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ARTICLE

Clustered 3-colouring graphs of bounded degree

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Abstract

A (not necessarily proper) vertex colouring of a graph has *clustering c* if every monochromatic component has at most c vertices. We prove that planar graphs with maximum degree Δ are 3-colourable with clustering $O(\Delta^2)$. The previous best bound was $O(\Delta^{37})$. This result for planar graphs generalises to graphs that can be drawn on a surface of bounded Euler genus with a bounded number of crossings per edge. We then prove that graphs with maximum degree Δ that exclude a fixed minor are 3-colourable with clustering $O(\Delta^5)$. The best previous bound for this result was exponential in Δ.

Keywords: clustered colouring, planar graphs, graph minors

2020 MSC Codes: Primary 05C10, 05C15, 05C83

1. Introduction

Consider a graph where each vertex is assigned a colour. A *monochromatic component* is a connected component of the subgraph induced by all the vertices assigned a single colour. A graph G is k-colourable with *clustering c* if each vertex can be assigned one of k colours so that each monochromatic component has at most c vertices. There have been several recent papers on clustered colouring [9, 21, 24, 26, 31–33, 37–40, 42, 44, 45, 53]; see [55] for a survey. The general goal of this paper is to prove that various classes of graphs are 3-colourable with clustering bounded by a polynomial function of the maximum degree.

First, consider clustered colouring of planar graphs. The 4-colour theorem [3, 47] says that every planar graph is 4-colourable with clustering 1. This result is best possible regardless of the clustering value: for every integer c, there is a planar graph G such that every 3-colouring of G has a monochromatic component with more than c vertices [1, 24, 34, 55]. All known examples of such graphs have unbounded maximum degree. This led Kleinberg et al. [34] to ask whether planar graphs with bounded maximum degree are 3-colourable with bounded clustering. This question was answered positively by Esperet and Joret [24].

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Three colours is best possible for $\Delta \geqslant 6$, since the Hex Lemma [28] implies that for every integer c, there is a planar graph G with maximum degree 6 such that every 2-colouring of G has a monochromatic component with more than c vertices [35,41]. Furthermore, this degree threshold is best possible, since Haxell et al. [31] proved that every graph with maximum degree 5 (regardless of planarity) is 2-colourable with clustering less than 20,000.

The following natural question arises: what is the least function $c(\Delta)$ such that every planar graph with maximum degree Δ has a 3-colouring with clustering $c(\Delta)$? The clustering function of Esperet and Joret [24] was $\Delta^{O(\Delta)}$. While Esperet and Joret [24] made no effort to optimise this function, exponential dependence on Δ is unavoidable using their method. Recently, Liu and Wood [39] improved this bound to $O(\Delta^{37})$. A primary contribution of this paper (Corollary 4) is to improve it further to $O(\Delta^2)$.

Like the above-mentioned works of Esperet and Joret [24] and Liu and Wood [39], our theorem generalises to graphs with bounded Euler genus.[a] In particular, we prove (in Corollary 5) that graphs with Euler genus g and maximum degree Δ are 3-colourable with clustering $O(g^3\Delta^2)$. The previous best clustering function was $O(g^{19}\Delta^{37})$ due to Liu and Wood [39]. In fact, our result and that of Liu and Wood [39] hold in the more general setting of bounded layered treewidth (defined in Section 2.3). This enables further generalisations. For example, we prove (in Corollary 6) that apex-minor-free graphs are 3-colourable with clustering $O(\Delta^2)$, and graphs that have a drawing on a surface of bounded Euler genus with a bounded number of crossings per edge are 3-colourable with clustering $O(\Delta^2)$. All these results are presented in Section 2.

Section 3 focuses on clustered colouring of graphs excluding a fixed minor. For K_t-minor-free graphs, at least $t-1$ colours are needed regardless of the clustering function; that is, for every integer c there is a K_t-minor-free graph G such that every $(t-2)$-colouring of G has a monochromatic component with more than c vertices [22, 55]. Again, all such examples have unbounded maximum degree. Indeed, in the setting of bounded-degree graphs, qualitatively different behaviour occurs. In particular, Liu and Oum [37] proved that bounded-degree graphs excluding a fixed minor are 3-colourable with bounded clustering (thus generalising the above result of Esperet and Joret [24] for planar graphs and graphs of bounded Euler genus).

Liu and Oum [37] did not state an explicit bound on the clustering function, but it is at least exponential in the maximum degree.[b] We prove (in Theorem 20) that graphs with maximum degree Δ that exclude a fixed minor are 3-colourable with clustering $O(\Delta^5)$. The proof of this result is much simpler than that of Liu and Oum [37], and is based on a new structural description of bounded-degree graphs excluding a minor that is of independent interest (Theorems 19 and 24).

Bounded maximum degree alone is not enough to ensure an absolute bound (independent of the degree) on the number of colours in a clustered colouring. In particular, for all integers $\Delta \geqslant 2$ and c, there is a graph G with maximum degree Δ such that every $\lfloor \frac{\Delta+2}{4} \rfloor$-colouring of G has a monochromatic component with more than c vertices; see [1, 31, 55]. This says that in all of the above results, to achieve an absolute bound on the number of colours, one must assume some structural property (such as bounded treewidth, being planar, or excluding a minor) in addition to assuming bounded maximum degree.

[a] The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer k such that G embeds in a surface of Euler genus k. Of course, a graph is planar if and only if it has Euler genus 0; see [43] for more about graph embeddings in surfaces.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class.

A graph H is *apex* if $H - v$ is planar for some vertex v.

[b] Chun-Hung Liu (private communication, 2020) believes that the method in [37] could be adapted to give a polynomial bound using more advanced graph structure theorems.

