MEAN ELECTROMOTIVE FORCE DUE TO MAGNETOCONVECTION IN ROTATING HORIZONTAL LAYER IN DEPENDENCE ON BOUNDARY CONDITIONS

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Abstract. The instability due to a vertical uniform temperature gradient is studied in a rapidly rotating horizontal layer of an electrically conducting fluid permeated by an azimuthal magnetic field linearly growing with the distance from the vertical axis of rotation. In addition to the boundary conditions used in Soward's study (1979), that is, force-free surface and perfect electrical and thermal conductivity outside, also other conditions more realistic for the Earth's core are considered, that is, rigid surface and electrically insulating walls. Using the concept of mean-field mhd mean electromotive and ponderomotive forces (E.M.F. and P.M.F.) are calculated and compared for various boundary conditions. The dependence of the E.M.F. and P.M.F. on the electromagnetic boundary conditions is strong (slight) if the boundaries are free (rigid).

Key words: Hydromagnetic instabilities, mean electromotive force, Earth's core

1. Introduction

The thermally and magnetically driven convection resulting from the instabilities is studied in various MHD-models. The model (Soward(1979), Ševčík(1989)) of electrically conducting Boussinesq rapidly rotating fluid layer with azimuthal magnetic field $\mathbf{B}_o = B_o(s/d)\,\hat{\varphi}$ and with temperature difference ΔT between perfectly thermally conducting boundaries is the basis of our problem. We consider various boundary conditions which better fit Earth's core conditions.

In Mean Field MHD various kinds of α-effect and other generation effects are usually apriori given and their consequences are studied, Krause, Rädler (1980), Brandenburg et al.(1989). It is useful to have information about the E.M.F. computed (and not apriori given) from the instabilities of the hydromagnetic model, Brestenský, Rädler (1989).

Our aim is therefore to study the dependence of the resulting E.M.F. (and also P.M.F.) on various boundary conditions of the investigated model.

2. Governing equations and method of solution

Governing linearized dimensionless equations in the magnetostrophic approximation for the perturbations of the velocity (w, ω), magnetic field (b, j) and temperature ϑ have a form

$$0 = -D\,w + 2\Lambda\,Db - im\Lambda\,j\,, \tag{1}$$

$$0 = -D\omega + 2\Lambda\,Dj + im\,(D^2 - k^2)\,b - Rk^2\,\vartheta\,, \tag{2}$$

$$\lambda\,b = im\,w + (D^2 - k^2)\,b\,, \tag{3}$$

$$\lambda\,j = im\,\omega + (D^2 - k^2)\,j\,, \tag{4}$$

$$q^{-1}\,\lambda\,\vartheta = w + (D^2 - k^2)\,\vartheta\,, \tag{5}$$

where the velocity and magnetic field

F. Krause et al. (eds.), The Cosmic Dynamo, 457–461.

$$\mathbf{u} = k^{-2}[\nabla\times(\nabla\times\tilde{w}\hat{\mathbf{z}}) + \nabla\times\tilde{\omega}\hat{\mathbf{z}}], \quad \mathbf{b} = k^{-2}[\nabla\times(\nabla\times\tilde{b}\hat{\mathbf{z}}) + \nabla\times\tilde{j}\hat{\mathbf{z}}], \tag{6}$$

are defined by unknown functions $(\tilde{w}, \tilde{\omega}, \tilde{b}, \tilde{j}, \tilde{\vartheta})$ in the form $\tilde{f}(z,s,\varphi,t) = \Re e\{f(z)J_m(ks)\exp(im\varphi + \lambda t)\}$, where $J_m(k\,s)$ is Bessel function, integer m, real k are azimuthal and radial wave numbers, λ is complex frequency. In the equations (1-5) $D = d/dz$, $q = \kappa/\eta$ (Roberts number), κ, η are thermal and magnetic diffusivities, d is width of the layer, Ω_0 is magnitude of angular velocity, $\Lambda = B_o^2\,/\,(2\Omega_0\mu\rho_c\eta)$ is Elsasser number, $R = g\,\alpha_T\,\Delta T\,d\,/\,(2\Omega_0\kappa)$ is Rayleigh number, α_T is the coefficient of thermal expansion and g is the acceleration due to gravity.

