), 173–179.

Troelstra, A. S. (1965). On intermediate propositional logics. *Nederlandse Akademie van Wetenschappen: Proceedings Series A, 68=Indagationes Mathematicae* **27**, 141–152.

Wroński, A. (1986). On factoring by compact congruences in algebras of certain varieties related to the intuitionistic logic. *Institute of the Polish Academy of Sciences, Sociology Bulletin of the Section of Logic* **15**(2), 48–51.

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