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To conclude our literature survey, we mention the results of Liu and Wood [38–40] that generalise the bounded-degree setting. First, Liu and Wood [39] proved that for all $s, t, k \in \mathbb{N}$, there exists $c \in \mathbb{N}$ such that every graph with layered treewidth k and with no $K_{s,t}$ subgraph is $(s+2)$-colourable with clustering c. The case $s = 1$ is equivalent to the bounded-degree setting; thus, this result generalises the above-mentioned 3-colouring results for graphs with bounded maximum degree. For $s \geqslant 2$, the clustering function here is very large, and the proof is 70+ pages long. In the setting of excluded minors, Liu and Wood [38] proved that for all $s, t \in \mathbb{N}$ and for every graph H, there is an integer c such that every graph with no H-minor and with no $K_{s,t}$-subgraph is $(s+2)$-colourable with clustering c. Similar results are obtained for excluded topological minors [40].

2. Planar graphs and generalisations

This section proves that planar graphs with maximum degree Δ (and other more general classes) are 3-colourable with clustering $O(\Delta^2)$. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$.

2.1 Treewidth

Tree decompositions and treewidth are used throughout this paper. For two graphs G and H, an *H-decomposition* of G consists of a collection $(B_x : x \in V(H))$ of subsets of $V(G)$, called *bags*, indexed by the nodes of H, such that:

- for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H and
- for every edge vw of G, there is a vertex $x \in V(H)$ for which $v, w \in B_x$.

The *width* of such an H-decomposition is $\max\{|B_x| : x \in V(H)\} - 1$. A *tree decomposition* is a T-decomposition for some tree T. Tree decompositions were introduced by Halin [29] and Robertson and Seymour [49]. The more general notion of H-decomposition was introduced by Diestel and Kühn [11]. The *treewidth* of a graph G is the minimum width of a tree decomposition of G. Treewidth measures how similar a given graph is to a tree. It is particularly important in structural and algorithmic graph theory; see [6, 30, 46] for surveys.

Our first tool, which was also used by Liu and Wood [39], is the following 2-colouring result for graphs of bounded treewidth due to Alon et al. [1]. The constant 20 comes from applying a result from [54].

Lemma 1 ([1]). *Every graph with maximum degree $\Delta \in \mathbb{N}$ and treewidth less than $k \in \mathbb{N}$ is 2-colourable with clustering $20k\Delta$.*

As an aside, it follows from the Lipton–Tarjan separator theorem [36] that n-vertex planar graphs have treewidth $O(\sqrt{n})$. Thus, Lemma 1 implies that n-vertex planar graphs with maximum degree $\Delta \in \mathbb{N}$ are 2-colourable with clustering $O(\Delta\sqrt{n})$, which answers an open problem raised by Linial et al. [35]. The same result holds for graphs excluding any fixed minor, using the separator theorem of Alon et al. [2].

2.2 Key lemma

The next lemma is a central result of the paper. Here, a *layering* of a graph G is an ordered partition $(V_0, V_1, \dots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. For example, if r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \text{dist}_G(r, v) = i\}$ for all $i \in \mathbb{N}_0$, then $(V_0, V_1, \dots)$ is called a *BFS layering* of G. The lemma assumes that for some layering

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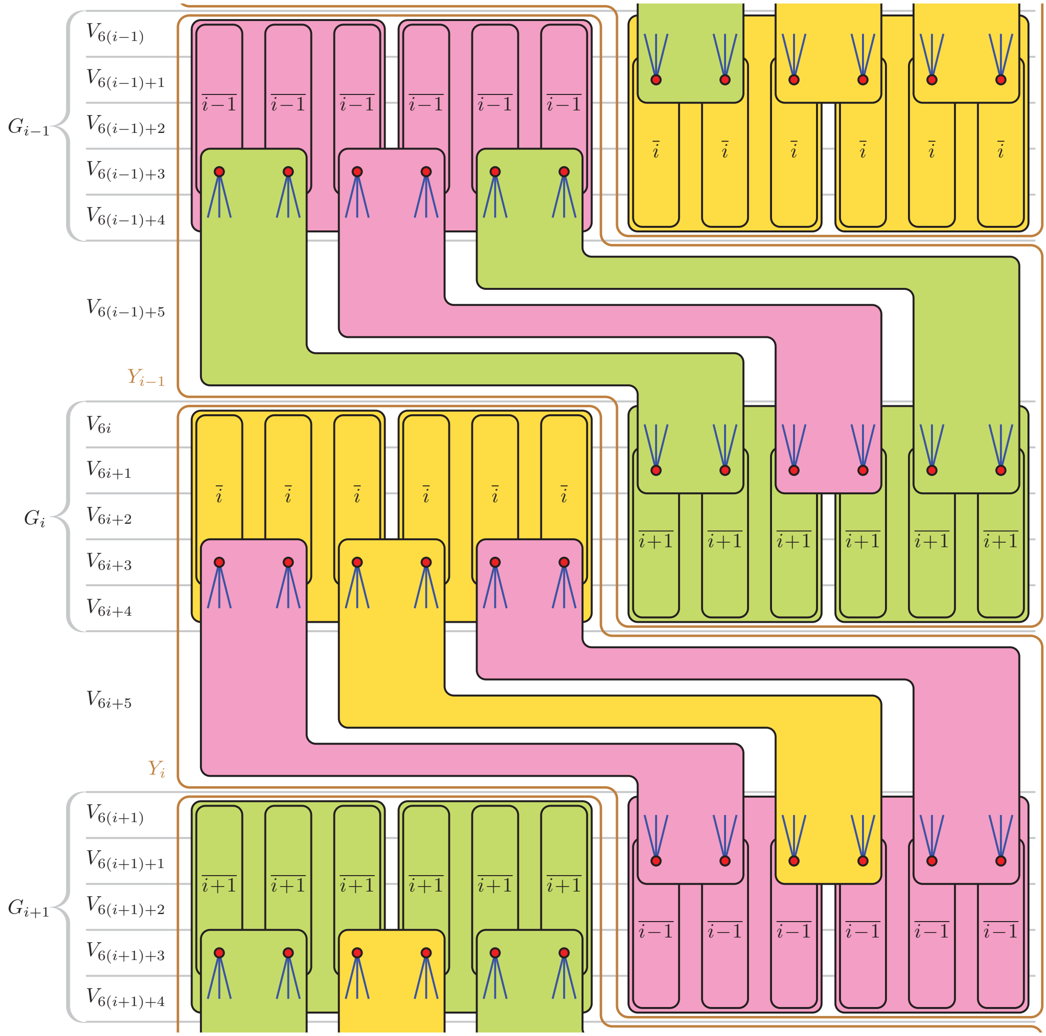