The set of equations (1-5) was solved for 6 combinations of boundary conditions corresponding to the electrical conductivities $\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}$ of the $\frac{\text{upper}}{\text{lower}}$ boundaries combined with both free FB or rigid RB boundaries[1]. In all cases both boundaries are perfectly thermally conductive. The mechanical boundary conditions for FB (RB) are

$$w = D^2 w = D\omega = 0\,, \qquad (w = D\,w = \omega = 0). \tag{7}$$

The electromagnetic boundary conditions for insulating (perfectly conductive) boundary are

$$j = 0\,, \qquad (b = 0). \tag{8}$$

In 4 various investigated kinds of boundaries, i.e. in FB0, FB∞, RB0, RB∞ the next electromagnetic conditions must be fulfilled $2\,D\,j - m\lambda\,b = 0$, $D\,b = 0$, $D\,b = 0$, $D\,j = 0$, respectively. The last ones are the consequences of the basic boundary conditions (7,8) and/or $\vartheta = 0$, and of the equations (1-5).

Following Soward's paper (1979) for the FB case and Chandrasekhar's method (1968) for the RB case the problem (1-5) can be reduced to the eigenvalue problem

$$\mathbf{A}\,\mathbf{v} = R\,\mathbf{v} \tag{9}$$

where $\mathbf{A}$ is complex matrix, R is Rayleigh number as eigenvalue and $\mathbf{v}$ is a set of coefficients $\{v_l\}$ as an eigenvector. The function $w(z)$ is given by the serie $w(z) = \sum v_l\,f_l\,(z)$, where $f_l(z)$ are the base functions satisfying conditions on the boundaries. The eigenvalue problem (9) was formulated for overstable marginal modes, $\Re e(\lambda) = 0$, the series were truncated for $l = 5$. Further functions $(\omega(z), b(z), j(z))$ were computed from $w(z)$ utilizing the boundary conditions.

For the set of parameters $(\Lambda,\, q,\, m,\, k)$ and for various values of frequency λ the eigenvalues $R(\lambda)$ with $\Im m(R) = 0$ and the smallest $|\Re e(R)|$ were found. Then we have found, by minimisation of $|\Re e(R)|$ over radial wave number k, critical values R_c, k_c, λ_c and resulting eigenvector $\{v_l\}$. R_c and $\{v_l\}$ $[f_l(z)]$ depend only on the mechanical [electromagnetic] boundary conditions.

The dimensionless[2] E.M.F. and P.M.F. related to Maxwell stresses, i.e.

$$\mathcal{E} = \langle\,\mathbf{u}\times\mathbf{b}\,\rangle, \quad \mathcal{P}_M = \frac{1}{\mu}\,\mathrm{div}\langle\,\mathbf{bb} - \frac{1}{2}\mathbf{b}^2\mathbf{I}\rangle, \ (\mathbf{I} \text{ is unit tensor}) \tag{10}$$

are computed by averaging over φ in the ranges of $z \in \langle\,0,1\,\rangle$ and $s \in \langle\,0, j_x/k_c\,\rangle$. The quantity $j_x = j(1,3)$, $[j(m,2)]$ for $m = 1$, $[m > 1]$, where $j(m,r)\,(r = 2,\,3)$ is the $r-$th positive root of the equation $J_m(x) = 0$. $\mathcal{E}_s = \mathcal{E}_\varphi = 0$ on the chosen right $s-$boundary. Mean square velocity $\langle\,\mathbf{u}^2\,\rangle$ by which the E.M.F. is normalized, was computed by averaging over all coordinates z, s, φ in the determined ranges.

[1] In the whole following text we use abbrevations FB, RB for free, rigid boundaries; 0, ∞ for insulating, perfectly conductive boundary and $\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}$ for the cases of investigated possibilities of electrical conductivity of the $\frac{\text{upper}}{\text{lower}}$ boundaries.

[2] $\tilde{\mathcal{E}} = (B_o\eta/d)\,\mathcal{E}$, $\tilde{\mathcal{P}}_M = \Lambda\,\mathcal{P}_M$, where tilded quantities are dimensional ones.

