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SINR PERCOLATION FOR COX POINT PROCESSES WITH RANDOM POWERS

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Abstract

Signal-to-interference-plus-noise ratio (SINR) percolation is an infinite-range dependent variant of continuum percolation modeling connections in a telecommunication network. Unlike in earlier works, in the present paper the transmitted signal powers of the devices of the network are assumed random, independent and identically distributed, and possibly unbounded. Additionally, we assume that the devices form a stationary Cox point process, i.e., a Poisson point process with stationary random intensity measure, in two or more dimensions. We present the following main results. First, under suitable moment conditions on the signal powers and the intensity measure, there is percolation in the SINR graph given that the device density is high and interferences are sufficiently reduced, but not vanishing. Second, if the interference cancellation factor γ and the SINR threshold τ satisfy $\gamma \geq 1/(2\tau)$, then there is no percolation for any intensity parameter. Third, in the case of a Poisson point process with constant powers, for any intensity parameter that is supercritical for the underlying Gilbert graph, the SINR graph also percolates with some small but positive interference cancellation factor.

Keywords: Signal-to-interference ratio; Cox point process; Poisson point process; continuum percolation; SINR percolation; Gilbert graph; Boolean model; stabilization; random power; degree bound

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1. Introduction

The study of percolation properties of infinite random graphs traces back many decades, and many of the classical results are already available in textbook form; see for example the monographs [3, 15, 25]. The first results for percolation in the continuum $\mathbb{R}^d$ were presented in the landmark paper by Gilbert [11], where nontrivial regimes of existence and absence of percolation (i.e., existence of an unbounded connected component) were established for the *Poisson–Gilbert graph*, i.e., where the set of nodes is given by a homogeneous *Poisson point process* (PPP), and edges are drawn between two nodes whenever their distance is below a certain fixed positive connectivity threshold. The context of telecommunication systems was already mentioned there.

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In order to make the model more flexible, instead of using a fixed connectivity threshold, the nodes in the PPP can also be marked with independent random radii, drawn from a common distribution, with two nodes connected by an edge whenever their distance is below the sum of the associated radii. In view of our topic in this paper, we call the resulting model a *Poisson–Gilbert graph with random radii*, whose percolation properties can equivalently be expressed in terms of the corresponding *Poisson Boolean model with random radii*. We note that a wide range of results for percolation for this model are available; see for example [1, 12–14, 25] and references therein. Starting from the seminal book [24], many results about the original Poisson Boolean model with spherical grains have been extended to more general grains, but in the present paper we focus on the classical case when the Boolean model is a union of balls (with random radii).

In view of applications in wireless telecommunication systems, the extension of Poisson–Gilbert graphs to Gilbert graphs based on *Cox point processes* (CPPs), i.e., PPPs with random intensity measure, allows one to study long-range communication properties in device-to-device networks where devices are placed according to a PPP in a *random environment* that is represented by the intensity measure of the CPP. Recently, continuum percolation and further properties of such *Cox–Gilbert graphs* have been studied under certain conditions of stabilization and connectedness; see [5, 16] and below. However, in these works, the edge-drawing rule remained, as in the classical case of the Poisson–Gilbert graph, based on a fixed connectivity threshold.

In the very recent manuscript [18], continuum percolation results are first presented for Cox–Gilbert graphs where, as in the Poisson–Gilbert graph with random radii, each node is equipped with a random radius, and edges are placed between any two nodes whenever their distance is below the sum of the radii. In this case, again under certain stabilization and connectedness assumptions, most of the percolation properties of the Poisson–Gilbert graph with random radii can be reproduced also for this *Cox–Gilbert graph with random radii*. We note that here, similarly to [25], percolation properties of the Gilbert graphs are again expressed in terms of the underlying *Cox Boolean models with random radii*.

Moving beyond a setting where edges are placed between pairs of points based on their mutual distance and their associated radii, another line of research aims towards a different kind of extension of Gilbert graphs with respect to the edges. Starting with the papers [8, 9], still based on a homogeneous PPP in $\mathbb{R}^2$, the edge-drawing mechanism is replaced by a highly non-local rule using the *signal-to-interference-plus-noise ratio* (SINR), which we describe precisely and more generally in (2.1). In words, very roughly, a pair of Poisson points is connected by an edge only if the weighted distance between the points is sufficiently small compared to the accumulated weighted distances of all the other points, the so-called *interference*. This definition is very much inspired by applications in wireless telecommunication networks, where the success of a transmission between network components is highly dependent on the relative signal strength between the components compared to the other (unwanted) signals present in the medium. In the simplest case, only the relative distances between points enter the SINR, giving rise to the SINR graph on PPPs, or the *Poisson SINR graph*. Let us note that this definition introduces long-range, or even infinite-range, dependencies for the construction of edges into the system. This is the setting considered in [8, 9], where, using comparison techniques with the Poisson–Gilbert graph, again nontrivial percolation properties are established. Let us mention that the SINR graph has very different monotonicity properties compared to the Poisson–Gilbert graph, which makes it more interesting but also harder to analyze. To see this, note that due to the definition of the SINR, an increase in the number of points also leads to an increase in the unwanted interference and thus to the potential loss of edges. On the other hand, for the Poisson–Gilbert graph, the connectivity increases if points are added into the system.

Now, similarly to the generalization of a Poisson–Gilbert graph to a Poisson–Gilbert graph with random radii, SINR graphs can also be generalized in the sense that each point is equipped with an independent power random variable. These powers enter the definition of the SINR as presented in (2.1) and represent the individual signal strengths of the network components. For the case of the SINR graph with random powers based on PPPs, or the *Poisson SINR graph with random powers*, the paper [21] presents first results similar to the assertions presented in [8, 9] under very strong boundedness assumptions on the powers. Let us note that the definition of an SINR graph with random powers already appears in [8], but the only proven result of this paper for this setting is about degree bounds (cf. Section 4.2). The first steps towards understanding the case of unbounded powers in the Poisson SINR graph with random powers were made recently in [22]. In this master's thesis, supervised by the authors, results in the spirit of one of our main theorems, Theorem 2.1, are presented under much stronger assumptions and only for the case of an underlying homogeneous PPP. The thesis [22] also provides sufficient conditions for the absence of percolation for small intensities of the PPP. Before [22], no positive assertions about percolation in an SINR graph with unbounded random powers had been known in the literature; regarding the case of bounded random powers, see also Section 4.

On the other hand, in [28], the two extensions described above were for the first time considered jointly, giving rise to the SINR graph based on CPPs, the *Cox SINR graph*, but without random powers. In [28] it was established that for sufficiently large intensities and sufficiently connected environments, the Cox SINR graph percolates almost surely at least for nonvanishing interference. In [27, Section 4.2.3.4] it was anticipated that the case of random but bounded powers might easily be handled via the same methods; see Section 4.1 for further details.

The present paper now completes this line of research by analyzing the *Cox SINR graph with random powers*, which are also not necessarily bounded. More precisely, in our first main result, Theorem 2.1, we present sufficient conditions for the existence of a supercritical percolation phase, i.e., a nontrivial regime for the intensity of the underlying CPP and nonvanishing interference, such that the Cox SINR graph with random powers percolates. This substantially extends the results of [8–10] from the case of a homogeneous PPP in $\mathbb{R}^2$ with constant powers to that of a CPP in $\mathbb{R}^d$, $d \geq 2$, with random and possibly unbounded powers, combining the methods of [28] for the case of a CPP with constant powers and those of [22] for the case of a PPP with random powers (both in dimension 2 or higher). We will discuss the relationship of Theorem 2.1 to these results in detail in Section 4.

Our second main result, Theorem 2.2, establishes a uniform bound on the strength of the interference, above which no percolation is possible. In essence, this theorem claims that there is no percolation in the SINR graph whenever the degree of its vertices is bounded by 2. The fact that SINR graphs with nonvanishing interference have bounded degrees originates from [8, Theorem 1]; however, in that paper, only the simple assertion that SINR graphs with degrees bounded by 1 do not percolate was proven, and this has not been improved until the present paper.

Finally, in our third main result, Theorem 2.3, we state that in the case of the Poisson SINR graph with constant powers, indeed, the critical intensity for percolation in the presence of interference can be represented via the critical intensity of an associated Poisson–Gilbert graph. This result extends the two-dimensional statement [9, Theorem 1] to higher dimensions, although its proof is rather different from the proof in [9].

The organization of the manuscript is as follows. In Section 2, we present the setting and main results but postpone the introduction of our main technical conditions for the CPP, namely stabilization and asymptotic essential connectedness. These conditions are presented in Section 3 together with examples of CPPs for which our main results are applicable. In Section 4 we present the proof strategies for our main results and give further background on how they relate to previous results in the literature. Finally, in Section 5, we give the detailed proofs.

2. Setting and main results

For $\lambda > 0$, let $\mathbf{X}^\lambda = \{(x_i, P_i)\}_{i\in\mathbb{N}}$ be an independent and identically distributed (i.i.d.) marked CPP in $\mathbb{R}^d \times [0, \infty)$ for $d \geq 1$, with directing measure $\lambda\Lambda \otimes \mu$ where Λ is stationary with $\mathbb{E}[\Lambda(Q_1)] = 1$ and $Q_n = [-n/2, n/2]^d$ for $n > 0$. Here, Λ is a random element in the space $\mathbb{M}$ of Borel measures on $\mathbb{R}^d$ equipped with the usual evaluation σ-algebra, and μ is a Borel probability measure on $[0, \infty)$, the common distribution of the marks $P = \{P_i\}_{i\in\mathbb{N}}$. We consider the *SINR graph* with vertex set given by the first component of $\mathbf{X}^\lambda$, which we denote by $X^\lambda = \{x_i\}_{i\in\mathbb{N}}$. Here, the pair of vertices $x_i, x_j \in X^\lambda$, with $x_i \neq x_j$, is connected by an edge if and only if

$$P_i\ell(|x_i - x_j|) > \tau\Big(N_o + \gamma \sum_{k\in\mathbb{N}\setminus\{i,j\}} P_k\ell(|x_k - x_j|)\Big) \quad \text{and}$$
$$P_j\ell(|x_i - x_j|) > \tau\Big(N_o + \gamma \sum_{k\in\mathbb{N}\setminus\{i,j\}} P_k\ell(|x_k - x_i|)\Big). \tag{2.1}$$

In (2.1), $\tau > 0$ is fixed and called the *SINR threshold*, the constant $N_o \geq 0$ represents *noise*, $r \mapsto \ell(r) \in [0, \infty)$ is referred to as the *path-loss function*, and $\gamma \geq 0$ is called the *interference-cancellation factor*. The random variables P_i, $i \in \mathbb{N}$, are often called random *powers*, and the term

$$I(x_i, x_j, \mathbf{X}^\lambda) = \sum_{k\in\mathbb{N}\setminus\{i,j\}} P_k\ell(|x_k - x_j|)$$

is referred to as *interference*. We will use the notation $G_\gamma(\mathbf{X}^\lambda)$ to indicate the SINR graph, suppressing the dependencies on τ, N_o, and ℓ, but highlighting the dependence on γ; see Figure 1 for an illustration. We refer to [8, Section 1] for further interpretation of the modeling parameters.

The SINR graph has a nice interpretation in the study of device-to-device telecommunication systems where the devices of X^λ can communicate directly with each other if their mutual distance, represented by the path-loss function, and their individual powers are sufficiently strong to overcome thermal noise plus all the interference coming from the other devices. If this is the case, then the ability to communicate is represented by an undirected edge.

Our main interest lies in percolation properties of the SINR graph, as first studied in [8–10]. A *cluster* in $G_\gamma(\mathbf{X}^\lambda)$ is a maximal connected component. We say that $G_\gamma(\mathbf{X}^\lambda)$ *percolates* if $G_\gamma(\mathbf{X}^\lambda)$ contains an unbounded connected component. Clusters and percolation are defined analogously in the case of any graph having a vertex set that is included in $\mathbb{R}^d$, $d \geq 1$, and is locally finite. Here we focus on the following key quantities. First, the *critical interference-cancellation factor* is defined as

$$\gamma(\lambda) = \sup\big\{\gamma > 0 \colon \mathbb{P}(G_\gamma(\mathbf{X}^\lambda) \text{ percolates}) > 0\big\}. \tag{2.2}$$

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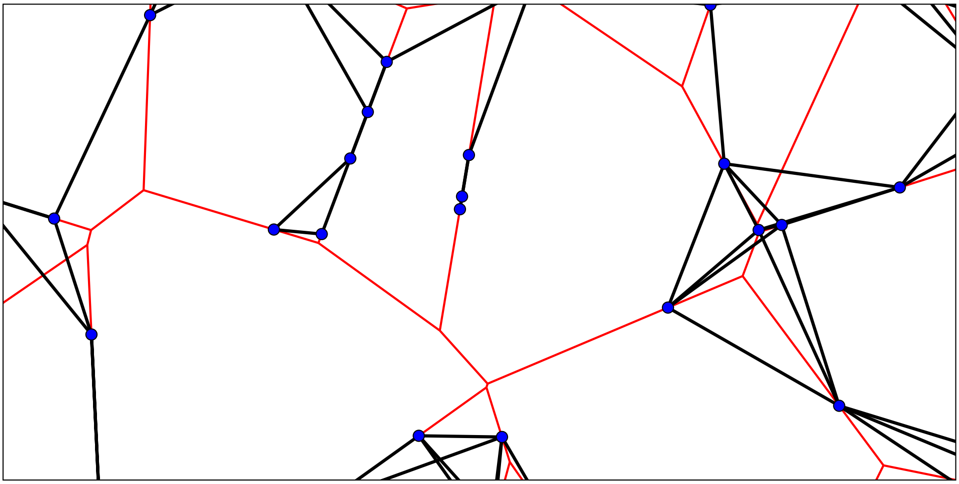

FIGURE 1. A typical realization of a Cox SINR graph (with blue vertices and black edges) with directing measure given by the edge-length measure of a two-dimensional Poisson–Voronoi tessellation (in red) in a box, with $N_o = 1$, $\gamma = 0.3$, $\tau = 0.2$, constant powers equal to 1, and a suitable path-loss function ℓ. The interference-cancellation factor is set to $\gamma = 1/(2\tau)$. We see only a few vertices having degree two, the largest connected component is of size three, and there are no cycles in the graph.