Figure 1. Proof of Lemma 2.

of a graph, every subgraph induced by a bounded number of consecutive layers has bounded treewidth. This property dates to the seminal work of Baker [4], who used it to obtain efficient approximation algorithms for various NP-hard problems on planar graphs. We show that graphs that satisfy this property and have small maximum degree are 3-colourable with small clustering.

Lemma 2. *Let G be a graph with maximum degree $\Delta \in \mathbb{N}$. Let $(V_0, V_1, \ldots)$ be a layering of G such that $G[\bigcup_{j=0}^{10} V_{i+j}]$ has treewidth less than $k \in \mathbb{N}$ for all $i \in \mathbb{N}_0$. Then G is 3-colourable with clustering* $8000k^3\Delta^2$.

Proof. No attempt is made to improve the constant 8000. We may assume (by renaming the layers) that $V_0 = V_1 = V_2 = V_3 = V_4 = \emptyset$.

Let $\overline{i} := i \bmod 3$ for $i \in \mathbb{N}_0$. As shown in Figure 1, for $i \in \mathbb{N}_0$, let G_i be the induced subgraph $G[V_{6i} \cup V_{6i+1} \cup \cdots \cup V_{6i+4}]$. Thus G_i has maximum degree at most Δ and treewidth less than k. By Lemma 1, G_i has a 2-colouring c_i with clustering $20k\Delta$. Use colours $\overline{i}$ and $\overline{i+1}$ for this colouring of G_i. We now define the desired colouring c of G. Vertices in $V_{6i} \cup V_{6i+1}$ coloured $\overline{i}$ in

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c_i keep this colour in c. Vertices in V_{6i+2} keep their colour from c_i in c. Vertices in $V_{6i+3} \cup V_{6i+4}$ coloured $\overline{i+1}$ in c_i keep this colour in c. Other vertices are assigned a new colour, as we now explain.

For $j \in \mathbb{N}_0$ and $\ell \in \{0, 1, 2\}$, let $V_{j,\ell}$ be the set of vertices in V_j coloured ℓ in the colouring c_i of the corresponding graph G_i (which is well defined since $G_0, G_1, \ldots$ are pairwise disjoint). For $i \in \mathbb{N}_0$, let

$$A_i := \bigcup_{j=0}^{3} V_{6i+j,\bar{i}} \quad , \quad B_i := \bigcup_{j=1}^{4} V_{6(i+1)+j,\overline{i-1}} \quad , \quad Y_i := A_i \cup V_{6i+4,\bar{i}} \cup V_{6i+5} \cup V_{6(i+1),\overline{i-1}} \cup B_i.$$

Note that $\{Y_i : i \in \mathbb{N}_0\}$ is a partition of $V(G)$ (since $V_0 = V_1 = V_2 = V_3 = V_4 = \emptyset$). In fact, $(Y_0, Y_1, \ldots)$ is a layering of G, since V_{6i-1} separates $Y_0 \cup \cdots \cup Y_{i-2}$ and $Y_i \cup Y_{i+1} \cup \cdots$.

For $i \in \mathbb{N}_0$, let Z_i be the graph obtained from $G[Y_i]$ as follows: for each component X of $G[A_i]$ or $G[B_i]$, contract X into a single vertex v_X. The neighbours of v_X in Z_i are contained within a monochromatic component of G_i or G_{i+1}; thus v_X has degree at most the size of the corresponding monochromatic component of G_i or G_{i+1}, which is at most $20k\Delta$. Since Z_i is a minor of $G[\bigcup_{j=0}^{10} V_{6i+j}]$ and treewidth is a minor-monotone parameter, Z_i has treewidth less than k. By Lemma 1, Z_i has a 2-colouring c'_i with clustering $400k^2\Delta$. Use colours $\bar{i}$ and $\overline{i-1}$ for this colouring of Z_i.

We now assign colours to the remaining vertices of G in the colouring c. Vertices in $V_{6i+4,\bar{i}} \cup V_{6i+5} \cup V_{6(i+1),\overline{i-1}}$ keep their colour from the colouring c'_i of Z_i. Note that these vertices were not contracted in the construction of Z_i. For each component X of $G[A_i]$, assign the colour given to v_X in c'_i to each vertex in $X \cap V_{6i+3}$. Similarly, for each component X of $G[B_i]$, assign the colour given to v_X in c'_i to each vertex in $X \cap V_{6i+7}$. This completes the definition of the colouring c of G.