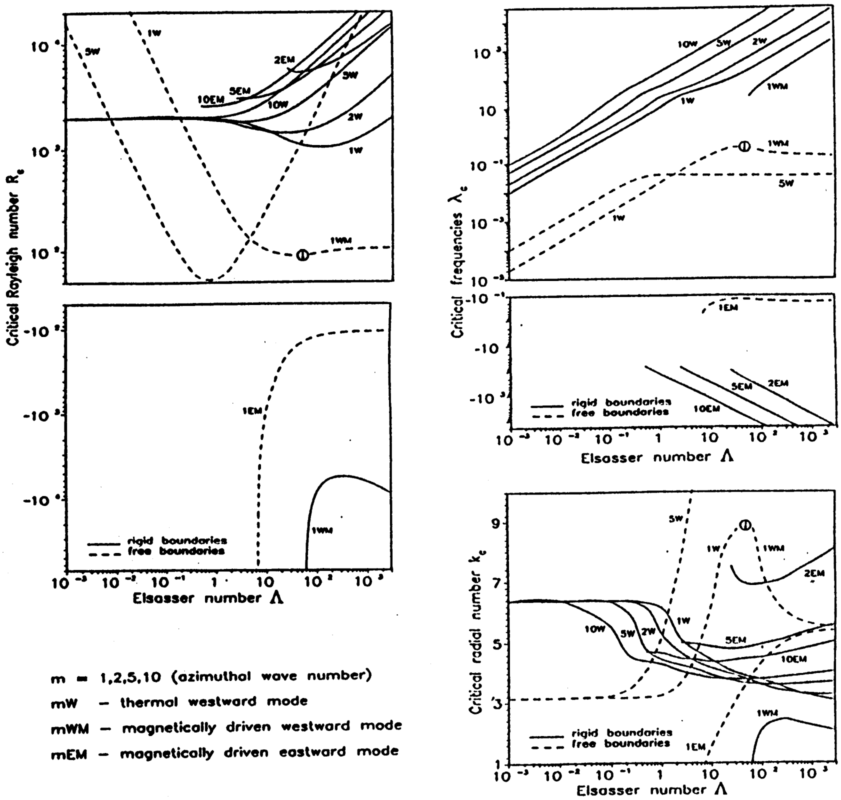

Fig. 1. Dependence of critical Rayleigh number R_c, frequency λ_c and radial wave number k_c on Elsasser number Λ for RB and FB cases. The mode $m = 1$ (for FB and $R > 0$) is thermally driven for the values of Λ smaller than $\Lambda \sim 50.$ and magnetically driven for values greater than $\Lambda \sim 50.$ (marked by symbol $\oslash$ which is used in all further figures).

3. Numerical results and conclusions

Thermal T and magnetically driven modes M were found for both kinds of mechanical boundaries FB, RB (Fig.1). The curves for FB are the same as Soward's results (1979). T-modes are generated only for unstable stratification ($R_c > 0$) and they propagate westward. M-modes (catalysed by buoyancy) require sufficiently strong magnetic field. They propagate for unstable stratification westward (eastward) for FB (RB) and for stable stratification eastward (westward) for FB (RB). The preferred convection is generated more easily for FB than for RB for all values of Λ. For the case of FB the smaller Λ the greater azimuthal wave number m for the preferred modes whereas for RB the modes $m \geq 1$ are generated almost for the same R_c. Critical $R_c \sim O(\Lambda)$ for strong magnetic field except for $m = 1$ for FB (Soward,1979). Frequencies λ_c are greater for RB and they are increasing with Λ. Radial wave numbers k_c are $O(1)$ for RB also for T-modes and $m > 1$. It is different from the case of FB where k_c is increasing with Λ ($m > 1$).

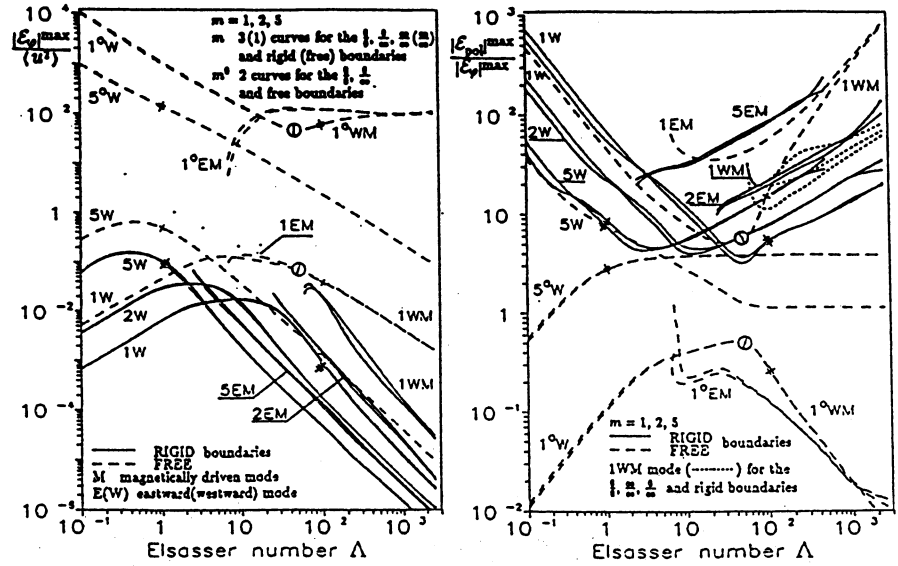

Fig. 2. Dependence of the $|\mathcal{E}_\varphi|^{max}$ and the poloidal component $|\mathcal{E}_{pol}|^{max} = (\sqrt{\mathcal{E}_z^2 + \mathcal{E}_\vartheta^2})^{max}$ related to the $|\mathcal{E}_\varphi|^{max}$ on Λ for T and M-modes of various azimuthal wave numbers m in 6 combinations of boundary conditions. The symbols + ++ +++ ++++ (for 1, 2, 3, 4 curves, respectively) in abscissas of $\Lambda = 1.$, 100. show the modes for which the space distribution of the E.M.F. is depicted in fig.3.