In words, it represents the maximal amount of interference that can be added to the system and still maintain percolation. Second, the *critical intensity* is defined as

$$\lambda_c^* = \inf\{\lambda > 0 \colon \gamma(\lambda') > 0, \ \forall \lambda' > \lambda\}, \tag{2.3}$$

which describes the smallest intensity such that for all larger intensities, the addition of a small amount of interference does not destroy percolation.

For the statement of our first main result, we assume certain decorrelation and connectivity properties for the directing measure Λ of the underlying CPP. The precise definitions for Λ to be *stabilizing*, *b-dependent*, or *asymptotically essentially connected* are technical and will be presented in Definitions 3.1 and 3.2 in Section 3, where we will also mention a number of relevant examples of random measures satisfying these definitions. We denote by P_o a generic power random variable distributed according to μ and put $P_{\sup} = \operatorname{ess\,sup} \mu$. We call the path-loss function ℓ *well-behaved* if ℓ is continuous, constant on $[0, d_o]$ for some $d_o \geq 0$, and strictly decreasing on $[d_o, \infty) \cap \operatorname{supp}(\ell)$, and satisfies $\int_0^\infty r^{d-1}\ell(r)\mathrm{d}r < \infty$. Our first result establishes percolation for the SINR graph based on CPPs with random powers.

Theorem 2.1. *Let $d \geq 2$, N_o, $\tau > 0$, $P_{\sup} = \infty$; let Λ be stabilizing and ℓ well-behaved. Then $\lambda_c^* < \infty$ holds if at least one of the following conditions is satisfied:*

1. *The path-loss function ℓ has unbounded support, Λ is b-dependent for some $b > 0$, and both $\mathbb{E}[\exp(\alpha\Lambda(Q_1))] < \infty$ and $\mathbb{E}[\exp(\alpha P_o)] < \infty$ hold for some $\alpha > 0$.*
2. *The path-loss function ℓ has bounded support, $\mathbb{E}[P_o] < \infty$, and Λ is asymptotically essentially connected.*
3. *The path-loss function ℓ has bounded support, $\mathbb{E}[P_o] < \infty$, and $\sup\operatorname{supp}(\ell)$ is larger than c, where c is a finite constant depending on Λ, τ, and N_o.*

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As discussed in the introduction, Theorem 2.1 extends similar results known for the case of a homogeneous PPP in $\mathbb{R}^2$ with constant powers to the case of a CPP in $\mathbb{R}^d$, $d \geq 2$, with random and possibly unbounded powers. Let us note that the complementary assertion that $\lambda_c^* > 0$ can be deduced in certain cases based on recent results on Cox–Gilbert graphs with random radii; see [18] and Section 4.1 for more details.

Our second main result establishes a uniform upper bound on the critical interference-cancellation factor. For this assertion we assume the basic nondegeneracy property that X^λ is *nonequidistant*, which is satisfied for a very large class of CPPs, including many examples relevant to wireless telecommunication systems; see Section 3. This means that for all $i, j, k, l \in \mathbb{N}$, $|x_i - x_j| = |x_k - x_l| > 0$ implies $\{i, j\} = \{k, l\}$ and $|x_i| = |x_j|$ implies $i \neq j$, almost surely. Clearly, this property implies that the point process X^λ is simple; furthermore, if X^λ is nonequidistant for some $\lambda > 0$, then it is nonequidistant for every $\lambda > 0$. As for a (pathological) counterexample, note that if Λ is the sum of Dirac measures at the points of the randomly shifted lattice $\mathbb{Z}^d + U$, where U is a uniform random variable in $[0, 1]^d$, the associated CPP is simple, stationary, but not nonequidistant.

Theorem 2.2. *Let $d \geq 1$, $N_o \geq 0$, and $\tau, \lambda > 0$, and assume that X^λ is nonequidistant for all $\lambda > 0$. Then $\gamma(\lambda) \leq 1/(2\tau)$.*

Note that we do not require any stabilization or connectedness, and also we impose no direct restrictions on μ and ℓ. The proof of Theorem 2.2 rests on showing absence of percolation in the SINR graph with a maximal degree given by 2. The fact that SINR graphs with $\gamma > 0$ have degrees less than $1 + 1/(\tau\gamma)$ is already stated in [8, Theorem 1]; an immediate consequence of this assertion is that there is no percolation in the case $\gamma \geq 1/\tau$ when degrees are at most 1. These claims can easily be seen to hold for any simple point process in any dimension, although in [8] only the case of a two-dimensional homogeneous PPP is considered. Theorem 2.2 is the first improvement of this bound since then, applicable to stationary CPPs and thus in particular also covering the case of homogeneous PPPs in all dimensions.

Finally, our third main result states that the critical intensity parameter for the SINR graph can be represented as the critical threshold for percolation of an associated Gilbert graph in any dimension. For this we assume a simpler setting in which Λ equals the Lebesgue measure, i.e., the CPP is in fact a PPP, and the powers are non-random and given by $p > 0$. Note that for $\gamma = 0$, the SINR graph is in fact a *Poisson–Gilbert graph* (cf. [11]) with connection radius given by

$$r_B = \ell^{-1}(\tau N_o/p), \tag{2.4}$$

which is a well-defined quantity if $\ell(0) > \tau N_o/p$ and the conditions of Theorem 2.1 on ℓ hold.

Recall that the Gilbert graph based on a simple point process Y with connection radius $r > 0$ has vertex set Y and an edge between two different points of Y whenever the distance between the two points is less than r, and the name 'Poisson–Gilbert graph' corresponds to the case when Y is a homogeneous PPP. It is a standard result in continuum percolation that for the Poisson–Gilbert graph with connection radius $r \in (0, \infty)$ in $d \geq 2$ dimensions, there exists a unique critical intensity $\lambda_c(r) \in (0, \infty)$ that separates a supercritical regime, where $\lambda > \lambda_c(r)$, in which the Gilbert graph percolates with probability one, from a subcritical regime, where $\lambda < \lambda_c(r)$, in which the Gilbert graph does not percolate almost surely; see for example [25, Section 3].

Theorem 2.3. *Let $d \geq 2$, $N_o, \tau, p > 0$, and $\Lambda(\mathrm{d}x) = \mathrm{d}x$; let ℓ be well-behaved with $\ell(0) > \tau N_o/p$. Then $\lambda_c^* = \lambda_c(r_B)$.*

Theorem 2.3 extends the result [9, Theorem 1] to dimensions $d \geq 3$ using new techniques; see Section 4 for details.

In the following section we present our main technical conditions together with examples for which our main theorems are applicable.

3. Stabilization, asymptotic essential connectedness, and examples

The following definitions were recently introduced in [16] in order to prove existence of a unique nontrivial critical intensity threshold for Cox–Gilbert graphs with fixed connectivity threshold. Let us recall that $Q_n = [-n/2, n/2]^d$ for $n > 0$ and $d \in \mathbb{N}$; let $Q_n(x) = Q_n + x$ denote the box with side length n, centered at $x \in \mathbb{R}^d$, and let $\mathrm{dist}(x, A) := \inf\{|x - y| : y \in A\}$ for $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. We start with the definition of stabilization, which can be understood as a quantitative spatial mixing property of the directing measure of a CPP.

Definition 3.1. *(Stabilization.)* The random measure Λ is called stabilizing if there exists a random field of *stabilization radii* $R = \{R_x\}_{x\in\mathbb{R}^d}$ defined on the same probability space as Λ such that, writing

$$R(Q_n(x)) = \sup_{y\in Q_n(x)\cap\mathbb{Q}^d} R_y, \qquad n \geq 1,\ x \in \mathbb{R}^d,$$

the following hold:

1. (Λ, R) is jointly stationary.
2. We have $\lim_{n\uparrow\infty} \mathbb{P}(R(Q_n) < n) = 1$.
3. For all $n \geq 1$, non-negative bounded measurable functions $f : \mathbb{M} \to [0, \infty)$, and finite $\varphi \subset \mathbb{R}^d$ with $\mathrm{dist}(x, \varphi \setminus \{x\}) > 3n$ for all $x \in \varphi$, the following random variables are independent:
$$f(\Lambda_{Q_n(x)})\mathbb{1}\{R(Q_n(x)) < n\}, \qquad x \in \varphi,$$
where for a measurable set $A \subseteq \mathbb{R}^d$, Λ_A denotes the restriction of Λ to A.

A stronger form of stabilization is when Λ is *b-dependent*. That is, the restrictions Λ_A and Λ_B of Λ to the measurable sets $A, B \subset \mathbb{R}^d$ are independent whenever $\mathrm{dist}(A, B) > b$ for some $b > 0$. For b-dependence of subsets of $\mathbb{Z}^d$ we will use the analogous definition but with dist replaced by the ℓ^∞-distance.

Next we give a definition of asymptotic essential connectedness, a suitable way of capturing connectedness of the support of the directing measure of a CPP with high probability.

Definition 3.2. *(Asymptotic essential connectedness.)* The stabilizing random measure Λ with stabilization radius field R is *asymptotically essentially connected* if for all $n \geq 1$, whenever $R(Q_{2n}) < n/2$, we have that

i. $\mathrm{supp}(\Lambda_{Q_n})$ contains a connected component of diameter at least $n/3$, and
ii. any two connected components of $\mathrm{supp}(\Lambda_{Q_n})$ of diameter at least $n/9$ are contained in the same connected component of $\mathrm{supp}(\Lambda_{Q_{2n}})$.

The class of stabilizing random measures includes a number of interesting and relevant examples, for instance directing measures given via random tessellations based on PPPs. As already proven in [16, Section 3.1], for example, the edge-length measures of *Poisson–Voronoi tessellations* are asymptotically essentially connected (and hence also stabilizing), and it was also pointed out there that the same property for *Poisson–Delaunay tessellations* can be proven

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very similarly. It is nevertheless easy to see that these intensity measures are not b-dependent for any $b > 0$. However, let us note that the edge-length measures of *Poisson line tessellations* in $\mathbb{R}^2$ are not even stabilizing.

Stabilizing random measures that are absolutely continuous with respect to the Lebesgue measure include the directing measure of some *modulated PPPs* or *shot-noise fields* with compactly supported kernel. For the purpose of the present paper, a modulated PPP is defined with directing measure $\Lambda(\mathrm{d}x) = \lambda \mathbb{1}\{x \in \Xi\}\mathrm{d}x + \lambda' \mathbb{1}\{x \notin \Xi\}\mathrm{d}x$, for some Poisson Boolean model Ξ with constant connection radii, where the definition of a Poisson Boolean model (with constant connection radii) will be presented at the beginning of Section 5.3, and $\lambda, \lambda' \geq 0$. As noted in [16, Section 2.1], the intensity measure that this definition yields is easily seen to be b-dependent for some $b > 0$, and if λ and λ' are positive, then Λ is asymptotically essentially connected. There exist examples, both for $\lambda > 0$ and $\lambda' = 0$ and for $\lambda = 0$ and $\lambda' > 0$, such that asymptotic essential connectedness fails; see [28, Section 2.5.1] for details. However, if Ξ is in the supercritical regime for percolation and $\lambda > 0$, then Λ is asymptotically essentially connected, which follows from [26, Theorems 2 and 5], as was observed in [16, Section 2.1]. The general definition of a modulated PPP can be found in [6, Section 5.2.2]; there, the construction is similar to the case presented in our paper, but Ξ can be a general random closed subset of $\mathbb{R}^d$, and hence the arising directing measure need not even be stabilizing, as explained in [28, Section 2.5.1]. Finally, without going into details, let us mention that in the case when Ξ is a Poisson Boolean model with random radii, it is possible that the corresponding directing measure is stabilizing but not b-dependent for any $b > 0$; see [18, Example 3.4].

Shot-noise fields have directing measures of the form $\Lambda(\mathrm{d}x) = \sum_{i \in \mathbb{N}} \kappa(y_i - x)\mathrm{d}x$, with $(y_i)_{i \in \mathbb{N}}$ a homogeneous PPP and $\kappa : \mathbb{R}^d \to [0, \infty)$ compactly supported; cf. [16, Example 2.2]. They are always b-dependent for some $b > 0$, but not asymptotically essentially connected in general (see [16, Section 2.1]); however, in some relevant cases they are (see [28, Section 2.5.1]).

In the following section, we lay out the strategies for the proofs of our main results, and comment on limitations and further extensions of the statements presented.

4. Methods and discussion

4.1. Strategy of proof and discussion for Theorem 2.1

As mentioned in the introduction, the statement of Theorem 2.1 is an extension of the results of [22] to the case of stabilizing CPPs. For the proof, we combine the approach used in [22, Theorem 4.5] for handling random radii and the approach used in [28, Theorem 2.4] for dealing with the spatial correlations of the directing measure Λ of the CPP. To begin with, an easy coupling argument (see [27, Section 4.2.3.4]) implies that as long as the powers are bounded, all positive results of [28] about percolation in the Cox SINR graph for asymptotically essentially connected Λ are applicable. More precisely, we have the following proposition for the Cox SINR graph with random bounded powers.