Consider a monochromatic component M in the 3-colouring c of G. Suppose that M contains an edge vw with $v \in Y_{i-1}$ and $w \in Y_i$ for some $i \in \mathbb{N}_0$. The only colour used by both Y_{i-1} and Y_i is $\overline{i-1}$; thus M is coloured $\overline{i-1}$. But V_{6i+2} does not use colour $\overline{i-1}$, and it separates Y_{i-1} and Y_i. This contradiction shows that M contains no such edge vw. Since $(Y_0, Y_1, \ldots)$ is a layering of G and M is connected, M is contained in some Y_i. The only colours used in Y_i are $\bar{i}$ and $\overline{i-1}$. By symmetry we may assume that M is coloured $\bar{i}$.

If M is contained in $V_{6i} \cup V_{6i+1} \cup V_{6i+2}$, then M is contained in some monochromatic component of G_i (with respect to the colouring c_i), and thus $|V(M)| \leqslant 20k\Delta$. Otherwise, M is contained in the graph obtained from a monochromatic component C of Z_i (with respect to the colouring c'_i) by replacing each contracted vertex v_X in C by X. Since $|V(C)| \leqslant 400k^2\Delta$ and $|V(X)| \leqslant 20k\Delta$, we conclude that $|V(M)| \leqslant 8000k^3\Delta^2$. □

2.3 Layered treewidth

Dujmović et al. [19] and Shahrokhi [52] independently introduced the following concept. The *layered treewidth* of a graph G is the minimum integer k such that G has a tree decomposition $(B_x : x \in V(T))$ and a layering $(V_0, V_1, \ldots)$ such that $|B_x \cap V_i| \leqslant k$ for every bag B_x and layer V_i. Applications of layered treewidth include graph colouring [19,39,53], graph drawing [5,19], book embeddings [17], boxicity [51], and intersection graph theory [52]. The related notion of layered pathwidth has also been studied [5,13]. In a graph with layered treewidth k, the subgraph induced by the union of any 11 consecutive layers has treewidth less than $11k$. Thus Lemma 2 implies:

Corollary 3. *Every graph with layered treewidth $k \in \mathbb{N}$ and maximum degree $\Delta \in \mathbb{N}$ is 3-colourable with clustering $O(k^3\Delta^2)$.*

This corollary improves on a result of Liu and Wood [39] who proved an upper bound of $O(k^{19}\Delta^{37})$ on the clustering function.

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Many classes of graphs are known to have bounded layered treewidth. For example, Dujmović et al. [19] proved that every planar graph has layered treewidth at most 3, every graph with Euler genus g has layered treewidth at most $2g+3$, and that any apex-minor-free class of graphs has bounded layered treewidth. Corollary 3 thus implies the following results.

Corollary 4. *Every planar graph with maximum degree $\Delta \in \mathbb{N}$ is 3-colourable with clustering $O(\Delta^2)$.*

Corollary 5. *Every graph with Euler genus $g \in \mathbb{N}_0$ and maximum degree $\Delta \in \mathbb{N}$ is 3-colourable with clustering $O(g^3\Delta^2)$.*

Corollary 6. *For every fixed apex graph H, every H-minor-free graph with maximum degree $\Delta \in \mathbb{N}$ is 3-colourable with clustering $O(\Delta^2)$.*

The above corollaries can also be deduced from Lemma 2 without considering layered treewidth. First, consider a planar graph G, which we may assume is connected. Let $(V_0, V_1, \dots)$ be a BFS layering of G. For $i \in \mathbb{N}_0$, let G_i be obtained from $G[V_0 \cup V_1 \cup \cdots \cup V_{i+10}]$ by contracting $G[V_0 \cup \cdots \cup V_{i-1}]$ (which is connected) into a single vertex. Thus G_i is planar and has radius at most 11. Robertson and Seymour [48] proved that every planar graph with radius d has treewidth at most $3d$. Thus $G[V_i \cup \cdots \cup V_{i+10}]$, which is a subgraph of G_i, has treewidth at most 33. Corollary 4 then follows from Lemma 2. The same proof works in any minor-closed class for which the treewidth of any graph G in the class is bounded by a function of the radius of G. For example, Eppstein [23] proved that every graph with Euler genus g and radius d has treewidth at most $O(gd)$. Corollary 5 follows. More generally, Eppstein [23] proved that for every apex graph H, every H-minor-free graph with bounded radius has bounded treewidth. Corollary 6 follows.

Finally, note that one can also prove that every graph with Euler genus g and maximum degree Δ is 3-colourable with clustering $O(g\Delta^6)$ using Lemma 2 and a result of Esperet and Joret [24].[c]

2.4 Examples

One advantage for considering layered treewidth is that several non-minor-closed classes of interest have bounded layered treewidth. We give three examples:

(g,k)-Planar Graphs: A graph is (g,k)-*planar* if it has a drawing on a surface of Euler genus at most g such that each edge is involved in at most k crossings (with other edges). Dujmović et al. [14] proved that every (g,k)-planar graph has layered treewidth $O(gk)$. Corollary 3 implies that every (g,k)-planar graph with maximum degree Δ is 3-colourable with clustering $O(g^3k^3\Delta^2)$. This improves on a result of Liu and Wood [39] who proved an upper bound of $O(g^{19}k^{19}\Delta^{37})$ on the clustering function.