In figures we present results related only to the E.M.F. The dependence of the E.M.F.-components on Λ is in fig.2. The space distribution of the E.M.F. is for some examples of modes in fig.3.

We can make the following conclusions.
The dependence of all components of the E.M.F. and P.M.F. is strong on the electromagnetic boundary conditions if the boundaries are free, FB. For RB this dependence is slight.

$\mathcal{E}_\varphi$ normalized by $\langle \mathbf{u}^2 \rangle$ is much more greater in the FB cases than in RB ones in all cases of electrical conductivity ($\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{0}{\infty}$) and especially when at least one of the boundaries is electric isolant ($\frac{0}{0}$, $\frac{0}{\infty}$).

The poloidal component of the E.M.F. strongly prevails the φ−component especially in the RB case. Non conductive boundaries $\frac{0}{0}$, $\frac{0}{\infty}$ support dominancy of the φ-component of the E.M.F. for FB.

The space distribution of the $\mathcal{E}_\varphi$−sign near rotation axis is also very sensitive on mechanical boundary conditions and on stratification of the layer, too. When whole layer is unstably stratified the $\mathcal{E}_\varphi$−sign distribution is $\pm$ ($\frac{\mp}{\mp}$) for FB (RB). When whole layer is stably stratified and boundaries are FB$\frac{0}{0}$ or FB$\frac{0}{\infty}$ the $\mathcal{E}_\varphi$−sign distribution is $\mp$ as was determined for the case when the layer is non-uniformly stratified, i.e. when the upper (lower) sublayer is stably (unstably) stratified; Brestenský, Rädler(1989), Brestenský(1991), Brestenský, Rädler, Fuchs(1992).

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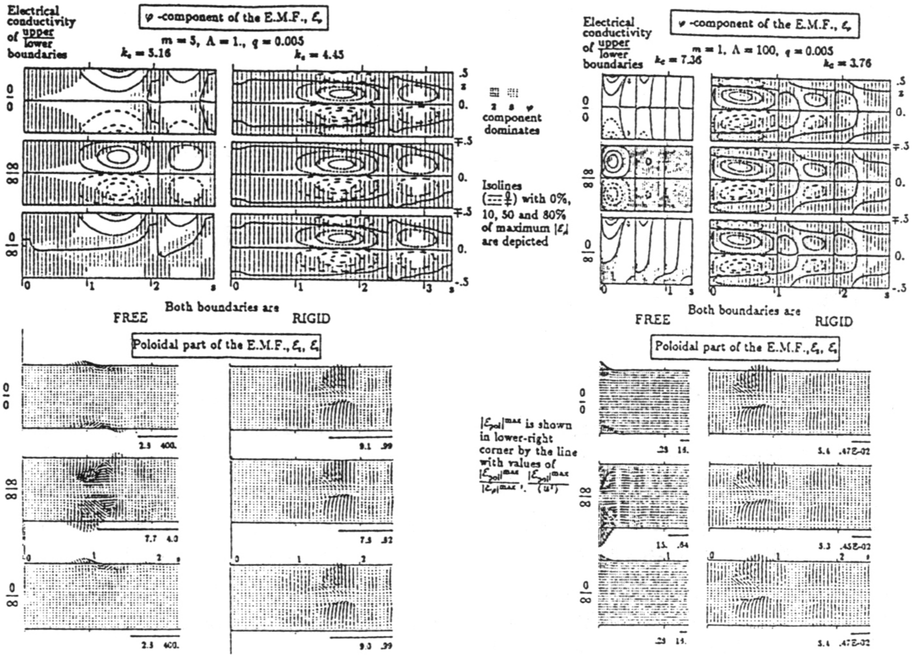

Fig. 3. The space distribution of the E.M.F for T-modes [FB, RB ($m = 5, \Lambda = 1.$); RB ($m = 1, \Lambda = 100.$)] and for M-modes [FB ($m = 1, \Lambda = 100.$)] for all investigated cases of boundary conditions. The space distribution for T-modes [FB ($m = 1, \Lambda < 50.$)] is very similar to depicted case of M-modes.

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