Proposition 4.1. ([28, Theorem 2.4 and Proposition 2.7].) *Let $d \geq 2$, $N_o, \tau > 0$, and $\mathbb{P}(P_o > 0) > 0$; let Λ be stabilizing and ℓ well-behaved. If $P_{\sup} < \infty$ and $\ell(0) > \tau N_o / P_{\sup}$, then $\lambda_c^* < \infty$ holds if at least one of the following conditions is satisfied:*

1. *The path-loss function ℓ has unbounded support, Λ is b-dependent for some $b > 0$, $\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty$ holds for some $\alpha > 0$, and at least one of the following conditions holds: Λ is asymptotically essentially connected, or $P_{\sup}$ is sufficiently large.*

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2. *The path-loss function ℓ has bounded support, and Λ is asymptotically essentially connected.*

3. *The path-loss function ℓ has bounded support, and* $\sup \mathrm{supp}(\ell)$ *and* $P_{\sup}$ *are both sufficiently large.*

Note that we have formulated Condition 1 in Proposition 4.1 more generally than the statement in [28], and Condition 3 does not appear in [28]. However, the proof from [28] can also be adapted to these more general cases. Indeed, let us first explain how the case of a constant power $\mu = \delta_p, p > 0$, can be handled using the methods of [28]. The cases where Λ is asymptotically essentially connected in Proposition 4.1 for constant powers are covered by [28, Theorem 2.4, Part (2)]. Furthermore, the methods of the proof of [28, Proposition 2.7] apply to the case when Λ is stabilizing but not necessary asymptotically essentially connected. To see this, note that the arguments of that proof require the connection radii r_{B} (see (2.4)) to be large enough. Now, if $\mathrm{supp}(\ell)$ is unbounded, then one can always make r_{B} arbitrarily large by choosing the power value p sufficiently large, which corresponds to the case of large $P_{\sup} = p$ in Condition 1. Otherwise, this is not always possible, because $\sup_{p>0} \ell^{-1}(\tau N_0/p)$ equals the finite number $\sup \mathrm{supp}(\ell)$. However, once $\sup \mathrm{supp}(\ell)$ is sufficiently large, one can make r_{B} sufficiently large so that the proof of [28, Proposition 2.7] becomes applicable. Hence, we see that Proposition 4.1 indeed follows from [28] for fixed $p > 0$. Now, if μ is not concentrated at one point, then one can always choose $p_2 \geq p_1 > 0$ such that $\mu([p_1, \infty)) > 0$ and $\mu((p_2, \infty)) = 0$. Then, if p_1 is sufficiently large, the above arguments imply that there exists an infinite connected component in the subgraph of the SINR graph spanned by all vertices x_i where $i \in \mathbb{N}$ is such that $P_i > p_1$, for all sufficiently large $\lambda > 0$ and all sufficiently small $\gamma > 0$. Here, we bound all power values corresponding to the interferences by p_2 from above. See Section 5.1, Step 1, for further details of a very similar argument. We conclude that $\lambda_{\mathrm{c}}^* < \infty$ holds under the assumptions of Proposition 4.1. Given Proposition 4.1, in the present paper it suffices to prove the case when $P_{\sup} = \infty$. We prove Theorem 2.1 in Section 5.1.

Let us comment on some further aspects of Theorem 2.1. First, as for Condition 2 in Theorem 2.1, an extension to the general stabilizing case is not possible in general. Indeed, even if P_o has very heavy tails, as soon as $\mathrm{supp}(\ell)$ is bounded, the radii of the associated Cox–Gilbert graph with random radii are bounded. Then it is not hard to exhibit examples of stabilizing directing measures Λ such that $\lambda_{\mathrm{c}}^* = \infty$; see the examples in [28, Section 2.5.1].

Second, if Λ is such that $\Lambda(Q_1)$ is almost surely bounded, then the exponential-moment condition

$$\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty \tag{4.1}$$

of Condition 1 in Theorem 2.1 clearly holds for all $\alpha > 0$. For example, this is the case for the modulated PPP with $\lambda, \lambda' \geq 0$. Furthermore, (4.1) holds for shot-noise fields for all $\alpha > 0$; see e.g. [28, Section 2.5.1]. For Poisson–Voronoi and Poisson–Delaunay tessellations, the b-dependence assumption in Condition 1 fails for all $b > 0$, and hence percolation in the SINR graph can only be concluded for compactly supported ℓ. On the other hand, it was verified in [17] that for these two kinds of tessellations in two dimensions, $\mathbb{E}[\exp(\alpha \Lambda(Q_1))] < \infty$ holds for all $\alpha > 0$; it is not known whether the same holds in higher dimension.

Third, the moment conditions on P_o may look surprising at first. Indeed, why do we need to upper-bound moments of P_o in order to guarantee percolation in an SINR graph? This is indeed counterintuitive in view of the Gilbert graph since there larger radii would lead to better connectivity. However, in the SINR graph, as mentioned above, larger powers also increase interference and thus also might decrease connectivity. The classical approach used in [2, 9,

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28] to establish percolation in SINR graphs is to show that the underlying Gilbert graph satisfies some strong connectivity properties, and that at the same time the interferences can be uniformly bounded on large connected areas. We follow this approach as well; however, the random powers dictate several workarounds.

Fourth, Condition 1 in Theorem 2.1 is not necessarily optimal. However, we believe that if percolation with unbounded $\mathrm{supp}(\ell)$ and without exponential moments of P_o is possible, then the proof for this statement must be rather different from ours. An interference-control argument may not be possible at all; instead one should be able to show that the SINR values are sufficiently large for many transitions, yielding satisfactory connectivity of the network for percolation. Let us mention a similar problem. It was conjectured in [8] that in the case with constant powers, in order to have percolation in the SINR graph for large λ, ℓ has only to have integrable tails but may explode at zero. However, the setting where $\lim_{r\downarrow 0}\ell(r)=\infty$ is such that the classical interference-control argument, as exhibited in [9], certainly cannot work. Indeed, the interferences are almost surely finite, but they have infinite expectation (see [7]); hence there is no hope of applying a version of the exponential Markov inequality. Let us also note that the results of [7] imply that, if the tails of ℓ are not integrable, then SINR graphs with $\gamma>0$ have no edges.

Finally, under the assumptions of Theorem 2.1 on ℓ, for $\gamma=0$, the SINR graph $G_0(\mathbf{X}^\lambda)$ is a Gilbert graph with i.i.d. random radii $R_i=\ell^{-1}(\tau N_o/P_i)$. Let R_o denote a generic random variable having the same distribution as R_i. Now, if all other parameters are kept fixed, it is easy to see that $\gamma\mapsto\mathbb{P}(G_\gamma(\mathbf{X}^\lambda)$ percolates) is decreasing. Hence, if almost surely there is no percolation in the SINR graph for $\gamma=0$, then the same holds for all $\gamma>0$. Furthermore, as already mentioned, percolation properties of Gilbert graphs can equivalently be expressed in terms of the corresponding Boolean models. Thus, the recent result [18, Theorem 2.6, Part (2)] about existence of a subcritical phase in Cox Boolean models immediately implies the following assertion. If Λ is ϕ-stabilizing and R_o is unbounded with $\mathbb{E}[R_o^d]<\infty$, then $\lambda_c^*>0$. Here, the notion of ϕ-stabilization (cf. [18, Definition 2.5]) is very similar to our definition of stabilization, and many relevant stabilizing examples are also ϕ-stabilizing. This observation complements the result $\lambda_c^*<\infty$ in Theorem 2.1. Moreover, it improves the assertion of [27, Section 4.2.3.4] that $\lambda_c^*<\infty$ holds for bounded P_o (equivalently, bounded R_o) if Λ is stabilizing and ℓ satisfies the assumptions of Theorem 2.1 with $\ell(0)>\tau N_o/\mathrm{essinf}\mu$ or $\ell(0)\le\tau N_o/P_{\mathrm{sup}}$.

4.2. Strategy of proof and discussion for Theorem 2.2

Recall that we call a maximal connected component in a graph a *cluster*. As already pointed out in [8, Theorem 1], for $\gamma>0$, all degrees in $G_\gamma(\mathbf{X}^\lambda)$, where X^λ is a PPP, are less than $1+1/(\tau\gamma)$ for any choice of λ, $\tau>0$ and $N_o\ge 0$. In other words, each vertex in $G_\gamma(\mathbf{X}^\lambda)$ has at most $1+1/(\tau\gamma)$ neighbors. It is not hard to see that this property remains true if the PPP is replaced by a CPP, or even any simple point process; see [27, Section A.3]. Thanks to the degree bounds, any such Cox SINR graph with random powers for which $\gamma\ge 1/\tau$ has no infinite cluster since it has degrees bounded by 1. For $\gamma\in[1/(2\tau),\,1/\tau)$, we have an *a priori* degree bound of 2, which implies that all maximal connected components of SINR graphs are finite cycles or paths that are infinite in zero, one, or two directions. This is reminiscent of a one-dimensional percolation model, and thus the conjecture is that it contains no infinite clusters under general assumptions on the directing measure of the CPP; see Figure 2 for an illustration. The following proposition shows that this is indeed true for the Cox SINR graph with random powers.

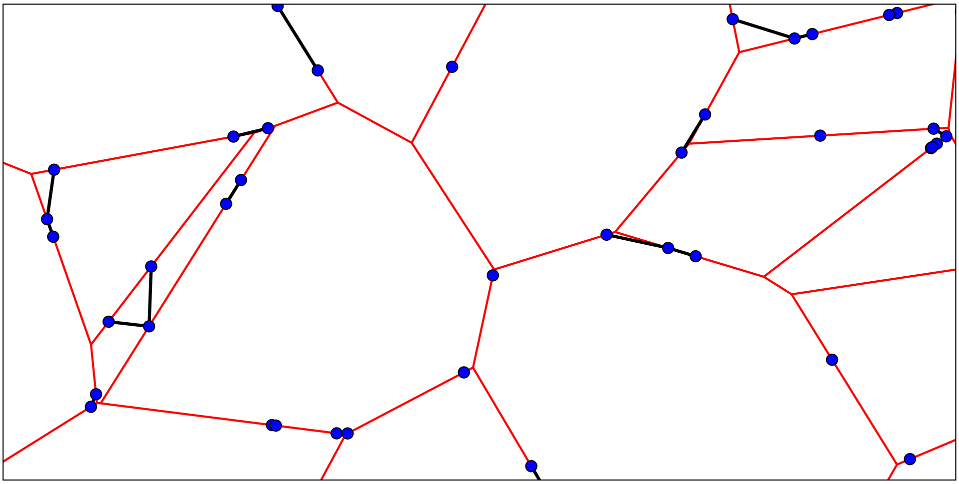

FIGURE 2. A typical realization of a Cox SINR graph (with blue vertices and black edges) with directing measure given by the edge-length measure of a two-dimensional Poisson–Voronoi tessellation (in red) in a box, with $N_o = P_o = \tau = 1$ and a suitable path-loss function ℓ. The interference-cancellation factor is set to $\gamma = 1/(2\tau)$. We see only a few vertices having degree two, the largest connected component is of size three, and there are no cycles in the graph. As indicated by Proposition 4.2 the graph is highly disconnected.

Proposition 4.2. *Let $d \geq 1$, $N_o \geq 0$, $\tau > 0$, and $\gamma \geq 1/(2\tau)$; then for Λ nonequidistant,*

$$\mathbb{P}(G_\gamma(\mathbf{X}^\lambda) \text{ percolates}) = 0.$$

The statement of Theorem 2.2 is an immediate consequence of Proposition 4.2, the proof of which can be found in Section 5.2. The proof employs a delicate configuration-wise analysis of the SINR graph, which seems to be new in the literature. Moreover, we expect the proof to hold for SINR graphs based on a large class of simple nonequidistant stationary point processes.

Let us comment on a further aspect of Theorem 2.2. It can be observed that the proof of Proposition 4.2 does not use the precise numerical relation $\gamma \geq 1/(2\tau)$, but rather just the fact that the SINR graph has degrees at most 2. Hence, once Λ is nonequidistant, the result holds as soon as the SINR graph has degrees bounded by 2. Note for example that if $N_o > 0$, P_o is bounded, and ℓ is continuous on $[0, \infty)$, then one can derive a stricter upper bound on the degrees (depending on several parameters) along the lines of the proof of [8, Theorem 1].

4.3. Strategy of proof for Theorem 2.3

As mentioned previously, we have $G_0(X^\lambda) = g_{r_B}(X^\lambda)$ for all $\lambda > 0$ in the Poisson SINR graph with fixed powers p, where r_B is defined in (2.4). We use X^λ instead of $\mathbf{X}^\lambda$ since the marks are non-random. Moreover, note that the increase of the interference-cancellation factor γ can only lead to edges being removed from the graph, and hence there is a monotonicity of $G_\gamma(X^\lambda)$ with respect to γ. Additionally, there is a monotonicity of $g_{r_B}(X^\lambda)$ with respect to λ, which together implies that $\lambda_c^* \geq \lambda_c(r_B)$.

Theorem 2.3 states that for PPPs and with constant powers, one actually has $\lambda_c^* = \lambda_c(r_B)$. This result is already known in two dimensions: see [9, Theorem 1]. In words, this result states that for any $\lambda > 0$ such that the Poisson–Gilbert graph $g_{r_B}(X^\lambda)$ is supercritical, there exists $\gamma > 0$ such that the Poisson SINR graph $G_\gamma(X^\lambda)$ also percolates. See [2, Section 3.4]

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for extensions of this result in the two-dimensional case in the context of sub-Poisson point processes.

The proof of [9, Theorem 1] employs Russo–Seymour–Welsh-type arguments for the Poisson–Gilbert graph in two dimensions; see [25, Section 4] and [9, Section 3]. These arguments have no known analogue in the Poisson case for $d \ge 3$, or in the general Cox case even for $d = 2$. Note that the results of [28] imply only that $\lambda_c^* < \infty$ for $d \ge 3$ and $\Lambda(\mathrm{d}x) = \mathrm{d}x$. However, [16, Section 2.1] includes some further observations about Gilbert graphs in $d \ge 3$ dimensions, originating from results of [26], that allow us to conclude Theorem 2.3 in higher dimensions. We will carry out the proof of Theorem 2.3 in Section 5.3, recalling also the corresponding results of [26].