Map Graphs: Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a (g,d)-*map graph*. A $(0,d)$-map graph is called a (plane) d-*map graph*; see [8, 27], for example. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most g; see [14]. So (g,d)-map graphs generalise

[c]Esperet and Joret [24] proved that if every plane graph with maximum degree Δ has a 3-colouring with clustering $f(\Delta)$, where one colour is not used on the outerface, then graphs with Euler genus g and maximum degree Δ are 3-colourable with clustering $O(\Delta^2 f(\Delta)^2 g)$. Now, let G be a plane graph with maximum degree Δ. Let G^+ be the plane graph obtained by adding one new vertex r adjacent to the vertices on the outerface of G. For $i \in \mathbb{N}_0$, let V_i be the set of vertices in G^+ at distance i from r in G^+. By the above contraction argument, $(V_1, \dots, V_n)$ is a layering of G such that any set of 11 consecutive layers induces a subgraph with bounded treewidth. By Lemma 2, G is 3-colourable with clustering $O(\Delta^2)$. Moreover, only two colours are used on V_1 and thus on the outerface of G. By the above-mentioned result of Esperet and Joret [24] with $f(\Delta) = O(\Delta^2)$, all graphs with Euler genus g and maximum degree Δ are 3-colourable with clustering $O(\Delta^6 g)$.

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graphs embedded in a surface. Dujmović et al [14] showed that every (g, d)-map graph has layered treewidth at most $(2g + 3)(2d + 1)$. Corollary 3 then implies that every (g, d)-map graph with maximum degree Δ is 3-colourable with clustering $O(g^3d^3\Delta^2)$. This improves on a result of Liu and Wood [39] who proved an upper bound of $O(g^{19}d^{19}\Delta^{37})$ on the clustering function.

Graph Powers: For $p \in \mathbb{N}$, the *p-th power* of a graph G is the graph G^p with vertex set $V(G^p) := V(G)$, where $vw \in E(G^p)$ if and only if $\text{dist}_G(v, w) \leqslant p$. It follows from the work of Dujmović et al. [20] that powers of graphs with bounded layered treewidth and bounded maximum degree have bounded layered treewidth. Here we give a direct proof with better bounds.

Lemma 7. *If G is a graph with layered treewidth $k \in \mathbb{N}$ and maximum degree $\Delta \in \mathbb{N}$, then G^p has layered treewidth less than $2pk\Delta^{\lfloor p/2 \rfloor}$.*

Proof. The result is trivial if $\Delta = 1$, so assume that $\Delta \geqslant 2$. Let $(V_1, V_2, \ldots)$ be a layering of G and let $(B_x : x \in V(T))$ be a tree decomposition of G such that $|V_i \cap B_x| \leqslant k$ for each $i \in \mathbb{N}_0$ and $x \in V(T)$. For each vertex $v \in V(G)$, let $X_v := \{w \in V(G) : \text{dist}_G(v, w) \leqslant \lfloor \frac{p}{2} \rfloor\}$. For each node $x \in V(T)$, let $B'_x := \bigcup_{v \in B_x} X_v$.

We now prove that $(B'_x : x \in V(T))$ is a tree decomposition of G^p. Consider a vertex $\alpha \in V(G^p)$. Since $\alpha \in X_v$ if and only if $v \in X_\alpha$,

$$\{x \in V(T) : \alpha \in B'_x\} = \bigcup_{v \in X_\alpha} \{x \in V(T) : v \in B_x\}.$$

Since $\{x \in V(T) : v \in B_x\}$ induces a connected subtree of T, and X_α induces a connected subgraph of G, it follows that $\{x \in V(T) : \alpha \in B'_x\}$ also induces a connected subtree of T. Now, consider an edge $\alpha\beta \in E(G^p)$. There is an edge vw of G (in the 'middle' of a shortest $\alpha\beta$-path) such that $\alpha \in X_v$ and $\beta \in X_w$. Now $v, w \in B_x$ for some node $x \in V(T)$. By construction, $\alpha, \beta \in B'_x$. This shows that $(B'_x : x \in V(T))$ is a tree decomposition of G^p.

For $i \in \mathbb{N}_0$, let $W_i := V_{ip} \cup V_{ip+1} \cup \cdots \cup V_{(i+1)p-1}$. For each edge $\alpha\beta \in E(G^p)$, if $\alpha \in V_i$ and $\beta \in V_j$, then $|i - j| \leqslant p$. Thus if $\alpha \in W_{i'}$ and $\beta \in W_{j'}$, then $|i' - j'| \leqslant 1$. This shows that $(W_1, W_2, \ldots)$ is a layering of G^p. Since $|X_v| < 2\Delta^{\lfloor p/2 \rfloor}$ for each vertex $v \in V(G)$, for each node $x \in V(T)$ and $i \in \mathbb{N}_0$, we have $|B'_x \cap V_i| < 2k\Delta^{\lfloor p/2 \rfloor}$, implying $|B'_x \cap W_i| < 2pk\Delta^{\lfloor p/2 \rfloor}$. Therefore G^p has layered treewidth less than $2pk\Delta^{\lfloor p/2 \rfloor}$. □

Corollary 3 and Lemma 7 imply that for every graph with layered treewidth k and maximum degree Δ, the p-th power G^p (which has maximum degree less than $2\Delta^p$) is 3-colourable with clustering $O(k^3\Delta^{3\lfloor p/2 \rfloor + 2p})$. For example, for every (g, k)-planar graph G with maximum degree Δ, the p-th power G^p has a 3-colouring with clustering $O(g^3k^3\Delta^{3\lfloor p/2 \rfloor + 2p})$.

3. Excluded minors

This section shows that graphs excluding a fixed minor and with maximum degree Δ are 3-colourable with clustering $O(\Delta^5)$. The starting point is Robertson and Seymour's Graph Minor Structure Theorem, which we now introduce.

3.1 Graph minor structure theorem

For a graph G_0 embedded in a surface, and a facial cycle F of G_0 (thought of as a subgraph of G_0), an *F-vortex* (relative to G_0) is an F-decomposition $(B_x \subseteq V(H) : x \in V(F))$ of a graph H such that $V(G_0 \cap H) = V(F)$ and $x \in B_x$ for each $x \in V(F)$.