5. Proofs

For the proofs it will be convenient to define the SINR of $x_i, x_j \in X^\lambda$, $x_i \neq x_j$, via

$$\mathrm{SINR}(x_i, x_j, \mathbf{X}^\lambda) = \frac{P_i \ell(|x_i - x_j|)}{N_o + \gamma \sum_{k \in \mathbb{N} \setminus \{i,j\}} P_k \ell(|x_k - x_j|)}. \qquad (5.1)$$

5.1. Proof of Theorem 1

In this section, we carry out the proof under Condition 1 of the theorem. The proofs under Conditions 2 and 3 are rather easy extensions of this proof that use some additional arguments [16, 28], which are omitted from our paper but included in the extended online version [19].

Assume for the rest of this section that Condition 1 holds. For fixed λ and γ, in order to show that $G_\gamma(\mathbf{X}^\lambda)$ percolates, it suffices to verify that a subgraph of it contains an infinite cluster. Our proof consists of four steps. First, for $\gamma, \lambda > 0$, we define a subgraph that is included in a Cox–Gilbert graph (with constant connection radii). Second, we map this subgraph to a lattice percolation model and show that this discrete model percolates for large λ for a suitable choice of auxiliary parameters. In particular, since Λ is assumed only to be stabilizing, the connection radius of the Gilbert graph must be large enough so that the graph percolates for large λ. In this step, we are able to employ multiple arguments of [9, 16, 28]. Our interference-control assertion, Proposition 5.1, is presented here. Third, using the subgraph, we make a choice of $\gamma > 0$ such that percolation in the discrete model implies percolation in the SINR graph $G_\gamma(\mathbf{X}^\lambda)$, which is done analogously to [9]. Fourth, we carry out the proof of Proposition 5.1, combining arguments of [9, 28] for SINR graphs with constant powers and arguments used in [22] for Poisson SINR graphs with random powers.

STEP 1. *A subgraph of the SINR graph.*

We first present a general construction of a subgraph of $G_\gamma(\mathbf{X}^\lambda)$ for $\gamma, \lambda > 0$. Let $r_o > d_o$. Since both P_o and $\mathrm{supp}(\ell)$ are unbounded, we have

$$p(r_o) := \mathbb{P}\big(\ell^{-1}(\tau N_o / P_o) \ge r_o\big) = \mathbb{P}\Big(P_o \ge \tau N_o / \ell(r_o)\Big) > 0.$$

Let us define the independent thinning

$$X^{\lambda,-} := \{x_i \in X^\lambda : P_o \ge \tau N_o / \ell(r_o)\}$$

of X^λ with survival probability $p(r_o)$. According to [20, Colouring Theorem], $X^{\lambda,-}$ is a CPP with directing measure $\lambda p(r_o)\Lambda$. Now, let us define a subgraph $G_\gamma^-(\mathbf{X}^\lambda)$ of $G_\gamma(\mathbf{X}^\lambda)$ as follows.

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The vertex set is $X^{\lambda,-}$, and two vertices $x_i, x_j \in X^{\lambda,-}$, $x_i \neq x_j$, are connected by an edge if and only if

$$\mathrm{SINR}^-(x_i, x_j, \mathbf{X}^\lambda) := \frac{\big(\tau N_o/\ell(r_o)\big)\ell(|x_i - x_j|)}{N_o + \gamma \sum_{k\in\mathbb{N}\setminus\{i,j\}} P_k \ell(|x_k - x_j|)} > \tau \tag{5.2}$$

and the analogously defined $\mathrm{SINR}^-(x_j, x_i, \mathbf{X}^\lambda)$ also exceeds τ. Note that for $x_i, x_j \in X^{\lambda,-}$, in the numerator of $\mathrm{SINR}(x_i, x_j, \mathbf{X}^\lambda)$, for the power of x_i we have $P_i \geq \tau N_o/\ell(r_o)$, whereas the denominators of (5.1) and (5.2) are equal, and the same holds with the roles of i and j interchanged. Hence, $G_\gamma^-(\mathbf{X}^\lambda)$ is indeed a subgraph of $G_\gamma(\mathbf{X}^\lambda)$ for any $\gamma \geq 0$. As for $\gamma = 0$, $G_0^-(\mathbf{X}^\lambda)$ equals the Cox–Gilbert graph $g_{r_o}(X^{\lambda,-})$ with connection radius r_o and vertex set $X^{\lambda,-}$. In words, in order to obtain $G_\gamma^-(\mathbf{X}^\lambda)$ from $G_\gamma(\mathbf{X}^\lambda)$, one first thins out vertices with small powers, so as to get rid of vertices with small values of the connection radius r_{B}^i, where

$$r_{\mathrm{B}}^i = \ell^{-1}(\tau N_o/P_i). \tag{5.3}$$

Then one bounds the powers of the remaining vertices by $\tau N_o/\ell(r_o)$ from below.

STEP 2. *Mapping the subgraph to a lattice-percolation problem and percolation on the lattice.*

Now we are in a position to adapt to the setting of [28, Section 3.2.2] and use strong connectivity of $g_{r_o}(X^{\lambda,-})$ in case r_o is sufficiently large and λ is chosen according to r_o. Together with an interference-control argument presented below, this will allow us to verify Condition 1 of Theorem 2.1.

First, let us recall the definition of rescalings of a Gilbert graph, which were also used in [28]. For $c > 0$ and a Gilbert graph G with connection radius $r > 0$, deterministic vertex set $V \subset \mathbb{R}^d$, $d \geq 1$, and edge set $E = \{(x, y) \in V \times V : x \neq y, |x - y| < r\}$, the graph cG is defined with vertex set $cV = \{cx : x \in V\}$ and edge set $cE = \{(cx, cy) : (x, y) \in E\}$. It is easy to see that cG is a Gilbert graph with vertex set cV and connection radius cr. For Gilbert graphs with random vertex sets (e.g., if the vertex set is given by a random simple point process), rescalings of the graph are defined realization-wise. From the proof of Theorem 2.9 (Convergence in Bounded Domains) in [16, Section 7.1], we know that in the coupled limit $\widetilde{r} \uparrow \infty$, $\widetilde{\lambda} \downarrow 0$ and $\widetilde{\lambda}\widetilde{r}^d = \widetilde{\varrho} > 0$, we have that $\widetilde{r}^{-1} g_{\widetilde{r}}(X^{\widetilde{\lambda}})$ converges weakly to the graph $g_1(Y^{\widetilde{\varrho}})$, where Y^ϱ is a homogeneous PPP with intensity ϱ. Let us note that in [16, Section 7.1] this convergence is formulated equivalently for the Boolean model, and the crucial point is that the convergence is guaranteed only in compact domains.

Let $\varrho_{\mathrm{c}}(1)$ be such that the Poisson–Gilbert graph $g_1(Y^{\varrho_{\mathrm{c}}(1)})$ is critical. Then, due to the scale invariance of Poisson–Gilbert graphs [25, Section 2.2], for $\varrho > \varrho_{\mathrm{c}}(1)$, we can choose a smaller intensity $\varrho' < \varrho$ such that $g_1(Y^{\varrho'})$ is still supercritical. Now, for $r > d_o$, we define $r_o(r) = r(\varrho/\varrho')^{1/d}$, $\lambda(r) = \varrho' r^{-d}(p(r_o(r)))^{-1}$, and $p(r) = \tau N_o/\ell(r_o(r))$. Noting that $g_r(X^{\lambda(r),-})$ is a Cox–Gilbert graph with connection radius r and stabilizing intensity $p(r_o(r))\lambda(r) = \varrho' r^{-d}$, we have that $r^{-1} g_r(X^{\lambda(r),-})$ converges to the supercritical graph $g_1(Y^{\varrho'})$ on compact sets, as r tends to infinity.

Furthermore, recalling that R denotes the stabilization radii of Λ, we put $R(Q) = \sup_{x\in Q\cap\mathbb{Q}^d} R_x$ for any measurable set $Q \subseteq \mathbb{R}^d$.

Using these notions, we construct a renormalized percolation process on $\mathbb{Z}^d$ as follows. For $n \geq 1$ and $r > d_o$, the site $z \in \mathbb{Z}^d$ is *(r, n)-good* if the following conditions hold:

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1. $R(Q_{6rn}(rnz)) < rn/2$.
2. $X^{\lambda(r),-} \cap Q_{rn}(rnz) \neq \varnothing$.
3. Every pair $x_i, x_j \in X^{\lambda(r),-} \cap Q_{3rn}(rnz)$ is connected by a path in $g_r(X^{\lambda(r),-}) \cap Q_{6rn}(rnz)$.

The site $z \in \mathbb{Z}^d$ is *(r, n)-bad* if it is not (r, n)-good. Note that the process of (r, n)-good sites is 7-dependent thanks to the definition of stabilization. The following lemma is verified in [28, Section 3.2.2]. However, since in [28] it is not formulated as a lemma, and two different proofs are presented for $d = 2$ and $d \geq 3$, we provide a self-contained proof here for the reader's convenience.

Lemma 5.1. ([28].) *Assume that the general conditions of Theorem 2.1 plus Condition 1 hold. Then, for all sufficiently large $\lambda > 0$ and for all $n \geq 1$ and $r > d_o$ with rn sufficiently large, there exists $q_A = q_A(\lambda, rn) < 1$ such that for any $N \in \mathbb{N}$ and pairwise distinct $z_1, \ldots, z_N \in \mathbb{Z}^d$,*

$$\mathbb{P}(z_1, \ldots, z_N \text{ are all } (r, n)\text{-bad}) \leq q_A^N. \tag{5.4}$$

Furthermore, for any $\varepsilon > 0$, one can choose λ and rn sufficiently large so that $q_A < \varepsilon$.

Proof. For $z \in \mathbb{Z}^d$, we write $J_{n,r}(z)$ for the event that z satisfies Parts 2 and 3 of the definition of (r, n)-goodness. Then, for any n, r under consideration, the process of (r, n)-good sites is 7-dependent by the definition of stabilization. Furthermore, we write $F_n(z)$ for the event that in the definition of $(1, n)$-goodness, the PPP $Y^{\varrho'}$ with intensity $\varrho' = \lambda(r)r^d$ satisfies Part 2 with $X^{\lambda(r),-}$ replaced by $Y^{\varrho'}$ and Part 3 with $g_r(X^{\lambda(r),-})$ replaced by $g_1(Y^{\varrho'})$ everywhere. The probability of $F_n(z)$ is independent of the choice of z and tends to 1 as $n \to \infty$ thanks to the arguments of [16, Section 5.2], since the constant directing measure of the PPP $Y^{\varrho'}$ is certainly asymptotically essentially connected. Using a union bound and the well-known scale invariance of Poisson–Gilbert graphs, namely that for $\widetilde{\lambda}, \widetilde{r} > 0$, $\widetilde{r}^{-1} g_{\widetilde{r}}(X^{\widetilde{\lambda}})$ equals $g_1(X^{\widetilde{\lambda}\widetilde{r}^d})$ in distribution, we conclude that for $z \in \mathbb{Z}^d$,

$$\mathbb{P}(z \text{ is } (n, r)\text{-bad}) \leq \mathbb{P}\Big(R(Q_{6nr}(nrz)) \geq nr/2\Big) + \mathbb{P}(F_n(z)^{\mathrm{c}}) + |\mathbb{P}(F_n(z)^{\mathrm{c}}) - \mathbb{P}(J_{n,r}(z)^{\mathrm{c}})|.$$

This can be made arbitrarily close to zero by choosing first n large and then r large according to n, due to the weak convergence of $r^{-1} g_r(X^{\lambda})$ to $g_1(Y^{\varrho'})$ on $Q_{6n}(nz)$ as $r \to \infty$, $\lambda(r) \to 0$, $r^d \lambda(r) = \varrho'$.

Hence, applying [23, Theorem 0.0], for all sufficiently large n and large enough r chosen according to n, the 7-dependent process of (r, n)-good sites is stochastically dominated from below by a supercritical independent site percolation process. Moreover, the probability that a site of the independent site percolation process is closed can be made arbitrarily close to 0 via further increasing nr. This implies the lemma. □

We proceed similarly to [9, 28] by defining 'shifted' versions of the path-loss function ℓ. For $a \geq 0$, define

$$\ell_a(r) = \ell(0)\mathbb{1}\{r < a\sqrt{d}/2\} + \ell\big(r - a\sqrt{d}/2\big)\mathbb{1}\{r \geq a\sqrt{d}/2\}. \tag{5.5}$$

Note that $\ell_0 = \ell$. Now, we define the shot-noise processes

$$I_a(x) = \textstyle\sum_{i \in \mathbb{N}} P_i \ell_a(|x - x_i|), \qquad I(x) = \textstyle\sum_{i \in \mathbb{N}} P_i \ell(|x - x_i|), \qquad x \in \mathbb{R}^d,$$

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and note that $I_0(x) = I(x)$. By the triangle inequality, for $a \geq 0$, $I(x) \leq I_a(z)$ holds for any $z \in \mathbb{R}^d$ and $x \in Q_a(z)$. Now, the interference-control argument consists in verifying the following proposition. For $z \in \mathbb{Z}^d$, let us write $B_{r,n,M}(z) = \{I_{6rn}(rnz) \leq M\}$.

Proposition 5.1. *Assume that the general conditions of Theorem 2.1 plus Condition 1 hold. Then, for all $\lambda > 0$, for all $n \geq 1$ and $r > d_o$ with rn sufficiently large, and for all $M > 0$ sufficiently large, there exists $q_B = q_B(\lambda, rn, N) < 1$ such that for all $N \in \mathbb{N}$ and for all pairwise distinct $z_1, \ldots, z_N \in \mathbb{Z}^d$ we have*

$$\mathbb{P}(B_{r,n,M}(z_1)^{\mathrm{c}} \cap \ldots \cap B_{r,n,M}(z_N)^{\mathrm{c}}) \leq q_B^N. \tag{5.6}$$

Furthermore, for any $\varepsilon > 0$ and $\lambda > 0$, one can choose rn and M sufficiently large so that $q_B < \varepsilon$.

The proof of this proposition is postponed until Step 4. Once we have proved Proposition 5.1, we can derive the following corollary using a standard argument (see e.g. the proof of [9, Proposition 3] or that of [28, Proposition 3.1]). For $z \in \mathbb{Z}^d$ let us define $C_{r,n,M}(z) = \{z \text{ is } (r, n)\text{-good}\} \cap \{I_{6rn}(rnz) \leq M\}$.

Corollary 5.1. *Assume that the general conditions of Theorem 2.1 plus Condition 1 hold. Then, for all sufficiently large $\lambda > 0$, for all $r > d_o$ and $n \geq 1$ with rn sufficiently large, and for all $M > 0$ sufficiently large, there exists $q_C = q_C(\lambda, rn, M) < 1$ such that for all $N \in \mathbb{N}$ and for all pairwise distinct $z_1, \ldots, z_N \in \mathbb{Z}^d$ we have*

$$\mathbb{P}(C_{r,n,M}(z_1)^{\mathrm{c}} \cap \ldots \cap C_{r,n,M}(z_N)^{\mathrm{c}}) \leq q_C^N. \tag{5.7}$$

Furthermore, for any $\varepsilon > 0$, one can choose λ, rn, M sufficiently large so that $q_C < \varepsilon$.

STEP 3. *Percolation in the subgraph of the SINR graph.*

Having Corollary 5.1 and employing a Peierls argument (cf. [15, Section 1.4]), we conclude that for λ, rn, M sufficiently large, the process of (r, n)-good sites $z \in \mathbb{Z}^d$ such that $I_{6rn}(rnz) \leq M$ percolates. Thanks to exactly the same arguments as in the proof of Theorem 2.6 in [16, Section 5.2], this implies percolation of the Cox–Gilbert graph $G_0^-(\mathbf{X}^{\lambda(r)}) = g_{r_o(r)}(X^{\lambda(r),-})$. From this point of the proof it is classical to derive that $G_\gamma^-(\mathbf{X}^{\lambda(r)})$ percolates for small $\gamma > 0$; see [9, Section 3.3]. For the convenience of the reader, let us give the details here. We define

$$\gamma' = \frac{N_o}{p(r)M}\Big(\frac{\ell(r)}{\ell(r_o(r))} - 1\Big) = \frac{\ell(r_o(r))}{\tau M}\Big(\frac{\ell(r)}{\ell(r_o(r))} - 1\Big) > 0,$$

where the strict inequality holds because $r_o(r) > r > d_o$ and ℓ has unbounded support. Then we have

$$\frac{p(r)\ell(r)}{N_o + \gamma' p(r)M} = \tau.$$

Now, let $x_i, x_j \in X^{\lambda(r),-}$ be situated in $Q_{rn}(rnz)$ and $Q_{rn}(rnz')$, respectively, for some sites $z, z' \in \mathbb{Z}^d$ included in the same infinite cluster of the process of (r, n)-good sites $z \in \mathbb{Z}^d$ satisfying $I_{6rn}(rnz) \leq M$, with $|x_i - x_j| < r$. Then, for $\gamma < \gamma'$, we have

$$\mathrm{SINR}(x_i, x_j, \mathbf{X}^\lambda) \geq \mathrm{SINR}^-(x_i, x_j, \mathbf{X}^\lambda) > \frac{p(r)\ell(r)}{N_o + \gamma' p(r)M} = \tau.$$

Thus, x_i and x_j are connected by an edge in $G_\gamma^-(\mathbf{X}^\lambda)$. Hence, $G_\gamma(\mathbf{X}^\lambda)$ also percolates. Thus, we can conclude Theorem 2.1 as soon as we have verified Proposition 5.1.

STEP 4. *Proof of Proposition 5.1: the interference-control argument.*

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Similarly to [28, Section 3.1.1], we split the interference into two parts. For $x \in \mathbb{R}^d$, $n \geq 1$, and $r > 0$, we put

$$I^{\text{in}}_{6rn}(x) = \sum_{x_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(x)} \ell_{6rn}(|x_i - x|),$$

$$I^{\text{out}}_{6rn}(x) = \sum_{x_i \in X^\lambda \setminus Q_{12rn\sqrt{d}}(x)} \ell_{6rn}(|x_i - x|).$$

Then, for $M > 0$, if $I_{6rn}(x) > M$, then $I^{\text{in}}_{6rn}(x) > M/2$ or $I^{\text{out}}_{6rn}(x) > M/2$. Using a union bound and the fact that M can be chosen arbitrarily large in Proposition 5.1, it suffices to conclude the proposition both with $B_{r,n,M}(z_i)$ replaced by $B^{\text{in}}_{r,n,M}(z_i)$ and with $B_{r,n,M}(z_i)$ replaced by $B^{\text{out}}_{r,n,M}(z_i)$ everywhere in (5.6) for all $i \in \{1, \ldots, N\}$, where for $z \in \mathbb{Z}^d$ we write $B^{\text{in}}_{r,n,M}(z) = \{I^{\text{in}}_{6rn}(rnz) \leq M\}$ and $B^{\text{out}}_{r,n,M}(z) = \{I^{\text{out}}_{6rn}(rnz) \leq M\}$. Indeed, having these assertions, we can combine them similarly to Corollary 5.1.

We now verify Proposition 5.1 with $B_{r,n,M}(\cdot)$ replaced by $B^{\text{in}}_{r,n,M}(\cdot)$ everywhere. For this assertion, instead of the assumption that P_o and $\Lambda(Q_1)$ have some exponential moments, it suffices to assume they have a first moment (for $\Lambda(Q_1)$ this is automatic since $\mathbb{E}[\Lambda(Q_1)] = 1$ by assumption). To be more precise, we prove the following lemma.

Lemma 5.2. *Assume that the general conditions of Theorem 2.1 plus Condition 1 hold. Furthermore, let Λ be stabilizing and let $\mathbb{E}[P_o] < \infty$. Then, for all $\lambda > 0$, for all $n \geq 1$ and $r > d_o$ with rn sufficiently large, and for all $M > 0$ sufficiently large, there exists $q_B = q_B(\lambda, rn, N) < 1$ such that for all $N \in \mathbb{N}$ and for all pairwise distinct $z_1, \ldots, z_N \in \mathbb{Z}^d$ we have*

$$\mathbb{P}\big(B^{\text{in}}_{r,n,M}(z_1)^{\text{c}} \cap \ldots \cap B^{\text{in}}_{r,n,M}(z_N)^{\text{c}}\big) \leq q_B^N.$$

Furthermore, for any $\varepsilon > 0$ and $\lambda > 0$, one can choose rn and M sufficiently large so that $q_B < \varepsilon$.

Proof. We use the following auxiliary discrete percolation process. A site $z \in \mathbb{Z}^d$ is *(r,n)-tame* if the following hold:

1. $R\big(Q_{12rn\sqrt{d}}(rnz)\big) < rn/2$.
2. $I^{\text{in}}_{6rn}(rnz) \leq M$.

A site $z \in \mathbb{Z}^d$ is *(r, n)-wild* if it is not (r, n)-tame. The process of (r, n)-tame sites is $\lceil 12\sqrt{d} + 1 \rceil$-dependent according to the definition of stabilization. Thus, it follows from dependent-percolation theory [23, Theorem 0.0] that, in order to verify Lemma 5.2, it suffices to show that for all $\lambda > 0$, $\mathbb{P}(o$ is (r, n)-wild$)$ can be made arbitrarily close to zero by choosing first rn sufficiently large and then M large enough accordingly. We have

$$\mathbb{P}(o \text{ is } (r,n)\text{-wild}) \leq \mathbb{P}\big(R\big(Q_{12rn\sqrt{d}}(rnz)\big) \geq rn/2\big) + \mathbb{P}\big(I^{\text{in}}_{6rn}(rnz) > M\big).$$

The first term can be made arbitrarily small by choosing rn large enough, thanks to the definition of stabilization. Moreover, by the definition of ℓ_a (see (5.5)),

$$I^{\text{in}}_{6rn}(o) = \sum_{x_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(o)} P_i \ell_{6rn}(|x_i|) \leq \ell(0) \sum_{x_i \in X^\lambda \cap Q_{12rn\sqrt{d}}(o)} P_i.$$

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In particular, using that the point process $\mathbf{X}^\lambda$ is independently marked with P_i having marginal distribution μ, and that Λ is stationary with $\mathbb{E}[\Lambda(Q_1)] = 1$, it follows that

$$\mathbb{E}\big[I^{\mathrm{in}}_{6rn}(o)\big] \le \ell(0)\lambda\mathbb{E}[P_o]\mathbb{E}\big[\Lambda\big(Q_{12rn\sqrt{d}}\big)\big] = \Big(12rn\sqrt{d}\Big)^d \ell(0)\lambda\mathbb{E}[P_o].$$

Thus, for any $n \ge 1$ and $r > 0$, $\mathbb{P}\big(I^{\mathrm{in}}_{6rn}(o) > M\big)$ can be made arbitrarily small by choosing M large enough, given that $\mathbb{E}[P_o] < \infty$. Thus, the statement of the lemma follows. □

It remains to verify Proposition 5.1 with $B_{r,n,M}(\,\cdot\,)$ replaced by $B^{\mathrm{out}}_{r,n,M}(\,\cdot\,)$ everywhere. More precisely, thanks to the exponential-moment and b-dependence assumption on Λ, the proof can be completed analogously to the proof of [28, Proposition 3.3] starting from [28, Equation (3.15)], as soon as we have verified the following lemma. (In [28] it is also assumed that $\ell(0) \le 1$, but since M can be made arbitrarily large in Proposition 5.1, $\ell(0) \le 1$ can be assumed without loss of generality since for ℓ continuous, the function $\widetilde{\ell} = \ell/\ell(0)$ satisfies $\widetilde{\ell}(0) = 1$, and for $a \ge 0$, we have $\ell_a = \ell(0)\widetilde{\ell}_a$ and hence $I_a(x) = \ell(0)\sum_{i\in\mathbb{N}} P_i\widetilde{\ell}_a(|x - x_i|)$.)

Lemma 5.3. *Under the general assumptions of Theorem 2.1 plus Condition 1, there exists a constant $c_o = c_o(\mu, \ell) > 0$ such that for all sufficiently small $s > 0$, for all $\lambda > 0$, for all $n \ge 1$ and $r > d_o$ with $rn > 0$ sufficiently large, and for all large enough $M > 0$, for all $N \in \mathbb{N}$ and pairwise distinct $z_1, \dots, z_N \in \mathbb{Z}^d$ we have*

$$\mathbb{P}\big(B^{\mathrm{out}}_{r,n,M}(z_1)^{\mathrm{c}} \cap \dots \cap B^{\mathrm{out}}_{r,n,M}(z_N)^{\mathrm{c}}\big) \le \mathbb{E}\left[\exp\left(c_o\lambda s\sum_{i=1}^N \int_{\mathbb{R}^d\setminus Q_{12rn\sqrt{d}}(rnz_i)} \ell_{6rn}(|rnz_i - x|)\Lambda(\mathrm{d}x)\right)\right]. \tag{5.8}$$

Indeed, the right-hand side of (5.8) is the same as that of [28, Equation (3.15)], and the assumptions on Λ in the two proofs are also the same.

Proof of Lemma 5.2. We start with an estimate originating from [9, Section 3.2]. By Markov's inequality, for any $s > 0$,

$$\begin{aligned}\mathbb{P}\big(B^{\mathrm{out}}_{r,n,M}(z_1)^{\mathrm{c}} \cap \dots \cap B^{\mathrm{out}}_{r,n,M}(z_N)^{\mathrm{c}}\big) &= \mathbb{P}\big(I^{\mathrm{out}}_{6rn}(rnz_1) > M, \dots, I^{\mathrm{out}}_{6rn}(rnz_N) > M\big) \\ &\le \mathbb{P}\left(\sum_{i=1}^N I^{\mathrm{out}}_{6rn}(rnz_i) > NM\right) \\ &\le \varepsilon^{-sNM}\mathbb{E}\left[\exp\left(s\sum_{i=1}^N \sum_{x_k\in X^\lambda\setminus Q_{12rn\sqrt{d}}(nz_i)} P_k\ell_{6rn}(|rnz_i - x_k|)\right)\right].\end{aligned} \tag{5.9}$$

The randomness of the power values P_k prevents us from continuing the proof analogously to [9, 28]. On the other hand, similarly to [22, Section 4.3] in the Poisson case, we can argue as follows. According to the Marking Theorem [20, Section 5.2], the independently marked CPP $\mathbf{X}^\lambda = \{(x_i, P_i)\}_{i\in\mathbb{N}}$ is a CPP in $\mathbb{R}^d \times [0, \infty)$ with directing measure $\Lambda \otimes \mu$, where we recall that $\mu = \mathbb{P} \circ P_o^{-1}$ is the distribution of P_o. Hence, applying the Laplace functional of a CPP (cf. [20, Sections 3.2 and 6]) to the function $f\colon \mathbb{R}^d \times [0, \infty) \to [0, \infty)$,

$$f(x, p) = s\sum_{i=1}^N p\ell_{6rn}(|x - rnz_i|)\mathbb{1}\big\{x \in \mathbb{R}^d \setminus Q_{12rn\sqrt{d}}(rnz_i)\big\},$$