For $k \in \mathbb{N}_0$, a graph G is *k-almost embeddable* if for some set $A \subseteq V(G)$ with $|A| \leqslant k$ and for some $s \in \{0, \ldots, k\}$, there are graphs $G_0, G_1, \ldots, G_s$ such that:

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- $G - A = G_0 \cup G_1 \cup \cdots \cup G_s$,
- $G_1, \ldots, G_s$ are pairwise vertex-disjoint;
- G_0 is embedded in a surface of Euler genus at most k,
- there are s pairwise vertex-disjoint facial cycles $F_1, \ldots, F_s$ of G_0, and
- for $i \in \{1, \ldots, s\}$, there is an F_i-vortex $(B_x \subseteq V(G_i) : x \in V(F_i))$ of G_i (relative to G_0) of width at most k.

The vertices in A are called *apex* vertices. They can be adjacent to any vertex in G.

It is not clear whether the class of k-almost embeddable graphs is hereditary, so it will be convenient to define a graph to be *k-almost$^{\downarrow}$ embeddable* if it is an induced subgraph of some *k-almost embeddable* graph.

In a tree decomposition $(B_x : x \in V(T))$ of a graph G, the *torso* of a bag B_x is the graph obtained from $G[B_x]$ as follows: for every edge $xy \in E(T)$, add every edge vw where $v, w \in B_x \cap B_y$.

The following graph minor structure theorem by Robertson and Seymour [50] is at the heart of graph minor theory.

Theorem 8 ([50]). *For every graph H, there exists $k \in \mathbb{N}_0$ such that every graph G that does not contain H as a minor has a tree decomposition $(B_x : x \in V(T))$ such that the torso G_x of B_x is k-almost embeddable for each node $x \in V(T)$.*

In Theorem 8, we have $|B_x \cap B_y| \leqslant 8k$ for each edge xy of T because of the following lemma.

Lemma 9 ([19, Lemma 21]). *Every clique in a k-almost embeddable graph has size at most $8k$.*

We need the following slight strengthening of Theorem 8.

Theorem 10. *For every graph H, there exists $k \in \mathbb{N}$ such that every graph G that does not contain H as a minor and has maximum degree at most $\Delta \in \mathbb{N}$ has a tree decomposition $(B_x : x \in V(T))$ such that for each node $x \in V(T)$, the torso G_x of B_x is k-almost$^{\downarrow}$ embeddable and has maximum degree less than $8k\Delta$.*

Proof. Let $(B_x : x \in V(T))$ be a tree decomposition of G such that each torso is k-almost$^{\downarrow}$ embeddable, and subject to this condition, $\sum_{x \in V(T)} |B_x|$ is minimum. This is well defined by Theorem 8.

Consider an edge $xy \in E(T)$. Let $T_{x,y}$ be the component of $T - xy$ containing x. Let $V_{x,y} := \bigcup \{B_z \setminus B_y : z \in V(T_{x,y})\}$. Suppose for the sake of contradiction that some vertex $v \in B_x \cap B_y$ has no neighbour in $V_{y,x}$. Let $B'_z := B_z \setminus \{v\}$ for each $z \in V(T_{y,x})$, and let $B'_z := B_z$ for each $z \in V(T_{x,y})$. Since induced subgraphs of k-almost$^{\downarrow}$ embeddable graphs are k-almost$^{\downarrow}$ embeddable, $(B'_z : z \in V(T))$ is a tree decomposition of G such that each torso is k-almost$^{\downarrow}$ embeddable. (This is the reason we define k-almost$^{\downarrow}$ embeddability.) Since $v \in B_y$, we have $|B'_y| < |B_y|$, implying $\sum_{z \in V(T)} |B'_z| < \sum_{z \in V(T)} |B_z|$. This contradicts the choice of $(B_x : x \in V(T))$. Hence every vertex in $B_x \cap B_y$ has a neighbour in $V_{y,x}$.

Consider a node $x \in V(T)$, a vertex $v \in B_x$, and some edge vw of the torso G_x that is not in $G[B_x]$. By definition of the torso, $v, w \in B_x \cap B_y$ for some edge $xy \in E(T)$. As shown above, there is an edge vu in G with $u \in V_{y,x}$; let $\phi_x(v, w) := (v, u)$. Since $u \notin B_x$ and $|B_x \cap B_y| \leqslant 8k$ (by Lemma 9), we have $|\phi_x^{-1}(v, u)| < 8k$ (all the elements in the pre-image of (v, u) with respect to ϕ_x are of the form (v, z) with $z \in V_x \cap V_y$). Thus $\deg_{G_x}(v) < 8k \deg_G(v) \leqslant 8k\Delta$. □

Let $C_1 = \{v_1, \ldots, v_k\}$ be a k-clique in a graph G_1. Let $C_2 = \{w_1, \ldots, w_k\}$ be a k-clique in a graph G_2. Let G be the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for $i \in \{1, \ldots, k\}$, and possibly deleting some edges in C_1 $(= C_2)$. Then G is a *clique-sum* of G_1 and G_2.

The following is a direct consequence of Theorem 10.

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Corollary 11. *For every proper minor-closed class $\mathcal{G}$, there exists $k \in \mathbb{N}$ such that every graph G in $\mathcal{G}$ with maximum degree at most $\Delta \in \mathbb{N}$ is obtained by clique-sums of k-almost$^{\downarrow}$ embeddable graphs of maximum degree less than $8k\Delta$.*

3.2 Partitions

A *vertex-partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of non-empty sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B.