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we obtain

$$\mathbb{E}\left[\exp\left(s\sum_{i=1}^{N}\sum_{x_k\in X^\lambda\setminus Q_{12rn\sqrt{d}}(rnz_i)}P_k\ell_{6rn}(|rnz_i-x_k|)\right)\right]$$

$$=\mathbb{E}\left[\exp\left(\lambda\int_{\mathbb{R}^d\setminus Q_{12rn\sqrt{d}}(rnz_i)}\int_0^\infty\left(\exp\left(sp\sum_{i=1}^{N}\ell_{6rn}(|rnz_i-x|)\right)-1\right)\mu(\mathrm{d}p)\Lambda(\mathrm{d}x)\right)\right]. \tag{5.10}$$

Thanks to the exponential-moment assumption on P_o from Condition 1, the moment-generating function

$$\alpha\mapsto\mathbb{E}[\exp(\alpha P_o)]=\int_0^\infty \varepsilon^{\alpha p}\mu(\mathrm{d}p)$$

is infinitely differentiable at $\alpha=0$, with first derivative $\int_0^\infty p\mu(\mathrm{d}p)=\mathbb{E}[P_o]<\infty$. Note that $\sum_{i=1}^N\ell_{6rn}(|rnz_i-x|)$ is uniformly bounded in $x\in\mathbb{R}^d$, rn, N, and pairwise distinct $z_1,\ldots,z_N$; see [28, Lemma 3.6]. Consequently, for any $C>1$, the following holds for all sufficiently small $s>0$ (depending on C):

$$\int_0^\infty\left(\exp\left(sp\sum_{i=1}^{N}\ell_{6rn}(|rnz_i-x|)\right)-1\right)\mu(\mathrm{d}p)\le Cs\mathbb{E}[P_o]\sum_{i=1}^{N}\ell_{6rn}(|rnz_i-x|). \tag{5.11}$$

For such s, plugging (5.11) back into (5.10), starting from (5.9) we obtain

$$\mathbb{P}\big(B^{\mathrm{out}}_{r,n,M}(z_1)^{\mathrm{c}}\cap\ldots\cap B^{\mathrm{out}}_{r,n,M}(z_N)^{\mathrm{c}}\big)$$

$$\le\mathbb{E}\left[\exp\left(C\mathbb{E}[P_o]\lambda s\sum_{i=1}^{N}\int_{\mathbb{R}^d\setminus Q_{12rn\sqrt{d}}(rnz_i)}\ell_{6rn}(|rnz_i-x|)\Lambda(\mathrm{d}x)\right)\right], \tag{5.12}$$

which is (5.8) with $c_o=C\mathbb{E}[P_o]$. With this we conclude the proof of the lemma. □

5.2. Proof of Proposition 4.2

The strategy of the proof of Proposition 4.2 is the following. We first show that up to $\mathbb{P}$-null sets, clusters are either finite or infinite in both directions, i.e., they contain no vertex of degree 1 in the case where they are infinite; see Lemma 5.4 below. Next, we assume for a contradiction that there exists an infinite cluster with positive probability. We then introduce a procedure that removes points from the infinite cluster that is closest to the origin in a certain sense. Thanks to elementary properties of the SINR graph, in the resulting configuration, the infinite cluster still remains infinite, but it contains a vertex of degree 1. Hence, the probability that the process takes values in the set of the resulting configurations is zero. What remains to show afterwards is that the probability that the process takes place in the set of original configurations is also zero, which leads to the desired contradiction. At this point it will be useful to compare the resulting configuration with an independent thinning of the original configuration in a certain ball, and this is where we make use of the fact that the underlying point process is a stationary CPP.

We assume throughout the proof that $\gamma\ge 1/(2\tau)$, so that degrees in $G_\gamma(\mathbf{X}^\lambda)$ are bounded by 2, and that X^λ is nonequidistant (for all $\lambda>0$). We can also assume that $\mathbb{P}(P_o>0)>0$ in what

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follows, since otherwise the statement is trivially true. We start the proof with the following lemma, which excludes infinite paths that have an endpoint in the case where the degrees are bounded by 2, in a substantially more general setting.

Lemma 5.4. *Let $g(\mathbf{X})$ be a random graph based on a stationary marked point process $\mathbf{X} = \{(x_i, P_i)\}_{i\in\mathbb{N}}$ with values in $\mathbb{R}^d \times Z$, where the mark space $(Z, \mathcal{Z})$ is an arbitrary measurable space, $X = \{x_i\}_{i\in\mathbb{N}}$ is the vertex set, and the degree of all $x_i \in X$, $\deg(x_i)$, is bounded by 2, almost surely. Let X have a finite intensity and consider the point process of degree-one points in infinite clusters,*

$$\mathcal{X}_0 = \sum_{i\in\mathbb{N}} \delta_{x_i} \mathbb{1}\{\deg(x_i) = 1,\ x_i \text{ is part of an infinite cluster in } g(\mathbf{X})\}.$$

Then $\mathbb{P}(\mathcal{X}_0(\mathbb{R}^d) = 0) = 1$.

We will apply this lemma to the SINR graph $g(\mathbf{X}) = G_\gamma(\mathbf{X}^\lambda)$ with λ arbitrary, $\gamma \geq 1/(2\tau)$, and $Z = [0, \infty)$. The proof is based on a variant of the mass-transport principle (cf. [4, Section 4.2] for instance).

Proof of Lemma 5.4. First, using the union bound and stationarity, it is enough to show that $\mathbb{E}[\mathcal{X}_0(Q_1)] = 0$. Let us define the point process of points in infinite clusters that are at distance equal to $k \in \mathbb{N}_o$ from a point in $\mathcal{X}_0$,

$$\mathcal{X}_k = \sum_{i\in\mathbb{N}} \delta_{x_i} \mathbb{1}\{x_i \text{ is part of an infinite cluster and has graph distance } k \text{ from } \mathcal{X}_0\}.$$

Thanks to the degree bound, every infinite cluster has at most one point in $\mathcal{X}_0$, and $\mathbb{E}[\mathcal{X}_k(Q_1)] = \mathbb{E}[\mathcal{X}_0(Q_1)]$ for all $k \in \mathbb{N}_o$, by stationarity. However, $\sum_{k\geq 0} \mathbb{E}[\mathcal{X}_k(Q_1)] \leq \mathbb{E}[X(Q_1)] < \infty$, and thus $\mathbb{E}[\mathcal{X}_0(Q_1)] = 0$. □

Let us denote by $(\mathcal{C}_i)_{0\leq i<L}$ the L-many infinite clusters in $G_\gamma(\mathbf{X}^\lambda)$, where $L \in \mathbb{N} \cup \{\infty\}$. For the proof of Proposition 4.2, it then suffices to show that

$$\mathbb{P}(L \geq 1) = 0. \tag{5.13}$$

We view $\mathbf{X}^\lambda$ as the canonical process $\mathbf{X}^\lambda(\boldsymbol{\omega}) = \boldsymbol{\omega}$ on the set $\mathbf{N}$ of marked point configurations $\boldsymbol{\omega}$ in $\mathbb{R}^d \times [0, \infty)$ such that $\omega = \{x_i : (x_i, p_i) \in \boldsymbol{\omega}\}$ is an infinite, locally-finite, nonequidistant point configuration on $\mathbb{R}^d$. The set of such point configurations ω will be denoted by N. Note that $\mathbf{N}$ and N are equipped with the corresponding evaluation σ-fields.

Now we introduce an ordering in $\mathbb{R}^d \times [0, \infty)$, which orders the points of the set according to the received signal power at a given point $y \in \mathbb{R}^d$ (or equivalently, according to the received SINR values $\mathrm{SINR}(\cdot, y, \boldsymbol{\omega})$ at y).

Definition 5.1. Let $(x, p), (z, r) \in \mathbb{R}^d \times [0, \infty)$ and $y \in \mathbb{R}^d$. We say that (x, p) transmits a stronger signal to y than (z, r) does if one of the following conditions is satisfied:

i. $p\ell(|x - y|) > r\ell(|z - y|)$, or

ii. $p\ell(|x - y|) = r\ell(|z - y|)$ and $|x - y| < |z - y|$.

When talking about the marked CPP $\mathbf{X}^\lambda$, we will always assume that transmitters are associated with their own transmitted signal powers, and hence we will say 'x_i transmits a stronger signal to x_j than x_l does' instead of '(x_i, P_i) transmits a stronger signal to x_j than (x_l, P_l) does',

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for any $i, j, l \in \mathbb{N}$ such that $i \neq j$ and $l \neq j$. It is easy to see that for $\mathbf{X}^\lambda$ such that X^λ is nonequidistant, almost surely the following holds. For all $i \in \mathbb{N}$, the relation 'x_i transmits a stronger signal to x_j than x_l does' is a total ordering (i.e., irreflexive, antisymmetric, and transitive, with any two elements being comparable) on the set $\{(i, l) \in \mathbb{N}^2 : i \neq j \text{ and } l \neq j\}$, which we call the *ordering of signal-weighted distance* from receiver x_i. This fact indeed relies on the tiebreaking mechanism (ii): e.g., if ℓ is constant on some interval (which is possible under the assumption of Proposition 4.2 and even under the stronger assumption of Theorem 2.1), then (i) does not define a total ordering on its own.

For $\boldsymbol{\omega} \in \mathbf{N}$ and $x_o \in \omega$, we can consider the vector $\mathbf{V}(x_o, \boldsymbol{\omega}) = (\mathbf{V}_n(x_o, \boldsymbol{\omega}))_{n \in \mathbb{N}_0}$ of the marked points of $\boldsymbol{\omega}$ ordered increasingly according to signal-weighted distance from receiver x_o. Then, we define $\mathrm{V}_i(x_o, \boldsymbol{\omega})$ as the first component of $\mathbf{V}_i(x_o, \boldsymbol{\omega})$, which we call the *ith nearest neighbor of x_o in signal-weighted order*. In particular, $\mathrm{V}_0(x_o, \boldsymbol{\omega}) = x_o$. Note that if the distribution μ is concentrated in one point $p > 0$, i.e., $\mu = \delta_p$, then the ith nearest neighbor of x_o in signal-weighted order is just the ith nearest neighbor of x_o with respect to Euclidean distance.

Now, if x_o has degree two in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}))$, then x_o must be connected by an edge to both $\mathrm{V}_1(x_o, \boldsymbol{\omega})$ and $\mathrm{V}_2(x_o, \boldsymbol{\omega})$, since the degree bound applies already for the edges towards x_o. Moreover, both $\mathrm{V}_1(x_o, \boldsymbol{\omega})$ and $\mathrm{V}_2(x_o, \boldsymbol{\omega})$ must also have x_o as one of their first two nearest neighbors in signal-weighted order; that is,

$$x_o \in \left\{\mathrm{V}_1(\mathrm{V}_i(x_o, \boldsymbol{\omega}), \boldsymbol{\omega}),\, \mathrm{V}_2(\mathrm{V}_i(x_o, \boldsymbol{\omega}), \boldsymbol{\omega})\right\}$$

for all $i \in \{1, 2\}$. These signal-weighted nearest neighbor relations hold almost surely, in particular for every nonequidistant configuration $\boldsymbol{\omega}$. The goal of using the configuration space $\mathbf{N}$ is to entirely exclude configurations that violate the degree bound or the signal-weighted nearest neighbor relations or that are not nonequidistant.

In the event $\{L \geq 1\}$, let $\mathbf{Z} = (Z, R)$ denote the closest point to the origin that has degree two and is contained in an infinite cluster. Without loss of generality, we will assume that this cluster is always equal to $\mathcal{C}_0$. Now, Proposition 4.2 immediately follows once we have verified the following proposition.

Proposition 5.2. *Consider the event $\{L \geq 1\}$ and define the random variable*

$$I = \inf\{i \geq 3 : \mathrm{V}_i(Z, \mathbf{X}^\lambda) \in \mathcal{C}_0\}.$$

Then, under the assumptions of Proposition 4.2, for any $i \geq 3$, we have

$$\mathbb{P}(\{L \geq 1\} \cap \{I = i\}) = 0. \tag{5.14}$$

Proof of Proposition 4.2. Using a union bound and noting that $\{L \geq 1\} \subset \{I < \infty\}$, Proposition 5.2 implies $\mathbb{P}(L \geq 1) = 0$, which is (5.17), and thus the proof of Proposition 4.2 is finished. □

Proof of Proposition 5.2. For $\boldsymbol{\omega} \in \{L \geq 1\}$, by definition, we have that $Z(\boldsymbol{\omega})$ is connected by an edge to both $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}))$. Furthermore, thanks to the degree bound of two, in the event $\{L \geq 1\}$, $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ have no further joint neighbor in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}))$, since otherwise $\mathcal{C}_0(\boldsymbol{\omega})$ has a loop and cannot be infinite by the degree bound. Thus, for any $i \geq 3$, there exists $l \in \{1, 2\}$ such that $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ and $\mathrm{V}_l(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ are not connected by an edge in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}))$. Let us denote the corresponding $\mathrm{V}_l(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ by $M_i(\boldsymbol{\omega})$, and define $M_i(\boldsymbol{\omega}) = \mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ if neither $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ nor $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ is connected to $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ by an edge. The element of $\{\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega}), \mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega})\}$ not equal to $M_i(\boldsymbol{\omega})$ is denoted by $N_i(\boldsymbol{\omega})$. We will write Q for the signal power transmitted by $M_i(\boldsymbol{\omega})$.

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Let us fix $i \geq 3$. Let $\boldsymbol{\omega} \in \{L \geq 1\}$ be such that $I(\boldsymbol{\omega}) = i$. Let us define a thinned configuration

$$\boldsymbol{\omega}^i = \boldsymbol{\omega} \setminus \{(M_i(\boldsymbol{\omega}), Q), \mathbf{V}_3(Z(\boldsymbol{\omega}), \boldsymbol{\omega}), \ldots, \mathbf{V}_{i-1}(Z(\boldsymbol{\omega}), \boldsymbol{\omega})\}.$$

We claim for $\mathbb{P}$-almost all $\boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\}$ that also $\boldsymbol{\omega}^i \in \{L \geq 1\}$. For this, first note that the removal of finitely many points and their associated edges from an infinite cluster does not change the property that the cluster is infinite. However, the removal of points can still change the edge structure of the remaining points. In order to exclude this, we use the following fundamental property of the SINR graph. Assume that $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ are elements of $\mathbf{N}$ such that $\boldsymbol{\omega} \subseteq \boldsymbol{\omega}'$. Then, for all $x, y \in \omega$, if $\mathrm{SINR}(x, y, \boldsymbol{\omega}') > \tau$, then $\mathrm{SINR}(x, y, \boldsymbol{\omega}) > \tau$, which is clear from (5.1). In words, if we remove some vertices from an SINR graph, then edges of the SINR graphs between the remaining points stay preserved.

Our next claim is that for $\boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\}$, $\boldsymbol{\omega}^i$ is contained in

$$B = \{\boldsymbol{\eta} : L(\boldsymbol{\eta}) \geq 1 \text{ and } \mathcal{C}_0(\boldsymbol{\eta}) \text{ contains a point of degree one}\} \subset \{L \geq 1\}.$$

The proof of this claim in the simplest case $i = 3$ is illustrated in Figure 3. For general $i \geq 3$, recall that Z cannot have degree higher than two in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$, whereas it has degree at least one and its cluster $\mathcal{C}_0(\boldsymbol{\omega}^i)$ is infinite in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$. Note also that the edge between $Z(\boldsymbol{\omega})$ and $N_i(\boldsymbol{\omega})$ still exists in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$. Furthermore, if $Z(\boldsymbol{\omega})$ has degree two in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$, then it is connected to the second-nearest neighbor of $Z(\boldsymbol{\omega})$ in signal-weighted order in $\boldsymbol{\omega}^i$, which is $\mathrm{V}_2(Z(\boldsymbol{\omega}), \boldsymbol{\omega}^i) = \mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$, whereas $\mathrm{V}_1(Z(\boldsymbol{\omega}), \boldsymbol{\omega}^i) = N_i(\boldsymbol{\omega})$. Now, since $\boldsymbol{\omega} \notin B$, $\boldsymbol{\omega} \in \{L \geq 1\}$, and $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in \mathcal{C}_0(\boldsymbol{\omega})$, it follows that $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ has degree equal to two in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}))$. Furthermore, it is neither connected to $M_i(\boldsymbol{\omega})$ by an edge nor to $Z(\boldsymbol{\omega})$ in this graph. Hence, both edges adjacent to $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ also exist in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$. But since $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ has degree at most two in $G_\gamma(\mathbf{X}^\lambda(\boldsymbol{\omega}^i))$, it follows that $Z(\boldsymbol{\omega})$ and $\mathrm{V}_i(Z(\boldsymbol{\omega}), \boldsymbol{\omega})$ are not connected by an edge in this graph. Hence, $\boldsymbol{\omega}^i \in B$, which implies the claim.

Note that by Lemma 5.4, the set B is a $\mathbb{P}$-null set, i.e.,

$$\mathbb{P}\big(\big\{\boldsymbol{\omega}^i : \boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\}\big\}\big) = 0. \tag{5.15}$$

This implies (5.14) and concludes the proof of Proposition 5.2 as soon as the following lemma is verified.

Lemma 5.5. *Under the assumptions of Proposition 4.2, for any $i \geq 3$, $\mathbb{P}(\{L \geq 1\} \cap \{I = i\}) > 0$ implies $\mathbb{P}\big(\big\{\boldsymbol{\omega}^i : \boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\}\big\}\big) > 0$.*

By Lemma 5.5—where we show that if the collection of thinned configurations is contained in a $\mathbb{P}$-null set, then the non-thinned configurations also form a $\mathbb{P}$-null set—we see that (5.15) implies (5.14), which concludes the proof of Proposition 5.2.

Proof of Lemma 5.5. Let us fix $i \geq 3$ and assume that $\mathbb{P}(\{L \geq 1\} \cap \{I = i\}) > 0$. Then, by continuity of measures, there exists $K > 0$ such that

$$\mathbb{P}\big(\big\{\boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\} : \mathrm{V}_j(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o),\ \forall j \in \{1, \ldots, i\}\big\}\big) > 0,$$

where $B_K(o)$ denotes the open Euclidean ball of radius K in $\mathbb{R}^d$. Hence, there exists $n \geq i$ such that $\mathbb{P}(C_{i,K,n}) > 0$, where

$$C_{i,K,n} = \big\{\boldsymbol{\omega} \in \{L \geq 1\} \cap \{I = i\} : \#\big(\omega \cap B_K(o)\big) = n + 1$$
$$\text{and } \mathrm{V}_j(Z(\boldsymbol{\omega}), \boldsymbol{\omega}) \in B_K(o), \forall j \in \{1, \ldots, i\}\big\}.$$

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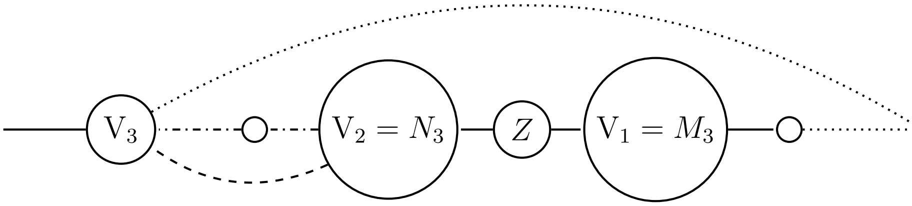

FIGURE 3. A visualization of the case $I(\omega) = 3$ for some realization $\omega \in \{L \geq 1\}$. $V_3 = V_3(Z(\omega), \omega)$ is contained in the infinite cluster $\mathcal{C}_0 = \mathcal{C}_0(\omega)$ of the SINR graph $G_\gamma(\omega)$ including $Z = Z(\omega)$, and it is not a neighbor of $M_3 = M_3(\omega)$, which in this example equals $V_1 = V_1(Z(\omega), \omega)$, while $V_2 = V_2(Z(\omega), \omega) = N_3 = N_3(\omega)$. Hence, if V_3 has degree two in $\mathcal{C}_0$, then there are various possibilities respecting the degree bound of two to connect V_3 to $\mathcal{C}_0$ so that it is not connected to M_3 by an edge. V_3 can be either a direct neighbor of V_2 (dashed line), or a later point of the path from Z to infinity starting with the edge from Z to V_2 (dash-dotted lines), or a non-direct neighbor of V_1 on the path from Z to infinity starting with the edge from Z to V_1 (dotted lines). Now, if M_3 is removed from the realization, both edges adjacent to V_3 are preserved. Also, all edges from Z to infinity starting with the edge from Z to V_2 are preserved, so that Z is still contained in an infinite cluster, but the edge from Z to V_1 is removed. In the resulting configuration, the second-nearest neighbor of Z in signal-weighted order is V_3, and hence this is the only point of the configuration that could be connected to Z by an edge. But V_3 still cannot have degree 3 or more, so it cannot be connected to Z, which implies that in the new configuration Z is in an infinite cluster containing a point of degree one.

Conditional on the event $C_{i,K,n}$, the marked point process $(\mathbf{X}^\lambda \setminus \{\mathbf{Z}\}) \cap B_K(o)$ has precisely n points.

Now, for some fixed $q \in (0, 1)$, we can represent $\mathbf{X}^\lambda$ as $\mathbf{X}^{\lambda,1} \cup \mathbf{X}^{\lambda,2}$ as follows. For $K > 0$, let $B_K(o)$ denote the open ℓ^2-ball of radius K around o. Let $\mathbf{X}^{\lambda,1}$ be given as the union of $\mathbf{X}^\lambda \setminus (B_K(o) \times [0, \infty))$ and the independent thinning of $\mathbf{X}^\lambda \cap (B_K(o) \times [0, \infty))$ with survival probability q, and let $\mathbf{X}^{\lambda,2}$ be the complementary thinning. That is, conditional on $\mathbf{X}^\lambda$, $\mathbf{X}^{\lambda,1} \cap (B_K(o) \times [0, \infty))$ contains each point of $\mathbf{X}^\lambda \cap (B_K(o) \times [0, \infty))$ with probability q independent of the other points of this point process, and it contains no other points. Note further that $\mathbf{X}^{\lambda,2}$ and $\mathbf{X}^{\lambda,1} \cap (B_K(o) \times [0, \infty))$ are independent thinnings of $\mathbf{X}^\lambda \cap (B_K(o) \times [0, \infty))$ with survival probabilities $1 - q$ and q, respectively; moreover, $\mathbf{X}^{\lambda,1} = \mathbf{X}^\lambda \setminus \mathbf{X}^{\lambda,2}$, $\mathbf{X}^{\lambda,2} \setminus (B_K(o) \times [0, \infty)) = \varnothing$, and $\mathbf{X}^{\lambda,1} \setminus (B_K(o) \times [0, \infty)) = \mathbf{X}^\lambda \setminus (B_K(o) \times [0, \infty))$. In order to provide a precise construction of the thinned processes, we choose a sequence $(J_m)_{m\in\mathbb{N}}$ of i.i.d. Bernoulli random variables with parameter q that is independent of $\mathbf{X}^\lambda$, and given the realization $\omega = \mathbf{X}^\lambda(\omega) = (\mathbf{V}_i(Z(\omega), \omega))_{i\in\mathbb{N}_0}$, the realizations of $\mathbf{X}^{\lambda,1}(\omega)$ and $\mathbf{X}^{\lambda,2}(\omega)$ are defined as follows, depending also on $(J_m)_{m\in\mathbb{N}}$:

$$\mathbf{X}^{\lambda,1}(\omega) = \mathbf{X}^{\lambda,1}(\omega, (J_m)_{m\in\mathbb{N}}) = \{\mathbf{V}_m(Z(\omega), \omega) \colon J_m = 1, \mathbf{V}_m(Z(\omega), \omega) \in B_K(o)\}$$
$$\cup \{\mathbf{Z}(\omega)\} \cup \{\mathbf{V}_m(Z(\omega), \omega) \colon \mathbf{V}_m(Z(\omega), \omega) \in \mathbb{R}^d \setminus B_K(o)\}$$

and

$$\mathbf{X}^{\lambda,2}(\omega) = \mathbf{X}^{\lambda,2}(\omega, (J_m)_{m\in\mathbb{N}}) = \{\mathbf{V}_m(Z(\omega), \omega) \colon J_m = 0, \mathbf{V}_m(Z(\omega), \omega) \in B_K(o)\}.$$

It is clear that the respective projections $X^{\lambda,1}$ and $X^{\lambda,2}$ of $\mathbf{X}^{\lambda,1}$ and $\mathbf{X}^{\lambda,2}$ to the $\mathbb{R}^d$-coordinate are nonequidistant; furthermore, $\mathbf{X}^{\lambda,1}$ can be represented as a random variable with values in

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$\mathbf{N}$, defined on an enlarged probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ governing both the point process $\mathbf{X}^\lambda$ and the sequence $(J_m)_{m\in\mathbb{N}}$. In particular, $\mathbb{P}'(\mathbf{X}^\lambda \in \cdot) = \mathbb{P}(\mathbf{X}^\lambda \in \cdot)$.

The next property of stationary and nonequidistant CPPs is crucial for completing the proof of Lemma 5.5.

Lemma 5.6. *Let Λ be stationary and nonequidistant. Then, for any $K > 0$, the law of $\mathbf{X}^{\lambda,1}$ is absolutely continuous with respect to that of $\mathbf{X}^\lambda$.*

To be more precise, the absolute continuity is meant in this lemma in the following way, with respect to the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ on which $\mathbf{X}^{\lambda,1}$ and $\mathbf{X}^\lambda$ are jointly defined with $\mathbb{P}'(\mathbf{X}^\lambda \in \cdot) = \mathbb{P}(\mathbf{X}^\lambda \in \cdot)$. Let $G \in \mathcal{F}'$ be any event such that $\mathbb{P}'(\mathbf{X}^{\lambda,1} \in G) > 0$; then we have $\mathbb{P}'(\mathbf{X}^\lambda \in G) > 0$.