A partition $\mathcal{P}$ of a graph G is called an *H-partition* if H is a graph that contains a spanning subgraph isomorphic to the quotient $G/\mathcal{P}$. Alternatively, an *H-partition* of a graph G is a partition $(A_x : x \in V(H))$ of $V(G)$ indexed by the vertices of H, such that for every edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ or $xy \in E(H)$. The *width* of such an H-partition is $\max\{|A_x| : x \in V(H)\}$. Note that a layering is equivalent to a path partition.

Dujmović et al. [18] introduced a layered variant of partitions (analogous to layered treewidth being a layered variant of treewidth). The *layered width* of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, \dots)$ of G, each part in $\mathcal{P}$ has at most ℓ vertices in each layer V_i. A partition $\mathcal{P}$ of a graph G is a *(k, ℓ)-partition* if $\mathcal{P}$ has layered width at most ℓ and $G/\mathcal{P}$ has treewidth at most k. A class $\mathcal{G}$ of graphs *admits bounded layered partitions* if there exist $k, \ell \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has a (k, ℓ)-partition.

Several recent results show that various graph classes admit bounded layered partitions. The first results were for minor-closed classes by Dujmović et al. [18], who proved that planar graphs admit bounded layered partitions; more generally, that graphs of bounded Euler genus admit bounded layered partitions; and most generally, a minor-closed class admits bounded layered partitions if and only if it excludes some apex graph. Some results for non-minor-closed classes were recently obtained by Dujmović et al. [20]. For example, they proved that (g, k)-planar graphs and (g, d)-map graphs admit bounded layered partitions amongst other examples.

Dujmović et al. [18] showed that this property implies bounded layered treewidth.

Lemma 12 **([18]).** *If a graph G has a (k, ℓ)-partition, then G has layered treewidth at most $(k+1)\ell$.*

What distinguishes layered partitions from layered treewidth is that layered partitions lead to constant upper bounds on the queue-number and non-repetitive chromatic number, whereas for both these parameters, the best known upper bounds obtainable via layered treewidth are $O(\log n)$. This led to the positive resolution of two old open problems; namely, whether planar graphs have bounded queue-number [18] and whether planar graphs have bounded non-repetitive chromatic number [16]. Other applications include p-centred colouring [10] and graph encoding/universal graphs [7, 15, 25].

Our next tool is the following result by Dujmović et al. [18].

Lemma 13 **([18]).** *Every k-almost embeddable graph with no apex vertices has an $(11k + 10, 6k)$-partition.*

3.3 Excluding a minor

We now prove that a result like Lemma 13 also holds for k-almost embeddable graphs in which all the apex vertices have bounded degree (and in particular if the graph has bounded degree).

Lemma 14. *Let G be a graph such that, for some $A \subseteq V(G)$, every vertex in A has degree at most $\Delta \in \mathbb{N}$, and $G - A$ has a (k, ℓ)-partition. Then G has a $(k + 1, 2\ell\Delta|A|)$-partition.*

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Proof. Let $\mathcal{P}$ be a (k, ℓ)-partition of $G - A$, where $\mathcal{P}$ has layered width at most ℓ with respect to a layering $(V_0, V_1, \dots)$ of $G - A$. Let I be the set of integers i such that some vertex in A has a neighbour in V_i. Thus $|I| \leqslant \Delta|A|$. Let P be the path graph $(0, 1, \dots)$. For $j \in \mathbb{N}_0$, let d_j be the minimum distance in P from j to a vertex in I. For $i \in \mathbb{N}_0$, let W_i be the union of the sets V_j such that $d_j = i$. For each edge vw of G, if $v \in V_a$ and $w \in V_b$ then $|a - b| \leqslant 1$, implying $|d_a - d_b| \leqslant 1$. Thus $(W_0, W_1, \dots)$ is a layering of $G - A$. Observe that each layer W_i is the union of at most $2|I|$ original layers (at most two layers between each pair of consecutive elements in I, plus one layer before $\min I$ and one layer after $\max I$). Thus $\mathcal{P}$ has layered width at most $2\ell|I| \leqslant 2\ell\Delta|A|$ with respect to $(W_0, W_1, \dots)$. By construction, the vertices of $G - A$ that are neighbours of vertices in A are all in W_0. Add A to W_0. We thus obtain a layering of G. Let $\mathcal{Q}$ be the partition of G obtained from $\mathcal{P}$ by adding one new part A. Thus $\mathcal{Q}$ has layered width at most $2\ell\Delta|A|$ with respect to $(W_0, W_1, \dots)$. Since $G/\mathcal{Q}$ has only one more vertex than $(G - A)/\mathcal{P}$, the treewidth of $G/\mathcal{Q}$ is at most $k + 1$. □

Lemma 13 and 14 lead to the next result.

Lemma 15. *Every k-almost*$^{\downarrow}$ *embeddable graph G such that every apex vertex has degree at most* $\Delta \in \mathbb{N}$ *has an* $(11k + 11, 12k^2\Delta)$*-partition.*

Proof. By definition, G is an induced subgraph of a k-almost embeddable graph G'. Since deleting an apex vertex in a k-almost embeddable graph produces another k-almost embeddable graph, we may assume that G and G' have the same set A of apex vertices. By Lemma 13, $G' - A$ has an $(11k + 10, 6k)$-partition $\mathcal{P}'$. Let $\mathcal{P}$ be obtained by restricting $\mathcal{P}'$ to $V(G - A)$. Thus $\mathcal{P}$ is an $(11k + 10, 6k)$-partition of $G - A$. Since every vertex in A has degree at most Δ in G, the result follows from Lemma 14. □

Dujmović et al. [18] introduced (an equivalent version of) the following definitions and lemmas as a way to handle clique-sums. Let C be a clique in a graph G, and let $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ be partitions of C. A (k, ℓ)-partition $\mathcal{P}$ of G is $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$*-friendly* if $P_1, \dots, P_c \in \mathcal{P}$ and $\mathcal{P}$ has layered width at most ℓ with respect to some layering $(V_0, V_1, \dots)$ of G with $C_0 \subseteq V_0$ and $C_1 \subseteq V_1$.

Lemma 16 **([18]).** *Let G be a graph that has a* (k, ℓ)*-partition. Let C be a clique in G, and let* $\{C_0, C_1\}$ *and* $\{P_1, \dots, P_c\}$ *be partitions of C such that* $|C_j \cap P_i| \leqslant 2\ell$ *for each* $j \in \{0, 1\}$ *and each* $i \in \{1, \dots, c\}$*. Then G has a* $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$*-friendly* $(k + c, 2\ell)$*-partition.*

A graph G *admits clique-friendly* (k, ℓ)*-partitions* if for every clique C in G, and for all partitions $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ of C, there is a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly (k, ℓ)-partition of G. A graph class $\mathcal{G}$ *admits clique-friendly* (k, ℓ)*-partitions* if every graph in $\mathcal{G}$ admits clique-friendly (k, ℓ)-partitions.

Lemma 17 **([18]).** *Let* $\mathcal{G}$ *be a graph class that admits clique-friendly* (k, ℓ)*-partitions. Then the class of graphs obtained from clique-sums of graphs in* $\mathcal{G}$ *admits clique-friendly* (k, ℓ)*-partitions.*

Lemma 15 and 16 lead to the next result.

Lemma 18. *Every k-almost*$^{\downarrow}$ *embeddable graph G of maximum degree at most* $\Delta \in \mathbb{N}$ *admits clique-friendly* $(19k + 11, 24k^2\Delta)$*-partitions.*

Proof. By Lemma 15, G has an $(11k + 11, 12k^2\Delta)$-partition. It follows from Lemma 16 and 9 that G admits clique-friendly $(19k + 11, 24k^2\Delta)$-partitions. □

The following result, of independent interest, says that bounded-degree graphs excluding a fixed minor admit bounded layered partitions.

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Theorem 19. *For every fixed graph H, there is a constant $k \in \mathbb{N}$ such that every H-minor-free graph with maximum degree $\Delta \in \mathbb{N}$ has a $(k, k\Delta)$-partition.*

Proof. Let G be an H-minor-free graph with maximum degree Δ. By Corollary 11, there is a constant k_0 (depending only on H) such that G can be obtained by clique-sums of k_0-almost$^{\downarrow}$ embeddable graphs with maximum degree at most $8k_0\Delta$. By Lemma 18, each such graph admits clique-friendly $(19k_0 + 11, 24k_0^2 \cdot 8k_0\Delta)$-partitions. It follows from Lemma 17 that G also admits clique-friendly $(19k_0 + 11, 192k_0^3\Delta)$-partitions. The result follows where $k \coloneqq \max\{19k_0 + 11, 192k_0^3\}$. □

With these tools, we are now ready to prove the main result of this section.

Theorem 20. *For every fixed graph H, every H-minor-free graph G with maximum degree $\Delta \in \mathbb{N}$ is 3-colourable with clustering $O(\Delta^5)$.*

Proof. Let G be an H-minor-free graph with maximum degree Δ. By Theorem 19, for some constant k (depending only on H), G has a $(k, k\Delta)$-partition. Lemma 12 implies that G has layered treewidth at most $(k+1)k\Delta$. By Corollary 3, G has a 3-colouring with clustering $O(k^6\Delta^5)$. □

3.4 Strong products

Some of the above structural results can be interpreted in terms of products. The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if:

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

Lemma 1 was proved using the following result by an anonymous referee of the paper by Ding and Oporowski [12] (refined in [54]).

Lemma 21 ([12, 54]). *Every graph with maximum degree $\Delta \in \mathbb{N}$ and treewidth less than $k \in \mathbb{N}$ is a subgraph of $T \boxtimes K_{20k\Delta}$ for some tree T.*

Lemma 1 follows from Lemma 21 by first properly 2-colouring T and then colouring each vertex of the graph by the colour of the corresponding vertex of T.

The next observation by Dujmović et al. [18] follows immediately from the definitions.

Observation 22 ([18]). *A graph G has an H-partition of layered width at most $\ell \in \mathbb{N}$ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

Dujmović et al. [18] also showed that if one does not care about the exact treewidth bound, then it suffices to consider partitions with layered width 1.

Observation 23 ([18]). *If a graph $G \subseteq H \boxtimes P \boxtimes K_\ell$ for some graph H of treewidth at most k and for some path P, then $G \subseteq H' \boxtimes P$ for some graph H' of treewidth at most $(k+1)\ell - 1$.*

By these two observations, Theorem 19 can be restated as follows:

Theorem 24. *For every fixed graph X, every X-minor-free graph with maximum degree $\Delta \in \mathbb{N}$ is a subgraph of $H \boxtimes P$ for some graph H of treewidth $O(\Delta)$ and for some path P.*

It is worth highlighting the similarity of Lemma 21 and Theorem 24. Lemma 21 says that graphs of bounded treewidth and bounded degree are subgraphs of the product of a tree and a complete

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graph of bounded size, whereas Theorem 24 says that bounded-degree graphs excluding a fixed minor are subgraphs of the product of a bounded treewidth graph and a path.

4. Open problem

We conclude with a natural open problem that arises from this work. Are planar graphs with maximum degree Δ 3-colourable with clustering $O(\Delta)$? A construction of Kleinberg et al. [34] shows a lower bound of $\Omega(\Delta^{1/3})$, while a slightly different construction by Esperet and Joret [24] shows $\Omega(\Delta^{1/2})$.

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