Proof of Lemma 5.6. Let F be an element of the evaluation σ-algebra of $\mathbf{N}$ such that $\mathbb{P}'(\mathbf{X}^{\lambda,1} \in F) > 0$. We have to show that $\mathbb{P}(\mathbf{X}^\lambda \in F) > 0$ as well. Under the assumption that $\mathbb{P}'(\mathbf{X}^{\lambda,1} \in F) > 0$, by continuity of measures, we can find $K, l \in \mathbb{N}$ such that

$$\varepsilon := \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in F, \#\big(\mathbf{X}^{\lambda,1} \cap (B_K(o) \times [0, \infty))\big) = l\big) > 0. \tag{5.16}$$

In other words, we have $0 < \varepsilon = \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G\big)$, where $G = \{\boldsymbol{\omega} \in F : \#(\boldsymbol{\omega} \cap (B_K(o) \times [0, \infty))) = l\}$. Thus,

$$\begin{aligned}\mathbb{P}\big(\mathbf{X}^\lambda \in F\big) &\geq \mathbb{P}'\big(\mathbf{X}^\lambda \in G, \mathbf{X}^{\lambda,1} = \mathbf{X}\big) \geq \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G\big)\mathbb{P}'\big(\mathbf{X}^{\lambda,1} = \mathbf{X}^\lambda | \mathbf{X}^{\lambda,1} \in G\big)\\ &= \varepsilon \mathbb{P}'\big(\mathbf{X}^{\lambda,1} = \mathbf{X}^\lambda | \mathbf{X}^{\lambda,1} \in G\big),\end{aligned}$$

and further,

$$\mathbb{P}'\big(\mathbf{X}^{\lambda,1} = \mathbf{X}^\lambda | \mathbf{X}^{\lambda,1} \in G\big) \geq \mathbb{P}'\big(\mathbf{X}^{\lambda,1} = \mathbf{X}, \mathbf{X}^{\lambda,1} \in G\big) = \mathbb{P}'\big(\mathbf{X}^{\lambda,2} = \emptyset, \mathbf{X}^{\lambda,1} \in G\big). \tag{5.17}$$

According to (5.16), we have

$$0 < \varepsilon = \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G\big) = \sum_{n=0}^{\infty} a_n,$$

where $a_n = \mathbb{E}'\big[\mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G|\Lambda\big)\mathbb{1}\{\Lambda(B_K(o)) \in [n, n+1)\}\big]$, and thus there exists $m \in \mathbb{N}_0$ with $a_m > 0$. Now, conditional on Λ, $\mathbf{X}^{\lambda,1}$ is an i.i.d. marked PPP, and hence a PPP on $\mathbb{R}^d \times [0, \infty)$, which also implies that the complementary thinnings $\mathbf{X}^{\lambda,1}$ and $\mathbf{X}^{\lambda,2}$ are independent given Λ; see the Colouring Theorem and the Marking Theorem in [20]. Hence, we obtain

$$\begin{aligned}\mathbb{P}'\big(\mathbf{X}^{\lambda,2} = \emptyset, \mathbf{X}^{\lambda,1} \in G\big) &= \mathbb{E}'\big[\mathbb{P}'\big(\mathbf{X}^{\lambda,2} = \emptyset|\Lambda\big)\mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G|\Lambda\big)\big]\\ &= \sum_{n=0}^{\infty} \mathbb{E}'\big[\varepsilon^{-(1-q)\Lambda(B_K(o))}\mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in G|\Lambda\big)\mathbb{1}\{\Lambda(B_K(o)) \in [n, n+1)\}\big]\\ &\geq \sum_{n=0}^{\infty} \varepsilon^{-(1-q)(n+1)} a_n \geq \varepsilon^{-(1-q)(m+1)} a_m > 0,\end{aligned}$$

which verifies the lemma that the distribution of $\mathbf{X}^{\lambda,1}$ is absolutely continuous with respect to that of $\mathbf{X}^\lambda$. □

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Given Lemma 5.6, we now finish the proof of Lemma 5.5. Thanks to the assumption that $\mathbb{P}(C_{i,K,n}) > 0$ and using the definition of $\mathbf{X}^{\lambda,1}$,

$$\begin{aligned}\mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in \{\omega^i : \omega \in \{L \geq 1\} \cap \{I = i\}\}\big) &\geq \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in \{\omega^i : \omega \in C_{i,K,n}\}\big) \\ &\geq \mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in \{\omega^i : \omega \in C_{i,K,n}\}, \mathbf{X}^{\lambda} \in C_{i,K,n}\big) \\ &= \mathbb{P}(C_{i,K,n})\mathbb{P}'\big(\mathbf{X}^{\lambda,1} \in \{\omega^i : \omega \in C_{i,K,n}\} | \mathbf{X}^{\lambda} \in C_{i,K,n}\big) \\ &= \mathbb{P}(C_{i,K,n}) q^{n-i+2}(1-q)^{i-2} > 0.\end{aligned} \tag{5.18}$$

Finally, by Lemma 5.6, under $\mathbb{P}'$ the distribution of $\mathbf{X}^{\lambda,1}$ is absolutely continuous with respect to that of $\mathbf{X}^{\lambda}$. Hence, it follows from (5.18) that

$$\mathbb{P}\big(\mathbf{X}^{\lambda} \in \{\omega^i : \omega \in \{L \geq 1\} \cap \{I = i\}\}\big) > 0,$$

which implies the lemma.

5.3. Proof of Theorem 2.3

This proof is similar to that of Theorem 2.1, Condition 1, but simpler. The new proof ingredient that we use here is the strong connectivity of *any* supercritical Poisson Boolean model [28, Theorems 2 and 5] in the case of $d \geq 2$, which allows us to improve the result that $\lambda_c^* < \infty$ to $\lambda_c^* = \lambda_c(r_B)$. First we introduce an adequate discrete percolation model and then we control the interferences.

Throughout the proof, $X^{\lambda} = \{x_i\}_{i \in \mathbb{N}}$ denotes a homogeneous PPP with intensity λ in $\mathbb{R}^d$, and we write X^{λ} instead of $\mathbf{X}^{\lambda}$ since marks are non-random. Let us introduce the notion and elementary properties of Boolean models with (constant) radius $r > 0$. The *Poisson Boolean model* $B(X^{\lambda}, r)$ (with constant connection radii r) is defined as

$$B(X^{\lambda}, r) = \bigcup_{i \in \mathbb{N}} B_r(x_i) = X^{\lambda} \oplus B_r(o).$$

Connecting any two different points $x_i, x_j \in X^{\lambda}$ by an edge whenever

$$|x_i - x_j| < 2r, \tag{5.19}$$

we obtain the Poisson–Gilbert graph $g_{2r}(X^{\lambda})$ with connection radius $2r$. Percolation in this Gilbert graph is equivalent to the existence of an unbounded connected component in $B(X^{\lambda}, r)$, which we also refer to as percolation. Thus, one can speak about subcritical, critical, and supercritical Poisson Boolean models.

Recall the definition of the radius r_B from (2.4), and let us fix $\lambda > \lambda_c(r_B)$ for the remainder of this section. Thanks to scale invariance of Poisson Boolean models [25, Section 2.2] and the well-behavedness of ℓ, we can fix $r \in (d_o, r_B)$ such that the Poisson Boolean model $B(X^{\lambda}, r/2)$ associated to $g_r(X^{\lambda})$ is still supercritical. The next lemma is an immediate consequence of the results in [26, Section 1].

Lemma 5.7. ([26].) *Let $B(X^{\lambda}, r/2)$ be a supercritical Poisson Boolean model and let $x \in \mathbb{R}^d$. With probability tending to one as $n \uparrow \infty$, the following two statements hold:*

1. *$B(X^{\lambda}, r/2) \cap Q_n(x)$ contains a connected component of diameter at least $n/3$.*

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2. *Any two connected components of* $B(X^\lambda, r/2) \cap Q_n(x)$ *of diameter at least* $n/9$ *each are contained in the same connected component of* $B(X^\lambda, r/2) \cap Q_{2n}(x)$.

Using Lemma 5.7, we construct a renormalized percolation process on $\mathbb{Z}^d$. For $z \in \mathbb{Z}^d$, let $\Xi_n(z)$ denote the union of all connected components of $B(X^\lambda, r/2) \cap Q_n(z)$ that are of diameter at least $n/3$. For $n \geq 1$, we say that the site $z \in \mathbb{Z}^d$ is *n-good* if

i. $\Xi_n(nz) \neq \emptyset$, and

ii. for any $z' \in \mathbb{Z}^d$ with $|z - z'|_\infty \leq 1$, it holds that all pairs of connected components C of $\Xi_n(nz)$ and C' of $\Xi_n(nz')$ are contained in the same connected component of $B(X^\lambda, r/2) \cap Q_{6n}(nz)$.

The site $z \in \mathbb{Z}^d$ is *n-bad* if z is not n-good. We have the following lemma.

Lemma 5.8. *Under the assumptions of Theorem 2.3, for all* $n \geq 1$ *sufficiently large, there exists* $q_A = q_A(\lambda, n) \in (0, 1)$ *such that for any* $N \in \mathbb{N}$ *and pairwise distinct* $z_1, \ldots, z_N \in \mathbb{Z}^d$ *we have*

$$\mathbb{P}(z_1, \ldots, z_N \text{ are all } n\text{-bad}) \leq q_A^N.$$

Furthermore, for any $\varepsilon > 0$*, for all large enough* n *one can choose* q_A *such that* $q_A < \varepsilon$.

Proof. For $z \in \mathbb{Z}^d$, $\mathbb{1}\{z \text{ is } n\text{-good}\}$ is measurable with respect to $X^\lambda \cap (Q_{6n}(nz) \oplus B_{r/2}(o))$, which is contained in $X^\lambda \cap Q_{7n}(nz)$ for all n large enough; hence for all sufficiently large n the process of n-good sites is 7-dependent thanks to the independence property of the PPP X^λ. Hence, by a standard argument (using dependent percolation theory [23], as in the proof of Lemma 5.2), it suffices to verify that

$$\limsup_{n\uparrow\infty} \mathbb{P}(o \text{ is } n\text{-bad}) = 0. \tag{5.20}$$

The limit (5.20) can be verified along the lines of the proof of [16, Theorem 2.6] using an adequate interpretation of the Poisson Boolean model. More precisely, in view of Definition 3.2, the assertion of Lemma 5.7 is equivalent to the statement in [16, Section 2.1] that the (for all sufficiently large $b > 0$) b-dependent directing random measure Λ given as $\Lambda(\mathrm{d}x) = \lambda_1 \mathbb{1}\{x \in B(X^\lambda, r/2)\}\mathrm{d}x$ is asymptotically essentially connected, where $\lambda_1 > 0$ is such that $\mathbb{E}[\Lambda(Q_1)] = 1$. □

The other essential proof ingredient is the interference control. We recall the 'shifted' path-loss functions ℓ_a (5.5) and the shot-noise processes $I_a(x)$, $I(x)$ from Section 5.1, and also that by the triangle inequality, for $a \geq 0$, $I(x) \leq I_a(z)$ holds for any $z \in \mathbb{R}^d$ and $x \in Q_a(z)$.

For $n \geq 1$ and $M > 0$, we say that $z \in \mathbb{Z}^d$ is *(n,M)-tame* if $I_{7n}(nz) \leq M$ and *(n,M)-wild* otherwise. Then we have the following assertion, which holds for all λ such that $B(X^\lambda, r/2)$ is supercritical.

Lemma 5.9. ([28].) *Under the assumptions of Theorem 2.3, for fixed* $n \geq 1$*, for all sufficiently large* $M > 0$*, there exists* $q_B = q_B(\lambda, n, M) \in (0, 1)$ *such that for any* $N \in \mathbb{N}$ *and pairwise distinct* $z_1, \ldots, z_N \in \mathbb{Z}^d$ *we have*

$$\mathbb{P}(z_1, \ldots, z_N \text{ are all } (n, M)\text{-wild}) \leq q_B^N.$$

Furthermore, for $\varepsilon > 0$*, for any* $n \geq 1$*, for all sufficiently large* M *one can choose* q_B *such that* $q_B < \varepsilon$.

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Proof. Clearly, the Lebesgue measure Λ is asymptotically essentially connected and b-dependent for any $b > 0$, and $\Lambda(Q_1)$ has all exponential moments. Hence the lemma can be proven very similarly to [28, Proposition 3.3] under the condition (2b) in [28, Theorem 2.4]. The only difference is that in [28], $I_{6n}(nz)$ was considered instead of $I_{7n}(nz)$, but this makes no qualitative difference for the proof. Furthermore, the additional condition in [28, Theorem 2.4] that $\ell(0) \leq 1$ can be assumed to hold without loss of generality, for the same reason as in the proof of Proposition 5.1 (see the beginning of Step 4 in the proof of Theorem 2.1). □

Equipped with these results, we can now prove our main theorem.

Proof of Theorem 2.3. For $n \geq 1$ and $M > 0$, we say that the site $z \in \mathbb{Z}^d$ is (n, M)-*nice* if it is both n-good and (n, M)-tame. We claim that for all sufficiently large n and accordingly chosen large enough M, the process of (n, M)-nice sites percolates. Indeed, this follows from combining the estimates of Lemmas 5.7 and 5.9, similarly to Corollary 5.1, and carrying out a Peierls argument.

We claim that this assertion implies percolation in $G_\gamma(X^\lambda)$ for small $\gamma > 0$. Indeed, let n, M be so large that the process of (n, M)-nice sites percolates, and such that $Q_{6n}(o) \oplus B_{r/2}(o) \subseteq Q_{7n}(o)$. Using a standard argument (cf. [9] or Step 3 in the proof of Theorem 2.1, Condition 1), one can choose $\gamma > 0$ sufficiently small so that for any (n, M)-tame site z, all connections in $g_r(X^\lambda) \cap Q_{7n}(nz)$ also exist in $G_\gamma(X^\lambda) \cap Q_{7n}(nz)$.

Now, analogously to [16, Section 5.2], we can argue as follows. Let $\mathcal{C}$ be an infinite connected component of the process of sites that are (n, M)-nice. Let $z, z' \in \mathcal{C}$ and let $\{z_0 = z, z_1, \ldots, z_{k-1}, z_k = z'\}$ be a path in $\mathcal{C}$ connecting z and z'. Then, thanks to n-goodness, for any $j = 0, \ldots, k$ and for any $x_j \in X^\lambda$ such that $B_{r/2}(x_j) \cap Q_n(nz_j) \subseteq \Xi_n(nz_j)$ we have that x_j and x_{j+1} are in the same connected component of $B(X^\lambda, r/2) \cap Q_{6n}(nz_j)$. In other words, x_j and x_{j+1} are connected in the Poisson–Gilbert graph $g_r(X^\lambda)$ via a path in $Q_{7n}(nz_j)$, where the additional unit of n comes from the fact that the centers of balls in the Boolean model might lie in a neighboring box. Hence, using (n, M)-tameness, we conclude that all edges of this path in $g_r(X^\lambda)$ also exist in $G_\gamma(X^\lambda)$. Thus, $G_\gamma(X^\lambda)$ also percolates. Since $\lambda > \lambda_c(r_B)$ was arbitrary, the theorem follows. □

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